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This work concerns the solution of generalized Riemann problems. To this end, we consider the ADER scheme of Titarev & Toro (2002), which relies on a generalization of the classical Godunov scheme. Another solution method is the power series expansion of LeFloch & Raviart (1988). We analyze the two resulting approximation schemes, where we show that for scalar 1d problems the Toro-Titarev solver and the LeFloch-Raviart expansion yield the same Taylor series expansions in time. The full analysis for the Burgers equation is finally provided.

1 INTRODUCTION

We consider a system of hyperbolic conservation laws

$$\partial_t q(x,t) + \partial_x f(q(x,t)) = 0 \quad \text{for } x \in \mathbb{R}, t > 0, \quad (1)$$

in one spatial dimension, where $q : \mathbb{R} \times [0,\infty) \to \mathbb{R}^n$ is a vector of conserved quantities and where the flux $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector-valued function.

We are interested in a Cauchy problem for (1), with discontinuous initial data of the form

$$q(x,0) = \begin{cases} \hat{q}_L(x) & \text{for } x < 0, \\ \hat{q}_R(x) & \text{for } x > 0, \end{cases} \quad (2)$$

where the two functions $\hat{q}_L$ and $\hat{q}_R$ are smooth. The Cauchy problem (1)-(2) is called *generalized Riemann problem*.

Due to the pioneering work of (van Leer 1979) and (Ben-Artzi & Falcovitz 1984), the generalized Riemann problem has been successfully used in the construction of high order extensions to the classical Godunov scheme. A state of the art variant of this approach is the ADER scheme of (Titarev & Toro 2002; Toro & Titarev 2006). The basic idea of the ADER scheme is to use a high order spatial reconstruction of the solution from cell averages and to use the generalized Riemann problem to design a time discretisation of matching order. The key strategy in the Toro-Titarev solver is the reduction of the generalized Riemann problem to a series of classical Riemann problems.

On the other hand, (LeFloch & Raviart 1988) have shown that, for $t > 0$ sufficiently small, the solution q of the generalized Riemann problem can be expanded into a power-series of self-similar functions,

$$q(x,t) = \sum_{k \geq 0} t^k u^k \left(\frac{x}{t}\right) \quad (3)$$

with polynomial functions $\xi \mapsto u^k(\xi)$. LeFloch & Raviart have given an explicit method to construct the expansion (3). We re-interpret the method of Toro & Titarev in the context of that LeFloch-Raviart series expansion for the generalized Riemann problem.

The outline of this paper is as follows. In Section 2 we briefly review generalized Godunov schemes and the Toro-Titarev solver for the ADER scheme. Then, we recall well-known results on the solution of classical and generalized Riemann problems in Section 3. In Section 4, we discuss the key steps of the LeFloch-Raviart expansion, where we show that for scalar 1d problems the Toro-Titarev solver and the LeFloch-Raviart expansion yield the same the Taylor series expansions in time. The full analysis for a scalar 1d example concerning Burgers equation is finally provided in Section 6.

2 THE ADER SCHEME

2.1 *Generalized Godunov Schemes*

To numerically solve the Cauchy problem

$$\left.\begin{array}{ll} \partial_t q + \partial_x f(q) = 0 & \text{for } x \in \mathbb{R},\ t > 0, \\ q(x,0) = \hat{q}(x) & \text{for } x \in \mathbb{R}, \end{array}\right\}$$

we use a Godunov-type finite volume scheme. To this end, we work with control volumes (cells) of the form

$$[x_{i-1/2}, x_{i+1/2}] \times [t^n, t^{n+1}] \quad \text{for } i \in \mathbb{Z},\ n \in \mathbb{N}.$$

For the sake of simplicity, we assume uniform grids, so that $x_{i+1/2} = (i+1/2)\Delta x,\ i \in \mathbb{Z}$, and $t^n = n\Delta t$ with $\Delta x, \Delta t > 0$. The cell average in the i-th cell at initial time $t^0 = 0$ is then given by

$$q_i^0 = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \hat{q}(x)\, \mathrm{d}x.$$

Now, the generalized Godunov scheme works as follows. At any time step $t^n \to t^{n+1}$, for $n \geq 0$,

- reconstruct a piecewise smooth function

  $$v^n(x) = \mathcal{R}\left(\{q_i^n\}_{i\in\mathbb{Z}}\right)(x)$$

  from the cell averages $\{q_i^n\}_i$, where $\mathcal{R}$ is a suitable conservative nonlinear reconstruction operator, e.g., WENO reconstruction. Denote the restriction of v^n to the cell $[x_{i-1/2}, x_{i+1/2}]$ by v_i^n;

- use v^n as initial data and evolve for one time step

  $$\tilde{v}^{n+1}(x) = \mathcal{E}(\Delta t) v^n(x),$$

  where $\mathcal{E}$ is the exact entropy evolution operator associated with (1);

- update the cell averages by averaging $\tilde{v}^{n+1}$,

  $$q_i^{n+1} = \mathcal{A}_i \tilde{v}^{n+1},$$

  where $\mathcal{A}_i$ is the cell averaging operator, given as

  $$\mathcal{A}_i v = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} v(x)\, \mathrm{d}x.$$

In a finite volume framework, evolution and averaging can be done in one step by the update formula

$$q_i^{n+1} = q_i^n - \frac{\Delta t}{\Delta x}\left(\bar{f}_{i+1/2}^n - \bar{f}_{i-1/2}^n\right),$$

if we can compute the flux f through the cell boundaries exactly, i.e., we need to compute the integral

$$\bar{f}_{i+1/2}^n = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f\left(\mathcal{E}(\tau) v^n(x_{i+1/2})\right) \mathrm{d}\tau \tag{4}$$

exactly. However, this may be exceedingly complicated, if not impossible. Therefore, we are looking for an approximation to (4), being based on the approximate solution of a generalized Riemann problem.

