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**Optimal control of fluid - structure interaction
problem: linear viscous fluid is coupled
with elastic body**

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Abstract

In the present paper we analyze the optimal control problem governed by a dynamical fluid-solid interaction model. We consider linear fluid and solid models, described by the Stokes equation and linear elasticity equation, respectively. Along the boundary, on which the fluid and the solid interact, we assume the velocity and stress vectors to be continuous. Moreover, we assume the interface boundary to be fixed. For a justification and an analysis of such type of fluid-solid interaction model see [3]. The cost function for the optimal control problem is considered to be of the tracking type.

In the first, theoretical, part of the paper we prove that the optimal control problem, introduced above, admits at least one solution $u \in U_{ad}$. Further, using both, sensitivity and adjoint approach, we derive the first order necessary optimality conditions for the optimal control problem.

The second, practical, part of the paper is devoted to the description of the numerical realization for finding solution of the optimal control problem and numerical examples. Theoretical results from the first part of the paper, in particular, the presentation of the adjoint system, are here exploited.

For the spatial discretization of both, primal and adjoint problems, we apply the Finite Volume Particle Method (FVPM) (see e.g. [4, 9]). During the implementation we also use an extension of this method for incompressible flows, i.e. an adaptation of the FVPM for the solution of the Poisson equation and the pressure-correction algorithm (see [6]). For the time discretization of the Stokes and linear elasticity problem we apply the implicit and the Newmark time stepping schemes, respectively. The solution of the optimal control problem is determined using the Gradient Algorithm.

In the final part of the paper we present some numerical examples for the solution of the optimal control problem using theoretical and numerical techniques introduced above.

Keywords: Optimal control, Fluid-structure interaction, Finite Volume Particle Method

1 Introduction

In the present paper we analyze the optimal control problem governed by a dynamical fluid-solid interaction model. To begin with, we consider linear fluid and solid models, described by the Stokes equation and linear elasticity equation, respectively, and derive the first order necessary optimality conditions for a specific form of the objective function.

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Let $\Omega = \Omega_1 \cup \Omega_2$ be a bounded set in \mathbb{R}^d , $d = 2$, with the boundary $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_0$. Γ is a C^∞ - manifold of dimension $(d - 1)$. We assume, that Ω lies locally on one side of Γ . Fluid occupies the Ω_1 domain, with the boundary $\Gamma_1 \cup \Gamma_0$. The solid occupies the domain Ω_2 with the boundary $\Gamma_2 \cup \Gamma_0$. Γ_0 is the common boundary for Ω_1 and Ω_2 , so that alongside the boundary Γ_0 the fluid and the solid interact. For the better understanding the situation is schematically represented in Figure 1.

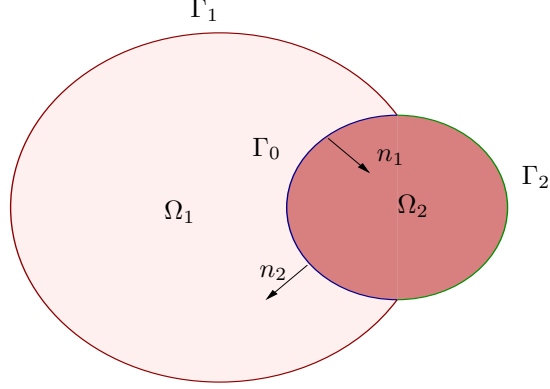


Figure 1: Schematic illustration of physical domains for the fluid-structure interaction

The mathematical model for the Stokes problem is given by:

$$\left. \begin{aligned} \rho_1 y_t + \nabla p - \mu_1 \nabla \cdot (\nabla y + \nabla y^T) &= \rho_1 f_1 + B_y u_y \text{ in } (0, T) \times \Omega_1, \\ -\operatorname{div} y &= 0 \text{ in } (0, T) \times \Omega_1, \\ y &= 0 \text{ on } (0, T) \times \Gamma_1, \\ y|_{t=0} &= y_0 \text{ in } \Omega_1. \end{aligned} \right\} \quad (\text{SP})$$

Here $y = (y_1, y_2)$ denotes the fluid velocity, p the fluid pressure, f_1 the given body force per unit mass, B_y a given control operator, $u_y \in U_y$ denotes a control variable of the fluid, ρ_1 and μ_1 the constant fluid density and viscosity, y_0 the given initial velocity, $T > 0$ the terminal time.

The following equations describe the linear elasticity problem:

$$\left. \begin{aligned} \rho_2 x_{tt} - \mu_2 \nabla \cdot (\nabla x + \nabla x^T) - \lambda_2 \nabla (\nabla \cdot x) &= \rho_2 f_2 + B_x u_x \text{ in } (0, T) \times \Omega_2, \\ x &= 0 \text{ on } (0, T) \times \Gamma_2, \\ x|_{t=0} &= x_0 \text{ in } \Omega_2, \\ x_t|_{t=0} &= x_1 \text{ in } \Omega_2. \end{aligned} \right\} \quad (\text{LE})$$

In the model (LE) $x = (x_1, x_2)$ denotes the displacement of the solid, f_2 the given loading force per unit mass, B_x a given control operator, $u_x \in U_x$ denotes a control variable for the solid, μ_2 and λ_2 are the Lamé constants, ρ_2 the constant solid density, x_0 and x_1 the given initial data. U_y and U_x are the Hilbert spaces of controls.

