Optimal Control of the Laplace-Beltrami operator on compact surfaces - concept and numerical treatment

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Abstract: We consider optimal control problems of elliptic PDEs on hypersurfaces \( \Gamma \) in \( \mathbb{R}^n \) for \( n = 2, 3 \). The leading part of the PDE is given by the Laplace-Beltrami operator, which is discretized by finite elements on a polyhedral approximation of \( \Gamma \). The discrete optimal control problem is formulated on the approximating surface and is solved numerically with a semi-smooth Newton algorithm. We derive optimal a priori error estimates for problems including control constraints and provide numerical examples confirming our analytical findings.

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1 Introduction

We are interested in the numerical treatment of the following linear-quadratic optimal control problem on a \( n \)-dimensional, sufficiently smooth hypersurface \( \Gamma \subset \mathbb{R}^{n+1}, n = 1, 2 \).

\[
\min_{u \in L^2(\Gamma), y \in H^1(\Gamma)} J(u, y) = \frac{1}{2} \|y - z\|^2_{L^2(\Gamma)} + \frac{\alpha}{2} \|u\|^2_{L^2(\Gamma)}
\]

subject to \( u \in U_{ad} \) and

\[
\int_{\Gamma} \nabla y \nabla \varphi + c y \varphi \, d\Gamma = \int_{\Gamma} u \varphi \, d\Gamma, \forall \varphi \in H^1(\Gamma)
\]

with \( U_{ad} = \{ v \in L^2(\Gamma) \mid a \leq v \leq b \} \), \( a < b \in \mathbb{R} \). For simplicity we will assume \( \Gamma \) to be compact and \( c = 1 \). In section 4 we briefly investigate the case \( c = 0 \), in section 5 we give an example on a surface with boundary.

Problem (1.1) may serve as a mathematical model for the optimal distribution of surfactants on a biomembrane \( \Gamma \) with regard to achieving a prescribed desired concentration \( z \) of a quantity \( y \).

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It follows by standard arguments that (1.1) admits a unique solution \( u \in U_{ad} \) with unique associated state \( y = y(u) \in H^2(\Gamma) \).

Our numerical approach uses variational discretization applied to (1.1), see [Hin05] and [HPUU09], on a discrete surface \( \Gamma^h \) approximating \( \Gamma \). The discretization of the state equation in (1.1) is achieved by the finite element method proposed in [Dzi88], where a priori error estimates for finite element approximations of the Poisson problem for the Laplace-Beltrami operator are provided. Let us mention that uniform estimates are presented in [Dem09], and steps towards a posteriori error control for elliptic PDEs on surfaces are taken by Demlow and Dziuk in [DD07]. For alternative approaches for the discretization of the state equation by finite elements see the work of Burger [Bur08]. Finite element methods on moving surfaces are developed by Dziuk and Elliott in [DE07]. To the best of the authors knowledge, the present paper contains the first attempt to treat optimal control problems on surfaces.

We assume that \( \Gamma \) is of class \( C^2 \) with unit normal field \( \nu \). As an embedded, compact hypersurface in \( \mathbb{R}^{n+1} \) it is orientable and hence the zero level set of a signed distance function \( d(x) = \text{dist}(x, \Gamma) \). We assume w.l.o.g. \( \nabla d(x) = \nu(x) \) for \( x \in \Gamma \). Further, there exists an neighborhood \( N \subset \mathbb{R}^{n+1} \) of \( \Gamma \), such that \( d \) is also of class \( C^2 \) on \( N \) and the projection

\[
a : N \rightarrow \Gamma, \quad a(x) = x - d(x) \nabla d(x) \tag{1.2}
\]

is unique, see e.g. [GT98, Lemma 14.16]. Note that \( \nabla d(x) = \nu(a(x)) \).

Using \( a \) we can extend any function \( \phi : \Gamma \rightarrow \mathbb{R} \) to \( \mathcal{N} \) as \( \tilde{\phi}(x) = \phi(a(x)) \). This allows us to represent the surface gradient in global exterior coordinates \( \nabla_{\Gamma} \phi = (I - \nu \nu^T) \nabla \tilde{\phi} \), with the euclidean projection \( (I - \nu \nu^T) \) onto the tangential space of \( \Gamma \).

We use the Laplace-Beltrami operator \( \Delta_{\Gamma} = \nabla_{\Gamma} \cdot \nabla_{\Gamma} \) in its weak form i.e. \( \Delta_{\Gamma} : H^1(\Gamma) \rightarrow H^1(\Gamma)^* 
\]

\[
y \mapsto - \int_{\Gamma} \nabla_{\Gamma} y \nabla_{\Gamma}(\cdot) \, d\Gamma \in H^1(\Gamma)^*. 
\]

Let \( S \) denote the prolonged restricted solution operator of the state equation

\[
S : L^2(\Gamma) \rightarrow L^2(\Gamma), \quad u \mapsto y - \Delta_{\Gamma} y + cy = u, 
\]

which is compact and constitutes a linear homeomorphism onto \( H^2(\Gamma) \), see [Dzi88, 1. Theorem].

