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Stability concerning the Time-dependence of the Mean Life Expectancy with Respect to its Relevant Parameters

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Abstract:
In [1], Danielmeyer and Martinetz published a paper on the parallelism of the mean life expectancy of a society and its GDP per capita. And from the biological stability of the mean life expectancy they found a similar stability of the industrial evolution. The authors assumed the mean life expectancy $L(t)$ considered as a function of the time t (measured in years) to fulfill a logistic differential equation but they did not give reasons for the choice of the parameters which occur in this theory and did not ask for the stability of these parameters with respect to inaccuracies of $L(t)$ or vice versa. As a matter of fact, more or less exact values of the mean life expectancy $L(t)$ at the instant t of a population that lives in a certain area under consideration can not earlier be reported than after the death of all the individuals born in the year t. From this point of view we are going to complete the theory insignificantly and to confirm the choice of the parameters or to correct it and to look −vice versa− for the stability of $L(t)$ with respect to these parameters.

1 The Theory

Let L_0 describe the reproduction minimum of the population to be considered and assume that

$$f(t) := L(t) - L_0 \tag{1}$$

can be approximated by the solution of the logistic differential equation

$$\dot{f} = \frac{1}{A} f (1 - \frac{1}{\Delta L} f) \tag{2}$$

where

$$\Delta L = \lim_{t \to \infty} f(t) \tag{3}$$

so that

$$\bar{L} := L_0 + \Delta L \tag{4}$$

specifies the expected genetic limit.

A is a growth parameter.

We find (cf. [1])

$$f(t) = \frac{\Delta L}{1 + e^{(T_L - t)/A}} \tag{5}$$

where T_L, the so-called halftime parameter, gives the particular instant when f has grown up to $\Delta L/2$:

$$f(T_L) = \frac{\Delta L}{2} \quad . \tag{6}$$

(2) then shows

$$\dot{f}\ (T_L) = \frac{\Delta L}{4A} \tag{7}$$

which is the maximum speed of the development of f or L because of $\ddot{f}\ (T_L) = 0$.

From (5), $\lim_{t \to -\infty} f(t) = 0$, hence

$$\lim_{t \to -\infty} L(t) = L_0 \quad . \tag{8}$$

Besides the value of ΔL, the graph of $f(t)$, hence of $L(t)$, is characterized by the parameters T_L and A.

In order to estimate these two parameters more or less exactly, we need at least two pairs $(t_0, \hat{L}_0)$, $(t_1, \hat{L}_1)$ $\quad (t_1 > t_0)$ where

$$\hat{L}_i := L(t_i) = L_0 + f(t_i) \qquad (i = 0, 1) \quad . \tag{9}$$

Both pairs should be really confirmed by observations, i.e. they should stem from observations made a lifetime (or even more) before the actual instant t, and $t_1 - t_0$ should not be too small.

From (9) we find with (4),(5)

$$T_L = t_i + A\ \ln \frac{\bar{L} - \hat{L}_i}{\hat{L}_i - L_0} \qquad (i = 0, 1)\ . \tag{10}$$

Because T_L does not depend on t, (10) yields

$$A = \frac{t_1 - t_0}{\ln \frac{(\bar{L} - \hat{L}_0)\,(\hat{L}_1 - L_0)}{(\bar{L} - \hat{L}_1)\,(\hat{L}_0 - L_0)}} \quad . \tag{11}$$

2 Example

We look for the societies considered in [1], we accept the intention that the mean life expectancy can be described by a logistic graph $L(t) = L_0 + f(t)$ with f from (5). We also keep from [1] in a first step the value of $L_0 = 30$ years and of $\Delta L = 88$ years (thus, $\bar{L} = 118$ years). We are particularly interested in the behaviour of $L(t)$ in recent years and assume that the early pairs [1]

$$(t_0, \hat{L}_0) = (1900, 48.5) \quad ; \quad (t_1, \hat{L}_1) = (1930, 57) \tag{12}$$

are sufficiently correct [2]. (12) then yields $A = 59$ (so that the choice of $A = 61$ in [1] was not too bad) and (10) then leads to $T_L = 1978$ (so that also the choice of $T_L = 1981$ can more or less be confirmed), provided that ΔL is chosen sufficiently well. Let us therefore ask for the stability of ΔL with respect to changes of f. Herewith we keep the theory as well as $L_0 = 30$ years. (2) then leads to

$$\Delta L = \frac{f(t)}{1 - A\,\frac{\dot{f}(t)}{f(t)}} \quad . \tag{13}$$

The mean value theorem gives

$$f(t_1) = f(t_0) + (t_1 - t_0)\ \dot{f}\,(\tilde{t}) \qquad (t_0 < \tilde{t} < t_1) \quad , \tag{14}$$

i.e.

$$\dot{f}\,(\tilde{t}) = \frac{f(t_1) - f(t_0)}{t_1 - t_0} \ .$$

Thus, (13) leads to

$$\Delta L = \frac{f(\tilde{t})}{1 - A\,\frac{f(t_1) - f(t_0)}{t_1 - t_0} \cdot \frac{1}{f(\tilde{t})}} \quad . \tag{15}$$

[1] more than two generations back

[2] We picked out these values from the graphs in [1], but the computations can obviously very easyly be repeated if better values are available.