Furthermore, along the boundary Γ_0 on which the fluid and the solid interact we assume the velocity and stress vectors to be continuous. This can be described mathematically by the following interface conditions:

$$\left. \begin{aligned} x_t &= y \quad \text{on } \Gamma_0, \\ \mu_2 (\nabla x + \nabla x^T) \cdot n_2 + \lambda_2 (\nabla \cdot x) n_2 &= p n_1 - \mu_1 (\nabla y + \nabla y^T) \cdot n_1 \quad \text{on } \Gamma_0. \end{aligned} \right\} \quad (\text{IC})$$

Here, n_i , $i = 1, 2$, are the unit normal vectors to the boundary $\partial\Omega_i$ pointing toward the exterior of Ω_i , $i = 1, 2$. Throughout we assume the interface boundary Γ_0 to be fixed.

In addition to the state equations (SP) and (LE) we consider the cost function $J(y, x, u_y, u_x)$. Then, the optimal control problem will be read as:

$$\left. \begin{array}{l} \min J(y, x, u_y, u_x) \quad \text{over } (y, x, u_y, u_x) \\ \text{subject to} \quad (\text{SP}), (\text{LE}), (\text{IC}). \end{array} \right\} \quad (\text{OCP})$$

We assume that the objective function $J(y, x, u_y, u_x)$ is at least once Frechet-differentiable, weakly lower semi-continuous and radially unbounded in u_y and u_x . The last means that

$$J(y, x, u_y, u_x) \rightarrow \infty \quad \text{as} \quad \|u_y\|_{U_y} \rightarrow \infty \quad \text{or} \quad \|u_x\|_{U_x} \rightarrow \infty.$$

In the following we specify the objective as

$$J = \frac{\alpha_1}{2} \int_0^T \left\{ \int_{\Omega_1} |y - y_d|^2 dx + \int_{\Omega_2} |x - x_d|^2 dx \right\} dt + \frac{\alpha_{4y}}{2} \|u_y\|_{U_y}^2 + \frac{\alpha_{4x}}{2} \|u_x\|_{U_x}^2. \quad (1)$$

This function is of tracking type and obviously satisfy the required properties. The task is to establish the necessary optimality conditions for the optimal control problem (OCP) with the objective function defined by (1).

To begin with, we introduce the required functional spaces:

$$V_y = \{f \in H_1^0(\Omega_1), \text{div } f|_{\Omega_1} = 0\}, \quad (2)$$

$$H_y = \{f \in L^2(\Omega_1), \text{div } f|_{\Omega_1} = 0\}, \quad (3)$$

$$V_x = \{f \in H_1^0(\Omega_2)\}, \quad (4)$$

$$H_x = \{f \in L^2(\Omega_2)\}, \quad (5)$$

$$W_y(0, T) = \{f \in L^2(0, T; V_y); f' \in L^2(0, T; V_y^*)\}, \quad (6)$$

$$W_x(0, T) = \{f \in L^2(0, T; V_x); f' \in L^2(0, T; H_x); f'' \in L^2(0, T; V_x^*)\}, \quad (7)$$

$$V = \{f \in H_1^0(\Omega), \text{div } f|_{\Omega_1} = 0\}, \quad (8)$$

$$H = \{f \in L^2(\Omega), \text{div } f|_{\Omega_1} = 0\}, \quad (9)$$

$$W = W_y \times W_x, \quad (10)$$

$$U_y, U_x = \text{Hilbert spaces of controls}, \quad (11)$$

$$U = U_y \times U_x. \quad (12)$$

$$V \subset H \subset V^*, \quad H^* = H.$$

Similar to [3], we introduce continuous bilinear forms

$$a_y(t, \phi, \psi) = \frac{1}{2} \int_{\Omega_1} \mu_1 (\nabla \phi + \nabla \phi^T) : (\nabla \psi + \nabla \psi^T) d\Omega, \quad \forall \phi, \psi \in V_y, \quad (13)$$

$$a_x(t, \phi, \psi) = \int_{\Omega_2} \frac{\mu_2}{2} (\nabla \phi + \nabla \phi^T) : (\nabla \psi + \nabla \psi^T) + \lambda_2 (\nabla \cdot \phi) (\nabla \cdot \psi) d\Omega, \quad \forall \phi, \psi \in V_x, \quad (14)$$

where the relation $:$ denotes $A : B = \text{tr}(A^T B)$. $a_y(t, \phi, \psi)$ and $a_x(t, \phi, \psi)$ are continuous, coercive, bilinear forms on V_y and V_x , respectively. Moreover, they satisfy the following relations:

$$a_y(t, \phi, \phi) \geq K_y \|\phi\|_{V_y}^2, \quad \forall \phi \in V_y, \quad \text{if } \text{meas}(\Gamma_1) \neq 0, \quad (15)$$

$$a_x(t, \phi, \phi) \geq K_x \|\phi\|_{V_x}^2, \quad \forall \phi \in V_x, \quad \text{if } \text{meas}(\Gamma_2) \neq 0, \quad (16)$$

$$a_y(t, \phi, \phi) + (\phi, \phi)_{\Omega_1} \geq K_y \|\phi\|_{V_y}^2, \quad \forall \phi \in V_y, \quad \text{if } \text{meas}(\Gamma_1) = 0, \quad (17)$$

$$a_x(t, \phi, \phi) + (\phi, \phi)_{\Omega_2} \geq K_x \|\phi\|_{V_x}^2, \quad \forall \phi \in V_x, \quad \text{if } \text{meas}(\Gamma_2) = 0. \quad (18)$$

(These conditions will be needed to prove the existence of y and x , compare [7], e.g. p.105).

