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ON COMBINING DEFLATION AND ITERATION TO LOW-RANK APPROXIMATIVE SOLUTION OF LUR'E EQUATIONS

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Abstract. We present an approach to the determination of the stabilizing solution of Lur'e matrix equations. We show that the knowledge of a certain deflating subspace of an even matrix pencil may lead to Lur'e equations which are defined on some subspace, the so-called “projected Lur'e equations”. These projected Lur'e equations are shown to be equivalent to projected Riccati equations, if the deflating subspace contains the subspace corresponding to infinite eigenvalues. This result leads to a novel numerical algorithm that basically consists of two steps. First we determine the deflating subspace corresponding to infinite eigenvalues using an algorithm based on the so-called “neutral Wong sequences”, which requires a moderate number of kernel computations. The second step consists of low-rank iterative solution of the projected Riccati equation via a generalization of the Newton-Kleinman-ADI iteration. Altogether this method delivers solutions in low-rank factored form, is applicable for large-scale Lur'e equations and exploits possible sparsity of the matrix coefficients.

Key words. Lur'e equations, Riccati equations, deflating subspaces, even matrix pencils, Newton-Kleinman method, ADI iteration

1. Introduction. For given matrices $A \in \mathbb{C}^{n,n}$, $B, S \in \mathbb{C}^{n,m}$ and Hermitian $Q \in \mathbb{C}^{n,n}$, $R \in \mathbb{C}^{m,m}$, we consider *Lur'e equations*

$$\begin{aligned} A^*X + XA + Q &= K^*K, \\ XB + S &= K^*L, \\ R &= L^*L, \end{aligned} \tag{1.1}$$

which have to be solved for the triple $(X, K, L) \in \mathbb{C}^{n,n} \times \mathbb{C}^{p,n} \times \mathbb{C}^{p,m}$ with Hermitian X , and p as small as possible. For sake of simplicity, we will call X a solution of the Lur'e equations, if there exist K and L such that (1.1) holds true.

This type of equations e.g. arises in linear-quadratic optimal control, that is, the minimization (resp. “infimization”) of the cost functional

$$\mathcal{J}(u(\cdot), x_0) = \frac{1}{2} \int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

subject to the constraint defined by the ordinary differential equation $\dot{x}(t) = Ax(t) + Bu(t)$ with initial and end conditions $x(0) = x_0$, $\lim_{t \rightarrow \infty} x(t) = 0$ [32]. In the case where the input is “fully weighted”, i.e., the matrix R is invertible, then the unknown matrices K and L can be eliminated and one obtains an *algebraic Riccati equation (ARE)*

$$A^*X + XA - (XB + S)R^{-1}(XB + S)^* + Q = 0. \tag{1.2}$$

While the invertibility of R is often a reasonable assumption in linear-quadratic optimal control, there exist various other important applications for Lur'e equations with possibly singular R : balancing-related model order reduction, in particular *positive*

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real balanced truncation and *bounded real balanced truncation* [10, 15, 20, 22, 26], requires numerical solution of large-scale Lur'e equations. Here the singularity of R is often a structural property of the system to be analyzed [25] and can therefore not be excluded by arguments of genericity.

Though the numerical solution of (especially large-scale) algebraic Riccati equations is still subject of present research, this field can be considered as widely well understood [4]. In particular, the *Newton-Kleinman method* [18] is popular mainly because of two reasons. First, this method is, under certain slight additional assumptions, quadratically convergent. Second, it can be reformulated such that the iterates X_i appear in *low rank factored form* $X^{(i)} = Z^{(i)}(Z^{(i)})^*$ for some $Z^{(i)} \in \mathbb{C}^{n, k_i}$ with $k_i \ll n$ [5]. The latter property enables a significantly less memory-consuming implementation, and, furthermore, factorizations of the solutions are required anyway in many applications, such as balancing-related model order reduction [6, 15].

However, while numerical analysis for algebraic Riccati equations has achieved a considerably advanced level, the case of singular R has been treated stepmotherly. Almost merely analytical results have been achieved so far [12, 13, 24]. In this work we present a numerical method for the solution of Lur'e equations. Before our approach is presented, let us briefly review some known approaches:

- a) The most common approach to the solution of Lur'e equations is *regularization*, i.e., the slight perturbation of R by εI_m for some $\varepsilon > 0$. Then by using the invertibility of $R + \varepsilon I$, the corresponding perturbed Lur'e equations are now equivalent to the Riccati equation

$$A^* X_\varepsilon + X_\varepsilon A - (XB + S)(R + \varepsilon I)^{-1}(X_\varepsilon B + S)^* + Q = 0. \quad (1.3)$$

It is shown in [17, 30] that convergence of desired solutions X_ε then converge as ε tends to zero.

- b) Recently, the *structure-preserving doubling algorithm (SDA)* [11] was extended to a certain class of Lur'e equations [23]. Roughly speaking, the problem is transformed via Cayley transformation to the discrete-time case, and a power iteration leads to the desired solution. It is shown that this iteration converges linearly.
- c) The works [16, 31] present an successive technique for the elimination of variables corresponding to $\ker R$. By performing an orthogonal transformation of R , and an accordant transformation of L , the equations can be divided into a 'regular part' and a 'singular part'. The latter leads to an explicit equation for a part of the matrix K . Plugging this part into (1.1), one obtains Lur'e equations of slightly smaller size. After a finite number of steps this leads to an algebraic Riccati equation. This also gives an equivalent solvability criterion that is obtained by the feasibility of this iteration. A related deflation approach for structured matrix pencils is presented in [9].

The regularization approach has two essential disadvantages: so far, no estimates for the perturbation error $\|X - X_\varepsilon\|$ have been found, and even convergence rates are unknown. Furthermore, the numerical sensitivity of the Riccati equation (1.3) increases drastically as ε tends to 0.

The structure-preserving doubling algorithm can only be applied successfully to dense, small-scale Lur'e equations, since sparsity and low-rank properties are not preserved among the iterates. Moreover, it is not applicable to all kinds of solvable Lur'e equations, since an essential requirement is that the associated even pencil is regular.

The approach presented in this work is related to c) in the sense that the 'singular part' of the Lur'e equation is extracted and, afterwards, an 'inherent algebraic Riccati

equation' is set up and solved. We make use of the results in [24], where it is shown that there exists a one-to-one correspondence between the solutions of Lur'e equations and certain deflating subspaces of the matrix pencil

$$s\mathcal{E} - \mathcal{A} = \begin{bmatrix} 0 & -sI + A & B \\ sI + A^* & Q & S \\ B^* & S^* & R \end{bmatrix}. \quad (1.4)$$

Based on these results, we show that the determination of deflating subspaces of the pencil (1.4) leads to the knowledge of the action of X on some subspace, that is,

$$X\check{V}_x = \check{V}_\mu \quad (1.5)$$

for some matrices $\check{V}_\mu, \check{V}_x \in \mathbb{C}^{n, \check{n}}$, which are constructed from a matrix spanning a deflating subspace of $s\mathcal{E} - \mathcal{A}$. Furthermore, we show that using the partial information in (1.5) we can reduce (1.1) to a system of *projected Lur'e equations*

$$\begin{aligned} \tilde{A}^* \tilde{X} + \tilde{X} \tilde{A} + \tilde{Q} &= \tilde{K}^* \tilde{K}, & \tilde{X} &= \tilde{X}^* = \Pi^* \tilde{X} \Pi \in \mathbb{C}^{n, n} \\ \tilde{X} \tilde{B} + \tilde{S} &= \tilde{K}^* \tilde{L}, \\ \tilde{R} &= \tilde{L}^* \tilde{L}, \end{aligned} \quad (1.6)$$

where $\Pi \in \mathbb{C}^{n, n}$ is a projector matrix (i.e., $\Pi^2 = \Pi$), the coefficients satisfy

$$\begin{aligned} \tilde{A} &= \Pi \tilde{A} \Pi \in \mathbb{C}^{n, n}, & \tilde{B} &= \Pi \tilde{B} \in \mathbb{C}^{n, \tilde{m}}, & \tilde{S} &= \Pi^* \tilde{S} \in \mathbb{C}^{n, \tilde{m}}, \\ \tilde{Q} &= \tilde{Q}^* = \Pi^* \tilde{Q} \Pi \in \mathbb{C}^{n, n}, & \tilde{R} &= \tilde{R}^* \in \mathbb{C}^{\tilde{m}, \tilde{m}}. \end{aligned} \quad (1.7)$$

and \tilde{p} as small as possible. We prove that these projected Lur'e equations are implicitly equivalent to a Riccati equation as long as our deflating subspace contains a certain part of the deflating subspace corresponding to the infinite eigenvalues. For the solution of this implicit algebraic Riccati equation, we present a generalized Newton-Kleinman-ADI approach.

This article is organized as follows. In the forthcoming section, we arrange the basic notation and present the fundamental facts about matrix pencils and their normal forms. In particular, we present fundamentals of deflating subspaces, give a constructive approach via so-called *Wong sequences*, and develop some extensions which are useful in later parts. Thereafter, in Section 3, we briefly repeat some results about solution theory for Lur'e equations. In particular, the connection between solutions and deflating subspaces of the even matrix pencil $s\mathcal{E} - \mathcal{A}$ as in (1.4) is highlighted. As well, we slightly extend this theory to projected Lur'e equations. In Section 4 we develop the main theoretical preliminaries for the numerical method introduced in this work: Based on the concept of *partial solution* we present some results on the structure of the corresponding projected Lur'e equations. In particular, we give equivalent criteria on the deflated subspace for the possibility to reformulate the projected Lur'e equations (1.6) as projected Riccati equations. This theory enables us to formulate in Section 5 a numerical algorithm for solution of Lur'e equations which consists first in determining a "critical deflating subspace of $s\mathcal{E} - \mathcal{A}$ ", and then an iterative solution of the obtained projected algebraic Riccati equation. The article ends with Section 6, where the presented numerical approach is tested by means of several numerical examples.

2. Matrix theoretic preliminaries.

2.1. Nomenclature.

We adopt the following notations.

\mathbb{N}, \mathbb{N}_0	set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\lfloor x \rfloor$	$= \max \{ n \in \mathbb{N} \mid n \leq x \} \cup \{0\}$, integer part of $x \in \mathbb{R}$
$\mathbb{C}^+, \mathbb{C}^-, i\mathbb{R}$	set of complex numbers with negative (positive, zero) real part, resp.
\bar{z}, A^*, A^T	conjugate of $z \in \mathbb{C}$, Hermitian and transpose of $A \in \mathbb{C}^{m,n}$, resp.
$\text{Gl}_n(\mathbb{C})$	the group of invertible real $n \times n$ matrices
$\mathbb{C}[s]$	the ring of polynomials with coefficients in \mathbb{C}
$\mathbb{C}(s)$	the quotient field of $\mathbb{R}[s]$
$R^{n,m}$	the set of $n \times m$ matrices with entries in a ring R
$\ M\ _F$	Frobenius norm of $M \in \mathbb{C}^{n,m}$
M^+	Moore-Penrose inverse of $M \in \mathbb{C}^{n,m}$, i.e., a matrix such that $M^-MM^- = M^-$, $MM^-M = M$, $M^-M = (M^-M)^*$, $MM^- = (MM^-)^*$
$M \geq N$ ($>, \leq, <$)	$M - N$ is Hermitian and positive semi-definite (positive definite, negative semi-definite, negative definite).
$M^{-1}\mathcal{Y}$	$= \{ x \in \mathbb{C}^m \mid Mx \in \mathcal{Y} \}$, the pre-image of the set $\mathcal{Y} \subseteq \mathbb{C}^n$ under $M \in \mathbb{C}^{n,m}$
$\mathcal{Y}^{M\perp}$	$\{ x \in \mathbb{R}^n \mid x^*My = 0 \forall y \in \mathcal{Y} \}$, the M -orthogonal complement of the subspace $\mathcal{Y} \subseteq \mathbb{R}^n$ (where $M \in \mathbb{R}^{n,m}$)
$\text{diag}(A_1, \dots, A_k)$	block-diagonal matrix with entries $A_i \in \mathbb{C}^{m_i, n_i}$ for $m_i, n_i \in \mathbb{N}_0, i = 1, \dots, k$ ($A \in \mathbb{C}^{m,n}$ for $n = n_1 + \dots + n_k, m = m_1 + \dots + m_k$)

Moreover, an identity matrix of size $n \times n$ is denoted by I_n or simply by I , the zero $n \times m$ matrix is by $0_{n,m}$ or simply by 0 . The symbol $e_i^{(n)}$ (or simply e_i) stands i -th canonical unit vector of size n . We further introduce the special matrices $J_k, M_k, N_k \in \mathbb{R}^{k,k}$, $K_k, L_k \in \mathbb{R}^{k-1,k}$ for $k \in \mathbb{N}$, which are given by

$$\begin{aligned}
 J_k &= \begin{bmatrix} & & 1 \\ & \ddots & \\ 1 & & \end{bmatrix}, \quad K_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}, \quad L_k = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix}, \\
 M_k &= \begin{bmatrix} & & 1 & 0 \\ & \ddots & \ddots & \\ 1 & \ddots & & \\ 0 & & & \end{bmatrix}, \quad N_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & & 0 \end{bmatrix}.
 \end{aligned} \tag{2.1}$$

2.2. Matrix pencils. Here we introduce some fundamentals of matrix pencils, i.e., first order matrix polynomials $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{M,N}$ with $\mathcal{E}, \mathcal{A} \in \mathbb{C}^{M,N}$.

DEFINITION 2.1. A matrix pencil $P(s) = s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{M,N}$ is called

- (i) regular if $M = N$ and $\text{rank}_{\mathbb{C}(s)} P(s) = N$, and
- (ii) even if $P(\bar{s})^* = P(s)$, i.e., $\mathcal{E} = -\mathcal{E}^T$ and $\mathcal{A} = \mathcal{A}^T$.

Many properties of a matrix pencil can be characterized in terms of the *Kronecker canonical form (KCF)*.