By standard arguments we get the following necessary (and here also sufficient) conditions for optimality of \( u \in U_{ad} \)

\[
\langle \nabla u J(u, y(u)), v - u \rangle_{L^2(\Gamma)} = \langle \alpha u + S^*(Su - z), v - u \rangle_{L^2(\Gamma)} \geq 0 \quad \forall v \in U_{ad}, \tag{1.3}
\]

We rewrite (1.3) as

\[
u = P_{U_{ad}} \left( - \frac{1}{\alpha} S^*(Su - z) \right), \tag{1.4}
\]

where \( P_{U_{ad}} \) denotes the \( L^2 \)-orthogonal projection onto \( U_{ad} \).

2 Discretization

We now discretize (1.1) using an approximation \( \Gamma^h \) to \( \Gamma \) which is globally of class \( C^{0,1} \).

Following Dziuk, we consider polyhedral \( \Gamma^h = \bigcup_{i \in I_h} T^i_h \) consisting of triangles \( T^i_h \) with corners
on $\Gamma$, whose maximum diameter is denoted by $h$. With FEM error bounds in mind we assume the family of triangulations $\Gamma^h$ to be regular in the usual sense that the angles of all triangles are bounded away from zero uniformly in $h$.

We assume for $\Gamma^h$ that $a(\Gamma^h) = \Gamma$, with $a$ from (1.2). For small $h > 0$ the projection $a$ also is injective on $\Gamma^h$. In order to compare functions defined on $\Gamma^h$ with functions on $\Gamma$ we use $a$ to lift a function $y \in L^2(\Gamma^h)$ to $\Gamma$:

$$y'(a(x)) = y(x) \quad \forall x \in \Gamma^h,$$

and for $y \in L^2(\Gamma)$ and sufficiently small $h > 0$ we define the inverse lift

$$y_i(x) = y(a(x)) \quad \forall x \in \Gamma^h.$$

For small mesh parameters $h$ the lift operation $(\cdot)_i : L^2(\Gamma) \to L^2(\Gamma^h)$ defines a linear homeomorphism with inverse $(\cdot)'$. Moreover, there exists $c_{\text{int}} > 0$ such that

$$1 - c_{\text{int}} h^2 \leq \| (\cdot)' \|^2_{L^2(\Gamma^h),L^2(\Gamma)}, \| (\cdot)' \|^2_{L^2(\Gamma^h),L^2(\Gamma)} \leq 1 + c_{\text{int}} h^2,$$

where $\det(\cdot)$ represents the Derivative $a(x) : T_x \Gamma^h \to T_{a(x)} \Gamma$ with respect to arbitrary orthonormal bases of the respective tangential space. For small $h > 0$ there holds

$$\sup_{\Gamma'} \left| 1 - \frac{d\Gamma'}{d\Gamma} \right| \leq c_{\text{int}} h^2,$$

Now let $\frac{d\Gamma}{d\Gamma}$ denote $|\det(M^{-1})|$, so that by the change of variable formula

$$\left| \int_{\Gamma^h} v_1 \, d\Gamma^h - \int_{\Gamma} v \, d\Gamma \right| = \left| \int_{\Gamma} v \frac{d\Gamma}{d\Gamma} - v \, d\Gamma \right| \leq c_{\text{int}} h^2 \| v \|_{L^1(\Gamma)}.$$

Proof. see [DE07, Lemma 5.1]

Problem (1.1) is approximated by the following sequence of optimal control problems

$$\min_{u \in L^2(\Gamma^h), y \in H^1(\Gamma^h)} J(u,y) = \frac{1}{2} \| y - z \|^2_{L^2(\Gamma^h)} + \frac{\alpha}{2} \| u \|^2_{L^2(\Gamma^h)}$$

subject to $u \in U_{ad}^h$ and

$$y = S_h u,$$

with $U_{ad}^h = \{ v \in L^2(\Gamma^h) \mid a \leq v \leq b \}$, i.e. the mesh parameter $h$ enters into $U_{ad}$ only through $\Gamma^h$. Problem (2.2) may be regarded as the extension of variational discretization introduced in [Hin05] to optimal control problems on surfaces.

