Hamburger Beiträge
zur Angewandten Mathematik

A note on the approximation of elliptic control problems with bang-bang controls

Klaus Deckelnick and Michael Hinze

Nr. 2009-10
June 2009

A note on the approximation of elliptic control problems with bang-bang controls

Klaus Deckelnick*& Michael Hinze†

Abstract: In the present work we use the variational approach in order to discretize elliptic optimal control problems with bang-bang controls. We prove error estimates for the resulting scheme and present a numerical example which supports our analytical findings.

Mathematics Subject Classification (2000): 49J20, 49K20, 35B37

Keywords: Elliptic optimal control problem, control constraints, bang-bang controls, error estimates

1 Introduction

In this note we consider an elliptic optimal control problem subject to pointwise control constraints. In many cases the underlying cost functional is chosen to be of tracking type, i.e.

$$J(y,v) = \frac{1}{2}\int_\Omega |y-y_0|^2 + \frac{\alpha}{2}\int_\Omega v^2, \tag{1.1}$$

where $y:\Omega\to\mathbb{R}$ is the solution of the state equation and $v:\Omega\to\mathbb{R}$ denotes the control. Furthermore, $\alpha>0$ and y_0 is a given target function. In situations, in which the cost of the control is negligible or in which the focus primarily lies on tracking y_0 one might prefer to study the control problem obtained from setting $\alpha=0$. To be more specific, let $\Omega\subset\mathbb{R}^d$ ($d=1,2,3$) be a bounded domain. We assume that Ω is either convex and polyhedral or that $\partial\Omega$ belongs to C^2. Then, given $v\in L^2(\Omega)$, the boundary value problem

$$\begin{aligned} -\Delta y &= v \quad \text{in } \Omega \\ y &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

has a unique solution $y\in H^2\cap H^1_0(\Omega)$ which we denote by $\mathcal{G}(v)$. Next, for $a,b\in\mathbb{R}, a<b$ we introduce

$$U_{ad} = \{v\in L^\infty(\Omega)\,|\, a\le v\le b \text{ a.e. in } \Omega\}$$

*Institut für Analysis und Numerik, Otto–von–Guericke–Universität Magdeburg, Universitätsplatz 2, 39106 Magdeburg, Germany

†Schwerpunkt Optimierung und Approximation, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany.

and consider the following optimal control problem:

$$\min_{v \in U_{ad}} J(v) = \frac{1}{2}\int_\Omega |y - y_0|^2 \quad \text{subject to } y = \mathcal{G}(v+f). \tag{1.2}$$

Here, $f \in L^2(\Omega)$ and $y_0 \in L^2(\Omega)$ are given functions.
It is not difficult to establish the existence of a unique solution $u \in U_{ad}$ to this problem which can be characterized as follows:

Theorem 1.1. *A function $u \in U_{ad}$ is a solution of* (1.2) *if and only if there exists an adjoint state p such that $y = \mathcal{G}(u+f), p = \mathcal{G}(y - y_0)$ and*

$$(p, v-u) \geq 0 \quad \textit{for all } v \in U_{ad}. \tag{1.3}$$

Remark 1.2. The relation (1.3) implies that

$$u(x) \begin{cases} = a, & p(x) > 0, \\ \in [a,b], & p(x) = 0, \\ = b, & p(x) < 0. \end{cases} \tag{1.4}$$

For later purposes we make the following assumption on the adjoint state p:

$$\exists\, C \geq 0 \;\; \forall \epsilon > 0 \quad |\{x \in \Omega \,|\, |p(x)| \leq \epsilon\}| \leq C\epsilon. \tag{1.5}$$

Here, $|A|$ denotes the d–dimensional Lebesgue measure of a set A. It follows in particular that the set $\{x \in \Omega \,|\, p(x) = 0\}$ has measure zero so that the control is of bang-bang type. Our aim in this note is to use (1.5) in order to carry out an error analysis for a suitable discretization of problem (1.2). Such an analysis is available for control problems involving functionals of the form (1.1) with $\alpha > 0$, see e.g. [5], Chapter 3.2 and the references therein. However, a closer look at the corresponding arguments shows that the constant in the error estimate blows up as $\alpha \to 0$ so that they cannot be used in order to study the limit problem. Instead we shall pursue a different approach based on an estimate for the L^1–norm between continuous and discrete optimal control. Let us remark that numerical experiments for bang-bang control of elliptic equations are conducted by Maurer and Mittelmann in [6, 7]. Problems of this kind which even involve state constraints also appear in the context of optimization of plates, see [1].
We close this section by presenting a criterion that ensures (1.5).