Since the forms $a_y(t, \phi, \psi)$ and $a_x(t, \phi, \psi)$ are linear and continuous, they can be interpreted using the operators $A_y(t)$ and $A_x(t)$ in the following way:

$$\begin{aligned} a_y(t, \phi, \psi) &= (A_y(t)\phi, \psi), & A_y(t)\phi &\in V_y^*, \\ a_x(t, \phi, \psi) &= (A_x(t)\phi, \psi), & A_x(t)\phi &\in V_x^*. \end{aligned}$$

This defines linear operators:

$$A_y \in \mathcal{L}(V_y, V_y^*), \quad A_x \in \mathcal{L}(V_x, V_x^*).$$

The control operators B_y and B_x are defined as:

$$B_y \in \mathcal{L}(U_y, L^2(0, T; V_y^*)), \quad B_x \in \mathcal{L}(U_x, L^2(0, T; H_x)) \quad (\text{h1})$$

Moreover, we assume that the body forces f_1 and f_2 and the initial data y_0 , x_0 and x_1 satisfy:

$$\begin{aligned} f_1 &\in L^2(0, T; V_y^*), & f_2 &\in L^2(0, T; H_x), \\ y_0 &\in H_y, & x_0 &\in V_x, & x_1 &\in H_x. \end{aligned} \quad (\text{h2})$$

Using the introduced linear operators A_y and A_x the evolution problems (SP) and (LE) can be rewritten in the form:

$$\left. \begin{aligned} \rho_1 y_t + A_y(t)y &= \rho_1 f_1 + B_y u_y \text{ in } L^2(0, T; V_y^*), \\ y|_{t=0} &= y_0, \quad y_0 \text{ given in } H_y. \end{aligned} \right\} \quad (\text{SPo})$$

$$\left. \begin{aligned} \rho_2 x_{tt} + A_x(t)x &= \rho_2 f_2 + B_x u_x \text{ in } L^2(0, T; H_x), \\ x|_{t=0} &= x_0, \quad x_0 \text{ given in } V_x, \\ x_t|_{t=0} &= x_1, \quad x_1 \text{ given in } H_x. \end{aligned} \right\} \quad (\text{LEo})$$

Using the results in [3], we can prove:

Theorem 1.1. *Assuming the hypotheses (h1) and (h2) be satisfied, the fluid-structure interaction problem (SPo) + (LEo) + (IC) admits a unique solution $(y, x) \in W_y \times W_x$. The mapping*

$$\{f_1, u_y, y_0, f_2, u_x, x_0, x_1\} \rightarrow \left\{ y, x, \frac{dx}{dt} \right\}$$

is a linear continuous map of

$$L^2(0, T; V_y^*) \times U_y \times H_y \times L^2(0, T; H_x) \times U_x \times V_x \times H_x \rightarrow L^2(0, T; V_y) \times L^2(0, T; V_x) \times L^2(0, T; H_x).$$

2 Set of inequalities defining the optimal control

Consider the sets of admissible controls $U_{y_{ad}}$ for the fluid and $U_{x_{ad}}$ for the solid to be closed convex subsets of U_y and U_x , respectively. Introduce the abbreviations

$$U = U_y \times U_x, \quad U_{ad} = U_{y_{ad}} \times U_{x_{ad}}, \quad u = (u_y, u_x) \in U_y \times U_x,$$

and

$$\begin{aligned} \Lambda_u &= \text{canonical isomorphism of } U \text{ into } U^*, \\ \Lambda_y &= \text{canonical isomorphism of } V_y \text{ into } V_y^*, \\ \Lambda_x &= \text{canonical isomorphism of } V_x \text{ into } V_x^*. \end{aligned}$$

Using the introduced notations and the existence results of Theorem 1.1 we rewrite the (OCP) into the reduced setting as:

$$\left. \begin{aligned} \min \hat{J}(u) &= J(y(u), x(u), u_y, u_x) & \text{over } & (u_y, u_x) \\ \text{subject to } & u = (u_y, u_x) \in U_{ad}. \end{aligned} \right\} \quad (\text{OCPr})$$

Here, $y(u_y)$ and $x(u_x)$ is the unique solution of the problem (SPo) + (LEo) + (IC) corresponding to the controls u_y and u_x . This solution exists due to Theorem 1.1. Now, we are in a position to prove the following theorem:

Theorem 2.1. *There exists an element $u \in U_{ad}$ being a solution of the reduced optimal control problem (OCPr).*

Proof. Since $\hat{J}(u) \geq 0$ there exists an infimum $j = \inf_{u \in U_{ad}} \hat{J}(u)$. Therefore, there is a minimizing sequence $\{u_n\}_{n=1}^{\infty} \subset U_{ad}$, such that

$$\hat{J}(u_n) \rightarrow j \quad \text{as } n \rightarrow \infty.$$

Because of the radial unboundness of $\hat{J}(u)$ relative to u and the fact that $\hat{J}(u_n) \rightarrow j$ as $n \rightarrow \infty$, the sequence $\{u_n\}_{n=1}^{\infty}$ is bounded, i.e.

$$\|u_n\|_U \leq \text{const.}$$

Since $U = U_y \times U_x$ is a Hilbert space, it is reflexive. Therefore, from the sequence $\{u_n\}_{n=1}^{\infty}$ we may extract a weakly convergent subsequence $\{u_{n_k}\}_{k=1}^{\infty}$, such that

$$u_{n_k} \rightharpoonup \bar{u} \quad \text{in } U.$$

Because U_{ad} is closed and convex, it is weakly closed. Therefore, $\bar{u} \in U_{ad}$. Due to the weakly lower semi-continuity of $\hat{J}(u)$ we obtain

$$\hat{J}(\bar{u}) \leq \liminf_{k \rightarrow \infty} \hat{J}(u_{n_k}) = j.$$

Since j is the infimum of all possible controls on U_{ad} , we have $\hat{J}(\bar{u}) = j$. Thus, $\bar{u} \in U_{ad}$ is the optimal control for (OCPr). \square