In [Dzi88] it is explained, how to implement a discrete solution operator $S_h : L^2(\Gamma^h) \to L^2(\Gamma^h)$, such that

$$\| (\cdot)' S_h (\cdot) - S \|^2_{L^2(\Gamma^h),L^2(\Gamma)} \leq C_{\text{FE}} h^2,$$

which we will use throughout this paper. See in particular [Dzi88, Equation (6)] and [Dzi88, 7, Lemma]. For the convenience of the reader we briefly sketch the method. Consider the space

$$V_h = \{ \varphi \in C^0(\Gamma^h) \mid \forall i \in I_h : \varphi_{| T_i^h } \in P^1(T_i^h) \} \subset H^1(\Gamma^h)$$
Theorem 2.2 (Order of Convergence)

Let us now investigate the relation between the optimal control problems (1.1) and (2.2).

Proof. From (2.6) it follows that the projection of \( y_h \) onto \( U_{ad}^h \) for any \( y_h \in V_h \). We choose \( L^2(\Gamma^h) \) as control space, because in general we cannot evaluate \( \int_{\Gamma^h} v \, d\Gamma \) exactly, whereas the expression \( \int_{\Gamma^h} v_l \, d\Gamma^h \) for piecewise polynomials \( v_l \) can be computed up to machine accuracy. Also, the operator \( S_h \) is self-adjoint, while \( (\cdot)^* S_h(\cdot) \) is not. The adjoint operators of (1.1) and (2.2) have the shapes

\[
\forall v \in L^2(\Gamma^h) : (\cdot)^* v = \frac{d\Gamma^h}{d\Gamma} \cdot v', \quad \forall v \in L^2(\Gamma) : (\cdot)^* v = \frac{d\Gamma}{d\Gamma} \cdot v',
\]

hence evaluating \( (\cdot)^* \) requires knowledge of the Jacobians \( \frac{d\Gamma^h}{d\Gamma} \) and \( \frac{d\Gamma}{d\Gamma} \) which may not be known analytically.

Similar to (1.1), problem (2.2) possesses a unique solution \( u_h \in U_{ad}^h \) which satisfies

\[
u_h = P_{U_{ad}^h} \left(-\frac{1}{\alpha} p_h(u_h)\right).
\]

(2.5)

Here \( P_{U_{ad}^h} : L^2(\Gamma^h) \to U_{ad}^h \) is the \( L^2(\Gamma^h) \)-orthogonal projection onto \( U_{ad}^h \) and for \( v \in L^2(\Gamma^h) \) the adjoint state is \( p_h(v) = S_h^*(y_h - z) \in H^1(\Gamma^h) \).

Observe that the projections \( P_{U_{ad}} \) and \( P_{U_{ad}^h} \) coincide with the point-wise projection \( P_{[a,b]} \) on \( \Gamma \) and \( \Gamma^h \), respectively, and hence

\[
\left(P_{U_{ad}^h}(v_l)\right)^* = P_{U_{ad}}(v)
\]

(2.6)

for any \( v \in L^2(\Gamma) \).

Let us now investigate the relation between the optimal control problems (1.1) and (2.2).

**Theorem 2.2 (Order of Convergence)**. Let \( u \in L^2(\Gamma) \), \( u_h \in L^2(\Gamma^h) \) be the solutions of (1.1) and (2.2), respectively. Then for sufficiently small \( h > 0 \) there holds

\[
\alpha \left\| u_h - u \right\|^2_{L^2(\Gamma)} + \left\| y_h - y \right\|^2_{L^2(\Gamma)} \leq \frac{1}{1 + c_{\text{int}} h^2} \left( \frac{1}{\alpha} \left\| (\cdot)^* S_h(\cdot) - S^* \right\|_{L^2(\Gamma)} \right) \left\| y - z \right\|^2_{L^2(\Gamma)} + \left\| (\cdot)^* S_h(\cdot) - S^* \right\|_{L^2(\Gamma)} \left\| u \right\|^2_{L^2(\Gamma)}
\]

(2.7)

with \( y = Su \) and \( y_h = S_h u_h \).