Lemma 1.3. *Suppose that the adjoint solution p belongs to $C^1(\bar{\Omega})$ and satisfies*

$$\min_{x \in K} |\nabla p(x)| > 0, \qquad \textit{where } K = \{x \in \bar{\Omega} \,|\, p(x) = 0\}. \tag{1.6}$$

Then, (1.5) is satisfied.

Proof. We give a sketch of the proof noting first that it is sufficient to verify (1.5) for small $\epsilon > 0$. Let us introduce for $t \in \mathbb{R}$

$$F_t := \{x \in \bar{\Omega} \,|\, p(x) = t\}.$$

Using (1.6) together with a continuity argument we can show that there exist constants $C \geq 0$, $c_0 > 0$ such that for $|t| \leq \epsilon_0$

$$|\nabla p(x)| \geq c_0 \text{ on } F_t, \quad \mathcal{H}^{d-1}(F_t) \leq C.$$

Here, $\mathcal{H}^{d-1}$ is the $(d-1)$–dimensional Hausdorff measure. Combining the above estimates with the coarea formula we derive for $0 < \epsilon \leq \epsilon_0$

$$c_0|\{x \in \Omega \,|\, |p(x)| \leq \epsilon\}| \leq \int_{\{x\in\Omega||p(x)|\leq\epsilon\}} |\nabla p| dx = \int_{-\epsilon}^{\epsilon} \mathcal{H}^{d-1}(F_t) dt \leq 2C\epsilon,$$

which implies the assertion. ∎

Condition (1.6) can be compared to conditions which appear in the study of the stability of bang-bang type controls in control problems for systems of ODEs, cf. [3], Assumption 2, p. 1850.

2 Discretization and error estimate

Let $\mathcal{T}_h$ be a quasi–uniform triangulation of Ω with maximum mesh size $h := \max_{T\in\mathcal{T}_h} \operatorname{diam}(T)$. If necessary, we allow elements to be curved along $\partial\Omega$. We consider the space of linear finite elements

$$X_h := \{\phi_h \in C^0(\bar{\Omega}) \,|\, \phi_h \text{ is a linear polynomial on each } T \in \mathcal{T}_h\}$$

with isoparametric modifications in curved simplices. Furthermore, let $X_{h0} := X_h \cap H_0^1(\Omega)$. We denote by I_h the usual Lagrange interpolation operator and by $R_h : H_0^1(\Omega) \to X_{h0}$ the Ritz–projection, defined by the relation

$$(\nabla R_h z, \nabla \phi_h) = (\nabla z, \nabla \phi_h) \quad \forall \phi_h \in X_{h0}. \tag{2.1}$$

It is well–known that

$$\|z - R_h z\| + h\|\nabla(z - R_h z)\| \leq Ch^2\|z\|_{H^2} \quad \forall z \in H^2(\Omega) \cap H_0^1(\Omega). \tag{2.2}$$

For a given function $v \in L^2(\Omega)$ we denote by $y_h = \mathcal{G}_h(v) \in X_{h0}$ the unique solution of

$$(\nabla y_h, \nabla \phi_h) = (v, \phi_h) \quad \forall \phi_h \in X_{h0}.$$

We use the variational approach of [4] in order to discretize our optimal control problem as follows:

$$\min_{v\in U_{ad}} J_h(v) = \frac{1}{2}\int_\Omega |y_h - y_0|^2 \quad \text{subject to } y_h = \mathcal{G}_h(v + f). \tag{2.3}$$

Note that the set of admissible controls is not discretized. Similarly as above, the discrete optimal control problem (2.3) has a unique solution $u_h \in U_{ad}$ which is characterized by the existence of a discrete adjoint state $p_h = \mathcal{G}_h(y_h - y_0) \in X_{h0}$ such that

$$(p_h, v - u_h) \geq 0 \quad \text{for all } v \in U_{ad}. \tag{2.4}$$

This relation implies again

$$u_h(x) \begin{cases} = a, & p_h(x) > 0, \\ \in [a,b], & p_h(x) = 0, \\ = b, & p_h(x) < 0. \end{cases} \tag{2.5}$$