To obtain a numerical flux, we use ADER *state expansion*, i.e., a Taylor expansion of the solution q around time $t = t^n$ at the cell interface $x_{i+1/2}$,

$$q(x_{i+1/2}, \tau) \approx q(x_{i+1/2}, 0_+) + \sum_{k=1}^{r-1} \partial_t^k q(x_{i+1/2}, 0_+) \frac{\tau^k}{k!}, \tag{5}$$

where $r > 1$ is a fixed integer, $\tau = t - t^n$ is the local time, $0_+ = \lim_{\tau \searrow 0} \tau$, and $q(x_{i+1/2}, 0_+)$ evaluates the solution of the generalized Riemann problem

$$\left.\begin{array}{l} \partial_t q + \partial_x f(q) = 0 \quad \text{for } x \in \mathbb{R}, \tau > 0, \\ q(x,0) = \left\{ \begin{array}{ll} v_i^n(x) & \text{for } x < x_{i+1/2}, \\ v_{i+1}^n(x) & \text{for } x > x_{i+1/2}, \end{array} \right. \end{array}\right\}$$

right at the cell interface for time $\tau = 0_+$.

Recall that the solution q may contain discontinuities. But for fixed $x_{i+1/2}$, the function $q(x_{i+1/2}, \cdot)$ (of the time variable) is smooth for small $\tau > 0$. To solve the generalized Riemann problem, we work with a numerical flux approximating the time integral in (4) by a Gaussian quadrature of the form

$$f_{i+1/2}^n = \sum_{\gamma=1}^{N} \omega_\gamma f(q(x_{i+1/2}, \tau_\gamma)),$$

where $\omega_\gamma, \tau_\gamma$ are the Gaussian weights and nodes, and N is the number of nodes. The values $q(x_{i+1/2}, \tau_\gamma)$ are determined through (5).

2.2 *The Toro-Titarev Solver*

We now describe how to compute the coefficients in (5), according to (Toro & Titarev 2006). The key idea is to reduce the solution of the generalized Riemann problem to a series of classical Riemann problems. To find the sought value $q(x_{i+1/2}, 0_+)$, we solve a classical Riemann problem

$$\left.\begin{array}{l} \partial_t q + \partial_x f(q) = 0 \quad \text{for } x \in \mathbb{R}, \tau > 0, \\ q(x,0) = \left\{ \begin{array}{ll} \hat{q}_L^0 & \text{for } x < x_{i+1/2}, \\ \hat{q}_R^0 & \text{for } x > x_{i+1/2}, \end{array} \right. \end{array}\right\} \tag{6}$$

with the extrapolated values

$$\hat{q}_L^0 = \lim_{x \nearrow x_{i+1/2}} \hat{q}_L(x) \quad \text{and} \quad \hat{q}_R^0 = \lim_{x \searrow x_{i+1/2}} \hat{q}_R(x).$$

This problem has a similarity solution that we denote by $q^0((x - x_{i+1/2})/\tau)$. The leading term of the expansion (5) is then given by $q(x_{i+1/2}, 0_+) = q^0(0)$, called the *Godunov state* of (6). For nonlinear systems of conservation laws, computing the complete solution of the Riemann problem can be a quite difficult task, and so we may need to employ a numerical (approximative) Riemann solver to compute the leading term. However, as we are mainly interested in the analytical aspects of the scheme, we assume that the Godunov state of (6) can be computed exactly.

For the higher order terms we perform a standard Cauchy-Kowalewskaya-type procedure to express all time derivatives as functions of lower order spatial derivatives, relying on a recursive mapping

$$\partial_t^k q = \Phi^k\left(q, \partial_x q, \ldots, \partial_x^k q\right) \quad \text{for } k = 0, \ldots, r-1.$$

Recall that for piecewise smooth initial data, the classical Cauchy-Kowalewskaya theorem does not apply. But to illustrate the basic ideas, we assume that q is smooth. In this case, the following equations can be obtained by simple manipulations of derivatives.

Using Φ^k we can can compute the expansion (5), provided that we can find the spatial derivatives

$$q^{(k)}(x,t) = \partial_x^k q(x,t).$$

To do so, we first compute the one-sided derivatives

$$\hat{q}_L^k = \lim_{x \nearrow x_{i+1/2}} \partial_x^k \hat{q}_L(x) \text{ and } \hat{q}_R^k = \lim_{x \searrow x_{i+1/2}} \partial_x^k \hat{q}_R(x).$$

Then, we use these values as initial conditions for classical Riemann problems. For the evolution equations of the spatial derivatives we take inhomogeneous equations of the form

$$\partial_t q^{(k)} + A(q)\partial_x q^{(k)} = H^k(q^{(0)}, \ldots, q^{(k)}), \tag{7}$$

where $A(q) = Df(q)$ is the Jacobian of the flux. Again, if the solution q was smooth, equation (7) could be derived by straight forward computation. Note, however, that we do not have yet a rigorous analysis whether these equations also can be used for discontinuous solutions.