**Proof**. From (2.6) it follows that the projection of \( -\left(\frac{1}{\alpha} p(u)\right)_l \) onto \( U_{ad}^h \) is \( u_l \)

\[
u_l = P_{U_{ad}^h} \left(-\frac{1}{\alpha} p(u)_l\right)
\]

which we insert into the necessary condition of (2.2). This gives

\[
\langle \alpha u_h + p_h(u_h), u_l - u_h \rangle_{L^2(\Gamma^h)} \geq 0
\]

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On the other hand $u_l$ is the $L^2(\Gamma^h)$-orthogonal projection of $-\frac{1}{\alpha} p(u)_l$, thus

$$\langle -\frac{1}{\alpha} p(u)_l - u_l, u_h - u_l \rangle_{L^2(\Gamma^h)} \leq 0.$$  

Adding these inequalities yields

$$\alpha \| u_l - u_h \|^2_{L^2(\Gamma^h)} \leq \langle (p_h(u_h) - p(u)_l), u_l - u_h \rangle_{L^2(\Gamma^h)}$$

$$= \langle p_h(u_h) - S_h^*(y-z)_l, u_l - u_h \rangle_{L^2(\Gamma^h)} + \langle S_h^*(y-z)_l - p(u)_l, u_l - u_h \rangle_{L^2(\Gamma^h)}.$$  

The first addend is estimated via

$$\langle (\cdot)_l \rangle \leq \frac{\alpha}{2} \| u_l - u_h \|^2_{L^2(\Gamma^h)} + \frac{1}{2\alpha} \| S_h^*(y-z)_l - p(u)_l \|^2_{L^2(\Gamma^h)}.$$  

The second addend satisfies

$$\langle S_h^*(y-z)_l - p(u)_l, u_l - u_h \rangle_{L^2(\Gamma^h)} \leq \frac{\alpha}{2} \| u_l - u_h \|^2_{L^2(\Gamma^h)} + \| S_h^*(y-z)_l - p(u)_l \|^2_{L^2(\Gamma^h)}.$$  

Together this yields

$$\alpha \| u_l - u_h \|^2_{L^2(\Gamma^h)} + \| y_h - y_l \|^2_{L^2(\Gamma^h)} \leq \frac{\alpha}{2} \| S_h^*(y-z)_l - p(u)_l \|^2_{L^2(\Gamma^h)} + \| S_h u_l - y_l \|^2_{L^2(\Gamma^h)}.$$  

The claim follows using (2.1) for sufficiently small $h > 0$.  

Because both $S$ and $S_h$ are self-adjoint, quadratic convergence follows directly from (2.7). For operators that are not self-adjoint one can use

$$\| \langle \rangle \|_{L^2(\Gamma^h)} \leq C_{\text{FE}} h^2.$$  

which is a consequence of (2.3). Equation (2.4) and Lemma 2.1 imply

$$\| \langle \rangle \|_{L^2(\Gamma^h)} \leq C_{\text{int}} h^2,$$  

$$\| \langle \rangle \|_{L^2(\Gamma^h)} \leq C_{\text{int}} h^2.$$  

Combine (2.7) with (2.8) and (2.9) to proof quadratic convergence for arbitrary linear elliptic state equations.

3 Implementation

In order to solve (2.5) numerically, we proceed as in [Hin05] using the finite element techniques for PDEs on surfaces developed in [Dzi88] combined with the semi-smooth Newton techniques from [HIK03] and [Ulb03] applied to the equation

$$G_h(u_h) = \left( u_h - P_{[a,b]} \left( \frac{-1}{\alpha} p_h(u_h) \right) \right) = 0.$$  

Since the operator $p_h$ continuously maps $v \in L^2(\Gamma^h)$ into $H^1(\Gamma^h)$, Equation (3.1) is semi-smooth and thus is amenable to a semismooth Newton method. The generalized derivative of $G_h$ is given by

$$DG_h(u) = \left( I + \frac{\alpha}{\alpha} S_h^* S_h \right).$$
where \( \chi : \Gamma^h \to \{0, 1\} \) denotes the indicator function of the inactive set \( \mathcal{I}(-\frac{1}{\alpha}p_h(u)) = \{ \gamma \in \Gamma^h \mid a < -\frac{1}{\alpha}p_h(u)[\gamma] < b \} \)

\[
\chi = \begin{cases} 1 & \text{on } \mathcal{I}(-\frac{1}{\alpha}p_h(u)) \subset \Gamma^h \\ 0 & \text{elsewhere on } \Gamma^h \end{cases},
\]

which we use both as a function and as the operator \( \chi : L^2(\Gamma^h) \to L^2(\Gamma^h) \) defined as the point-wise multiplication with the function \( \chi \). A step semi-smooth Newton method for (3.1) then reads

\[
(I + \frac{\chi}{\alpha}S_h^*S_h)u^+ = -G_h(u) + DG_h(u)u = P_{[a,b]}\left(-\frac{1}{\alpha}p_h(u)\right) + \frac{\chi}{\alpha}S_h^*S_hu.
\]

Given \( u \) the next iterate \( u^+ \) is computed by performing three steps

1. Set \( ((1 - \chi)u^+)\eta = (1 - \chi)P_{[a,b]}\left(-\frac{1}{\alpha}p_h(u) + m\right)\eta \), which is either \( a \) or \( b \), depending on \( \gamma \in \Gamma^h \).