The control u is an optimal control for the (OCPr) and thus also for (OCP) if and only if the following optimality condition is satisfied:

$$\langle \hat{J}'(u), (v - u) \rangle_{U^*, U} \geq 0, \quad \forall v \in U_{ad}.$$

This means

$$\left. \begin{aligned} \langle \hat{J}_{u_y}'(u), (v_y - u_y) \rangle_{U_y^*, U_y} &\geq 0, & \forall v_y \in U_{y_{ad}}, \\ \langle \hat{J}_{u_x}'(u), (v_x - u_x) \rangle_{U_x^*, U_x} &\geq 0, & \forall v_x \in U_{x_{ad}}. \end{aligned} \right\} \quad (\text{OC})$$

Calculating the derivatives $\hat{J}_{u_y}'(u)$ and $\hat{J}_{u_x}'(u)$ we obtain

$$\left. \begin{aligned} \int_0^T \langle \alpha_1 \Lambda_y(y(u) - y_d), y(v) - y(u) \rangle_{V_y^*, V_y} dt + \alpha_{4y}(u_y, v_y - u_y)_{U_y} &\geq 0, & \forall v_y \in U_{y_{ad}}, \\ \int_0^T \langle \alpha_1 \Lambda_x(x(u) - x_d), x(v) - x(u) \rangle_{V_x^*, V_x} dt + \alpha_{4x}(u_x, v_x - u_x)_{U_x} &\geq 0, & \forall v_x \in U_{x_{ad}}. \end{aligned} \right\} \quad (\text{OC}')$$

We formally introduce the adjoint states $\bar{y}(u)$ and $\bar{x}(u)$ for the fluid and solid by:

$$\left. \begin{aligned} -\bar{y}_t + A_y^*(t)\bar{y} &= \alpha_1 \Lambda_y(y(u) - y_d) & \text{in } \Omega_1 \times (0, T), \\ \bar{y}(T, u) &= 0, & \text{in } \Omega_1. \end{aligned} \right\} \quad (\text{ASP})$$

$$\left. \begin{aligned} \bar{x}_{tt} + A_x^*(t)\bar{x} &= \alpha_1 \Lambda_x(x(u) - x_d) & \text{in } \Omega_2 \times (0, T), \\ \bar{x}(T, u) &= 0, & \text{in } \Omega_2, \\ \bar{x}_t(T, u) &= 0, & \text{in } \Omega_2. \end{aligned} \right\} \quad (\text{ALE})$$

If we impose a stronger hypotheses on $\Lambda_x(x(u) - x_d)$, namely:

$$\Lambda_x(x(u) - x_d) \in L^2(0, T; H_x), \quad (\text{instead of } L^2(0, T; V_x^*))$$

and reverse the flow of time (change t to $T-t$), then by applying Theorem 1.1 the adjoint problem (ASP) + (ALE) admits a unique solution $\{\bar{y}, \bar{x}, \frac{d\bar{x}}{dt}\} \in L^2(0, T; V_y) \times L^2(0, T; V_x) \times L^2(0, T; H_x)$.

To proceed further we multiply the both sides of (ASP) and (ALE) by $(y(v) - y(u))$ and $(x(v) - x(u))$, respectively. We integrate the obtained products over the time interval $[0, T]$. Noting that

$$\int_0^T \left(-\frac{d}{dt} \bar{y}(u), y(v) - y(u) \right) dt = \int_0^T \left(\bar{y}(u), \frac{d}{dt} y(v) - \frac{d}{dt} y(u) \right) dt,$$

$$\int_0^T (A_y^*(t) \bar{y}(u), y(v) - y(u)) dt = \int_0^T (\bar{y}(u), A_y(t) y(v) - A_y(t) y(u)) dt,$$

and

$$\int_0^T (\phi''(t), \psi(t)) dt = (\phi'(T), \psi(T)) - (\phi'(0), \psi(0)) - (\phi(T), \psi'(T)) + (\phi(0), \psi'(0)) + \int_0^T (\phi(t), \psi''(t)) dt.$$

The last equality is valid for e.g. $\phi, \psi \in L^2(0, T; V_x)$; $\phi', \psi' \in L^2(0, T; H_x)$; $\phi'', \psi'' \in L^2(0, T; V_x^*)$. Using these three relations we obtain for the (ASP)

$$\int_0^T \alpha_1 (\Lambda_y(y(u) - y_d), y(v) - y(u))_{V_y^*, V_y} dt = \int_0^T \left(\bar{y}(u), \left(\frac{d}{dt} + A_y(t) \right) (y(v) - y(u)) \right) dt =$$

$$= \int_0^T (\bar{y}(u), B_y v_y - B_y u_y) dt = (B_y^* \bar{y}(u), v_y - u_y)_{U_y^*, U_y} = (\Lambda_{U_y}^{-1} B_y^* \bar{y}(u), v_y - u_y)_{U_y}.$$

For the adjoint problem (ALE) the reformulations yield:

$$\int_0^T \alpha_1 (\Lambda_x(x(u) - x_d), x(v) - x(u))_{V_x^*, V_x} dt = \int_0^T \left(\bar{x}(u), \left(\frac{d^2}{dt^2} + A_x(t) \right) (x(v) - x(u)) \right) dt =$$

$$= \int_0^T (\bar{x}(u), B_x v_x - B_x u_x) dt = (B_x^* \bar{x}(u), v_x - u_x)_{U_x^*, U_x} = (\Lambda_{U_x}^{-1} B_x^* \bar{x}(u), v_x - u_x)_{U_x}.$$