Now we simplify the given problem as follows. Firstly, we neglect the source terms and secondly, we linearise the equations, so that we work with

$$\left.\begin{array}{l} \partial_t q^{(k)} + A_{LR}\partial_x q^{(k)} = 0 \quad \text{for } x \in \mathbb{R}, \tau > 0, \\ q(x,0) = \left\{ \begin{array}{ll} \hat{q}_L^k & \text{for } x < x_{i+1/2}, \\ \hat{q}_R^k & \text{for } x > x_{i+1/2}, \end{array} \right. \end{array}\right\}$$

where $A_{LR} = A(q(x_{i+1/2}, 0_+))$. Then the self-similar solutions $q^k((x - x_{i+1/2})/\tau)$ of these *linear* problems can be easily computed. Note that for all k we have the same A_{LR}.

These simplifications appear to be reasonable and, in fact, they have already been used in many practical applications. However, to the best of our knowledge, no theoretical justification concerning these simplifications has been given so far. Therefore, we show in the following analysis, that for nonlinear, scalar, 1d problems the proposed simplifications lead to a method, whose solution agrees with a series expansion of the exact solution. Thereby, we show that the resulting method does not reduce the accuracy order.

3 GENERALIZED RIEMANN PROBLEMS

To review some well-known results for the solution to the classical and the generalized Riemann problem, let us consider the system

$$\partial_t q + \partial_x f(q) = 0 \quad \text{for } x \in \mathbb{R},\ t > 0, \tag{8}$$

which we assume to be strictly hyperbolic, i.e., the Jacobian $A(q) = Df(q)$ has n distinct real eigenvalues

$$\lambda_1(q) < \lambda_2(q) < \cdots < \lambda_n(q) \quad \text{for all } q \in \mathbb{R}^n.$$

We further assume that all eigenvalues $\lambda_i(q(x,t))$ are uniformly bounded in a neighbourhood of the origin.

We then choose bases of left and right eigenvectors, $\{\ell_1(q), \ldots, \ell_n(q)\}$ and $\{r_1(q), \ldots r_n(q)\}$, i.e.,

$$\ell_i(q)^T A(q) = \lambda_i(q)\ell_i(q)^T, \quad A(q)r_i(q) = \lambda_i(q)r_i(q),$$

for $i = 1, \ldots, n$ and all $q \in \mathbb{R}^n$, here normalized as

$$\ell_j(q) \cdot r_i(q) = \left\{ \begin{array}{l} 1 \text{ for } i = j, \\ 0 \text{ for } i \neq j, \end{array} \right. \quad \text{for all } q \in \mathbb{R}^n.$$

We assume f to be a smooth function and thereby, all λ_i, ℓ_i, r_i have the same regularity.

We restrict our analysis to systems, where we assume that their characteristic fields are, for any $1 \leq i \leq n$, either genuinely nonlinear in the sense of (Lax 1957),

$$\nabla\lambda_i(q) \cdot r_i(q) \neq 0 \quad \text{for all } q \in \mathbb{R}^n,$$

or linearly degenerate,

$$\nabla\lambda_i(q) \cdot r_i(q) \equiv 0 \quad \text{for all } q \in \mathbb{R}^n.$$

Under these assumptions, we have the following well-known result: Given two states $\hat{q}_L^0,\ \hat{q}_R^0 \in \mathbb{R}^n$ with $|\hat{q}_R^0 - \hat{q}_L^0|$ sufficiently small, the classical Riemann problem

$$\left.\begin{array}{l} \partial_t q^0 + \partial_x f(q^0) = 0 \quad \text{for } x \in \mathbb{R}, t > 0, \\ q^0(x,0) = \left\{ \begin{array}{ll} \hat{q}_L^0 & \text{for } x < 0, \\ \hat{q}_R^0 & \text{for } x > 0, \end{array} \right. \end{array}\right\}$$

permits a unique entropy admissible weak solution that is self-similar,

$$q^0(x,t) = u^0\left(\frac{x}{t}\right).$$

The solution consists of at most $(n+1)$ constant states, separated by rarefaction waves, shock waves or contact discontinuities. For a comprehensive analysis on the classical Riemann problem and the properties of its solution, see (Bressan 2000).