2. Solve

\[
(I + \frac{\chi}{\alpha}S_h^*S_h)\chi u^+ = \frac{\chi}{\alpha}(S_h^*z - S_h^*S_h(1 - \chi)u^+)
\]

for \( \chi u^+ \) by CG iteration over \( L^2(\mathcal{I}(-\frac{1}{\alpha}p_h(u))) \).

3. Set \( u^+ = \chi u^+ + (1 - \chi)u^+ \).

Details can be found in [HV11].

4 The case \( c = 0 \)

In this section we investigate the case \( c = 0 \) which corresponds to a stationary, purely diffusion driven process. Since \( \Gamma \) has no boundary, in this case total mass must be conserved, i.e. the state equation admits a solution only for controls with mean value zero. For such a control the state is uniquely determined up to a constant. Thus the admissible set \( U_{ad} \) has to be changed to

\[
U_{ad} = \{ v \in L^2(\Gamma) \mid a \leq v \leq b \} \cap L^2_0(\Gamma), \text{ where } L^2_0(\Gamma) := \{ v \in L^2(\Gamma) \mid \int_\Gamma v \, d\Gamma = 0 \},
\]

and \( a < 0 < b \). Problem (1.1) then admits a unique solution \( (u, y) \) and there holds \( \int_\Gamma y \, d\Gamma = \int_\Gamma z \, d\Gamma \). W.l.o.g we assume \( \int_\Gamma z \, d\Gamma = 0 \) and therefore only need to consider states with mean value zero. The state equation now reads \( y = \tilde{S}u \) with the solution operator \( \tilde{S} : L^2_0(\Gamma) \to L^2_0(\Gamma) \) of the equation \( -\Delta \gamma y = u, \int_\Gamma y \, d\Gamma = 0 \).

Using the injection \( L^2_0(\Gamma) \to L^2(\Gamma) \), \( \tilde{S} \) is prolonged as an operator \( S : L^2(\Gamma) \to L^2(\Gamma) \) by \( S = i\tilde{S}r^* \). The adjoint \( r^* : L^2(\Gamma) \to L^2_0(\Gamma) \) of \( i \) is the \( L^2 \)-orthogonal projection onto \( L^2_0(\Gamma) \). The unique solution of (1.1) is again characterized by (1.4), where the orthogonal projection now takes the form

\[
P_{U_{ad}}(v) = P_{[a,b]}(v + m)
\]

with \( m \in \mathbb{R} \) chosen such that

\[
\int_\Gamma P_{[a,b]}(v + m) \, d\Gamma = 0.
\]
If \( v \in L^2(\Gamma) \) the inactive set \( \mathcal{I}(v + m) = \{ \gamma \in \Gamma \mid a < v|\gamma| + m < b \} \) is non-empty, the constant \( m = m(v) \) is uniquely determined by \( v \in L^2(\Gamma) \). Hence, the solution \( u \in U_{ad} \) satisfies

\[
u = P_{[a,b]} \left( -\frac{1}{\alpha} p(u) + m \left( -\frac{1}{\alpha} p(u) \right) \right),
\]

with \( p(u) = S^*(S u - v^* z) \in H^2(\Gamma) \) denoting the adjoint state and \( m(-\frac{1}{\alpha} p(u)) \in \mathbb{R} \) is implicitly given by \( \int_{\Gamma} \nu d\Gamma = 0 \). Note that \( v^* \) is the identity on \( L^0_0(\Gamma) \).

In (2.2) we now replace \( U_{ad}^h \) by \( U_{ad}^h = \{ v \in L^2(\Gamma^h) \mid a \leq v \leq b \} \cap L^2_0(\Gamma^h) \). Similar as in (2.5), the unique solution \( u_h \) then satisfies

\[
u_h = P_{U_{ad}^h} \left( -\frac{1}{\alpha} p_h(u_h) \right) = P_{[a,b]} \left( -\frac{1}{\alpha} p_h(u_h) + m_h \left( -\frac{1}{\alpha} p_h(u_h) \right) \right),
\]

with \( p_h(v_h) = S^h(S_h v_h - v^*_h z) \in H^1(\Gamma^h) \) and \( m_h(-\frac{1}{\alpha} p_h(u_h)) \in \mathbb{R} \) the unique constant such that \( \int_{\Gamma^h} u_h d\Gamma^h = 0 \). Note that \( m_h(-\frac{1}{\alpha} p_h(u_h)) \) is semi-smooth with respect to \( u_h \) and thus Equation (4.1) is amenable to a semi-smooth Newton method.