Type	Size	$\mathcal{C}_j(s)$	Parameters
K1	$k_j \times k_j$	$(s - \lambda)I_{k_j} - N_{k_j}$	$k_j \in \mathbb{N}, \lambda \in \mathbb{C}$
K2	$k_j \times k_j$	$sN_{k_j} - I_{k_j}$	$k_j \in \mathbb{N}$
K3	$(k_j - 1) \times k_j$	$sK_{k_j} - L_{k_j}$	$k_j \in \mathbb{N}$
K4	$k_j \times (k_j - 1)$	$sK_{k_j}^T - L_{k_j}^T$	$k_j \in \mathbb{N}$

TABLE 2.1

Block types in Kronecker canonical form (with matrices as defined in (2.1))

THEOREM 2.2. [14] For a matrix pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{M,N}$, there exist matrices $U_l \in \text{Gl}_M(\mathbb{C})$, $U_r \in \text{Gl}_N(\mathbb{C})$, such that

$$U_l(s\mathcal{E} - \mathcal{A})U_r = \text{diag}(\mathcal{C}_1(s), \dots, \mathcal{C}_k(s)), \quad (2.2)$$

where each of the pencils $\mathcal{C}_j(s)$ is of one of the types presented in Table 2.1.

The numbers λ appearing in the blocks of type K1 are called the (generalized) eigenvalues of $sE - A$. Blocks of type K2 are said to be corresponding to infinite eigenvalues.

A special modification of the KCF for even matrix pencils, the so-called *even Kronecker canonical form (EKCF)* is presented in [28]. Note that there is also an extension of this form such that realness is preserved [29].

Type	Size	$\mathcal{D}_j(s)$	Parameters
E1	$2k_j \times 2k_j$	$\begin{bmatrix} 0_{k_j, k_j} & (\lambda - s)I_{k_j} - N_{k_j} \\ (\bar{\lambda} + s)I_{k_j} - N_{k_j}^T & 0_{k_j, k_j} \end{bmatrix}$	$k_j \in \mathbb{N}, \lambda \in \mathbb{C}^+$
E2	$k_j \times k_j$	$\epsilon_j((-is - \omega)J_{k_j} + M_{k_j})$	$k_j \in \mathbb{N}, \omega \in \mathbb{R},$ $\epsilon_j \in \{-1, 1\}$
E3	$k_j \times k_j$	$\epsilon_j(isM_{k_j} + J_{k_j})$	$k_j \in \mathbb{N},$ $\epsilon_j \in \{-1, 1\}$
E4	$(2k_j - 1) \times (2k_j - 1)$	$\begin{bmatrix} 0_{k_j, k_j} & isK_{k_j}^T + L_{k_j}^T \\ isK_{k_j} + L_{k_j} & 0_{k_j - 1, k_j - 1} \end{bmatrix}$	$k_j \in \mathbb{N}$

TABLE 2.2

Block types in even Kronecker canonical form (with matrices as defined in (2.1))

THEOREM 2.3. [28] For an even matrix pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{N,N}$, there exists

some $U \in \text{Gl}_N(\mathbb{C})$ such that

$$U^*(s\mathcal{E} - \mathcal{A})U = \text{diag}(\mathcal{D}_1(s), \dots, \mathcal{D}_k(s)), \quad (2.3)$$

where each of the pencils $\mathcal{D}_j(s)$ is of one of the types presented in Table 2.2. The numbers ε_j in the blocks of type E2 and E3 are called the block signatures.

The KCF can be easily obtained from an EKCF by permuting rows and columns: The blocks of type E1 contains pairs $(\lambda, -\bar{\lambda})$ of generalized eigenvalues. In the case where $\mathcal{E}, \mathcal{A} \in \mathbb{R}^{n,n}$, non-imaginary eigenvalues even occur in quadruples $(\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda})$. The blocks of type E2 and E3 respectively correspond to the purely imaginary and infinite eigenvalues. Blocks of type E4 consist of a combination of blocks that are equivalent to those of type K3 and K4. Note that regularity of the pencil $s\mathcal{E} - \mathcal{A}$ is equivalent to the absence of blocks of type E4.

The following concept generalizes the notion of invariant subspaces to matrix pencils.

DEFINITION 2.4. *A subspace $\mathcal{V} \subset \mathbb{C}^N$ is called (right) deflating subspace for the pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{M,N}$ if for a matrix $V \in \mathbb{C}^{N,k}$ with full column rank and $\text{im } V = \mathcal{V}$, there exists some $l \in \mathbb{N}_0$, a matrix $W \in \mathbb{C}^{M,l}$ and a pencil $s\tilde{E} - \tilde{A} \in \mathbb{C}[s]^{l,k}$ with $\text{rank}_{\mathbb{C}(s)}(s\tilde{E} - \tilde{A}) = l$, such that*

$$(s\mathcal{E} - \mathcal{A})V = W(s\tilde{E} - \tilde{A}). \quad (2.4)$$

In the sequel we introduce special properties of matrix pencils $[sI - A, B] \in \mathbb{C}[s]^{n,n+m}$. In systems theory these properties correspond to trajectory design and stabilization of systems $\dot{x}(t) = Ax(t) + Bu(t)$ and are also known under the name *Hautus criteria*.

DEFINITION 2.5. *Let a pair $(A, B) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,m}$ be given. Then*

- (i) $\lambda \in \mathbb{C}$ is called an uncontrollable mode of (A, B) , if it is a generalized eigenvalue of $[sI - A, B]$;
- (ii) (A, B) is called controllable if it does not have any uncontrollable modes;
- (iii) (A, B) is called stabilizable, if all uncontrollable modes have non-positive real part.

Finally, we present some notations about (possibly indefinite) inner products induced by a Hermitian matrix.

DEFINITION 2.6. *Let an Hermitian matrix $\mathcal{M} \in \mathbb{C}^{N,N}$ be given.*

- (i) Two subspaces $\mathcal{V}_1, \mathcal{V}_2 \subset \mathbb{C}^N$ are called \mathcal{M} -orthogonal if $\mathcal{V}_2 \subset \mathcal{V}_1^{\mathcal{M}\perp}$.
- (ii) A subspace $\mathcal{V} \subset \mathbb{C}^N$ is called \mathcal{M} -neutral if \mathcal{V} is \mathcal{M} -orthogonal to itself.

2.3. Deflating subspaces and (neutral) Wong sequences. It is immediate that, in the KCF (2.2) and EKCF (2.3), the space spanned by the columns of U_τ (resp. U) that correspond to a single block defines a deflating subspace. Roughly speaking, we now give a characterization of these spaces without making use of the full KCF or EKCF. This is obtained by using the so-called Wong sequences [7, 8, 33].

The Wong sequence $(\mathcal{W}_\lambda^{(\ell)})$ of a pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{M,N}$ associated to a given $\lambda \in \mathbb{C}$ is the sequence of subspaces defined recursively by

$$\mathcal{W}_\lambda^{(0)} = \{0\}, \quad \mathcal{W}_\lambda^{(\ell)} = (\lambda\mathcal{E} - \mathcal{A})^{-1}(\mathcal{E}\mathcal{W}_\lambda^{(\ell-1)}), \quad \ell \in \mathbb{N}, \quad (2.5a)$$

while the Wong sequence for $\lambda = \infty$ is defined via

$$\mathcal{W}_\infty^{(0)} = \{0\}, \quad \mathcal{W}_\infty^{(\ell)} = \mathcal{E}^{-1}(\mathcal{A}\mathcal{W}_\infty^{(\ell-1)}), \quad \ell \in \mathbb{N}. \quad (2.5b)$$

It is shown in [7, 8, 33] that $(\mathcal{W}_\lambda^{(\ell)})$ is an increasing sequence of nested subspaces (i.e., $\mathcal{W}_\lambda^{(\ell-1)} \subseteq \mathcal{W}_\lambda^{(\ell)}$), and, by reasons of finite-dimensionality, we have stagnation of this sequence. We define

$$\mathcal{W}_\lambda := \bigcup_{\ell=0}^{\infty} \mathcal{W}_\lambda^{(\ell)}. \quad (2.6)$$

In the following, we show that \mathcal{W}_λ is exactly the sum of the deflating subspaces associated to blocks corresponding to the generalized eigenvalue $\lambda \in \mathbb{C} \cup \{\infty\}$ together with the space corresponding to blocks of type K3.

First we present an auxiliary result stating that Wong sequences of a block-diagonal pencil are formed by direct sums of separate Wong sequences. It is furthermore shown how the pre- and post-multiplication of a pencil by invertible matrices influences Wong sequences.

LEMMA 2.7. *Let $\lambda \in \mathbb{C} \cup \{\infty\}$ and a pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{M,N}$ be given.*

- (i) *If $(\mathcal{W}_\lambda^{(\ell)})$ is a Wong sequence for $s\mathcal{E} - \mathcal{A}$, and $U_l \in \text{Gl}_M(\mathbb{C})$, $U_r \in \text{Gl}_N(\mathbb{C})$, then the corresponding Wong sequence for $U_l(s\mathcal{E} - \mathcal{A})U_r$ is given by $(U_r^{-1}\mathcal{W}_\lambda^{(\ell)})$.*
- (ii) *Let $\mathcal{W}_\lambda^{(\ell)}$, $\widetilde{\mathcal{W}}_\lambda^{(\ell)}$ be Wong sequences for $s\mathcal{E} - \mathcal{A}$ and $s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}$ respectively. Then the corresponding Wong sequence for the pencil $s \text{diag}(\mathcal{E}, \widetilde{\mathcal{E}}) - \text{diag}(\mathcal{A}, \widetilde{\mathcal{A}})$ is given by $\mathcal{W}_\lambda^{(\ell)} \times \widetilde{\mathcal{W}}_\lambda^{(\ell)}$.*

This enables us to consider the Wong sequences of the blocks in the KCF separately. It is easy to work out directly what happens on a single block of a Kronecker canonical form. For instance, for $\lambda = \infty$, direct computation shows that $\mathcal{W}_\infty^{(\ell)} = \{0\}$ for all ℓ on a K1 or K4 block, while for either a block of type K2 with size $k_j \times k_j$, or a block of type K3 with size $(k_j - 1) \times k_j$ we obtain that

$$\mathcal{W}_\infty^{(\ell)} = \begin{cases} \text{span}\{e_1, \dots, e_\ell\} & \ell < k_j, \\ \mathbb{C}^{k_j} & \ell \geq k_j. \end{cases}$$

As a consequence of these computations and Lemma 2.7, we can formulate the subsequent result that connects the subspace \mathcal{W}_λ (which obviously does not depend on the particular choice of the matrices U_r and U_l as in (2.2)) to the space spanned by certain columns of U_r .

COROLLARY 2.8. *Let $\lambda \in \mathbb{C} \cup \{\infty\}$ and a pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{M,N}$ be given. Let $U_l \in \text{Gl}_M(\mathbb{C})$, $U_r \in \text{Gl}_N(\mathbb{C})$ such that $U_l(s\mathcal{E} - \mathcal{A})U_r$ is in KCF (2.2). Further, let U_r be partitioned conformably with the KCF as*

$$U_r = [U_1 \quad U_2 \quad \dots \quad U_k].$$

Then, for \mathcal{W}_λ as in (2.6), there holds

$$\mathcal{W}_\lambda = \sum_{j \in T_\lambda \cup S} \text{im } U_j,$$

where

$$S = \{ j \in \mathbb{N} \mid \mathcal{C}_j \text{ is of type K3} \},$$

$$T_\lambda = \begin{cases} \{ j \in \mathbb{N} \mid \mathcal{D}_j \text{ is of type K1 with eigenvalue } \lambda \}, & \text{if } \lambda \in \mathbb{C}, \\ \{ j \in \mathbb{N} \mid \mathcal{D}_j \text{ is of type K3} \}, & \text{if } \lambda = \infty. \end{cases}$$

A further consequence is that, for any $\lambda, \mu \in \mathbb{C} \cup \{\infty\}$ with $\lambda \neq \mu$, there holds $\mathcal{W}_\lambda \cap \mathcal{W}_\mu = \sum_{j \in S} \text{im } U_j$. Hence, the deflating subspace corresponding to all blocks of type K3 is well defined as well. We now present an auxiliary result on Wong sequences of upper triangular matrix pencils, which will be an essential ingredient for one of the main results of this article.

LEMMA 2.9. *Let the matrix pencils $s\mathcal{E}_{ij} - \mathcal{A}_{ij} \in \mathbb{C}[s]^{M_i, N_j}$ be given for $(i, j) \in \{(1, 1), (1, 2), (2, 2)\}$. For $\lambda \in \mathbb{C} \cup \{\infty\}$, denote $\mathcal{W}_{\lambda, 11}$ and \mathcal{W}_λ to be the spaces at which the Wong sequences of the pencils $s\mathcal{E}_{11} - \mathcal{A}_{11}$ and, respectively,*

$$s\mathcal{E} - \mathcal{A} = \begin{bmatrix} s\mathcal{E}_{11} - \mathcal{A}_{11} & s\mathcal{E}_{12} - \mathcal{A}_{12} \\ 0 & s\mathcal{E}_{22} - \mathcal{A}_{22} \end{bmatrix},$$

stagnate. Assume that the KCF of $s\mathcal{E}_{11} - \mathcal{A}_{11}$ does not contain any blocks of type K4 and, moreover, $\dim \mathcal{W}_{\lambda, 11} = \dim \mathcal{W}_\lambda$. Then $\mathcal{W}_\lambda = \mathcal{W}_{\lambda, 11} \times \{0\}$ and $\ker \lambda\mathcal{E}_{22} - \mathcal{A}_{22} = \{0\}$.

Proof. We only show the result for $\lambda \in \mathbb{C}$. The case of infinite eigenvalue can be proven by reversing the roles of \mathcal{E} and \mathcal{A} , and then setting $\lambda = 0$.

By the upper triangularity of $s\mathcal{E} - \mathcal{A}$ and the construction of the Wong sequences, we immediately obtain that $\mathcal{W}_{\lambda, 11} \times \{0\}$ is a subset of \mathcal{W}_λ . Since the dimensions of these spaces equal, we obtain $\mathcal{W}_{\lambda, 11} \times \{0\} = \mathcal{W}_\lambda$.