Therefore, the optimality conditions (OC') can be rewritten as

$$\left. \begin{aligned} (\Lambda_{U_y}^{-1} B_y^* \bar{y}(u) + \alpha_{4y} u_y, v_y - u_y)_{U_y} &\geq 0, & \forall v_y \in U_{y_{ad}}, \\ (\Lambda_{U_x}^{-1} B_x^* \bar{x}(u) + \alpha_{4x} u_x, v_x - u_x)_{U_x} &\geq 0, & \forall v_x \in U_{x_{ad}}. \end{aligned} \right\} \quad (\text{OC}'')$$

At this point we are in a position to formulate the following theorem:

Theorem 2.2. *First order necessary optimality conditions. The optimal control u for the (OCP) can be characterized by the following system of equations and inequalities:*

(SPo) + (LEo) + (ASP) + (ALE) + (OC'') with

$$y(u), \bar{y}(u) \in L^2(0, T; V_y);$$

$$x(u), \bar{x}(u) \in L^2(0, T; V_x); \quad x'(u), \bar{x}'(u) \in L^2(0, T; H_x).$$

The well-posed adjoint boundary value problem, i.e. with boundary and interface conditions, is given by:

$$\left. \begin{aligned} -\rho_1 \bar{y}_t + \nabla \bar{p} - \mu_1 \nabla \cdot (\nabla \bar{y} + \nabla \bar{y}^T) &= \alpha_1 \Lambda_y (y(u) - y_d) \text{ in } (0, T) \times \Omega_1, \\ -\text{div } \bar{y} &= 0 \text{ in } (0, T) \times \Omega_1, \\ \bar{y}(u) &= 0 \text{ on } (0, T) \times \Gamma_1, \\ \bar{y}(T, u) &= 0 \text{ in } \Omega_1. \end{aligned} \right\} \quad (\text{ASP}')$$

$$\left. \begin{aligned}
\rho_2 \bar{x}_{tt} - \mu_2 \nabla \cdot (\nabla \bar{x} + \nabla \bar{x}^T) - \lambda_2 \nabla (\nabla \cdot \bar{x}) &= \alpha_1 \Lambda_x (x(u) - x_d) \text{ in } (0, T) \times \Omega_2, \\
\bar{x} &= 0 \text{ on } (0, T) \times \Gamma_2, \\
\bar{x}(T, u) &= 0 \text{ in } \Omega_2, \\
\bar{x}_t(T, u) &= 0 \text{ in } \Omega_2.
\end{aligned} \right\} \text{ (ALE')}$$

$$\left. \begin{aligned}
\bar{p} n_1 - \mu_1 (\nabla \bar{y} + \nabla \bar{y}^T) \cdot n_1 &= -\bar{y} \quad \text{on } (0, T) \times \Gamma_0, \\
\mu_2 (\nabla \bar{x} + \nabla \bar{x}^T) \cdot n_2 + \lambda_2 (\nabla \cdot \bar{x}) n_2 &= \bar{x}_t \quad \text{on } (0, T) \times \Gamma_0.
\end{aligned} \right\} \text{ (AIC)}$$

3 Numerical realization

The discretization of the fluid-solid interaction problem (SP) + (LE) + (IC) is described firstly in this section. Then, an optimization algorithm is presented to solve the (OCP). A description for its numerical realization follows after that. Finally, a numerical example is presented.

3.1 Discrete solution of the fluid-solid interaction problem

At the beginning we concentrate on the fluid modeled by the Stokes problem. For convenience, we rewrite its mathematical formulation once more:

$$\left. \begin{aligned}
\rho_1 y_t + \nabla p - \mu_1 \nabla \cdot (\nabla y + \nabla y^T) &= \rho_1 f_1 + B_y u_y \text{ in } (0, T) \times \Omega_1, \\
-\text{div } y &= 0 \text{ in } (0, T) \times \Omega_1, \\
y &= 0 \text{ on } (0, T) \times \Gamma_1, \\
y|_{t=0} &= y_0 \text{ in } \Omega_1.
\end{aligned} \right\} (1) \quad \text{(SP)}$$

$$\left. \begin{aligned}
x_t &= y \quad \text{on } \Gamma_0, \\
\mu_2 (\nabla x + \nabla x^T) \cdot n_2 + \lambda_2 (\nabla \cdot x) n_2 &= p n_1 - \mu_1 (\nabla y + \nabla y^T) \cdot n_1 \quad \text{on } \Gamma_0.
\end{aligned} \right\} \begin{matrix} (2) \\ (3) \end{matrix} \quad \text{(IC)}$$

After division of the time interval $(0, T)$ into n equidistant parts, a single time step become the length Δt . The time discretization of the Stokes problem (SP) is of semi-implicit type, i.e. the velocity in the mass conservation equation as well as in the diffusive part of the momentum equation is considered in the new time point. The same is true for the pressure contribution in the momentum equation. The r.h.s. of this equation is considered in contrast at the old time point. Particularly, at each time step we solve the following Quasi-Stokes problem:

$$\left. \begin{aligned}
\rho_1 \frac{y^{i+1} - y^i}{\Delta t} + \nabla p^{i+1} - \mu_1 \nabla \cdot (\nabla y^{i+1} + \nabla (y^{i+1})^T) &= \rho_1 f_1^i + B_y u_y^i, \\
-\text{div } y^{i+1} &= 0.
\end{aligned} \right\} \text{ (DSP)}$$

For the solution of the discrete problem (DSP) a Schurcomplement-method is applied, which is realized using the preconditioned CG-Algorithm (for details see [5, 8]). Details of the implementation: For solving the Quasi-Stokes problem we use the BC (1) + (2).