Assume that the initial data

$$q(x,0) = \left\{ \begin{array}{ll} \hat{q}_L(x) & \text{for } x < 0, \\ \hat{q}_R(x) & \text{for } x > 0, \end{array} \right. \tag{9}$$

is piecewise smooth but discontinuous at $x = 0$. For the generalized Riemann problem (8)-(9) it is well-known that taking $\hat{q}_L^0 = \hat{q}_L(0),\ \hat{q}_R^0 = \hat{q}_R(0)$, for sufficiently small $|\hat{q}_R^0 - \hat{q}_L^0|$, there exists a neighbourhood

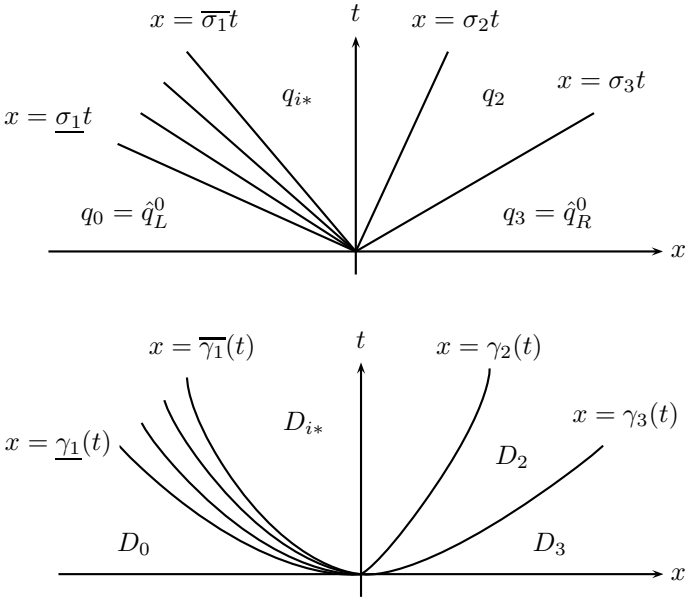

Figure 1: Corresponding wave patterns.

around the origin in which (8)-(9) has a unique entropy admissible weak solution, see (Li & Yu 1985). Moreover, the solution q consists of at most $(n+1)$ open domains of smoothness $D_i,\ 0 \le i \le n$, separated either by smooth curves $x = \gamma_j(t)$ passing through the origin, or by rarefaction zones of the form

$$R = \{(x,t) \in \mathbb{R} \times [0,\infty) \,\Big|\, \underline{\gamma}_j(t) < x < \overline{\gamma}_j(t)\},$$

where $x = \underline{\gamma}_j(t),\ x = \overline{\gamma}_j(t),\ 1 \le j \le n$, are smooth characteristic curves passing through the origin. In either case, we assume that these curves are defined for $t > 0$ sufficiently small. Then q has a shock or contact discontinuity across each curve $x = \gamma_j(t)$ and is continuous across the characteristic curves $x = \underline{\gamma}_j(t)$, $x = \overline{\gamma}_j(t)$. The solution of the generalized Riemann problem and the solution of the corresponding classical Riemann problem for q^0 with the initial states $\hat{q}_L^0 = \hat{q}_L(0)$ and $\hat{q}_R^0 = \hat{q}_R(0)$ have the same wave structure, at least for small time $t > 0$. That is, if the solution of the classical Riemann problem contains an i-shock moving to the right, the same is true for the solution of the generalized Riemann problem and so on. For illustration, a typical configuration of corresponding wave patterns is shown in Figure 1.

We finally remark that the generalized Riemann problem has been the subject of ongoing research. Special emphasis has been put on the global existence and the structural stability of solutions. We refer to (Chen, Huang, & Han 2009; Chen, Han, & Zhang 2009; Kong 2003; Kong 2005) and references therein for an up-to-date account on the generalized Riemann problem. However, for the analysis of the numerical schemes under consideration in this paper, we can rely the results on local existence and local structural stability.

4 ASYMPTOTIC EXPANSION

4.1 *Preliminary Discussion*

To further study the solution of the generalized Riemann problem

$$\left.\begin{array}{l} \partial_t q + \partial_x f(q) = 0 \quad \text{for } x \in \mathbb{R}, t > 0, \\ q(x,0) = \left\{ \begin{array}{ll} \hat{q}_L(x) & \text{for } x < 0, \\ \hat{q}_R(x) & \text{for } x > 0, \end{array} \right. \end{array}\right\}$$

we follow (LeFloch & Raviart 1988), where we want to find an asymptotic expansion of the form

$$q(x,t) = \sum_{k \ge 0} t^k u^k(\xi) \tag{10}$$

with $\xi = x/t$. This is possible in any domain of smoothness D_i, by simply taking a Taylor expansion, so that every u^k is a polynomial of degree k. We will further discuss this in Section 5.

It can be shown that such a series expansion can also be constructed inside a rarefaction zone R. However, for our numerical scheme we only need detailed information about the solution along the line segment $\{x = 0\} \times [0, \Delta t]$ (in local coordinates). We assume that the solution does *not* contain a transonic rarefaction wave. In that case, the solution along that line segment is given by some function q_{i*} inside a domain of smoothness $D_{i*},\ 0 \le i* \le n$, and we do not need the explicit construction of the expansion inside a rarefaction zone.