Our main result are the following error estimates:

Theorem 2.1. *Let u be the solution of (1.2), u_h the solution of (2.3) with corresponding states $y = \mathcal{G}(u+f)$ and $y_h = \mathcal{G}_h(u_h+f)$. Then*

$$\|y - y_h\|, \|u - u_h\|_{L^1}, \|p - p_h\|_{L^\infty} \le C\big(h^2 + \|p - R_h p\|_{L^\infty}\big).$$

Proof. Using $v = u_h$ in (1.3) and $v = u$ in (2.4) we obtain

$$0 \le (p - p_h, u_h - u) = (p - R_h p, u_h - u) + (R_h p - p_h, u_h - u) \equiv I + II. \tag{2.6}$$

Clearly,

$$|I| \le \|p - R_h p\|_{L^\infty} \|u - u_h\|_{L^1}. \tag{2.7}$$

Since $|\{x \in \bar{\Omega} \,|\, p(x) = 0\}| = 0$ we deduce with the help of (1.4) and (2.5) that

$$\|u - u_h\|_{L^1} = \int_{\{p>0\}} (u_h - a) + \int_{\{p<0\}} (b - u_h) = \int_{A_1} (u_h - a) + \int_{A_2} (b - u_h), \tag{2.8}$$

where $A_1 = \{x \in \Omega \,|\, p(x) > 0, p_h(x) \le 0\}$ and $A_2 = \{x \in \Omega \,|\, p(x) < 0, p_h(x) \ge 0\}$. For $x \in A_1$ we have

$$0 < p(x) = (p(x) - p_h(x)) + p_h(x) \le \|p - p_h\|_{L^\infty},$$

and similarly $0 > p(x) \ge -\|p - p_h\|_{L^\infty}$ for $x \in A_2$. Thus,

$$A_1 \cup A_2 \subset \{x \in \Omega \,|\, |p(x)| \le \|p - p_h\|_{L^\infty}\},$$

so that (2.8) and (1.5) yield

$$\|u - u_h\|_{L^1} \le (b-a)|A_1 \cup A_2| \le (b-a)|\{x \in \Omega \,|\, |p(x)| \le \|p - p_h\|_{L^\infty}\}| \le C\|p - p_h\|_{L^\infty}. \tag{2.9}$$

Inserting (2.9) into (2.7) we obtain

$$|I| \le C\|p - R_h p\|_{L^\infty} \|p - p_h\|_{L^\infty} \le C\|p - R_h p\|_{L^\infty} \big(\|p - R_h p\|_{L^\infty} + \|R_h p - p_h\|_{L^\infty}\big). \tag{2.10}$$

Let us fix

$$q \begin{cases} = 2, & d = 1; \\ > 2, & d = 2; \\ \in (3,6), & d = 3 \end{cases}$$

and define $p^h := \mathcal{G}(y_h - y_0)$, so that $p_h = R_h p^h$. The continuous embeddings $H^2(\Omega) \hookrightarrow W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$ together with the stability of the Ritz projection in $W^{1,q}(\Omega)$ and the fact that $p - p^h = \mathcal{G}(y - y_h)$ implies

$$\begin{aligned} \|R_h p - p_h\|_{L^\infty} &\leq C\|R_h p - p_h\|_{W^{1,q}} = \|R_h(p - p^h)\|_{W^{1,q}} \leq C\|p - p^h\|_{W^{1,q}} \\ &\leq C\|p - p^h\|_{H^2} \leq C\|y - y_h\|. \end{aligned} \tag{2.11}$$

Returning to (2.10) we obtain with the help of Young's inequality

$$|I| \leq \epsilon\|y - y_h\|^2 + C_\epsilon\|p - R_h p\|^2_{L^\infty}, \quad \epsilon > 0. \tag{2.12}$$

Next, recalling that $y = \mathcal{G}(u + f), y_h = \mathcal{G}_h(u_h + f)$ and the definition of R_h we have

$$II = \big(\nabla(R_h p - p_h), \nabla y_h\big) - \big(\nabla(R_h p - p_h), \nabla y\big) = \big(\nabla(p - p_h), \nabla y_h\big) - \big(\nabla(p - p_h), \nabla R_h y\big).$$