The spatial discretization of the Stokes problem (SP) is realized by the Finite Volume Particle Method (FVPM). Details of this method for conservation laws see e.g. in [4, 9]. An extension of this method for incompressible flows, i.e. an adaptation of the FVPM for the solution of the Poisson equation and the pressure-correction algorithm see [6]. Also, we give some details how this method should be applied for the PCG-Algorithm later for the concrete numerical example.

In the following we consider the solid which motion is modeled by the linear elasticity problem:

$$\left. \begin{aligned}
\rho_2 x_{tt} - \mu_2 \nabla \cdot (\nabla x + \nabla x^T) - \lambda_2 \nabla (\nabla \cdot x) &= \rho_2 f_2 + B_x u_x \text{ in } (0, T) \times \Omega_2, \\
x &= 0 \text{ on } (0, T) \times \Gamma_2, \\
x|_{t=0} &= x_0 \text{ in } \Omega_2, \\
x_t|_{t=0} &= x_1 \text{ in } \Omega_2.
\end{aligned} \right\} \text{ (LE)}$$

$$\left. \begin{aligned} x_t &= y && \text{on } \Gamma_0, \\ \mu_2(\nabla x + \nabla x^T) \cdot n_2 + \lambda_2(\nabla \cdot x)n_2 &= pn_1 - \mu_1(\nabla y + \nabla y^T) \cdot n_1 && \text{on } \Gamma_0. \end{aligned} \right\} \quad (\text{IC})$$

For its time discretization we apply the "Newmark" time integration scheme (follow description in the book [2]). The crucial idea is to use the Taylor expansion for the displacement x and its velocity x_t at the new time point $(t + \Delta t)$.

$$\begin{aligned} x(t + \Delta t) &\approx x(t) + \Delta t x_t(t) + \frac{\Delta t^2}{2} [(1 - \beta_2)x_{tt}(t) + \beta_2 x_{tt}(t + \Delta t)], \\ x_t(t + \Delta t) &\approx x_t(t) + \Delta t [(1 - \beta_1)x_{tt}(t) + \beta_1 x_{tt}(t + \Delta t)], \end{aligned}$$

where β_1 and β_2 are appropriate parameters, which monitor the implicitity or the explicitity of the integration scheme. In the following numerical computations we use parameters $\beta_1 = 1/2$ and $\beta_2 = 1/2$, which produce an unconditionally stable time stepping scheme. For simplicity of notations we rewrite the system (LE) in an operator setting form as:

$$\rho_2 x_{tt} + A_x(t)x = \rho_2 f_2 + B_x u_x$$

with a linear continuous operator $A_x \in \mathcal{L}(V_x, V_x^*)$ introduced in Section 1. Given the initial values for the displacement $x(0)$ and its velocity $x_t(0)$ at the time point $t = 0$, we obtain the acceleration $x_{tt}(0)$ from the equation:

$$x_{tt}(0) = -\frac{1}{\rho_2} A_x(0)x(0) + f_2 + \frac{1}{\rho_2} B_x u_x.$$

For the successive time steps we calculate $x_{tt}(t + \Delta t)$ from the equation:

$$\left. \begin{aligned} \rho_2 x_{tt}(t + \Delta t) + A_x(t + \Delta t)x(t + \Delta t) &= \\ \rho_2 x_{tt}(t + \Delta t) + A_x(t + \Delta t) \left[x(t) + \Delta t x_t(t) + \frac{\Delta t^2}{2} [(1 - \beta_2)x_{tt}(t) + \beta_2 x_{tt}(t + \Delta t)] \right] &= \\ \left[\rho_2 + \frac{\Delta t^2}{2} \beta_2 A_x(t + \Delta t) \right] x_{tt}(t + \Delta t) + A_x(t + \Delta t) \left[x(t) + \Delta t x_t(t) + \frac{\Delta t^2}{2} (1 - \beta_2)x_{tt}(t) \right] &= \\ \rho_2 f_2 + B_x u_x, \end{aligned} \right\} \quad (\star)$$

and substitute its value if required in:

$$\begin{aligned} x(t + \Delta t) &\approx x(t) + \Delta t x_t(t) + \frac{\Delta t^2}{2} [(1 - \beta_2)x_{tt}(t) + \beta_2 x_{tt}(t + \Delta t)], \\ x_t(t + \Delta t) &\approx x_t(t) + \Delta t [(1 - \beta_1)x_{tt}(t) + \beta_1 x_{tt}(t + \Delta t)]. \end{aligned}$$

The interface boundary condition on Γ_0 are considered once in (\star) by the application of $A_x(t + \Delta t)$ in the second addend as a flux contribution. The space discretization is implemented, similar to the fluid case, using the FVPM.

Now follows the discretization of the adjoint Stokes system (ASP'). Following the argumentation in [1] we apply the next scheme for the time discretization:

$$\left. \begin{aligned} \rho_1 \frac{\bar{y}^{i-1} - \bar{y}^i}{\Delta t} + \nabla \bar{p}^{i-1} - \mu_1 \nabla \cdot (\nabla \bar{y}^{i-1} + \nabla (\bar{y}^{i-1})^T) &= \alpha_1 \Lambda_y (y^{i-1} - y_d^{i-1}), \\ -\text{div } \bar{y}^{i-1} &= 0. \end{aligned} \right\} \quad (\text{DASP})$$

Starting with the final time $t = T$ we apply the discretization scheme (DASP) to obtain values for \bar{y} and \bar{p} for the preceding time point $(t - \Delta t)$, which corresponds to the notation $(i - 1)$ in (DASP). For their computation we apply also the preconditioned CG-algorithm with the schurcomplement-method (see above).