The discretization error between the problems (2.2) and (1.1) now decomposes into two components, one introduced by the discretization of \( U_{ad} \) through the discretization of the surface, the other by discretization of \( S \).

For the first error we need to investigate the relation between \( P_{U_{ad}^h}(u) \) and \( P_{U_{ad}^h}(u) \), which is now slightly more involved than in (2.6).

**Lemma 4.1.** Let \( h > 0 \) be sufficiently small. There exists a constant \( C_m > 0 \) depending only on \( \Gamma \), \( |a| \) and \( |b| \) such that for all \( v \in L^2(\Gamma) \) with \( \int_{\mathcal{I}(v+m(v))} d\Gamma > 0 \) there holds

\[
|m_v(v_l) - m(v)| \leq \frac{C_m}{\int_{\mathcal{I}(v+m(v))} d\Gamma} h^2.
\]

**Proof.** For \( v \in L^2(\Gamma) \), \( \epsilon > 0 \) choose \( \delta > 0 \) and \( h > 0 \) so small that the set

\[
\mathcal{I}_\delta = \left\{ \gamma \in \Gamma^h \mid a + \delta \leq v_l(\gamma) + m(\gamma) \leq b - \delta \right\}.
\]

satisfies \( \int_{\mathcal{I}_\delta} d\Gamma^h(1+\epsilon) \geq \int_{\mathcal{I}(v+m(v))} d\Gamma \). It is easy to show that hence \( m_v(v_l) \) is unique. Set \( C = c_{\text{int}} \max(|a|, |b|) \int_{\Gamma} d\Gamma \). Decreasing \( h \) further if necessary ensures

\[
\frac{Ch^2}{\int_{\mathcal{I}_\delta} d\Gamma^h} \leq (1+\epsilon) \frac{Ch^2}{\int_{\mathcal{I}(v+m(v))} d\Gamma} \leq \delta.
\]

For \( x \in \mathbb{R} \) let

\[
M_v^h(x) = \int_{\Gamma^h} P_{[a,b]} (v_l + x) d\Gamma^h.
\]

Since \( \int_{\Gamma} P_{[a,b]} (v + m(v)) d\Gamma = 0 \), Lemma 2.1 yields

\[
|M_v^h(m(v))| \leq c_{\text{int}} \| P_{[a,b]} (v + m(v)) \|_{L^1(\Gamma)} h^2 \leq Ch^2.
\]

Let us assume w.l.o.g. \( -Ch^2 \leq M_v^h(m(v)) \leq 0 \). Then

\[
M_v^h \left( m(v) + \frac{Ch^2}{\int_{\mathcal{I}_\delta} d\Gamma^h} \right) \geq M_v^h (m(v)) + Ch^2 \geq 0
\]
implies \( 0 \leq m(v) - m_h(v) \leq \frac{Ch^2}{\int_{x \in \Gamma} d\Gamma} \leq \frac{(1+\epsilon)c}{\int_{x \in \Gamma} d\Gamma} h^2 \), since \( M_h^2(x) \) is continuous with respect to \( x \). This proves the claim.

Because
\[
\left( P_{U_{ad}^h(v)} \right)^t - P_{U_{ad}}(v) = P_{[a,b]}(v + m_h(v)) - P_{[a,b]}(v + m(v)),
\]
we get the following corollary.

**Corollary 4.2.** Let \( h > 0 \) be sufficiently small and \( C_m \) as in Lemma 4.1. For any fixed \( v \in L^2(\Gamma) \) with \( \int_{\Gamma} d\Gamma > 0 \) we have
\[
\left\| \left( P_{U_{ad}^h(v)} \right)^t - P_{U_{ad}}(v) \right\|_{L^2(\Gamma)} \leq C_m \frac{\sqrt{\int_{\Gamma} d\Gamma}}{\int_{\Gamma} d\Gamma} h^2.
\]

Note that since for \( u \in L^2(\Gamma) \) the adjoint \( p(u) \) is a continuous function on \( \Gamma \), the corollary is applicable for \( v = -\frac{1}{\alpha}p(u) \).

The following theorem can be proofed along the lines of Theorem 2.2.