Using that the KCF of $s\mathcal{E}_{11} - \mathcal{A}_{11}$ does not contain any blocks of type K4, we may employ the KCF to obtain the identity

$$\mathcal{E}_{11}\mathcal{W}_{\lambda, 11} + \text{im}(\lambda\mathcal{E}_{11} - \mathcal{A}_{11}) = \mathbb{C}^{N_1}. \quad (2.7)$$

Now assume that $y \in \ker(\lambda\mathcal{E}_{22} - \mathcal{A}_{22})$. Then, by (2.7), there exists some $x \in \mathbb{C}^{N_1}$ with

$$(\lambda\mathcal{E}_{11} - \mathcal{A}_{11})x + (\lambda\mathcal{E}_{12} - \mathcal{A}_{12})y \in \mathcal{E}_{11}\mathcal{W}_{\lambda, 11}.$$

Hence,

$$\begin{bmatrix} \lambda\mathcal{E}_{11} - \mathcal{A}_{11} & \lambda\mathcal{E}_{12} - \mathcal{A}_{12} \\ 0 & \lambda\mathcal{E}_{22} - \mathcal{A}_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{E}_{11}\mathcal{W}_{\lambda, 11} \times \{0\},$$

i.e.,

$$\begin{bmatrix} x \\ y \end{bmatrix} \in \begin{bmatrix} \lambda\mathcal{E}_{11} - \mathcal{A}_{11} & \lambda\mathcal{E}_{12} - \mathcal{A}_{12} \\ 0 & \lambda\mathcal{E}_{22} - \mathcal{A}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ 0 & \mathcal{E}_{22} \end{bmatrix} \cdot (\mathcal{W}_{\lambda, 11} \times \{0\}) = \mathcal{W}_{\lambda, 11} \times \{0\}.$$

However, this implies $y = 0$. Altogether, we have $\ker(\lambda\mathcal{E}_{22} - \mathcal{A}_{22}) = \{0\}$, whence λ is no generalized eigenvalue of $s\mathcal{E}_{22} - \mathcal{A}_{22}$. \square

In the sequel we extend the theory of Wong sequences to obtain \mathcal{E} -neutral deflating subspaces of even matrix pencils, which are essential for our theoretical and algorithmic framework for Lur'e equations. By a closer look at the EKCF (2.3), it can be realized that for $\lambda \in \mathbb{C} \setminus i\mathbb{R}$, the space \mathcal{W}_λ is \mathcal{E} -neutral. However, this does not hold for imaginary or infinite generalized eigenvalues. The following modification of Wong sequences provides a suitable “ \mathcal{E} -neutral part” of these subspaces. We define the *neutral Wong sequence* $(\mathcal{V}_{i\omega}^{(\ell)})$ associated with the imaginary eigenvalue $\lambda \in i\mathbb{R}$

via

$$\mathcal{Z}_\lambda^{(0)} = \mathcal{V}_\lambda^{(0)} = \{0\}, \quad (2.8a)$$

$$\mathcal{Z}_\lambda^{(\ell)} = (\lambda \mathcal{E} - \mathcal{A})^{-1} \left(\mathcal{E} \mathcal{V}_\lambda^{(\ell-1)} \right), \quad (2.8b)$$

$$\mathcal{V}_\lambda^{(\ell)} = \mathcal{V}_\lambda^{(\ell-1)} + \left(\mathcal{Z}_\lambda^{(\ell)} \cap \left(\mathcal{Z}_\lambda^{(\ell)} \right)^{\mathcal{E}^\perp} \right), \quad \ell \in \mathbb{N}, \quad (2.8c)$$

and the corresponding sequence for the infinite eigenvalue as

$$\mathcal{Z}_\infty^{(0)} = \mathcal{V}_\infty^{(0)} = \{0\}, \quad (2.8d)$$

$$\mathcal{Z}_\infty^{(\ell)} = \mathcal{E}^{-1} \left(\mathcal{A} \mathcal{V}_\infty^{(\ell-1)} \right), \quad (2.8e)$$

$$\mathcal{V}_\infty^{(\ell)} = \mathcal{V}_\infty^{(\ell-1)} + \left(\mathcal{Z}_\infty^{(\ell)} \cap \left(\mathcal{Z}_\infty^{(\ell)} \right)^{\mathcal{E}^\perp} \right), \quad \ell \in \mathbb{N}. \quad (2.8f)$$

It is obvious from its definition that $(\mathcal{V}_\lambda^{(\ell)})$ is an increasing and eventually stagnating sequence of nested subspaces, and we may define the subspace

$$\mathcal{V}_\lambda := \bigcup_{\ell=0}^{\infty} \mathcal{V}_\lambda^{(\ell)}. \quad (2.9)$$

Furthermore, if for the ‘‘conventional Wong sequence’’ $(W_\lambda^{(\ell)})$ there holds that $W_\lambda^{(\ell)}$ is \mathcal{E} -neutral for $\ell = 0, 1, 2, \dots, h$, then $\mathcal{V}_\lambda^\ell = \mathcal{W}_\lambda^\ell$ for $\ell = 1, 2, \dots, h$.

The following statement (which is analogous to Lemma 2.7) applies to $(\mathcal{V}_{i\omega}^{(\ell)})$ and shows that we may consider separately the blocks in the EKCF when analysing the neutral Wong sequences.

LEMMA 2.10. *Let $\lambda \in i\mathbb{R} \cup \{\infty\}$ and an even matrix pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{N,N}$ be given.*

- (i) *If $(\mathcal{V}_\lambda^{(\ell)})$ is a neutral Wong sequence for $s\mathcal{E} - \mathcal{A}$ and $U \in \text{Gl}_N(\mathbb{C})$, then the corresponding neutral Wong sequence for $U^*(s\mathcal{E} - \mathcal{A})U$ is given by $(U^{-1}\mathcal{V}_\lambda^{(\ell)})$.*
- (ii) *If $(\mathcal{V}_\lambda^\ell)$, $(\tilde{\mathcal{V}}_\lambda^\ell)$ are neutral Wong sequences for $s\mathcal{E} - \mathcal{A}$ and $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ respectively, then the corresponding neutral Wong sequence for $s \text{diag}(\mathcal{E}, \tilde{\mathcal{E}}) - \text{diag}(\mathcal{A}, \tilde{\mathcal{A}})$ is given by $(\mathcal{V}_\lambda^{(\ell)} \times \tilde{\mathcal{V}}_\lambda^{(\ell)})$.*

Again, we can explicitly characterize the space at which neutral Wong sequences stagnate.

THEOREM 2.11. *Let $\mathcal{V}_\lambda^{(\ell)}$ be the neutral Wong sequence associated to $\lambda \in i\mathbb{R} \cup \{\infty\}$ for the even pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{N,N}$. Let U be a nonsingular matrix reducing it to EKCF as in (2.3), partitioned conformably as*

$$U = [U_1 \quad U_2 \quad \dots \quad U_k].$$

Then, for \mathcal{V}_λ as in (2.9), there holds

$$\mathcal{V}_\lambda = \sum_{j \in T_\lambda} \text{im} \left(U_j \begin{bmatrix} I_{h_j} \\ 0_{k_j-h_j, h_j} \end{bmatrix} \right) + \sum_{j \in S} \text{im} \left(U_j \begin{bmatrix} I_{k_j} \\ 0_{k_j-1, k_j} \end{bmatrix} \right),$$

where

$$\begin{aligned} S &= \{ j \in \mathbb{N} \mid \mathcal{D}_j \text{ is of type } E4 \}, \\ T_\lambda &= \begin{cases} \{ j \in \mathbb{N} \mid \mathcal{D}_j \text{ is of type } E2 \text{ with eigenvalue } \lambda \}, & \text{if } \lambda \in i\mathbb{R}, \\ \{ j \in \mathbb{N} \mid \mathcal{D}_j \text{ is of type } E3 \}, & \text{if } \lambda = \infty, \end{cases} \\ h_j &= \begin{cases} \lfloor \frac{k_j}{2} \rfloor & \text{if } \lambda \in i\mathbb{R}, \\ \lfloor \frac{k_j+1}{2} \rfloor & \text{if } \lambda = \infty. \end{cases} \end{aligned}$$

In particular, the subspaces \mathcal{V}_λ are all \mathcal{E} -neutral and do not depend on the choice of U .

Proof. Lemma 2.10 allows to restrict to the case where $s\mathcal{E} - \mathcal{A}$ is a single block of one of the four types in Table 2.2.

E1 Since $\lambda \in i\mathbb{R} \cup \{\infty\}$, both matrices \mathcal{E} , $\lambda\mathcal{E} - \mathcal{A}$ are nonsingular, whence $\mathcal{V}_\lambda^{(\ell)} = \{0\}$.

E2 $\mathcal{V}_\lambda^{(\ell)} = \{0\}$ unless λ coincides with the generalized eigenvalue associated to given block. It therefore suffices to only consider the latter case. Explicit computation shows that $\mathcal{V}_\lambda^{(\ell)} = \text{span}\{e_1, e_2, \dots, e_\ell\}$ for $\ell \leq h_j$. After that, $\mathcal{Z}_\lambda^{(h_j+1)}$ is not \mathcal{E} -neutral anymore. The computations in the case of even and odd k_j slightly differ, but in both we obtain $(\mathcal{Z}_\lambda^{(h_j+1)})^{\mathcal{E}\perp} \subseteq \mathcal{V}_\lambda^{(h_j)}$, thus $\mathcal{V}_\lambda^{(h_j+1)} = \mathcal{V}_\lambda^{(h_j)}$, and the sequence stagnates.

E3 Here we have $\mathcal{V}_\lambda^{(\ell)} = \{0\}$ unless $\lambda = \infty$, so we consider only this case: However, a similar argumentation to that described for the case of a block of type E2 can be applied here to obtain the desired result.

E4 A block of type E4 is anti-diagonally composed of the block of type K3 and a block of type K4. For the latter, the ‘‘conventional Wong sequence’’ is trivial, i.e., $\mathcal{W}_\lambda^{(\ell)} = \{0\}$; for the former, the conventional Wong sequence reaches \mathbb{C}^{k_j} after k_j steps. Therefore, for any $\lambda \in i\mathbb{R} \cup \{\infty\}$ the Wong sequence $\mathcal{W}_\lambda^{(\ell)}$ of an E4 block fulfills $\mathcal{W}_\lambda^{(k_j)} = \text{span}\{e_1, e_2, \dots, e_{k_j}\}$. Since this subspace is \mathcal{E} -neutral, we may apply the statement that for the conventional and neutral Wong sequences there holds $\mathcal{W}_\lambda^{(\ell)} = \mathcal{V}_\lambda^{(\ell)}$, if $\mathcal{W}_\lambda^{(\ell)}$ is \mathcal{E} -neutral. Hence, we have $\mathcal{V}_\lambda = \mathcal{W}_\lambda = \text{span}\{e_1, e_2, \dots, e_{k_j}\}$. \square

3. Lur’e equations and deflating subspaces of even matrix pencils. Solvability and structure of the solution set of the Lur’e equations (1.1) are described in [24]. In particular, the eigenstructure of the associated even matrix pencil $s\mathcal{E} - \mathcal{A}$ (1.4) can be related to solutions of (1.1) in a way that these define deflating subspaces via

$$\begin{bmatrix} 0 & -sI + A & B \\ sI + A^* & Q & S \\ B^* & S^* & R \end{bmatrix} \begin{bmatrix} X & 0 \\ I_n & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ -X & K^* \\ 0 & L^* \end{bmatrix} \begin{bmatrix} -sI + A & B \\ K & L \end{bmatrix}. \quad (3.1)$$

The property $X = X^*$ is equivalent to this deflating subspace being \mathcal{E} -neutral.

DEFINITION 3.1. *Let $A, Q \in \mathbb{C}^{n,n}$, $B, S \in \mathbb{C}^{n,m}$ and $R \in \mathbb{R}^{m,m}$ be given with $Q = Q^*$, $R = R^*$. Then a solution $X \in \mathbb{C}^{n,n}$ of the Lur’e equations (1.1) with $X = X^*$ is called stabilizing (anti-stabilizing), if*

$$\text{rank} \begin{bmatrix} -\lambda I + A & B \\ K & L \end{bmatrix} = n + p \quad \text{for all } \lambda \in \mathbb{C}^+ \ (\lambda \in \mathbb{C}^-). \quad (3.2)$$

Condition (3.2) is equivalent to the KCF of the pencil

$$\begin{bmatrix} -sI + A & B \\ K & L \end{bmatrix}$$

containing no blocks of type K4 and, moreover, all generalized eigenvalues having non-positive (non-negative) real part. Note that in the case of invertible R the concept of (anti-) stabilizing solution introduced above coincides with the corresponding notion for algebraic Riccati equations [19].

It is shown in [24] that a stabilizing solution X is maximal, where the word “maximal” has to be understood in terms of definiteness. More precisely, all other solutions Y of the Lur'e equations fulfill $X \geq Y$. In an analogous way, anti-stabilizing solutions are minimal with respect to definiteness. For sake of brevity and analogy, we focus only on stabilizing solutions.

As we have seen in (3.1), solutions to Lur'e equations define \mathcal{E} -neutral deflating subspaces of the even matrix pencil (1.4). It is shown in [24] that also the converse holds true; that is, the solutions of the Lur'e equations can be constructed from certain \mathcal{E} -neutral deflating subspaces of $s\mathcal{E} - \mathcal{A}$. First we relate the existence of a stabilizing solution of the Lur'e equations (1.1) to the structure of the EKCF of (1.4), namely the following:

P1 All blocks of type E2 in the EWCF of $s\mathcal{E} - \mathcal{A}$ have even size.

P2 All blocks of type E3 in the EWCF of $s\mathcal{E} - \mathcal{A}$ have odd size and negative sign.