For the time discretization of the adjoint linear elasticity problem we apply the Newmark scheme described above. The difference is only in the starting time point, $t = T$ and Δt is replaced by $-\Delta t$ in the computations concerning the adjoint problem. Spatial discretization for both adjoint problems is provided using FVPM.

Next, we describe the optimization strategy to solve the optimal control problem (OCP). For this purpose we apply the following gradient algorithm:

Algorithm 3.1. Optimization algorithm (*Gradient algorithm*)

$k = 0$, u_0 is given.

S0. If $\|\nabla \hat{J}(u^k)\|_2 \leq \text{tol}$ **Stop**, else:

S1. Compute y_k, p_k, x_k, \dot{x}_k for given u_k using state equations.

S2. Compute $\bar{y}_k, \bar{p}_k, \bar{x}_k, \dot{\bar{x}}_k$ using adjoint equations.

S3. Compute the descent direction v_k as an antigradient

$$v_k = -\hat{J}(u^k).$$

S4. Step length computation:

$$s_k = \operatorname{argmin}_{s>0} \hat{J}(u_k + sv_k).$$

S5. Set $u_{k+1} = u_k + s_k v_k$, $k = k + 1$, goto S0.

Details for application of Algorithm 3.1 will be described in the next subsection, where we consider a concrete numerical example.

3.2 Example

As a numerical example we consider the following two domains $\Omega_1 \cup \Omega_2 = \Omega$. For better understanding the situation is illustrated in Figure 2. The fluid occupies

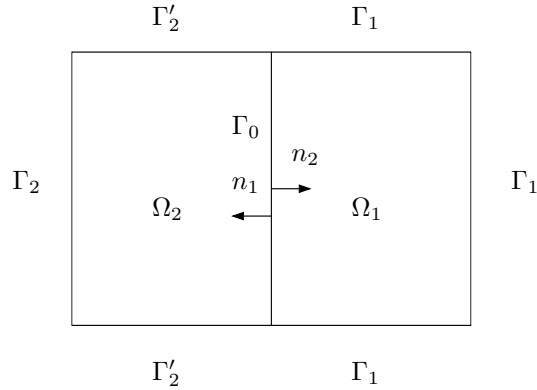


Figure 2: Schematic illustration of physical domains for the numerical example

the domain Ω_1 with the boundary $\Gamma_0 \cup \Gamma_1$, the solid occupies the domain Ω_2 with the boundary $\Gamma_0 \cup \Gamma_2 \cup \Gamma'_2$. Γ_0 is the common boundary for the fluid and for the solid. Each domain is a quadrat with the side length one. We consider $U_{ad} = U$, which simplifies the calculation of the descent direction v_k in the step S3 of the Optimization Algorithm 3.1.

For the realization of the spatial discretization with FVPM $N \times N$ particles $\psi_I(x, t)$, $I = (i, j)$, $i, j = 1, \dots, N$ are equidistantly positioned within each domain, i.e. into Ω_1 and Ω_2 . Theoretically, particles can move with their own velocities, but since the interface Γ_0 is fixed due to the assumptions, velocities of particles are zero. The flux function $\mathcal{F}(y, t)$ for the fluid is

$$\mathcal{F}(y, t) = -\frac{1}{\rho_1} \left(\frac{2\mu_1 \frac{\partial y_x}{\partial x} - p \quad \Big| \quad \mu_1 \left(\frac{\partial y_x}{\partial y} + \frac{\partial y_y}{\partial x} \right)}{\mu_1 \left(\frac{\partial y_x}{\partial y} + \frac{\partial y_y}{\partial x} \right) \quad \Big| \quad 2\mu_1 \frac{\partial y_y}{\partial y} - p} \right).$$

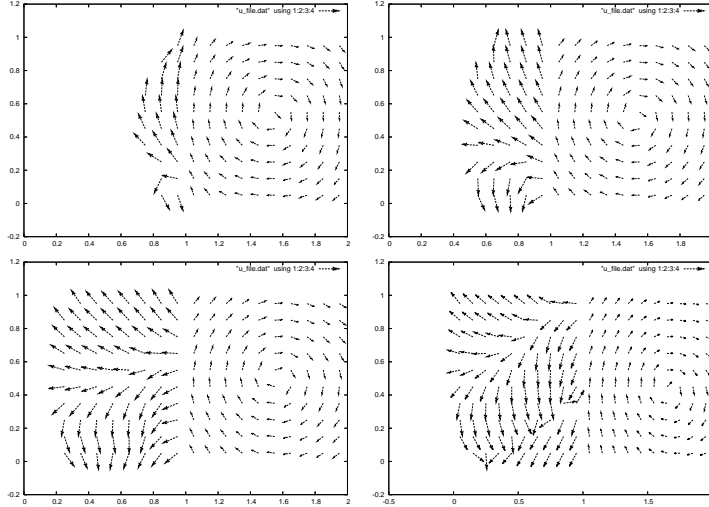


Figure 3: Displacement field for the desired solid and the velocity field for the desired fluid for $t = 0.25$, $t = 0.5$, $t = 0.75$, $t = 1.0$.