The above construction may be summarized as follows: Take a Taylor expansion in regions, where the solution q is smooth. Then, investigate the jump conditions at the boundaries of the smoothness domains. As we are looking for an expansion in terms of self-similar functions, it is useful to change the variables to $\xi = x/t$. We let $\tilde{q}(\xi,t) = q(\xi t, t)$ and see that

$$\partial_x = \frac{1}{t}\partial_\xi \quad \text{and} \quad \partial_t q = \partial_t \tilde{q} - \frac{\xi}{t}\partial_\xi \tilde{q}. \tag{11}$$

4.2 *Step I: Derivation of the Differential Equations*

To derive an explicit construction of the functions u^k in (10), we first analyze the equations satisfied by these functions inside the domains of smoothness. By using (11), the conservation law (1) can be written as

$$t\partial_t \tilde{q} - \xi \partial_\xi \tilde{q} + \partial_\xi f(\tilde{q}(\xi,t)) = 0.$$

Note that the expansion

$$\tilde{q}(\xi,t) = \sum_{k \ge 0} t^k u^k(\xi)$$

gives

$$t\partial_t \tilde{q} - \xi \partial_\xi \tilde{q} = -\xi \frac{du^0}{d\xi} + \sum_{k \ge 1} t^k \left(k u^k - \xi \frac{du^k}{d\xi} \right). \tag{12}$$

Inserting this expansion into the physical flux yields

$$f(\tilde{q}(\xi,t)) = f(u^0) + \sum_{k\geq 1} t^k (A(u^0)u^k + f^k(U^{k-1})). \tag{13}$$

Here, the function f^k depends only on the previous terms $U^{k-1} = (u^0, \ldots, u^{k-1})$. We get f^k by a Taylor expansion of the physical flux, so that f^k contains all higher order terms in that expansion, i.e., all but $A(u^0)u^k$. By using that Taylor expansion of f in powers of t, it is easy to see that f^k is a polynomial of degree at most k, if every u^ℓ is a polynomial (in ξ) of degree at most ℓ, for $0 \leq \ell \leq k-1$.

Combining (12) and (13), we get

$$-\xi \frac{du^0}{d\xi} + \frac{d}{d\xi} f(u^0)$$

$$+ \sum_{k\geq 1} t^k \left(k u^k - \xi \frac{du^k}{d\xi} + \frac{d}{d\xi}(A(u^0)u^k + f^k) \right) = 0.$$

For $k = 0$, we have

$$-\xi \frac{du^0}{d\xi} + \frac{d}{d\xi} f(u^0) = 0 \tag{14}$$

and for $k \geq 1$, we have

$$k u^k - \xi \frac{du^k}{d\xi} + \frac{d}{d\xi}(A(u^0)u^k + f^k) = 0.$$

Letting

$$h^k(\xi) = -\frac{d}{d\xi} f^k(u^0(\xi), \ldots, u^{k-1}(\xi)),$$

this becomes

$$k u^k - \xi \frac{du^k}{d\xi} + \frac{d}{d\xi}(A(u^0)u^k) = h^k. \tag{15}$$

Since f^k is a polynomial in ξ of degree at most k, the function h^k is a polynomial of degree at most $k-1$.

4.3 *Step II: Jump Conditions.*

The above construction is valid wherever q is smooth. So next we need to investigate the jump conditions satisfied by u^k at the points of discontinuity of q. Take a curve $x = \gamma(t)$ that separates two domains of smoothness of q, thus either a shock curve, a contact discontinuity or the boundary of a rarefaction zone. Since these curves are all smooth, we can use a Taylor expansion to write

$$\gamma(t) = \sigma^0 t + \sigma^1 t^2 + \cdots + \sigma^{k-1} t^k + \ldots .$$

It follows from (10) that

$$q(\gamma(t),t) = \sum_{k\geq 0} t^k u^k \left(\frac{\gamma(t)}{t} \right) = \sum_{k\geq 0} t^k u^k \left(\sum_{\ell \geq 0} t^\ell \sigma^\ell \right).$$

In fact, the solution q is smooth, not only in D_i, but also in the closure $\overline{D}_i$, see (Li & Yu 1985). So again we can use a Taylor expansion in powers of t around the origin to obtain

$$q(\gamma(t),t) =$$

$$u^0(\sigma^0) + \sum_{k\geq 1} t^k \left(u^k(\sigma^0) + \sigma^k \frac{du^0}{d\xi}(\sigma^0) \right)$$

$$+ \sum_{k\geq 0} t^k z^k(\Sigma^{k-1}, U^{k-1}). \tag{16}$$

Similar to the f^k in (13), the functions z^k depend only on $\Sigma^{k-1} = (\sigma^0, \ldots, \sigma^{k-1})$ and U^{k-1}. Again, we insert all higher order terms in an Taylor expansion into z^k.

We denote the jump of a function u at a point x_0 by

$$[u](x_0) = u(x_{0,+}) - u(x_{0,-}),$$

so that in the case where q is continuous across the curve $x = \gamma(t)$ we simply get

$$[u^0](\sigma^0) = 0 \quad \text{for } k = 0 \tag{17}$$

from (16), whereas for $k \geq 1$ we get

$$\left[u^k + \sigma^k \frac{du^0}{d\xi} + z^k(\Sigma^{k-1}, U^{k-1}) \right](\sigma^0) = 0.$$

Note that u^0 is continuous at the point σ^0, whereas, for $k \geq 1$, u^k is in general discontinuous at σ^0.