THEOREM 3.2 ([24]). *Let $A \in \mathbb{C}^{n,n}$, $B, S \in \mathbb{C}^{n,m}$ and $Q \in \mathbb{C}^{n,n}$, $R \in \mathbb{C}^{m,m}$ with $Q = Q^*$ and $R = R^*$ be given. Then the following holds true:*

- (a) *If a solution of the Lur'e equations exists, then the pencil $s\mathcal{E} - \mathcal{A}$ as in (1.4) fulfills **P1** and **P2**.*
- (b) *If a stabilizing solution of the Lur'e equations exists, then (A, B) is stabilizable and, the pencil $s\mathcal{E} - \mathcal{A}$ as in (1.4) fulfills **P1** and **P2**.*
- (c) *If **P1** and **P2** hold true and, moreover, at least one of the properties*
 - (i) *the pair (A, B) is stabilizable and the pencil $s\mathcal{E} - \mathcal{A}$ as in (1.4) is regular;*
 - (ii) *the pair (A, B) is controllable;**are fulfilled, then a stabilizing solution of the Lur'e equations exists.*

The stabilizing solution can indeed be explicitly constructed from deflating subspaces of the even matrix pencil (1.4):

It is shown in [24] that the *extended graph space*

$$\mathcal{G}_X = \text{im} \begin{bmatrix} I_n & 0 \\ X & 0 \\ 0 & I_m \end{bmatrix} \quad (3.3a)$$

of the stabilizing solution X fulfills

$$\left(\sum_{\lambda \in \mathbb{C}^-} \mathcal{W}_\lambda \right) + \left(\sum_{\lambda \in i\mathbb{R}} \mathcal{V}_\lambda \right) + \mathcal{V}_\infty = \mathcal{G}_X. \quad (3.3b)$$

In other words, for matrices $V_\mu, V_x \in \mathbb{C}^{n,n+m}$, $V_u \in \mathbb{C}^{m,n+m}$ with

$$\text{im} \begin{bmatrix} V_\mu \\ V_x \\ V_u \end{bmatrix} = \left(\sum_{\lambda \in \mathbb{C}^-} \mathcal{W}_\lambda \right) + \left(\sum_{\lambda \in i\mathbb{R}} \mathcal{V}_\lambda \right) + \mathcal{V}_\infty, \quad (3.3c)$$

there holds

$$X = V_\mu V_x^-, \quad (3.3d)$$

where $V_x^- \in \mathbb{C}^{n+m,n}$ is an arbitrary right inverse of V_x , that is, $V_x^- V_x = I$. Moreover, the value of p in the solutions is equal to $(\text{rank}_{\mathbb{C}(s)} s\mathcal{E} - \mathcal{A}) - 2n$.

Remark 3.3. In the case where the matrices A, Q, B, S and R are all real, then the space \mathcal{V}_∞ is real, too. Since the spaces $\mathcal{W}_\lambda + \mathcal{W}_{\bar{\lambda}}$ and $\mathcal{V}_\mu + \mathcal{V}_{\bar{\mu}}$ are real as well for any generalized eigenvalues $\lambda \in \mathbb{C}^-$, $\mu \in i\mathbb{R}$, it can be verified that the stabilizing solution is real in this case. Note that all numerical algorithms that will be introduced in this paper avoid complex arithmetic, if A, B, S, Q and R are all real.

The following result is a direct conclusion from the relations in (3.3). It is shown that the stabilizing solution of the Lur'e equations satisfies a certain identity with the matrices generating some deflating subspace $\check{\mathcal{V}}$ of $s\mathcal{E} - \mathcal{A}$ with

$$\check{\mathcal{V}} \subset \left(\sum_{\lambda \in \mathbb{C}^-} \mathcal{W}_\lambda \right) + \left(\sum_{\lambda \in i\mathbb{R}} \mathcal{V}_\lambda \right) + \mathcal{V}_\infty. \quad (3.4)$$

COROLLARY 3.4. Let $A \in \mathbb{C}^{n,n}$, $B, S \in \mathbb{C}^{n,m}$ and $Q \in \mathbb{C}^{n,n}$, $R \in \mathbb{C}^{m,m}$ with $Q = Q^*$ and $R = R^*$ be given. Assume that the Lur'e equations have a stabilizing solution. Let $\check{\mathcal{V}}$ be an r -dimensional deflating subspace of $s\mathcal{E} - \mathcal{A}$ such that (3.4) holds true. Then, for $\check{V}_\mu, \check{V}_x \in \mathbb{C}^{2n+m,r}$, $\check{V}_u \in \mathbb{C}^{2n+m,r}$ with

$$\check{\mathcal{V}} = \text{im} \begin{bmatrix} \check{V}_\mu \\ \check{V}_x \\ \check{V}_u \end{bmatrix}, \quad (3.5)$$

there holds $\ker \check{V}_x \subset \ker \check{V}_\mu$. Moreover, the stabilizing solution X of (1.1) satisfies (1.5).

Remark 3.5. Note that for any deflating subspace $\check{\mathcal{V}}$ with (3.4), the space

$$\check{\mathcal{V}} + (\{0\} \times \{0\} \times \mathbb{C}^m)$$

is an \mathcal{E} -neutral deflating subspace which is also a subset of G_X . Hence we can make use of Corollary 3.4 to see that it is no loss of generality to assume that

$$\check{\mathcal{V}} = \text{im} \underbrace{\begin{bmatrix} \check{V}_\mu & 0 \\ \check{V}_x & 0 \\ 0 & I_m \end{bmatrix}}_{=: \check{V}}. \quad (3.6)$$

Furthermore, \check{V} has full column rank if and only if \check{V}_x has full column rank. Therefore, we may assume in the following that \check{V}_x has a left inverse \check{V}_x^- , i.e., the relation $\check{V}_x^- \check{V}_x = I$ holds true.

3.1. Projected Lur'e equations. Now we extend some of the terminology and solution theory to projected Lur'e equations (1.6) with (1.7). These equations will occur in later parts after a certain transformation of standard Lur'e equations.

In theory, we may change coordinates so that the equations are equivalent to a system of Lur'e equations of smaller dimension. Namely, for $T \in \text{Gl}_n(\mathbb{C})$ with

$$\Pi = T \text{diag}(I, 0) T^{-1}, \quad (3.7a)$$

conditions (1.7) imply

$$T^{-1}\tilde{A}T = \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad T^*\tilde{X}T = \begin{bmatrix} \tilde{X}_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad T^*\tilde{Q}T = \begin{bmatrix} \tilde{Q}_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad (3.7b)$$

$$T^{-1}\tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}, \quad T^*\tilde{S} = \begin{bmatrix} \tilde{S}_1 \\ 0 \end{bmatrix} \quad (3.7c)$$

and we are led back to Lur'e equations in standard form

$$\begin{aligned} \tilde{A}_{11}^*\tilde{X}_{11} + \tilde{X}_{11}\tilde{A}_{11} + \tilde{Q}_{11} &= \tilde{K}_1^*\tilde{K}_1, \\ \tilde{X}_{11}\tilde{B}_1 + \tilde{S}_1 &= \tilde{K}_1^*\tilde{L}, \\ \tilde{R} &= \tilde{L}^*\tilde{L}. \end{aligned} \quad (3.8)$$

In practice, we would like to avoid this transformation for numerical reasons.

DEFINITION 3.6. *We say that \tilde{X} is a (stabilizing) solution of the projected Lur'e equations (1.6), if there holds*

$$\text{im} \begin{bmatrix} \tilde{X} & 0 \\ \Pi & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \Pi^* & 0 & 0 \\ 0 & \Pi & 0 \\ 0 & 0 & I \end{bmatrix} \cdot \left(\left(\sum_{\lambda \in \mathbb{C}^-} \tilde{W}_\lambda \right) + \left(\sum_{\lambda \in i\mathbb{R}} \tilde{V}_\lambda \right) + \tilde{V}_\infty \right), \quad (3.9)$$

where \tilde{W}_λ , \tilde{V}_λ and \tilde{V}_∞ are the corresponding (neutral) Wong sequences of the even pencil

$$s\tilde{\mathcal{E}} - \tilde{\mathcal{A}} = \begin{bmatrix} 0 & -s\Pi + \tilde{A} & \tilde{B} \\ s\Pi^* + \tilde{A}^* & \tilde{Q} & \tilde{S} \\ \tilde{B}^* & \tilde{S}^* & \tilde{R} \end{bmatrix}. \quad (3.10)$$

It follows immediately that \tilde{X} is the stabilizing solution of the projected Lur'e equations (1.6) with (1.7), if, and only if, \tilde{X}_{11} with (3.7) is the stabilizing solution of the reduced Lur'e equations (3.8). As a consequence, we may infer the following from Theorem 3.2:

COROLLARY 3.7. *Let $\tilde{A} \in \mathbb{C}^{n,n}$, $\tilde{B}, \tilde{S} \in \mathbb{C}^{n,m}$ and $\tilde{Q} \in \mathbb{C}^{n,n}$, $\tilde{R} \in \mathbb{C}^{m,m}$ with $Q = Q^*$ and $R = R^*$ be given. Furthermore, let $\Pi \in \mathbb{C}^{n,n}$ be a projector with*

$$\tilde{A} = \Pi\tilde{A}\Pi, \quad \tilde{B} = \Pi\tilde{B}, \quad \tilde{S} = \Pi^*\tilde{S}, \quad \tilde{Q} = \Pi^*\tilde{Q}\Pi.$$

Then the following holds true:

- (a) *If a solution of the projected Lur'e equations (1.6) with (1.7) exists, then the pencil $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ as in (1.4) fulfills **P1** and **P2**.*
- (b) *If a stabilizing solution of the projected Lur'e equations (1.6) with (1.7) exists, then the pencil $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ as in (3.10) fulfills **P1** and **P2**.*

4. Partial solutions and projected Lur'e equations. If we have computed some deflating subspace $\check{V} = \text{im } \check{V} \subset G_X$ for some matrix \check{V} as in (3.6) with full column rank, then Corollary 3.4 provides information on the action of X on a certain subspace.

In this section, we show that the remaining 'part' of the stabilizing solution X solves projected Lur'e equations. As explained in Remark 3.5, we may assume that for \check{V} as in (3.6), the submatrix $\check{V}_x \in \mathbb{C}^{n,\check{n}}$ possesses a left inverse $\check{V}_x^- \in \mathbb{C}^{\check{n},n}$. The

matrix $\Pi = I_n - \check{V}_x \check{V}_x^- \in \mathbb{C}^{n,n}$ is therefore a projector along $\text{im } \check{V}_x$. Expanding the stabilizing solution X of the Lur'e equations (1.1) as

$$X = \Pi^* X \Pi + (I - \Pi)^* X + \Pi^* X (I - \Pi),$$

the relation $X(I - \Pi) = X \check{V}_x \check{V}_x^- = \check{V}_\mu \check{V}_x^-$ gives rise to

$$\begin{aligned} X &= \Pi^* X \Pi + (\check{V}_x^-)^* \check{V}_\mu^* + \Pi^* \check{V}_\mu \check{V}_x^- \\ &= \Pi^* X \Pi + (\check{V}_x^-)^* \check{V}_\mu^* + \check{V}_\mu \check{V}_x^- - (\check{V}_x^-)^* \check{V}_x^* \check{V}_\mu \check{V}_x^-. \end{aligned} \quad (4.1)$$

As a consequence, the problem of solving the Lur'e equations for X is reduced to the problem of solving for X on a subspace complementary to $\text{im } \check{V}_x$. We therefore speak of *partially solving* the Lur'e equations. We describe in the sequel that the matrix $\Pi^* X \Pi$ is indeed a solution of certain projected Lur'e equations (1.6):

Multiplying $A^* X + X A + Q = K^* K$ a) from the left with Π^* and from the right with Π , b) from the left with \check{V}_x^* and from the right with Π , and c) from the left with \check{V}_x^* and from the right with \check{V}_x yields

$$\begin{aligned} &\Pi^* A^* \Pi^* X \Pi + \Pi^* X \Pi A \Pi \\ &+ \Pi^* (A^* (\check{V}_x^-)^* \check{V}_\mu^* + \check{V}_\mu \check{V}_x^- A + Q) \Pi = (K \Pi)^* (K \Pi), \end{aligned} \quad (4.2a)$$

$$\check{V}_x^* A^* \Pi^* X \Pi + \check{V}_x^* A^* (\check{V}_x^-)^* \check{V}_\mu^* \Pi + \check{V}_\mu^* A \Pi + \check{V}_x^* Q \Pi = (K \check{V}_x)^* (K \Pi), \quad (4.2b)$$

$$\check{V}_x^* A^* \check{V}_\mu + \check{V}_\mu^* A \check{V}_x + \check{V}_x^* Q \check{V}_x = (K \check{V}_x)^* (K \check{V}_x). \quad (4.2c)$$

Furthermore, a multiplication of $B^* X + S^* = L^* K$ from the right with a) Π and b) \check{V}_x gives

$$B^* \Pi^* X \Pi + (B^* (\check{V}_x^-)^* \check{V}_\mu^* + S^*) \Pi = L^* (K \Pi), \quad (4.3a)$$

$$B^* \check{V}_\mu + S^* \check{V}_x = L^* (K \check{V}_x). \quad (4.3b)$$

Then (4.2) and (4.3) imply that, by setting

$$\tilde{K} = K \Pi, \quad \tilde{L} = [K \check{V}_x \quad L],$$

then $\tilde{X} = \Pi^* X \Pi$ fulfills the projected Lur'e equation (1.6) with matrices

$$\begin{aligned} \tilde{A} &= \Pi A \Pi, \\ \tilde{Q} &= \Pi^* (A^* (\check{V}_x^-)^* \check{V}_\mu^* + \check{V}_\mu \check{V}_x^- A + Q) \Pi, \\ \tilde{B} &= [\Pi A \check{V}_x \quad \Pi B], \\ \tilde{S} &= [\Pi^* \check{V}_\mu \check{V}_x^- A \check{V}_x + \Pi^* A^* \check{V}_\mu + \Pi^* Q \check{V}_x \quad \Pi^* (S + \check{V}_\mu \check{V}_x^- B)], \\ \tilde{R} &= \begin{bmatrix} \check{V}_x^* A^* \check{V}_\mu + \check{V}_\mu^* A \check{V}_x + \check{V}_x^* Q \check{V}_x & \check{V}_x^* B + \check{V}_x^* S \\ B^* \check{V}_\mu + S^* \check{V}_x & R \end{bmatrix}. \end{aligned} \quad (4.4)$$

Conversely, the above computations imply that, if \tilde{X} solves the projected Lur'e equations, then X as in (4.1) solves the original Lur'e equations (1.1). In particular, there holds $\tilde{p} = p$.