Since $p = \mathcal{G}(y - y_0), p_h = \mathcal{G}_h(y_h - y_0)$ we may continue

$$II = (y - y_h, y_h - R_h y) = -\|y - y_h\|^2 + (y - y_h, y - R_h y) \leq -\frac{1}{2}\|y - y_h\|^2 + Ch^4 \tag{2.13}$$

by (2.2). Inserting (2.12), (2.13)into (2.6) and choosing $\epsilon = \frac{1}{4}$ we derive

$$\|y - y_h\|^2 \leq Ch^4 + C\|p - R_h p\|^2_{L^\infty}. \tag{2.14}$$

If we employ this estimate in (2.11) we obtain

$$\begin{aligned} \|p - p_h\|_{L^\infty} &\leq C\big(\|p - R_h p\|_{L^\infty} + \|R_h p - p_h\|_{L^\infty}\big) \leq C\big(\|p - R_h p\|_{L^\infty} + \|y - y_h\|\big) \\ &\leq C\big(h^2 + \|p - R_h p\|_{L^\infty}\big). \end{aligned}$$

The bound on $\|u - u_h\|_{L^1}$ then follows from (2.9). ∎

Remark 2.2. a) According to Theorem 2.1 the order of convergence is now determined by the behaviour of $\|p - R_h p\|_{L^\infty}$. For example, it is well known (cf. [2], Section 3.3) that

$$\|p - R_h p\|_{L^\infty} \leq Ch^2|\log h|^{\gamma(d)}$$

provided that $p \in W^{2,\infty}(\Omega)$.
b) An appropriate modification of the proof of Theorem 2.1 shows that it is still possible to derive error bounds under a more general condition of the form

$$\exists C \geq 0 \ \ \forall \epsilon > 0 \quad |\{x \in \Omega \,|\, |p(x)| \leq \epsilon\}| \leq C\epsilon^\beta$$

for some $\beta \in (0, 1]$.

3 A numerical experiment

The following one-dimensional example is adapted from [8], 2.9.1. Let $\Omega := (0, 1)$, $a := -1$, $b := 1$ and

$$f(x) := m^2\pi^2 \sin(m\pi x) + \operatorname{sign}(-\sin(m\pi x)) \text{ and } y_0(x) := (1 + m^2\pi^2)\sin(m\pi x),$$

where $m \in \mathbb{N}$. The exact control is given by $u(x) = -\text{sign}(-\sin(m\pi x))$ with corresponding optimal state $y(x) = \sin(m\pi x)$ and adjoint state $p(x) = -\sin(m\pi x)$. The optimal control has the switching points $x_k = \frac{k}{m}, k = 1, \ldots, m-1$ in Ω. Furthermore, recalling Lemma 1.3, Assumption (1.5) is satisfied since $|p'(\pm 1)| = |p'(x_k)| = m\pi, k = 1, \ldots, m-1$.
For the discretization of the state we use piecewise linear, continuous finite elements on a sequence of equidistant grids $\mathcal{T}_i$ with gridwidth $h_i := 2^{-i}, i = 1, \ldots, 9$. The numerical solution is obtained by the following fixed-point iteration: given u_h^0, $n = 0$, compute

$$y_h^n = \mathcal{G}_h(u_h^n + f), p_h^n = \mathcal{G}_h(y_h^n - y_0) \text{ and set } u_h^{n+1} = -\text{sign}(p_h^n),\ n = n + 1.$$

Table 1 presents our numerical findings for $m = 3$. The fixed-point iteration is initialized with $u_h^0 = 0$ and takes 5 iterations on Level $i = 9$ to drive the maximum distance of two consecutively computed switching points below $1.e - 16$. We note that changing u_h^0 does not affect the convergence behaviour of the iteration in the present example. As predicted by Theorem 2.1 we obtain convergence order 2 for $\|y - y_h\|$ and $\|p - p_h\|_{L^\infty}$, whereas the convergence order for $\|u - u_h\|_{L^1}$ and for the Lebesgue measure of the symmetric difference $(A \setminus A_h) \cup (A_h \setminus A)$ of the active sets A, A_h associated to the solutions u, u_h seems to be 3 in the present example. Figure 1 shows the numerical solution on a grid corresponding to the refinement level $i = 3$ together with a zoom on the first switching point. Note that the discrete switching points obtained with the help of the variational approach need not coincide with finite element grid points as it would be the case if the controls were discretized on the finite element grid by piecewise constant ansatz functions, say. Finally we note that the observed experimental orders of convergence for larger values of m are similar to those displayed in Tab. 1. We therefore omit the presentation of numerical results for other values of m.