We choose again for t_0, t_1 the more than two generations old values used in (12), and we approximate $f(\tilde{t})$ by $(f(t_0)+f(t_1))/2$. (15) then leads to

$$\Delta L \approx \frac{f(t_0)+f(t_1)}{2-2A\frac{f(t_1)-f(t_0)}{t_1-t_0}\frac{2}{f(t_0)+f(t_1)}}\ ,$$

i.e., taking the corrected value of A into account, to

$$\Delta L = 85.8 \text{ years}$$

or

$$\bar{L} = 116 \text{ years} .$$

These corrections are not really relevant within the limits of the possible accuracy of the theory presented in [1] so that the parameters of the paper have not necessarily to be criticized. In other words: The parameters are very stable with respect to inaccuracies of the observed values of $L(t)$. And this says that one can trust the forecasts of the mean life expectancy presented in [1] also for people born in the years of our lifetime. And this stability also holds if the theory will be applied to other clusters of living beings as far as the model of the logistic differential equation does also hold.

3 How do the parameters influence the solution ?

We investigate now −vice versa− the stability of $f(t)$ with respect to small disturbancess δ, τ, α of the parameters, i.e. we ask for the solution of (2) if the parameters $\Delta L, T_L, A$ are replaced by

$$\Delta L+\delta, \quad T_L+\tau, \quad A+\alpha. \tag{16}$$

Obviously, we find from (5) instead of the solution f the disturbed solution

$$\tilde{f}(t) = \frac{\Delta L+\delta}{1+e^{(T_L+\tau-t)/(A+\alpha)}}\ . \tag{17}$$

What one can easily see −also without any formalism− is the fact, that small values of δ and of τ seem to lead to only small disturbances of f. What about α ?

For fixed values of $\Delta L, T_L, A$ and of t, $\tilde{f}(t)$ is a function of δ, τ, α:

$$\tilde{f}(t) = g(\delta, \tau, \alpha) \tag{18}$$

with

$$g(0,0,0) = f(t) \ . \tag{19}$$

Taylor expansion leads to

$$g(\delta,\tau,\alpha) = g(0,0,0)+g_\delta(0,0,0)\,\delta+g_\tau(0,0,0)\,\tau+g_\alpha(0,0,0)\alpha+\mathcal{O}((\max(|\delta|,|\tau|,|\alpha|))^2)\ . \tag{20}$$

Because we restricted ourselves to only relatively small disturbances, we ignore the quadratic terms, what can particularly be justified if (2), (5) will be formulated in dimensionless setting, and because of

$$\left.\begin{array}{lcl} g_\delta(0,0,0) & = & \frac{1}{1+e^{(T_L-t)/A}} \\ g_\tau(0,0,0) & = & -\frac{\Delta L}{A}\frac{e^{(T_L-t)/A}}{\left[1+e^{(T_L-t)/A}\right]^2} \\ g_\alpha(0,0,0) & = & \frac{\Delta L(T_L-t)}{A^2}\frac{e^{(T_L-t)/A}}{\left[1+e^{(T_L-t)/A}\right]^2} \end{array}\right\} \tag{21}$$

we find

$$\begin{array}{rcl} g(\delta\,,\tau\,,\alpha) & \approx & g(0,0,0)+\frac{1}{1+e^{(T_L-t)/A}}\left\{\delta\,+\,\frac{\Delta L}{A}\,\tau-\frac{\Delta L}{A^2}(T_L-t)\alpha\right\} \\ & \leq & g(0,0,0)+|\delta|+\frac{\Delta L}{A}|\tau|+\frac{\Delta L}{A^2}|(T_L-t)\alpha|\ . \end{array} \tag{22}$$

Hence, the influence of α occurs only for great values of $|t|$.

For our values $\Delta L = 86$, $T_L = 1978$, $A = 59$, the disturbance for the year 2010 results in

$$|\tilde{f}(t) - f(t)| \stackrel{<}{\approx} 4.5\max(|\delta|,|\tau|,|\alpha|)\ . \tag{23}$$

References

[1] Danielmeyer, H.G.: *The biologic stability of the industrial evolution.* Eur. Rev. **18** (2010) and T. Martinetz

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