In the following, we show that this reduction even preserves the property of $\tilde{X} = \Pi^* X \Pi$ being stabilizing.

THEOREM 4.1. *Let $A \in \mathbb{C}^{n,n}$, $B, S \in \mathbb{C}^{n,m}$ and $Q \in \mathbb{C}^{n,n}$, $R \in \mathbb{C}^{m,m}$ with $Q = Q^*$ and $R = R^*$ be given. Assume that the Lur'e equations have a stabilizing solution. Let $\check{V} \subset \mathbb{C}^{2n+m}$ be a deflating subspace of the even matrix pencil (1.4) with*

$$\{0\} \times \{0\} \times \mathbb{C}^m \subset \check{V} \subset \left(\sum_{\lambda \in \mathbb{C}^-} \mathcal{W}_\lambda \right) + \left(\sum_{\lambda \in i\mathbb{R}} \mathcal{V}_\lambda \right) + \mathcal{V}_\infty,$$

and

$$\check{V} = \text{im } \check{V} = \text{im} \begin{bmatrix} \check{V}_\mu & 0 \\ \check{V}_x & 0 \\ 0 & I_m \end{bmatrix} \quad (4.5)$$

for some $\check{V}_x \in \mathbb{C}^{n,\check{n}}$ with full column rank. Let $\Pi = I_n - \check{V}_x \check{V}_x^-$, where $\check{V}_x^- \in \mathbb{C}^{\check{n},n}$ fulfills $\check{V}_x^- \check{V}_x = I_{\check{n}}$. Then X is the stabilizing solution of the Lur'e equations (1.1), if and only if $\tilde{X} = \Pi^* X \Pi$ is a stabilizing solution of the projected Lur'e equations (1.6) with matrices as in (4.4).

Proof. By the definition of deflating subspace, there exists a matrix $W \in \mathbb{C}^{2n+m,k}$ with $\text{rank } W = k$ and a pencil $s\check{E} - \check{A} \in \mathbb{C}[s]^{k,\check{n}+m}$ with $\text{rank}_{\mathbb{C}(s)}(s\check{E} - \check{A}) = k$ and $(s\mathcal{E} - \mathcal{A})\check{V} = \check{W}(-s\check{E} + \check{A})$. In particular, the equation

$$\begin{bmatrix} -\check{V}_x & 0 \\ \check{V}_\mu & 0 \\ 0 & 0 \end{bmatrix} = \mathcal{E}\check{V} = \check{W}\check{E}$$

implies that $\check{E} = [\check{E}_1 \quad 0_{k,m}]$ for some $\check{E}_1 \in \mathbb{C}^{k,\check{n}}$ with $\text{rank } \check{E}_1 = \check{n}$. By a suitable change of coordinates in W , we can therefore assume that

$$-s\check{E} - \check{A} = \begin{bmatrix} -sI_{\check{n}} + \check{A}_{11} & \check{A}_{12} \\ \check{A}_{21} & \check{A}_{22} \end{bmatrix},$$

and thereby, for some for some matrices $\check{W}_{12}, \check{W}_{22} \in \mathbb{C}^{n,k-\check{n}}$, $\check{W}_{32} \in \mathbb{C}^{m,k-\check{n}}$,

$$W = \begin{bmatrix} \check{V}_x & \check{W}_{12} \\ -\check{V}_\mu & \check{W}_{22} \\ 0 & \check{W}_{32} \end{bmatrix}.$$

Let $T_x \in \mathbb{C}^{n,n-\check{n}}$ with $T_x = \Pi T_x \in \mathbb{C}^{n,n-\check{n}}$. Then $[\check{V}_x \quad T_x]$ is a nonsingular (square) matrix and

$$\text{im} \begin{bmatrix} X & 0 \\ I_n & 0 \\ 0 & I_m \end{bmatrix} = \text{im} \begin{bmatrix} \check{V}_\mu & X\Pi & 0 \\ \check{V}_x & \Pi & 0 \\ 0 & 0 & I_m \end{bmatrix} = \text{im} \begin{bmatrix} \check{V}_\mu & XT_x & 0 \\ \check{V}_x & T_x & 0 \\ 0 & 0 & I_m \end{bmatrix},$$

and the rightmost matrix has full column rank. Hence there exist matrices $W_{13}, W_{23} \in \mathbb{C}^{n,n+p-k}$, $W_{33} \in \mathbb{C}^{n,n+p-k}$ and $E_{13}, A_{13} \in \mathbb{C}^{\check{n},n-\check{n}}$, $E_{23}, A_{23} \in \mathbb{C}^{k-\check{n},n-\check{n}}$

and $E_{33}, A_{33} \in \mathbb{C}^{n+p-k, n-\tilde{n}}$ with

$$\begin{aligned} & \begin{bmatrix} 0 & -sI + A & B \\ sI + A^* & Q & S \\ B^* & S^* & R \end{bmatrix} \underbrace{\begin{bmatrix} \check{V}_\mu & 0 & XT_x \\ \check{V}_x & 0 & T_x \\ 0 & I_m & 0 \end{bmatrix}}_{=: V \in \mathbb{C}^{2n+m, m+n}} \\ &= \underbrace{\begin{bmatrix} \check{V}_x & \check{W}_{12} & W_{13} \\ -\check{V}_\mu & \check{W}_{22} & W_{23} \\ 0 & \check{W}_{32} & W_{33} \end{bmatrix}}_{=: W \in \mathbb{C}^{2n+m, n+p}} \underbrace{\begin{bmatrix} -sI_{\tilde{n}} + \check{A}_{11} & \check{A}_{12} & -sE_{13} + A_{13} \\ \check{A}_{21} & \check{A}_{22} & -sE_{23} + A_{23} \\ 0 & 0 & -sE_{33} + A_{33} \end{bmatrix}}_{=: -s\check{E} + \check{A} \in \mathbb{C}[s]^{n+p, m+n}}. \end{aligned}$$

The solution X is stabilizing if and only if $-\lambda\check{E} + \check{A}$ has full column rank for all $\lambda \in \mathbb{C}^+$. Due to our choice of the subspace $\check{\mathcal{V}}$, this holds if and only if $-\lambda E_{33} + A_{33}$ has full row rank for all $\lambda \in \mathbb{C}^+$. Now consider the matrices

$$M_{\check{\mathcal{V}}} = \begin{bmatrix} \Pi^* & (\check{V}_x^-)^* \check{V}_\mu^* \Pi & \check{V}_\mu & 0 & (\check{V}_x^-)^* \\ 0 & \Pi & \check{V}_x & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 \end{bmatrix}, \quad M_{\check{\mathcal{V}}}^- = \begin{bmatrix} \Pi^* & -\Pi^* \check{V}_\mu \check{V}_x^- & 0 \\ 0 & \Pi & 0 \\ 0 & \check{V}_x^- & 0 \\ 0 & 0 & I_m \\ \check{V}_x^* & -\check{V}_\mu^* & 0 \end{bmatrix}$$

Then we have $M_{\check{\mathcal{V}}} M_{\check{\mathcal{V}}}^- = I$ and

$$M_{\check{\mathcal{V}}}^* (s\mathcal{E} - \mathcal{A}) M_{\check{\mathcal{V}}} = \begin{bmatrix} 0 & -s\Pi + \tilde{A} & \tilde{B}_1 & \tilde{B}_2 & 0 \\ s\Pi^* + \tilde{A}^* & \tilde{Q} & \tilde{S}_1 & \tilde{S}_2 & \tilde{M}_1 \\ \tilde{B}_1^* & \tilde{S}_1^* & \tilde{R}_{11} & \tilde{R}_{12} & \tilde{M}_2 \\ \tilde{B}_2^* & \tilde{S}_2^* & \tilde{R}_{12}^* & \tilde{R}_{22} & \tilde{M}_3 \\ 0 & \tilde{M}_1^* & \tilde{M}_2^* & \tilde{M}_3^* & 0 \end{bmatrix},$$

with \tilde{A} and \tilde{Q} as in (4.4),

$$\tilde{M}_1 = \Pi^* A V_x^-, \quad \tilde{M}_2 = sI + \check{V}_x^* A \check{V}_x, \quad \tilde{M}_3 = B^* (\check{V}_x)^*,$$

and

$$\tilde{B} = [\tilde{B}_1 \quad \tilde{B}_2], \quad \tilde{S} = [\tilde{S}_1 \quad \tilde{S}_2], \quad \tilde{R} = \begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} \\ \tilde{R}_{12}^* & \tilde{R}_{22} \end{bmatrix}.$$

Then an evaluation of the matrix products in

$$(M_{\check{\mathcal{V}}}^* (s\mathcal{E} - \mathcal{A}) M_{\check{\mathcal{V}}}) \cdot (M_{\check{\mathcal{V}}}^- V) = (M_{\check{\mathcal{V}}}^* W) \cdot (-s\check{E} + \check{A})$$

leads to

$$\begin{aligned}
 & \begin{bmatrix} 0 & -s\Pi + \tilde{A} & \tilde{B}_1 & \tilde{B}_2 & 0 \\ s\Pi^* + \tilde{A}^* & \tilde{Q} & \tilde{S}_1 & \tilde{S}_2 & \tilde{M}_1 \\ \tilde{B}_1^* & \tilde{S}_1^* & \tilde{R}_{11} & \tilde{R}_{12} & \tilde{M}_2 \\ \tilde{B}_2^* & \tilde{S}_2^* & \tilde{R}_{12}^* & \tilde{R}_{22} & \tilde{M}_3 \\ 0 & \tilde{M}_1^* & \tilde{M}_2^* & \tilde{M}_3^* & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & \Pi^* X T_x \\ 0 & 0 & T_x \\ I_{\tilde{n}} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & \tilde{W}_{12} & \tilde{W}_{13} \\ 0 & \tilde{W}_{22} & \tilde{W}_{23} \\ 0 & \tilde{W}_{32} & \tilde{W}_{33} \\ 0 & \tilde{W}_{42} & \tilde{W}_{43} \\ I_{\tilde{n}} & \tilde{W}_{52} & \tilde{W}_{53} \end{bmatrix} \cdot \begin{bmatrix} -sI_{\tilde{n}} + \check{A}_{11} & \check{A}_{12} & -sE_{13} + A_{13} \\ \check{A}_{21} & \check{A}_{22} & -sE_{23} + A_{23} \\ 0 & 0 & -sE_{33} + A_{33} \end{bmatrix}
 \end{aligned}$$

for some matrices $\tilde{W}_{12}, \tilde{W}_{13}, \tilde{W}_{22}, \tilde{W}_{23}, \tilde{W}_{32}, \tilde{W}_{33}, \tilde{W}_{42}, \tilde{W}_{43}, \tilde{W}_{52}, \tilde{W}_{53}$ of suitable dimensions.

Cancelling the last row of this equation and, furthermore, realizing that the last block column of the matrix pencil on the left hand side has now influence on the product, we obtain

$$\begin{aligned}
 & \begin{bmatrix} 0 & -s\Pi + \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ s\Pi^* + \tilde{A}^* & \tilde{Q} & \tilde{S}_1 & \tilde{S}_2 \\ \tilde{B}_1^* & \tilde{S}_1^* & \tilde{R}_{11} & \tilde{R}_{12} \\ \tilde{B}_2^* & \tilde{S}_2^* & \tilde{R}_{12}^* & \tilde{R}_{22} \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & \Pi^* X T_x \\ 0 & 0 & T_x \\ I_{\tilde{n}} & 0 & 0 \\ 0 & I_m & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & \tilde{W}_{12} & \tilde{W}_{13} \\ 0 & \tilde{W}_{22} & \tilde{W}_{23} \\ 0 & \tilde{W}_{32} & \tilde{W}_{33} \\ 0 & \tilde{W}_{42} & \tilde{W}_{43} \end{bmatrix} \cdot \begin{bmatrix} -sI_{\tilde{n}} + \check{A}_{11} & \check{A}_{12} & -sE_{13} + A_{13} \\ \check{A}_{21} & \check{A}_{22} & -sE_{23} + A_{23} \\ 0 & 0 & -sE_{33} + A_{33} \end{bmatrix} \quad (4.6) \\
 &= \begin{bmatrix} \tilde{W}_{12} & \tilde{W}_{13} \\ \tilde{W}_{22} & \tilde{W}_{23} \\ \tilde{W}_{32} & \tilde{W}_{33} \\ \tilde{W}_{42} & \tilde{W}_{43} \end{bmatrix} \cdot \begin{bmatrix} \check{A}_{21} & \check{A}_{22} & -sE_{23} + A_{23} \\ 0 & 0 & -sE_{33} + A_{33} \end{bmatrix}.
 \end{aligned}$$

By our choice of T_x , the matrix before the first equal sign has full column rank and spans the subspace in (3.9). Thus, \tilde{X} is a stabilizing solution of the projected Lur'e equations if and only if $-\lambda E_{33} + A_{33}$ has full row rank for all $\lambda \in \mathbb{C}^+$. \square

THEOREM 4.2. *Under the assumptions and notation of Theorem 4.1, the following statements hold true for the pencil $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ in (3.10):*

- If, for $\lambda \in \mathbb{C}^-$, there holds $\mathcal{W}_\lambda \subset \check{\mathcal{V}}$, then the EKCF of $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ does not have blocks of type E1 corresponding the the generalized eigenvalue λ . Moreover, all blocks of type E4 in the EKCF of $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ are of size 1×1 .
- If, for $\lambda \in i\mathbb{R}$, there holds $\mathcal{V}_\lambda \subset \check{\mathcal{V}}$, then the EKCF of $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ does not have blocks of type E2 corresponding the the generalized eigenvalue λ . Moreover, all blocks of type E4 in the EKCF of $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ are of size 1×1 .
- If there holds $\mathcal{V}_\infty \subset \check{\mathcal{V}}$, then all blocks of type E3 and E4 in the EKCF of $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ are of size 1×1 .