**Theorem 4.3.** Let \( u \in L^2(\Gamma) \), \( u_h \in L^2(\Gamma^h) \) be the solutions of (1.1) and (2.2), respectively, in the case \( c = 0 \). Let \( \tilde{u}_h = \left( P_{U_{ad}^h \left( -\frac{1}{\alpha}p(u) \right)} \right)^t \). Then there holds for \( \epsilon > 0 \) and \( 0 \leq h < h_* \)
\[
\alpha \| u_h^l - \tilde{u}_h \|_{L^2(\Gamma)}^2 + \| y_h^l - z \|_{L^2(\Gamma)}^2 \leq \left( 1 + \epsilon \right) \left( \frac{1}{\alpha} \left\| (\cdot)^t S_h^*(\cdot) - S^* \right\|_{L^2(\Gamma)} \right) \left\| y - z \right\|_{L^2(\Gamma)}^2 + \left\| (\cdot)^t S_h^*(\cdot) \tilde{u}_h - y \right\|_{L^2(\Gamma)}^2.
\]

Using Corollary 4.2 we conclude from the theorem
\[
\| u_h^l - u \|_{L^2(\Gamma)} \leq C \left( \frac{1}{\alpha} \left\| (\cdot)^t S_h^*(\cdot) - S^* \right\|_{L^2(\Gamma)} + \frac{1}{\sqrt{\alpha}} \left\| (\cdot)^t S_h^*(\cdot) - S \right\|_{L^2(\Gamma)} \right) \left\| y - z \right\|_{L^2(\Gamma)} + \left( 1 + \frac{\| S \|_{L(\Gamma^h),L(\Gamma^h)}}{\sqrt{\alpha}} \right) \frac{C_m \sqrt{\int_{\Gamma} d\Gamma} h^2}{\int_{\Gamma} p(u + m(-\frac{1}{\alpha}p(u))) d\Gamma},
\]
the latter part of which is the error introduced by the discretization of \( U_{ad} \). Hence one has \( h^2 \)-convergence of the optimal controls.

## 5 Numerical Examples

The figures show some selected Newton steps \( u^+ \). Note that jumps of the color-coded function values are well observable along the border between active and inactive set. For all examples Newton’s method is initialized with \( u_0 \equiv 0 \).

The meshes are generated from a macro triangulation through congruent refinement, new nodes are projected onto the surface \( \Gamma \). The maximal edge length \( h \) in the triangulation is not exactly halved in each refinement, but up to an error of order \( O(h^2) \). Therefore we just compute our estimated order of convergence (EOC) according to
\[
EOC_i = \frac{\ln \| u_{h_{i-1}} - u \|_{L^2(\Gamma^{h_{i-1}})} - \ln \| u_{h_i} - u \|_{L^2(\Gamma^{h_i})}}{\ln(2)},
\]

For different refinement levels, the tables show $L^2$-errors, the corresponding EOC and the number of Newton iterations before the desired accuracy of $10^{-6}$ is reached.

It was shown in [HU04], under certain assumptions on the behaviour of $-\frac{1}{\alpha}p(u)$, that the undamped Newton Iteration is mesh-independent. These assumptions are met by all our examples, since the surface gradient of $-\frac{1}{\alpha}p(u)$ is bounded away from zero along the border of the inactive set. Moreover, the displayed number of Newton-Iterations suggests mesh-independence of the semi-smooth Newton method.

**Example 5.1 (Sphere I).** We consider the problem

$$\min_{u \in L^2(\Gamma), y \in H^1(\Gamma)} J(u, y) \text{ subject to } -\Delta_{\Gamma} y + y = u - r, \quad -1 \leq u \leq 1$$

(5.1)

with $\Gamma$ the unit sphere in $\mathbb{R}^3$ and $\alpha = 1.5 \cdot 10^{-6}$. We choose $z = 52\alpha x_3(x_1^2 - x_2^2)$, to obtain the solution

$$\bar{u} = r = \min (1, \max (-1, 4x_3(x_1^2 - x_2^2)))$$

of (5.1).

**Example 5.2.** Let $\Gamma = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_3 = x_1x_2 \wedge x_1, x_2 \in (0, 1)\}$ and $\alpha = 10^{-3}$. For

$$\min_{u \in L^2(\Gamma), y \in H^1(\Gamma)} J(u, y) \text{ subject to } -\Delta_{\Gamma} y = u - r, \quad y = 0 \text{ on } \partial \Gamma, \quad -0.5 \leq u \leq 0.5$$

we get

$$\bar{u} = r = \max \left(-0.5, \min \left(0.5, \sin(\pi x) \sin(\pi y)\right)\right)$$

by proper choice of $z$ (via symbolic differentiation).

Example 5.2, although $c = 0$, is also covered by the theory in Sections 1-3, as by the Dirichlet boundary conditions the state equation remains uniquely solvable for $u \in L^2(\Gamma)$. In the last two examples we apply the variational discretization to optimization problems, that involve zero-mean-value constraints as in Section 4.

For different refinement levels, the tables show $L^2$-errors, the corresponding EOC and the number of Newton iterations before the desired accuracy of $10^{-6}$ is reached.