Now let q have a jump across the curve $x = \gamma(t)$. Then, by the Rankine-Hugoniot conditions, we have

$$\dot{\gamma}(t)[q](x) = [f(q)](x) \quad \text{for } x = \gamma(t).$$

Then we take expansions of both $\dot{\gamma}(t)q(\gamma(t),t)$ and $f(q(\gamma(t),t))$. By a similar technique as above, we find

$$\sigma^0 [u^0] = [f(u^0)] \quad \text{at } \sigma^0, \tag{18}$$

for $k = 0$, whereas for $k \geq 1$ we get

$$\left[(A(u^0) - \sigma^0) u^k \right] + \sigma^k \left[(A(u^0) - \sigma^0) \frac{du^0}{d\xi} \right]$$

$$- \sigma^k \left[(k+1) u^0 \right] + \left[w^k \right] = 0$$

with a function w^k depending only on Σ^{k-1}, U^{k-1}. For further details on the (rather technical) proof we refer to (LeFloch & Raviart 1988).

We remark that (by finite speed of propagation) for $|\xi|$ large enough, say $|\xi| \geq \xi_0$, we have

$$u^0(\xi) = \begin{cases} \hat{q}_R^0 & \text{for } \xi > \xi_0, \\ \hat{q}_L^0 & \text{for } \xi < -\xi_0. \end{cases} \tag{19}$$

We summarize our above construction as follows.

PROPOSITION. The function u^0 satisfies the relations (14), (17), (18) and (19), which characterize the piecewise continuous self-similar entropy solution $q^0(x,t) = u^0(\xi)$ of the classical Riemann problem

$$\left.\begin{array}{l} \partial_t q^0 + \partial_x f(q^0) = 0 \quad \text{for } x \in \mathbb{R}, t > 0, \\ q^0(x,0) = \begin{cases} \hat{q}_L^0 & \text{for } x < 0, \\ \hat{q}_R^0 & \text{for } x > 0, \end{cases} \end{array}\right\} \tag{20}$$

with initial states $\hat{q}_L^0 = \hat{q}_L(0)$ and $\hat{q}_R^0 = \hat{q}_R(0)$. □

This shows that the solution strategy of the ADER method sets up "the right problem" for computing the leading term of the expansion.

4.4 *Step III: Higher Order Terms.*

Assume that the solution of (20) contains no transonic rarefaction wave. Then line segment $\{x = 0\} \times [0, \Delta t]$ is contained in a domain of smoothness, say in D_{i*}, and u_{i*}^0 is the Godunov state of the Riemann problem (20). Since we do not explicitly need the expansion inside the rarefaction zones, we only consider the simplified case, where the solution u^0 contains only shock waves or contact discontinuities. The full problem requires similar techniques, although some of the details are more involved, see (LeFloch & Raviart 1988) for the full construction.

In situations, where we only have shocks and contact discontinuities, the solution u^0 of (20) has the form

$$u^0(\xi) = \begin{cases} u_0^0 = \hat{q}_L^0 & \text{for } \xi \in (-\infty, \sigma_1^0), \\ u_i^0 & \text{for } \xi \in (\sigma_i^0, \sigma_{i+1}^0), 1 \leq i < n, \\ u_n^0 = \hat{q}_R^0 & \text{for } \xi \in (\sigma_n^0, \infty). \end{cases}$$

In case of an i-shock, we have

$$\lambda_i(u_i^0) \geq \sigma_i^0 \geq \lambda_i(u_{i+1}^0),$$

and for an i-contact discontinuity, we have

$$\sigma_i^0 = \lambda_i(u_i^0) = \lambda_i(u_{i+1}^0).$$

Now consider the domains

$$D_i^0 = \{(x,t) | \sigma_i^0 < \xi < \sigma_{i+1}^0\} \quad \text{for } i = 0, \ldots, n,$$

in which u_0 takes the constant value u_i^0. As a convention, we let $\sigma_0^0 = -\infty$ and $\sigma_{n+1}^0 = +\infty$. Then equation (15) in D_i^0 becomes

$$k u^k + \left(A(u_i^0) - \xi\right) \frac{d}{d\xi} u^k = h^k. \tag{21}$$

Recall that h^k is a polynomial of degree at most $k-1$. It is straightforward to show that the general solution of (21) is given by

$$u^k(\xi) = \left(\xi - A(u_i^0)\right)^k u_i^k + p_i^k(\xi), \tag{22}$$

see Lemma 2 in (LeFloch & Raviart 1988), where $u_i^k \in \mathbb{R}^n$ is an arbitrary vector and $p_i^k : \mathbb{R} \to \mathbb{R}^n$ is a polynomial of degree at most $k-1$ with coefficients that depend only on $U^{k-1} = (u^0, \ldots, u^{k-1})$.

5 CONNECTING SOLVER AND EXPANSION

Now let us take a look at the Taylor expansion that we used to define the functions u^k. We consider the domains

$$D_i = \{\xi \in \mathbb{R} \,|\gamma_{i-1}(t)/t < \xi < \gamma_i(t)/t\}.$$

Since we have $\gamma_i(0) = 0, \gamma_i'(0) = \sigma_i^0$, the domains remain close to the domains D_i^0 in which u^0 is constant, for small $t > 0$. In every domain of smoothness D_i we can take some (x_0, t_0) close to the origin and write

$$q(x,t) = q(x_0,t_0) + \sum_{k=1}^{\infty} \sum_{\ell=0}^{k} \frac{\partial^\ell}{\partial x^\ell} \frac{\partial^{k-\ell}}{\partial t^{k-\ell}} q(x_0,t_0) \frac{(x-x_0)^\ell (t-t_0)^{k-\ell}}{\ell!(k-\ell)!}.$$