Moreover, \tilde{X} is the stabilizing solution of the projected Lur'e equations (1.6) if, and only if, \tilde{X} fulfills the projected algebraic Riccati equation

$$\tilde{A}^* \tilde{X} + \tilde{X} \tilde{A} - (\tilde{X} \tilde{B} + \tilde{S}) \tilde{R}^+ (\tilde{X} \tilde{B} + \tilde{S})^* + \tilde{Q} = 0, \quad \tilde{X} = \Pi^* \tilde{X} \Pi \quad (4.7a)$$

with the additional property that all generalized eigenvalues of the pencil

$$-s\Pi + \tilde{A} - \tilde{B}\tilde{R}^+(\tilde{X}\tilde{B} + \tilde{S})^* \quad (4.7b)$$

are in $i\mathbb{R} \cup \mathbb{C}^-$.

Proof. By Corollary 3.7, solvability of the projected Lur'e equations implies that, in the EKCF of $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$, all blocks of type E2 have even size and, further, all blocks of type E3 have negative sign and odd size. Assume that $\tilde{V} \in \mathbb{C}^{2n+m, n-\tilde{n}+m}$, $\tilde{W} \in \mathbb{C}^{2n+m, n-\tilde{n}+p}$, $s\hat{E} - \hat{A} \in \mathbb{C}[s]^{n-\tilde{n}+m, n-\tilde{n}+p}$ with $\text{im } \tilde{V} = \text{im } \tilde{X} \times \text{im } \Pi \times \mathbb{C}^{m+\tilde{n}}$ and

$$(s\tilde{\mathcal{E}} - \tilde{\mathcal{A}})\tilde{V} = \tilde{W}(s\hat{E} - \hat{A}).$$

Then the following connection between the EKCF of $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ and the KCF of $s\hat{E} - \hat{A}$ holds true:

- (i) The EKCF of $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ has a block of type E1 with size $2k_j \times 2k_j$ corresponding to the generalized eigenvalues $\lambda, -\bar{\lambda}$ with $\lambda \in \mathbb{C}^-$, if, and only if, the KCF of $s\hat{E} - \hat{A}$ has a block of type K1 with size k_j corresponding to the generalized eigenvalue λ .
- (ii) The EKCF of $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ has a block of type E2 with size $k_j \times k_j$ corresponding to the generalized eigenvalue $\lambda \in i\mathbb{R}$, if, and only if, the KCF of $s\hat{E} - \hat{A}$ has a block of type K1 with size $k_j/2 \times k_j/2$ corresponding to the generalized eigenvalues λ .
- (iii) The EKCF of $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ has a block of type E3 with size $k_j \times k_j$, if, and only if, the KCF of $s\hat{E} - \hat{A}$ has a block of type K2 with size $(k_j + 1)/2 \times (k_j + 1)/2$.
- (iv) The EKCF of $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ has a block of type E4 with size $(2k_j - 1) \times (2k_j - 1)$, if, and only if, the KCF of $s\hat{E} - \hat{A}$ has a block of type K3 with size $(k_j - 1) \times k_j$.

By (4.6), we may assume that

$$s\hat{E} - \hat{A} = \begin{bmatrix} \check{A}_{21} & \check{A}_{22} & -sE_{23} + A_{23} \\ 0 & 0 & -sE_{33} + A_{33} \end{bmatrix}.$$

Now using Lemma 2.9 we obtain the following facts:

- (i') If, for $\lambda \in \mathbb{C}^-$, there holds $\mathcal{W}_\lambda \subset \check{\mathcal{V}}$, then $\lambda E_{33} - A_{33}$ has full row rank.
- (ii') If, for $\lambda \in i\mathbb{R}$, there holds $\mathcal{V}_\lambda \subset \check{\mathcal{V}}$, then $\lambda E_{33} - A_{33}$ has full row rank.
- (iii') If $\mathcal{V}_\infty \subset \check{\mathcal{V}}$, then E_{33} has full row rank.

Statements a) and b) are then immediate consequences of (i), (ii), (iv), (i') and (ii'). It remains to show c): If $\mathcal{V}_\infty \subset \check{\mathcal{V}}$, then, by (iii'), we have that E_{33} has full row rank. Assuming that the EKCF of $s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ has a block of type E3 which is of size greater than 1×1 , we obtain by (ii) that the KCF of

$$\begin{bmatrix} \check{A}_{21} & \check{A}_{22} & -sE_{23} + A_{23} \\ 0 & 0 & -sE_{33} + A_{33} \end{bmatrix}$$

has a block of type K2 with size greater than 1×1 . Then there exist non-zero vectors

$$z_0 = \begin{bmatrix} z_{01} \\ z_{02} \end{bmatrix}, \quad z_1 = \begin{bmatrix} z_{11} \\ z_{12} \end{bmatrix}$$

with

$$\begin{bmatrix} z_{01} \\ z_{02} \end{bmatrix}^* \begin{bmatrix} 0 & 0 & E_{23} \\ 0 & 0 & E_{33} \end{bmatrix} = 0, \\ \begin{bmatrix} z_{01} \\ z_{02} \end{bmatrix}^* \begin{bmatrix} \check{A}_{21} & \check{A}_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} = \begin{bmatrix} z_{11} \\ z_{12} \end{bmatrix}^* \begin{bmatrix} 0 & 0 & E_{23} \\ 0 & 0 & E_{33} \end{bmatrix}.$$

Since $[\check{A}_{21} \quad \check{A}_{22}]$ has full row rank, the latter equation gives rise to $z_{01} = 0$. The first equation together with the full E_{33} having full row rank then implies $z_{02} = 0$, which is a contradiction.

To complete the result, we have to show that the projected Lur'e equations can be transformed into a projected Riccati equation, if $\mathcal{V}_\infty \subset \check{\mathcal{V}}$: Since the blocks of type E3 and E4 in the EKCF of $s\check{\mathcal{E}} - \check{\mathcal{A}}$ have size at most 1, the Wong sequence $\mathcal{V}_\infty^{(\ell)} = \mathcal{W}_\infty^{(\ell)}$ stagnates at $\ell = 1$. In particular, $\ker \check{L} = \ker \check{R} \subset \ker \check{B} \cap \ker \check{S}$, as otherwise the sequence would continue. This implies

$$\check{K}^* \check{K} = (\check{B}\check{X} + \check{S})^* \check{R}^+ (\check{B}\check{X} + \check{S}).$$

Plugging this into $\check{A}^* \check{X} + \check{X} \check{A} + \check{Q} = \check{K}^* \check{K}$, we obtain that \check{X} solves the projected Riccati equation (4.7a). Furthermore, since

$$\begin{bmatrix} -s\Pi + \check{A} & \check{B} \\ \check{K} & \check{L} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\check{L}^* \check{K} & I \end{bmatrix} = \begin{bmatrix} -s\Pi + \check{A} - \check{B}\check{R}^-(\check{X}\check{B} + \check{S})^* & 0 \\ \check{K} & \check{L} \end{bmatrix},$$

the finite generalized eigenvalues of $-s\Pi + \check{A} - \check{B}\check{R}^-(\check{X}\check{B} + \check{S})^*$ equal to those of $sE_X - A_X$, i.e., they are contained in $i\mathbb{R} \cup \mathbb{C}^-$. \square

Remark 4.3. In many cases of practical relevance, such as in the positive real lemma [2], the bounded real lemma [1] or the case of positive definite cost functional [32], that is

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0,$$

there is an a priori knowledge of the stabilizing solution X being semi-definite. Then, we can choose \check{V}_x^- in a special way that simplifies the expressions (4.4). Consider the matrix $S := \check{V}_x^* X \check{V}_x = \check{V}_\mu^* \check{V}_x = \check{V}_x^* \check{V}_\mu$. Since $X \geq (\leq) 0$, we have $\check{S} \geq (\leq) 0$ and $\ker \check{V}_\mu \subset \ker \check{S}$, and thus we can write $\check{V}_\mu = \check{W} \check{S}$ for some $\check{W} \in \mathbb{C}^{n, \check{n}}$. Then, for a left inverse \check{V}_x^- of \check{V}_x , we can verify that

$$\check{V}_x^- = (I_{\check{n}} - \check{S}^+ \check{S}) \check{V}_x^- + \check{S}^+ \check{V}_\mu^*$$

is another left inverse of \check{V}_x and satisfies

$$\check{V}_\mu^* (I_{\check{n}} - \check{V}_x \check{V}_x^-) = \check{V}_\mu^* - (\check{S} - \check{S} \check{S}^+ \check{S}) \check{V}_x^- - \check{S} \check{S}^+ \check{S} \check{W}^* = \check{V}_\mu^* - \check{S} \check{W}^* = 0.$$

Therefore, if we use \check{V}_x^- instead of \check{V}_x^- in our computations then $\check{V}_\mu^* \Pi = 0$ holds. With this additional property, the matrices in (4.4) simplify to

$$\begin{aligned} \check{A} &= \Pi A \Pi, & \check{Q} &= \Pi^* Q \Pi, & \check{B} &= [\Pi A \check{V}_x^- \quad \Pi B], \\ \check{S} &= [\Pi^* A^* \check{V}_\mu + \Pi^* Q \check{V}_x^- \quad \Pi^* S], \\ \check{R} &= \begin{bmatrix} \check{V}_x^* A^* \check{V}_\mu + \check{V}_\mu^* A \check{V}_x^- + \check{V}_x^* Q \check{V}_x^- & \check{V}_\mu^* B + \check{V}_x^* S \\ B^* \check{V}_\mu + S^* \check{V}_x^- & R \end{bmatrix} \end{aligned} \quad (4.8)$$

and, by (4.1), the stabilizing solution is given by

$$X = \check{X} + \check{V}_\mu \check{S}^+ \check{V}_\mu^*,$$

where \tilde{X} is the stabilizing solution of the projected Lur'e equations (1.6) with matrices as in (4.8). In particular, given a solution $\tilde{X} = \pm \tilde{Z} \tilde{Z}^*$ in factored form, we obtain a factorization $X = \pm ZZ^*$, where $Z = \begin{bmatrix} \tilde{Z} & \check{V}_\mu \check{Y} \end{bmatrix}$ for some matrix \check{Y} with $\pm \check{Y} \check{Y}^* = \check{S}$. Solutions in this factored form are essential in balancing-related model order reduction and are provided by several algorithms for the solution of algebraic Riccati equations [4, 6].

5. Numerical aspects for the determination of the stabilizing solution.

Theorem 4.1 and Theorem 4.2 results in the previous sections can be used to develop an algorithm for the computation of the stabilizing solutions of Lur'e equations. The raw procedure can be outlined as follows.

- (1) For $\ell = 1, 2, \dots$, determine matrices $V_\infty^{(\ell)}$ with full column rank and $\mathcal{V}_\infty^{(\ell)} = \text{im } V_\infty^{(\ell)}$, until they stagnate to $\mathcal{V}_\infty = \text{im } V_\infty$.
- (2) Solve the projected Riccati equation (4.7a).

More details of these two steps are described in the next subsections.

Remark 5.1.

- a) The first step could, by Theorem 4.1 and Theorem 4.2 be extended by a further computation of Wong sequences corresponding eigenvalues with negative real part, or neutral Wong sequences corresponding to purely imaginary generalized eigenvalues.
- b) In practically relevant examples, we often have $m \ll n$ and further, the Wong sequence corresponding to the infinite eigenvalue usually stagnates after only a couple of steps. Therefore, step (1) is extremely fast, and the bulk of the computation is in step (2).

Often, the kernels that need to be computed in step (1) can be obtained from considerations of structural properties of the system, e.g., in the equations of the generalized positive real lemma for equations of linear electrical circuits [25].

- c) We will solve the projected Riccati equation (4.7a) by a slight generalization of the Newton-Kleinman-ADI iteration. In particular, sparsity of A , and low rank of Q are exploited by this method. Further note this iteration is quadratically convergent, if the pencil

$$-s\Pi + \tilde{A} - \tilde{B}\tilde{R}^+(\tilde{X}\tilde{B} + \tilde{S})^*$$

does not have purely imaginary generalized eigenvalues or, equivalently, the EKCF of $s\tilde{\mathcal{E}} - \tilde{A}$ does not have any blocks of type E2. By Theorem 4.2, the latter property is fulfilled if $s\mathcal{E} - A$ does not have any imaginary generalized eigenvalues or

$$\sum_{\lambda \in i\mathbb{R}} \mathcal{V}_\lambda \subset \check{\mathcal{V}}.$$

- d) In principle, this approach can be also applied to algebraic Riccati equations. If some “critical generalized eigenvalues” of the even matrix pencil (or, equivalently, some “critical eigenvalues” of the Hamiltonian matrix) are known a priori, these can be deflated to obtain a projected algebraic Riccati equation with nicer structural properties.

5.1. Computational aspects for Wong sequences. While spanning matrices for the spaces of Wong sequences are in principle explicitly computable from (2.5) and (2.8), some care is required in the implementation, especially in the case of a large-scale problem.

An essential step in the computation of the Wong sequence corresponding to a generalized eigenvalue $\lambda \in \mathbb{C}$ (note that infinite eigenvalues can be treated analogously) is the determination of the preimage

$$\mathcal{W}_\lambda^{(\ell)} = (\lambda\mathcal{E} - \mathcal{A})^{-1}(\mathcal{E}\mathcal{W}_\lambda^{(\ell-1)}).$$

This can be done as follows: For matrices T , U and V with full column rank and

$$\text{im } T = \ker(\lambda\mathcal{E} - \mathcal{A})^*, \quad \text{im } U = \ker(\lambda\mathcal{E} - \mathcal{A}), \quad \text{im } V = \mathcal{E}\mathcal{W}_\lambda^{(\ell-1)},$$

consider an matrix S with full column rank matrix and $\text{im } S = \ker T^*V$. Notice that the equation $(\lambda\mathcal{E} - \mathcal{A})x = b$ is solvable if and only if $T^*b = 0$, thus $\text{im } VS = \text{im } V \cap \text{im}(\lambda\mathcal{E} - \mathcal{A})$. In particular, the equation $(\lambda\mathcal{E} - \mathcal{A})X = VS$ is solvable and for any solution X , there holds $\mathcal{W}_\lambda^{(\ell)} = \text{im } X + \text{im } U$. This computation is feasible whenever T and U are stably computable or explicitly available due to structural properties of the involved matrices.