The flux function $\mathcal{F}(x, t)$ for the solid is

$$\mathcal{F}(x, t) = -\frac{1}{\rho_2} \left(\frac{2\mu_2 \frac{\partial x_x}{\partial x} + \lambda_2 \left(\frac{\partial x_x}{\partial x} + \frac{\partial x_y}{\partial y} \right) \quad \Big| \quad \mu_2 \left(\frac{\partial x_x}{\partial y} + \frac{\partial x_y}{\partial x} \right)}{\mu_2 \left(\frac{\partial x_x}{\partial y} + \frac{\partial x_y}{\partial x} \right) \quad \Big| \quad 2\mu_2 \frac{\partial x_y}{\partial y} + \lambda_2 \left(\frac{\partial x_x}{\partial x} + \frac{\partial x_y}{\partial y} \right)} \right).$$

In the two formulas above the fluid velocity and solid displacement vectors y and x are defined as $y = \begin{pmatrix} y_x \\ y_y \end{pmatrix}$ and $x = \begin{pmatrix} x_x \\ x_y \end{pmatrix}$. Note that the derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ within the flux functions are evaluated at the consistency points x_{ij} in order to obtain a consistent approximation (for details see [6]). This approach is similar to the staggered grid approach.

Boundary conditions on Γ'_2 for this numerical example slightly differ from the BC assumed so far. For the sake of stability in this example we consider:

$$\mu_2(\nabla x + \nabla x^T) \cdot n_2 + \lambda_2(\nabla \cdot x)n_2 = 0 \quad \text{on } \Gamma'_2. \quad (IC')$$

Parameters α_1 , α_{4x} and α_{4y} are set to one, one and 0.01, respectively. Final time $T = 1$. The desired state is evaluated with controls u_x and u_y are set to zero and 0.1, respectively, for all $t \in [0, T]$. The parameter N for the spatial discretization is set to $N = 10$. The time step interval is $\Delta t = 0.25$. Initial state for the solid displacement and velocity is $x(0) = x_t(0) = 0$ in Ω_2 . Initial fluid velocity is

$$y(x, 0) = e \begin{bmatrix} (\cos 2\pi x_1 - 1) \sin 2\pi x_2 \\ -(\cos 2\pi x_2 - 1) \sin 2\pi x_1 \end{bmatrix} \text{ in } \Omega_1.$$

In Figure 3 the displacement field for the desired solid and the velocity field for the desired fluid are illustrated for several time points. For the solution of the (OCP) with the parameters described above, we apply the Gradient Algorithm 3.1 from the previous section. Starting control variables are $u_x(t) = 1.0$, $u_y(t) = 0$ for all $t \in [0, T]$. Velocity and displacement fields produced with these controls and serving as an initial state for the optimization are illustrated in Figure 4. The convergence behavior of this numerical test is listed in Table 1.

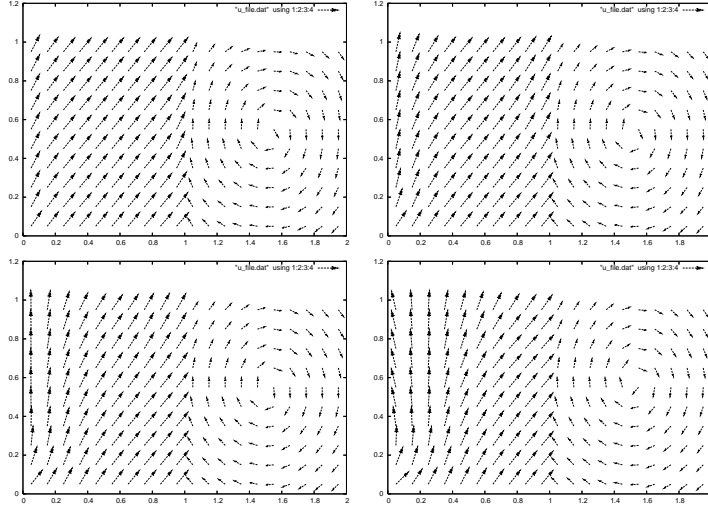


Figure 4: Displacement field for the initial solid and the velocity field for the initial fluid for $t = 0.25$, $t = 0.5$, $t = 0.75$, $t = 1.0$.

Iteration number	Step length	$J(x, y, u_x, u_y)$	$\ \nabla \hat{J}(u_x, u_y)\ _0$
0		0.168569	
1	1.00	0.158357	0.020238
2	1.00	0.158202	0.000175
3	1.00	0.158088	0.000129
4	1.00	0.158004	0.000095

Table 1: Convergence behavior of the Algorithm 3.1 for the numerical example

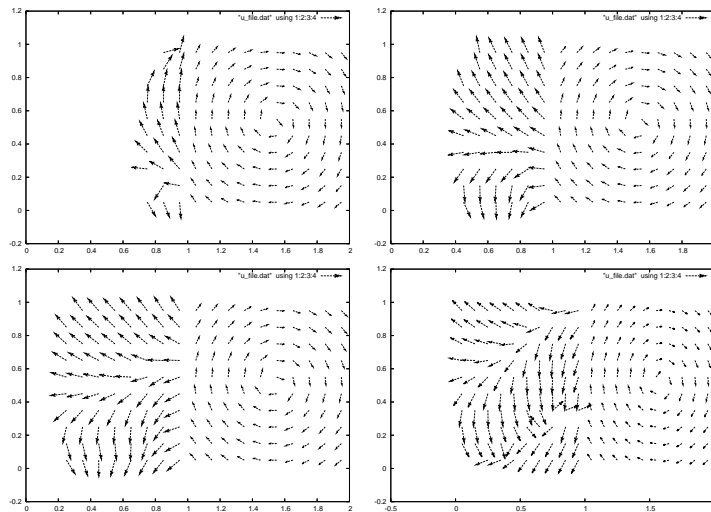


Figure 5: Displacement field for the optimized solid and the velocity field for the optimized fluid for $t = 0.25$, $t = 0.5$, $t = 0.75$, $t = 1.0$.

Figure 5 illustrates displacement and velocity fields for the solid and for the fluid, respectively, obtained after the optimization for different time points.

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