Let $i_* \in \{1, \ldots, n\}$ be the index for which the line segment $\{x = 0\} \times [0, \Delta t]$ is contained in D_{i_*}. Inside D_{i_*} we may take the limit $(x_0, t_0) \to (0, 0_+)$ and thus the Taylor expansion around the origin gives

$$\begin{aligned} q(x,t) &= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{\partial^\ell}{\partial x^\ell} \frac{\partial^{k-\ell}}{\partial t^{k-\ell}} \frac{q(0,0_+)}{\ell!(k-\ell)!} x^\ell t^{k-\ell} \\ &= u_{i_*}^0 + \sum_{k=1}^{\infty} t^k \underbrace{\sum_{\ell=0}^{k} \frac{\partial^\ell}{\partial x^\ell} \frac{\partial^{k-\ell}}{\partial t^{k-\ell}} \frac{q(0,0_+)}{\ell!(k-\ell)!} \cdot \left(\frac{x}{t}\right)^\ell}_{= u^k(x/t)} \end{aligned}$$

Thus, the vector $u_{i_*}^k$ in (22), which gives the leading coefficient of this polynomial, defines the value $\partial_x^k q(0, 0_+)$.

To determine the vectors u_i^k, we first describe u_0^k and u_n^k. Using the notation from Section 2, we can write for the initial data

$$\hat{q}_L(x) = \hat{q}_L^0 + \sum_{k=1}^{r-1} \frac{\hat{q}_L^k}{k!} x^k \text{ and } \hat{q}_R(x) = \hat{q}_R^0 + \sum_{k=1}^{r-1} \frac{\hat{q}_R^k}{k!} x^k.$$

In D_0, the solution is given by the functions

$$u^k(\xi) = (\xi - A(u_0^0))^k u_0^k + p_0^k(\xi).$$

Since p_0^k is a polynomial of degree at most $k-1$,

$$\lim_{\substack{t\to 0,\\ x<\gamma_1(t)}} t^k u^k\left(\frac{x}{t}\right) = x^k u_0^k.$$

Hence, it follows

$$q(x,0) = \lim_{\substack{t\to 0,\\ x<\gamma_1(t)}} q(x,t) = u_0^0 + \sum_{k=1}^{r-1} u_0^k x^k.$$

Therefore, $u_0^k = \hat{q}_L^k/k!$, and likewise, $u_n^k = \hat{q}_R^k/k!$, for $k = 0, \ldots, r-1$.

Now consider the scalar case. For a strictly convex flux, $f'' > 0$, we only have two domains of smoothness. In that case, all coefficients $u_i^k, i = 0, 1,$ and $k = 1, \ldots, r-1$, are uniquely determined by the initial data and its derivatives. Assuming that there is no transonic wave, solving linear Riemann problems merely means picking the left or the right side, depending on the sign of the coefficient in the evolution equation. Thus, to build the expansion, we first have to solve one nonlinear Riemann problem to determine which domain of smoothness contains the line segment $\{x = 0\} \times [0, \Delta t]$. Then we use the data from that side, which is equivalent to solving linear Riemann problems. Therefore, the Toro-Titarev solver reproduces the first $r-1$ terms of the LeFloch-Raviart expansion exactly, so that we can finally conclude:

THEOREM. Consider the generalized Riemann problem for a scalar, nonlinear hyperbolic conservation law with strictly convex flux in one spatial dimension. Let the initial data consist of piecewise polynomials of degree $r-1$. Assume that the solution does not contain a transonic wave. Then the numerical flux constructed with the solver of Toro and Titarev is accurate of order $\mathcal{O}(\Delta t^r)$ as $\Delta t \to 0_+$, in the sense that

$$\left| f(q(0,\tau)) - f\left(\sum_{k=0}^{r-1} u^k(0)\tau^k\right)\right| = \mathcal{O}(\Delta t^r).$$

□

6 BURGERS EQUATION

Consider Burgers equation,

$$\partial_t q + \partial_x(q^2/2) = 0$$

with initial data

$$q(x,0) = \begin{cases} \hat{q}_L(x) = x^2 + 2x + 1 & \text{for } x < 0, \\ \hat{q}_R(x) = 2x^2 - 4x + 2 & \text{for } x > 0. \end{cases}$$

In this case, we have $\hat{q}_L^0 = 1 < 2 = \hat{q}_R^0$. Therefore, the classical Riemann problem for the leading term contains a rarefaction wave. Therefore, the solution of the generalized Riemann problem is given as

$$q(x,t) = \begin{cases} q_0(x,t) & \text{for } x \le \underline{\gamma}(t), \\ x/t & \text{for } \underline{\gamma}(t) < x < \overline{\gamma}(t), \\ q_1(x,t) & \text{for } \overline{\overline{\gamma}}(t) \le x. \end{cases}$$

Using the method of characteristics, we obtain

$$q_0(x,t) = \frac{2t(x+1) + 1 - \sqrt{4t(x+1)+1}}{2t^2},$$

$$q_1(x,t) = \frac{4t(x-1) + 1 - \sqrt{8t(x-1)+1}}{4t^2}.$$

The boundaries of the rarefaction zone are given by the head-characteristic $\overline{\gamma}(t) = 2t$ and the tail-characteristic $\underline{\gamma}(t) = t$.