### Table 1: $L^2$-error, EOC and number of iterations for Example 5.1.

<table>
<thead>
<tr>
<th>reg. refs</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L^2$-error</td>
<td>5.8925e-01</td>
<td>1.4299e-01</td>
<td>3.5120e-02</td>
<td>8.7123e-03</td>
<td>2.2057e-03</td>
<td>5.4855e-04</td>
</tr>
<tr>
<td>EOC</td>
<td>-</td>
<td>2.0430</td>
<td>2.0255</td>
<td>2.0112</td>
<td>1.9818</td>
<td>2.0075</td>
</tr>
<tr>
<td># Steps</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>
Table 2: $L^2$-error, EOC and number of iterations for Example 5.2.

<table>
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<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L^2$-error</td>
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<td>6.6120e-02</td>
<td>1.5904e-02</td>
<td>3.6357e-03</td>
<td>8.8597e-04</td>
<td>2.1769e-04</td>
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<tr>
<td>EOC</td>
<td>-</td>
<td>2.4173</td>
<td>2.0557</td>
<td>2.1291</td>
<td>2.0369</td>
<td>2.0250</td>
</tr>
<tr>
<td># Steps</td>
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<td>12</td>
<td>12</td>
<td>11</td>
<td>13</td>
<td>12</td>
</tr>
</tbody>
</table>

Table 3: $L^2$-error, EOC and number of iterations for Example 5.3.

<table>
<thead>
<tr>
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<th>4</th>
<th>5</th>
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</thead>
<tbody>
<tr>
<td>$L^2$-error</td>
<td>6.7223e-01</td>
<td>1.6646e-01</td>
<td>4.3348e-02</td>
<td>1.1083e-02</td>
<td>2.7879e-03</td>
<td>6.9832e-04</td>
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<tr>
<td>EOC</td>
<td>-</td>
<td>2.0138</td>
<td>1.9412</td>
<td>1.9677</td>
<td>1.9911</td>
<td>1.9972</td>
</tr>
<tr>
<td># Steps</td>
<td>8</td>
<td>8</td>
<td>7</td>
<td>7</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

Table 4: $L^2$-error, EOC and number of iterations for Example 5.4.
<table>
<thead>
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<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
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<td>$L^2$-error</td>
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<td>9.8016e-02</td>
<td>2.6178e-02</td>
<td>6.6283e-03</td>
<td>1.6680e-03</td>
<td>4.1889e-04</td>
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<tr>
<td>EOC</td>
<td>-</td>
<td>1.8198e+00</td>
<td>1.9047e+00</td>
<td>1.9816e+00</td>
<td>1.9905e+00</td>
<td>1.9935e+00</td>
</tr>
<tr>
<td>$#$ Steps</td>
<td>9</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 4: $L^2$-error, EOC and number of iterations for Example 5.4.

**Example 5.3** (Sphere II). We consider

$$\min_{u \in L^2(\Gamma), y \in H^1(\Gamma)} J(u, y) \text{ subject to } -\Delta\Gamma y = u, \quad -1 \leq u \leq 1, \quad \int_{\Gamma} y \, d\Gamma = \int_{\Gamma} u \, d\Gamma = 0,$$

with $\Gamma$ the unit sphere in $\mathbb{R}^3$. Set $\alpha = 10^{-3}$ and

$$z(x_1, x_2, x_3) = 4\alpha x_3 + \begin{cases} 
\ln(x_3 + 1) + C, & \text{if } 0.5 \leq x_3 \\
x_3 - \frac{1}{4}\arctanh(x_3), & \text{if } -0.5 \leq x_3 \leq 0.5 \\
-C - \ln(1 - x_3), & \text{if } x_3 < -0.5
\end{cases},$$

where $C$ is chosen for $z$ to be continuous. The solution according to these parameters is

$$\bar{u} = \min \{1, \max (-1, 2x_3)\}.$$

**Example 5.4** (Torus). Let $\alpha = 10^{-3}$ and

$$\Gamma = \left\{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid \sqrt{x_3^2 + \left(\sqrt{x_1^2 + x_2^2} - 1\right)^2} = \frac{1}{2}\right\}$$

the 2-Torus embedded in $\mathbb{R}^3$. By symbolic differentiation we compute $z$, such that

$$\min_{u \in L^2(\Gamma), y \in H^1(\Gamma)} J(u, y) \text{ subject to } -\Delta\Gamma y = u - r, \quad -1 \leq u \leq 1, \quad \int_{\Gamma} y \, d\Gamma = \int_{\Gamma} u \, d\Gamma = 0$$

is solved by

$$\bar{u} = r = \max \{-1, \min (1, 5xyz)\}.$$

As the presented tables clearly demonstrate, the examples show the expected convergence behaviour.

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References