Data: a matrix pencil $sE - A \in \mathbb{C}[s]^{M,N}$, $\lambda \in \mathbb{C} \cup \{\infty\}$

Result: a matrix W_k with full column rank and $\text{im } W_k = \mathcal{W}_\lambda$

```

1  if  $\lambda = \infty$  then
2  |    $\mathcal{M} := \mathcal{E}$ ,  $\mathcal{N} := \mathcal{A}$ ;
3  else
4  |    $\mathcal{M} := \lambda\mathcal{E} - \mathcal{A}$ ,  $\mathcal{N} := \mathcal{E}$ ;
5  end
6  Determine a matrix  $T$  with full column rank and  $\text{im } T = \ker \mathcal{M}^*$ ;
7  Determine a matrix  $U$  with full column rank and  $\text{im } U = \ker \mathcal{M}$ ;
8   $W_0 := U$ ,  $k := 0$ ;
9  repeat
10 |    $k := k + 1$ ;
11 |    $Z_k := \mathcal{N}Z_{k-1}$ ;
12 |   Determine a matrix  $S$  with full column rank and  $\text{im } S_k = \ker T^*W_k$ ;
13 |   Solve  $\mathcal{M}X_k = Z_kS_k$  for the matrix  $X_k$ ;
14 |   Determine a matrix  $W_k$  with full column rank and  $\text{im } W_k = \text{im } [X_k \ U]$ ;
15 until  $\text{rank } W_k = \text{rank } W_{k-1}$ ;
    
```

Algorithm 1: Computation of \mathcal{W}_λ

Note that, if $s\mathcal{E} - \mathcal{A} \in \mathbb{C}[s]^{M,N}$ is a regular pencil, then $\text{im } X_k \cap \text{im } U = \{0\}$. In this case, line 14 in Algorithm 1 can be replaced with $W_k := [X_k \ U]$. Furthermore, since W_k, W_{k-1} have full column rank, the stop criterion $\text{rank } W_k = \text{rank } W_{k-1}$ reduces to a simple comparison of the numbers of rows of W_k and W_{k-1} .

For $\lambda \in i\mathbb{R} \cup \{\infty\}$ and an even matrix pencil $s\mathcal{E} - \mathcal{A}$, the computation of a matrix spanning \mathcal{V}_λ additionally involves the step $\mathcal{Z}_\lambda^{(\ell)} \cap (\mathcal{Z}_\lambda^{(\ell)})^{\mathcal{E}\perp}$. Computation of the latter subspace is based on the following result.

LEMMA 5.2. For skew-symmetric $\mathcal{E} \in \mathbb{C}^{N,N}$ and $U \in \mathbb{C}^{N,M}$, there holds

$$\text{im } U \cap (\text{im } U)^{\mathcal{M}\perp} = U \cdot \ker(U^*\mathcal{E}U).$$

Proof. For $w \in \ker(U^*\mathcal{E}U)$, simple arithmetics leads to $Uw \in \text{im } U \cap (\text{im } U)^{\mathcal{E}\perp}$. Hence, $\text{im } U \cap (\text{im } U)^{\mathcal{E}\perp} \supset U \cdot \ker(U^*\mathcal{E}U)$.

For showing the converse inclusion, let $u = Uw \in \text{im } U \cap (\text{im } U)^{\mathcal{E}\perp}$. Then, by $Uw \in (\text{im } U)^{\mathcal{E}\perp}$, there holds $(Uw)^* \mathcal{E}U = 0$, whence $w \in \ker(U^* \mathcal{E}U)$. This gives rise to

$$u = Uw \in U \cdot \ker(U^* \mathcal{E}U).$$

□

Using this result, we can extend Algorithm 1 to determine the \mathcal{E} -neutral deflating subspace corresponding to a generalized eigenvalue $\lambda \in i\mathbb{R} \cup \{\infty\}$. Note that, for $\lambda \in i\mathbb{R}$, the matrix $\lambda\mathcal{E} - \mathcal{A}$ is Hermitian. Since, moreover, \mathcal{E} is skew-Hermitian, we may choose $T = U$ in the notation of Algorithm 1.

Data: an even matrix pencil $sE - A \in \mathbb{C}[s]^{N,N}$, $\lambda \in i\mathbb{R} \cup \{\infty\}$

Result: a matrix V_k with full column rank and $\text{im } V_k = \mathcal{V}_\lambda$

```

1 if  $\lambda = \infty$  then
2   |  $\mathcal{M} := \mathcal{E}$ ,  $\mathcal{N} := \mathcal{A}$ ;
3 else
4   |  $\mathcal{M} := \lambda\mathcal{E} - \mathcal{A}$ ,  $\mathcal{N} := \mathcal{E}$ ;
5 end
6 Determine  $U$  with full column rank and  $\text{im } U = \ker \mathcal{M}$ ;
7  $Z_0 := U$ ,  $V_0 := U$ ,  $k := 0$ ;
8 repeat
9   |  $k := k + 1$ ;
10  |  $V_k := \mathcal{N}V_{k-1}$ ;
11  | Determine  $S_k$  with full column rank and  $\text{im } S_k = \ker U^*V_k$ ;
12  | Solve  $\mathcal{M}X_k = V_k S_k$  for the matrix  $X_k$ ;
13  | Determine  $Z_k$  with full column rank and  $\text{im } Z_k = \text{im } [X_k \ U]$ ;
14  | Determine  $Y_k$  with full column rank and  $\text{im } Y_k = \ker Z_k^* \mathcal{E} Z_k$ ;
15  | Determine  $V_k$  with full column rank and  $\text{im } V_k = \text{im } [V_k \ Z_k Y_k]$ ;
16 until  $\text{rank } V_k = \text{rank } V_{k-1}$ ;

```

Algorithm 2: Computation of \mathcal{V}_λ

Remark 5.3.

- a) Some further extensions are possible to further improve numerical efficiency in the computation of \mathcal{W}_λ and \mathcal{V}_λ . For instance, we may consider at every step only a basis of a space $\mathcal{P}^{(\ell)}$ such that $\mathcal{V}_\lambda^{(\ell-1)} \oplus \mathcal{P}^{(\ell)} = \mathcal{V}_\lambda^{(\ell)}$. However, in the case where $\dim \mathcal{V}_\lambda$ is small, this improvement is only very little.
- b) In the computation of \mathcal{V}_∞ for the even matrix pencil $s\mathcal{E} - \mathcal{A}$ as in (1.4), no computation of the nullspace of \mathcal{E} is necessary, and we may set $U = [0_{m,n} \ 0_{m,n} \ I_m]^T$. Further, the computation of Z_k , Y_k and V_k in Algorithm 2 can be done directly in terms of the matrices A , B , Q , S and R .

5.2. Low-rank iterative solutions of the projected Riccati equations.

For large-scale standard algebraic Riccati equations $A^*X + XA + H - XGX = 0$, the *Newton-Kleinman-ADI iteration* is known to be an efficient solution method. Besides numerical efficiency and exploitation of possible sparsity of A , it is memory economical in the case where $\text{rank } H$ and $\text{rank } G$ are low and the singular values of X decay rapidly. This is due to the fact that it provides a sequence $(X^{(k)}) = (X^{(1)} - Z^{(k)}(Z^{(k)})^*)$ for matrices $X^{(1)} \in \mathbb{C}^{n,n}$, $Z^{(k)} \in \mathbb{C}^{n,l_k}$ with $l_k \ll n$ [5, 6, 15, 18]. This method is essentially an implementation of the customary Newton iteration for the (nonlinear) algebraic Riccati equation. It is shown in [18] that this iteration

converges towards the stabilizing solution, if it is initialized with $X^{(0)}$ such that all eigenvalues of $A - GX^{(0)}$ have negative real part. The regularity of the Jacobian of the Riccati operator in the stabilizing solution is equivalent to the absence of purely imaginary eigenvalues of the corresponding Hamiltonian matrix. By the standard results for the general Newton method, we have quadratical convergence in this case. It turns out that in each step of Newton's iteration for Riccati equations, a Lyapunov equation has to be solved. This is done via the *alternating direction implicit (ADI)* iteration.

5.2.1. Newton-Kleinman iteration for projected Riccati equations. We consider projected Riccati equations

$$A_R^* \tilde{X} + \tilde{X} A_R + H_R - \tilde{X} G_R \tilde{X} = 0, \quad \tilde{X} = \Pi^* \tilde{X} \Pi \quad (5.1a)$$

with

$$A_R = \Pi A_R \Pi, \quad H_R = \Pi^* H_R \Pi, \quad G_R = \Pi G_R \Pi^*. \quad (5.1b)$$

These are obtained from Lur'e equations after the deflation process described in Theorem 4.1 and Theorem 4.2 (see Section 4 for computational issues) with matrices

$$A_R = \tilde{A} - \tilde{B} \tilde{R}^+ \tilde{S}^* \quad H_R = \tilde{Q} - \tilde{S} \tilde{R}^+ \tilde{S}^*, \quad G_R = \tilde{B} \tilde{R}^+ \tilde{B}^*. \quad (5.2)$$

In theory, a change of coordinates with T as in (3.7) transforms this equation into a conventional algebraic Riccati equation bordered by zero blocks:

$$\begin{aligned} \begin{bmatrix} A_{R11}^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{X}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \tilde{X}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{R11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} H_{R11} & 0 \\ 0 & 0 \end{bmatrix} \\ - \begin{bmatrix} \tilde{X}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G_{R11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{X}_{11} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \quad (5.3)$$

Indeed the presented method is algebraically equivalent to the Newton-Kleinman iteration for the algebraic Riccati equation $A_{R11}^* \tilde{X}_{11} + \tilde{X}_{11} A_{R11} + H_{R11} - \tilde{X}_{11} G_{R11} \tilde{X}_{11} = 0$. However, we aim to solve the projected algebraic Riccati equation (5.1) without actually changing coordinates as above. Namely, we use the iteration in Algorithm 3.

Determination of a suitable initial matrix $\tilde{X}^{(0)}$ can be led back to a stabilization problem [3, 4]. Stopping criteria may be chosen based on the norm of the residual of $N^{(k)}$, such as typically done in the Newton-Kleinman iteration for conventional algebraic Riccati equations [4, 5].

The bottleneck in the above iteration is the solution of the projected Lyapunov equation (5.6). Details about this problem are presented in Section 5.2.2. In particular, advantages of the projector approach in contrast to will be highlighted to the coordinate transformation approach (5.3) will be highlighted. Beforehand, we state a result about convergence of this iteration for matrices A_R , G_R and H_R obtained from the Lur'e equations (1.1) via (5.2).

THEOREM 5.4. *Let $A \in \mathbb{C}^{n,n}$, $B, C \in \mathbb{C}^{n,m}$ and $Q \in \mathbb{C}^{n,n}$, $R \in \mathbb{C}^{m,m}$ with $Q = Q^*$ and $R = R^*$ be given. Assume that the Lur'e equations have a stabilizing solution. Let $\check{\mathcal{V}} \subset \mathbb{C}^{2n+m}$ be a deflating subspace of the even matrix pencil (1.4) with*

$$\mathcal{V}_\infty \subset \check{\mathcal{V}} \subset \left(\sum_{\lambda \in \mathbb{C}^-} \mathcal{W}_\lambda \right) + \left(\sum_{\lambda \in i\mathbb{R}} \mathcal{V}_\lambda \right) + \mathcal{V}_\infty,$$

Data: matrices $A_R, G_R, H_R \in \mathbb{C}^{n,n}$ with $G_R = G_R^*$, $H_R = H_R^*$, a projector $\Pi \in \mathbb{C}^{n,n}$ such that $A_R = \Pi A_R \Pi$, $G_R = \Pi G_R \Pi^*$, $H_R = \Pi^* H_R \Pi$ and the projected algebraic Riccati equation (5.1a) possesses a stabilizing solution $\tilde{X} \in \mathbb{C}^{n,n}$.

Result: matrix $\tilde{X}^{(k)} \in \mathbb{C}^{n,l_k}$ with $\tilde{X}^{(k)} \approx \tilde{X}$

1 Determine a matrix $\tilde{X}^{(0)}$ such that

$$\operatorname{im}(-\lambda \Pi + A_R - G_R \tilde{X}^{(0)}) = \operatorname{im} \Pi \quad \text{for all } \lambda \in \mathbb{C}^+ \cup i\mathbb{R} \quad (5.4)$$

Solve

$$(A_R - G_R \tilde{X}^{(0)})^* \tilde{X}^{(1)} + \tilde{X}^{(1)} (A_R - G_R \tilde{X}^{(0)}) = -H_R - \tilde{X}^{(0)} G_R \tilde{X}^{(0)} \quad (5.5)$$

for Hermitian $\tilde{X}^{(1)} \in \mathbb{C}^{n,n}$ with $\tilde{X}^{(k)} = \Pi^* \tilde{X}^{(k)} \Pi$;

2 $N^{(1)} := \tilde{X}^{(1)} - \tilde{X}^{(0)}$;

3 $k = 0$;

4 **repeat**

5 $k := k + 1$;

6 Solve

$$(A_R - G_R \tilde{X}^{(k-1)})^* N^{(k)} + N^{(k)} (A_R - G_R \tilde{X}^{(k)}) = N^{(k-1)} G_R N^{(k-1)} \quad (5.6)$$

for Hermitian $N^{(k)} \in \mathbb{C}^{n,n}$ with $N^{(k)} = \Pi^* N^{(k)} \Pi$;

7 $\tilde{X}^{(k)} = \tilde{X}^{(k-1)} + N^{(k)}$;

8 **until** a suitable stopping criterion is fulfilled;

Algorithm 3: Solution of a projected algebraic Riccati equation via Newton-Kleinman method

and $\check{V} = \operatorname{im} \check{V}$ with $\check{V} \in \mathbb{C}^{2n+m, \check{n}+m}$ as in (4.5). Let $\Pi = I_n - \check{V}_x \check{V}_x^-$, where $\check{V}_x^- \in \mathbb{C}^{\check{n}, n}$ fulfills $\check{V}_x^- \check{V}_x = I_{\check{n}}$.