For sufficiently small time $t > 0$, we approximate the solution along the t-axis by

$$q(0,t) \approx q(0,0_+) + \partial_t q(0,0_+)t + \partial_t^2 q(0,0_+)\frac{t^2}{2}.$$

Note that the t-axis is contained in the domain D_0, so in the subsequent analysis we only need to consider the function q_0. We have

$$\partial_t q_0(0,0_+) = -2 \quad \text{and} \quad \partial_t^2 q_0(0,0_+) = 10.$$

We now compute the terms of the LeFloch-Raviart expansion up to the function $u^2(\xi)$. At first, consider the expansion of the flux around $t = 0$,

$$f(\tilde{q}(\xi,t)) \approx f(u^0) + tf'(u^0)u^1 + t^2\left(f'(u^0)u^2 + \frac{1}{2}f''(u^0)(u^1)^2\right).$$

Then, u^0 is the solution of the Riemann problem

$$\left.\begin{array}{l} \partial_t u^0 + \partial_x((u^0)^2/2) = 0 \quad \text{for } x \in \mathbb{R}, t > 0, \\ u^0(x,0) = \begin{cases} 1 & \text{for } x < 0, \\ 2 & \text{for } x > 0, \end{cases} \end{array}\right\}$$

in which case $u^0(\xi) = 1$ for all $\xi \in D_0$. Then, the equation for u^1 is

$$u_0^1 + (1-\xi)\frac{d}{d\xi}u_0^1 = 0,$$

and the solution consistent with the initial data is

$$u_0^1(\xi) = 2(\xi - 1).$$

Therefore, we have

$$h^2(\xi) = -\frac{f''(u_0^0)}{2}(u^1(\xi))^2 = -4(\xi-1).$$

Note that the inhomogeneous equation for u^2,

$$2u_0^2 + (1-\xi)\frac{d}{d\xi}u_0^2 = -4(\xi-1),$$

has the solution $u_0^2(\xi) = (\xi-1)^2 - 4(\xi-1)$. Thus,

$$\tilde{q}(\xi,t) \approx 1 + 2(\xi-1)t + ((\xi-1)^2 - 4(\xi-1))t^2,$$

$$q(x,t) \approx x^2 + 2x - 6xt + 5t^2 - 2t + 1,$$

and in particular

$$q(0,t) \approx 1 - 2t + 5t^2.$$

Now we use the solver of Toro and Titarev, where we find the leading term $q(0,0_+)$ by solving

$$\left.\begin{array}{l} \partial_t q + \partial_x(q^2/2) = 0 \quad \text{for } x \in \mathbb{R}, t > 0, \\ q(x,0) = \left\{ \begin{array}{ll} \hat{q}_L^0 = 1 & \text{for } x < 0, \\ \hat{q}_R^0 = 2 & \text{for } x > 0. \end{array} \right. \end{array} \right\}$$

This gives $q(0,0_+) = 1$.

Next, the Cauchy-Kowaleskaya procedure leads to

$$\begin{array}{l} \partial_t q = -q\partial_x q, \quad \partial_t^2 q = 2(\partial_x q)^2 + q\partial_x^2 q, \\ \partial_t(\partial_x q) = -(\partial_x q)^2 - q\partial_x^2 q, \\ \partial_t(\partial_x^2 q) = -3\partial_x q \partial_x^2 q - q\partial_x^3 q. \end{array} \tag{23}$$

For $q^{(1)} = \partial_x q$ and $q^{(2)} = \partial_x^2 q$, we have the evolution equations

$$\partial_t q^{(1)} + q\partial_x q^{(1)} = -(q^{(1)})^2$$

and

$$\partial_t q^{(2)} + q\partial_x q^{(2)} = -3q^{(1)}q^{(2)}$$

with the initial conditions

$$q^{(1)}(x,0) = \left\{ \begin{array}{ll} \hat{q}_L^{(1)} = 2 & \text{for } x < 0, \\ \hat{q}_R^{(1)} = -4 & \text{for } x > 0, \end{array} \right. \tag{24}$$

and

$$q^{(2)}(x,0) = \left\{ \begin{array}{ll} \hat{q}_L^{(2)} = 2 & \text{for } x < 0, \\ \hat{q}_R^{(2)} = 4 & \text{for } x > 0. \end{array} \right. \tag{25}$$

We drop the source terms and linearise around $q(0,0_+) = 1$, so that we have

$$\partial_t q^{(k)} + \partial_x q^{(k)} = 0 \quad \text{for } k = 1, 2,$$

together with the initial conditions (24) and (25), respectively. These *linear* problems are readily solved, where we find the Godunov states

$$q^{(1)}(0,0_+) = 2 \quad \text{and} \quad q^{(2)}(0,0_+) = 2.$$

By the Cauchy-Kowaleskaya procedure (23) we get

$$\partial_t q(0,0_+) = -q(0,0_+)q^{(1)}(0,0_+) = -2,$$

$$\partial_t^2 q(0,0_+) = 2(q^{(1)}(0,0_+))^2 + q(0,0_+)q^{(2)}(0,0_+) = 10,$$

and thus again we find

$$q(0,t) \approx 1 - 2t + 5t^2.$$

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