Level i	$\|u - u_h\|_{L^1}$	$\|y - y_h\|$	$\|y - y_h\|_{L^\infty}$	$\|p - p_h\|_{L^\infty}$	$|(A \setminus A_h) \cup (A_h \setminus A)|$
5	3.00254305	1.98745818	1.98859976	1.99608005	3.00254305
6	3.00063492	1.99687071	2.00092856	1.99848923	3.00063492
7	3.00015871	1.99921806	1.99926368	1.99989082	3.00015870
8	3.00003903	1.99980454	2.00036904	1.99983872	3.00003886
9	3.00000606	1.99995114	1.99995397	2.00001112	3.00000351

Table 1: Experimental order of convergence, 2 switching points

Acknowledgements

The authors gratefully acknowledge the support of the DFG Priority Program 1253 entitled Optimization With Partial Differential Equations.

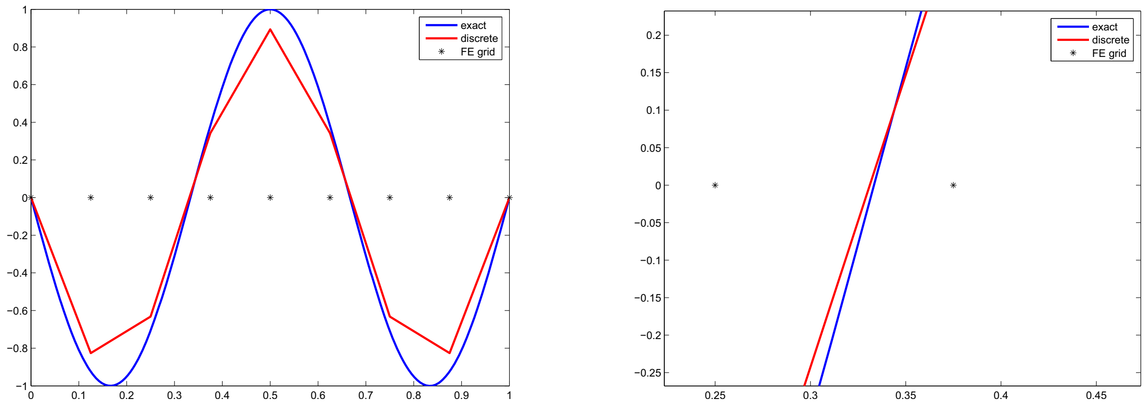

Figure 1: Exact versus discrete adjoint (left) with zoom at the first switching point (right) for $m = 3$.

References

[1] Arnautu, V., Langmach, H., Sprekels, J., Tiba, D.: *On the approximation and the optimization of plates*, Numer. Funct. Anal. Optim. **21**, 337–354 (2000).

[2] Ciarlet, P.G.: *The Finite Element Method for Elliptic Problems*, North–Holland, Amsterdam, New York, Oxford, 1978.

[3] Felgenhauer, U.: *On stability of bang-bang type controls*, SIAM J. Control Optim. **41**, 1843–1867 (2003).

[4] Hinze, M.: *A variational discretization concept in control constrained optimization: the linear-quadratic case*, Comput. Optim. Appl. **30**, 45-61 (2005).

[5] Hinze, M., Pinnau, R., Ulbrich, M., Ulbrich, S.: *Optimization with pde constraints*. Mathematical Modelling: Theory and Applications **23**, Springer, 2009.

[6] Maurer, H, Mittelmann, H.D.: *Optimization techniques for solving elliptic control problems with control and state constraints. Part 1: Boundary control*, Comput. Optim. Appl. **16**, 29-55 (2000).

[7] Maurer, H, Mittelmann, H.D.: *Optimization techniques for solving elliptic control problems with control and state constraints. Part 2: Distributed control*, Comput. Optim. Appl. **18**, 141-160 (2001).

[8] Tröltzsch, F.: *Optimale Steuerung mit partiellen Differentialgleichungen*. Vieweg, 2005.