Let A_R, B_R, G_R and H_R be defined as in (5.2) and let $(\tilde{X}^{(k)})$ be the sequence obtained by Algorithm 3. Then the sequence

$$(X^{(k)}) = \left(\tilde{X}^{(k)} + (\check{V}_x^-)^* \check{V}_\mu^* + \check{V}_\mu \check{V}_x^- - (\check{V}_x^-)^* \check{V}_x^* \check{V}_\mu \check{V}_x^- \right). \quad (5.7)$$

converges to the stabilizing X of the Lur'e equations (1.1).

If, moreover

$$\sum_{\lambda \in i\mathbb{R} \cup \{\infty\}} \nu_\lambda \subset \check{V}, \quad (5.8)$$

then the sequence $(X^{(k)})$ as in (5.7) converges quadratically to X .

Proof. By Theorem 4.1 and Theorem 4.2, the sequence $(\tilde{X}^{(k)})$ converges (quadratically) to \tilde{X} , if, and only if, $(X^{(k)})$ converges (quadratically) to X .

Convergence of the sequence $(\tilde{X}^{(k)})$ to \tilde{X} follows by a combination of the results in [18] with the fact that the iteration in Algorithm 3 is arithmetically equivalent to a coordinate transformation to (5.3), a Newton-Kleinman iteration of the upper left equation in (5.3), and a reverse coordinate transformation.

If (5.8) holds true, then Theorem 4.2 shows that the Riccati equation in the upper left

blocks of (5.3) has no purely imaginary eigenvalues, and thus the Newton-Kleinman method is quadratically convergent. \square

5.2.2. ADI iteration for projected Lyapunov equations. As presented in Algorithm 3, each step of the Newton-Kleinman iteration for projected Riccati equations requires the solution of a projected Lyapunov equation

$$A_L^* N + N A_L = F^* F, \quad N = \Pi^* N \Pi, \quad (5.9)$$

with $F = F \Pi \in \mathbb{C}^{l,n}$, $A_L = \Pi A_L \Pi \in \mathbb{C}^{n,n}$ and the eigenvalues of A_L have negative real part, except for one at the origin with geometric and algebraic multiplicity $n - \text{rank } \Pi$. Equations of similar type are for instance considered in [27]. We now generalize the ADI method to (5.9). As for conventional Lyapunov equations, the ADI iteration involves so-called *shift parameters* [5] in order to accelerate convergence. Their choice can be done via the non-zero spectrum of A_L , analogous to the standard case.

Data: $A_L \in \mathbb{C}^{n,n}$, $F \in \mathbb{C}^{l,n}$ and a sequence of shift parameters (p_j) .

Result: $S = \Pi S \in \mathbb{C}^{n,kl}$, such that $SS^* \approx N$, where $A_L^* N + N A_L = F^* F$

- 1 Solve $(A_L + p_1 I)^* Z = F^*$ for Z ;
- 2 $S_1 := \sqrt{-2 \text{Re}(p_1)} \cdot Z$, $V_1 := \sqrt{-2 \text{Re}(p_1)} \cdot Z$, $j := 1$;
- 3 **repeat**
- 4 $j := j + 1$;
- 5 Solve $(A_L + p_j I)^* Z = V_{j-1}$ for Z ;
- 6 $V_j := \sqrt{\text{Re}(p_j) / \text{Re}(p_{j-1})} \cdot (V_{j-1} - (p_j + \overline{p_{j-1}}) Z)$;
- 7 $S_j := [S_{j-1} \ V_j]$;
- 8 **until** a suitable stopping criterion is fulfilled;
- 9 $S := S_j$;

Algorithm 4: ADI iteration for projected Lyapunov equations.

Each matrix V_j fulfills $V_j = \Pi V_j$. In order to avoid numerical drift, one can additionally introduce a step $V_j := \Pi V_j$ in each iteration on Algorithm 4.

Indeed, the Lyapunov equation (5.5) is in general not in the form (5.9). However, we can split $-H_R - \tilde{X}^{(0)} G_R \tilde{X}^{(0)}$ into a positive semi-definite and a negative semi-definite part and factorize them separately, falling back into the case treated in Algorithm 4. The solution $\tilde{X}^{(1)}$ is now the difference between these two solutions. Note that, if (5.5) derives from the Lur'e equations with our approach, H_R and G_R are formed by (4.8) and (5.2), and thus the ‘‘right hand sides’’ of these two projected Lyapunov equations are of low rank, if m and $\text{rank } Q$ are small.

For brevity, we will only focus on the projected Lyapunov equation (5.6). According to (4.8) and (5.2), each step of the Newton-Kleinman iteration (Algorithm 3) for projected algebraic Riccati equations obtained after the reductional transformation of Lur'e equations consists of the solution of a projected Lyapunov equation with

$$\begin{aligned} A_L &= A_R - G_R \tilde{X}^{(k-1)} = (I_n - \check{V}_x \check{V}_x^-) A (I_n - \check{V}_x \check{V}_x^-) \\ &\quad - \tilde{B} \tilde{R}^+ \tilde{S}^* - \tilde{B} \tilde{R}^+ \tilde{B}^* \tilde{X}^{(k-1)}. \\ F &= (\tilde{L}^+)^* \tilde{B}^* N^{(k-1)}, \end{aligned}$$

where $\tilde{L} \in \mathbb{C}^{m+\tilde{n},p}$ is a matrix with $\tilde{L}^*\tilde{L} = \tilde{R}$. We have therefore to solve a couple linear equations $(A_R - G_R\tilde{X}^{(k-1)} + p_j I)^* Z = V_{j-1}$. However, due to possible sparsity of A , it is not desirable to forming A_R and Q_R explicitly. Instead we consider the extended system

$$\begin{bmatrix} A^* + \overline{p_j} I & (\tilde{X}^{(k-1)}\tilde{B} + \tilde{S})\tilde{L}^+ & (\check{V}_x^-)^* & A^*(\check{V}_x^-)^* \\ (\tilde{L}^+)^*\tilde{B}^* & I_p & 0 & 0 \\ \check{V}_x^* A^* - \check{V}_x^* A^*(\check{V}_x^-)^*\check{V}_x^* & 0 & I_{\tilde{n}} & 0 \\ \check{V}_x^* & 0 & 0 & I_{\tilde{n}} \end{bmatrix} \begin{bmatrix} Z \\ \tilde{Z}_2 \\ \tilde{Z}_3 \\ \tilde{Z}_4 \end{bmatrix} = \begin{bmatrix} V_{j-1} \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (5.10)$$

Using Schur complementation, we can see that Z solves $(A_R - G_R\tilde{X}^{(k-1)} + p_j I)^* Z = V_{j-1}$. If the dimension of the space \check{V} as in (3.6) is moderate and A is sparse, then the extended system matrix can be stored in sparse form and a suitable sparse solver be used.

6. Numerical examples. As a numerical experiment, we consider Lur'e equations arising in a slightly modified version of the *positive real lemma* [2]: Given are matrices $A \in \mathbb{R}^{n,n}$ and $B, C^T \in \mathbb{R}^{n,m}$ with the property that (A, B) is controllable and $G(s) = C(sI - A)^{-1}B \in \mathbb{R}(s)^{m,m}$ fulfills $G(\lambda) = G^*(\lambda) \geq 0$ for all $\lambda \in \mathbb{C}^+$, the Lur'e equations (1.1) with $R = 0$, $Q = 0$ and $S = C^T$ are known to be solvable for some negative definite stabilizing solution X .

In the considered examples, we have $A + A^* \leq 0$ and $B = C^T$, which implies positive realness of $G(s)$. We have taken dynamical systems in the benchmarks examples `demo_m1`, `demo_r1` and `demo_r3` from the LYAPACK library [21].

For the corresponding Lur'e equations, we compute the subspace $\check{V} = \mathcal{V}_\infty$ by Algorithm 2. The generalized inverse \check{V}_x^- has been chosen in a way that $\check{V}_\mu^* \Pi = 0$ (see Remark 4.3). The obtained projected algebraic Riccati equation (5.1) is solved with the Newton-Kleinman-ADI method as presented in Algorithm 3 and Algorithm 4.

Since Algorithm 4 differs from standard ADI only in the procedure used for solving the linear equations, we can simply use the implementation of ADI in the library LYAPACK, by providing a custom solver for both shifted and unshifted linear equations. In particular, we rely on the library's heuristic for the choice of the shift parameters. In the considered examples, the Newton-Kleinman iteration may be stably initialized with $X^{(0)} = 0$. After obtaining the solution \tilde{X} of the projected Riccati equation, we recover the Lur'e solution as $X = \tilde{X} + \check{V}_\mu \check{V}_x^-$.

Computations were done on Intel(R) Core(TM) 2 Duo CPU E6750 2.66GHz with machine precision $\varepsilon = 2.22 \times 10^{-16}$ using MATLAB 2010b. We report the results of the experiments in Table 6.1. The relative residual of the Lur'e equations is measured as

$$\text{Res} = \frac{\left\| \mathfrak{L}(X) - \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix} \right\|_F}{\left\| \mathfrak{L}(X) \right\|_F}, \quad \mathfrak{L}(X) = \begin{bmatrix} A^* X + X A + Q & X B + S \\ B^* X + S^* & R \end{bmatrix} \quad (6.1)$$

where the missing solution components K and L are computed by truncating to rank m an eigendecomposition of $\mathfrak{L}(X)$. Notice that in most application only X is needed, so we only need this expensive computation if we want to check the residual.

To check whether the computed solution is the stabilizing one, we construct the

TABLE 6.1
Results of the numerical experiments

	demo_m1	demo_r1	demo_r3
n	408	2500	821
m	1	1	6
\check{n}	1	1	6
Rtol (see (6.3))	$(0,1.4 \times 10^{-03})$	$(0,1.6 \times 10^{-04})$	$(0,1.1 \times 10^{-06})$
ADI itns for computing $X^{(1)}$	41	39	44
rank of $X^{(1)}$	25	24	138
rank of $X - X^{(1)}$	28	23	130
number of Newton steps	8	4	7
avg. ADI itns per Newton step	32.25	37.25	36.857
Res (see (6.1))	2.4×10^{-07}	2.6×10^{-15}	3.5×10^{-15}
Stab (see (6.2))	-2.9×10^{-09}	-1.8×10^{-15}	-1.3×10^{-08}
CPU time (seconds)	$5.5 \times 10^{+00}$	$1.7 \times 10^{+01}$	$6.5 \times 10^{+01}$

reduced pencil associated to this deflating subspace

$$s\widehat{\mathcal{E}} - \widehat{\mathcal{A}} = \begin{bmatrix} -sI + A & B \\ K & L \end{bmatrix}$$

according to (3.1), and we check whether its Cayley transform $s(\widehat{\mathcal{A}} + \widehat{\mathcal{E}}) - (\widehat{\mathcal{A}} - \widehat{\mathcal{E}})$ has only eigenvalues larger than 1, since this Cayley transform maps the left half-plane onto the exterior of the unit disc. We report on our table the value of

$$\text{Stab} = \min_{\lambda \in \sigma(s(\widehat{\mathcal{A}} + \widehat{\mathcal{E}}) - (\widehat{\mathcal{A}} - \widehat{\mathcal{E}}))} |\lambda| - 1; \quad (6.2)$$

we expect $\text{Stab} \geq -c \cdot 10^{-8}$ for a moderate constant $c > 0$ based on the preceding discussion. Indeed, due to $R = 0$, in all our problems the EKCF of the corresponding even matrix pencil $s\mathcal{E} - \mathcal{A}$ as in (1.4) has at least one block of type $E3$ with size greater or equal to 3×3 . This implies that the KCF of $s\widehat{\mathcal{E}} - \widehat{\mathcal{A}}$ contains a block $K2$ of size at least 2×2 ; therefore, the sensitivity of the eigenvalue 1 in the computation of Stab is $\sqrt{\varepsilon}$.

Moreover, we have reported a measure of the accuracy of the rank decisions performed in the computation of \mathcal{V}_∞ . The actual rank decisions are performed with a relative tolerance of $\sqrt{\varepsilon}$; the relative tolerance is used in the same way as in the MATLAB function `orth`; i.e., in the singular value decomposition $W = U\Sigma V^* \in \mathbb{R}^{k_1, k_2}$, all singular values $\sigma_1, \dots, \sigma_r$ smaller than $\max\{m, n\} \cdot \max\{\sigma_1, \dots, \sigma_r\} \cdot \text{tol}$ are set to zero. A good way to assess the stability of these rank decisions is checking what would be the smaller value tol_- and the larger value tol_+ that would result in truncating the same singular values and thus making the same rank decisions along the whole algorithm. We report the range

$$\text{Rtol} = (\text{tol}_-, \text{tol}_+) \quad (6.3)$$

in the table with the computational results. In all examples, the range is sufficiently large to ensure that the rank decisions are not affected significantly by numerical errors.

7. Conclusion. We have considered a constructive approach to the determination of the stabilizing solution of Lur'e equations. Based on the correspondence of the solution set to \mathcal{E} -neutral deflating subspaces of an associated even matrix pencils $s\mathcal{E} - \mathcal{A}$, the Lur'e equation has been transformed to a projected algebraic Riccati equation. For the latter one, an algorithm that generalizes the Newton-Kleinman-ADI method is presented. Altogether, this provides a new method for the low-rank approximative numerical solution of large-scale Lur'e equations.

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