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Abstract

Different concepts related to controllability of differential-algebraic equations are described. The class of systems considered consists of linear differential-algebraic equations with constant coefficients. Regularity, which is, loosely speaking, a concept related to existence and uniqueness of solutions for any inhomogeneity, is not required in this article. The concepts of impulse controllability, controllability at infinity, behavioral controllability, strong and complete controllability are described and defined in time-domain. Equivalent criteria that generalize the Hautus test are presented and proved.
Special emphasis is placed on normal forms under state space transformation and, further, under state space, input and feedback transformations. Special forms generalizing the Kalman decomposition and Brunovsky form are presented. Consequences for state feedback design and geometric interpretation of the space of reachable states in terms of invariant subspaces are proved.

Keywords: Differential-algebraic equations, controllability, stabilizability, Kalman decomposition, Brunovsky form, Hautus criterion, invariant subspaces

1 Introduction

Controllability is, roughly speaking, the property of a system that it can be moved from an arbitrary state to any other by applying certain admissible manipulations. The precise definition however depends on the specific framework, as quite a number of different concepts of controllability are present today.
Since the famous work by Kalman [75–77], who introduced the notion of controllability about fifty years ago, the field of mathematical control theory has been revived and rapidly growing ever since, emerging into one of the most important mathematical areas of the last decades, mainly due to its contributions to fields such as mechanical, electrical and chemical engineering, see e.g. [2, 44, 137]. For a good overview of standard mathematical control theory, i.e., involving ordinary differential equations (ODEs), and its history see e.g. [65, 70, 71, 74, 128, 132].
Just as mathematical control theory began to grow, Gantmacher published his famous book [56] and therewith laid the foundations for the rediscovery of differential-algebraic equations (DAEs), the first main theories of which have been developed by Weierstrass [144] and Kronecker [87] in terms of matrix pencils. DAEs have then been discovered to be the perfect tool for modeling a vast variety of problems in economics [104], demography [35], mechanical systems [7, 29, 55, 62, 120], multibody dynamics [51, 62, 129, 131], electrical networks [7, 34, 50, 99, 110], fluid mechanics [7, 60, 99] and chemical engineering [45, 47, 48, 119], which often cannot be modeled by standard ODE systems.

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In general, DAEs are implicit differential equations, and in the simplest case just a combination of differential equations along with algebraic constraints (from which the name DAE comes from). These algebraic constraints however may cause that the solutions of initial value problems are no longer unique, or that there do not exist solutions at all. Furthermore, when considering inhomogeneous problems, the inhomogeneity has to be "consistent" with the DAE in order for solutions to exist. Dealing with these problems a huge solution theory for DAEs has been developed, the most important contribution of which is the one by WILKINSON [145]. Nowadays, there are a lot of monographs and textbooks where the whole theory can be looked up, see e.g. [29, 35, 36, 46, 61, 90]. A comprehensive representation of the solution theory of general linear time-invariant DAEs, along with possible distributional solutions based on the theory developed in [133, 134], is given in [24]. A good overview of DAE theory and an historical background (until the publication of this paper) can also be found in [92].

DAEs found its way into control theory ever since the famous book by ROSENBROCK [126], in which he developed his ideas of the description of linear systems by polynomial system matrices. Then a rapid development followed with important contributions of ROSENBROCK himself [127] and LUENBERGER [100–103], not to forget the work by PUGH et al. [124], VERGHESE et al. [139, 141–143], Pandolfi [117, 118], COBB [40–43], YIP et al. [153] and BERNARD [25]. The most important of these contributions for the development of concepts of controllability are certainly [43, 143, 153]. Further developments were made by LEWIS and ÖZÇALDIRAN [94, 95] and by BENDER and LAUB [19, 20]. The first textbook which summarizes the development of control theory for DAEs so far was the one by DAI [46]. All these contributions deal with regular systems, i.e., systems of the form

$$E\dot{x}(t) = Ax(t) + f(t), \quad x(0) = x^0,$$

where for any inhomogeneity f there exits initial values x^0 for which the corresponding initial value problem has a solution and this solution is unique. This has been proved to be equivalent to the condition that E, A are square matrices and $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$.

The aim of the present paper is to state the different concepts of controllability for differential-algebraic systems which are not necessarily regular, i.e., E and A may be non-square. Applications with the need for non-regular DAEs turn up in the modeling of electrical circuits [50] for instance. Furthermore, the class of regular DAE systems is not closed under the action of a feedback group [11].

The paper is organized as follows: The concepts of impulse controllability, controllability at infinity, R-controllability, controllability in the behavioral sense, strong and complete controllability, as well as strong and complete reachability and stabilizability in the behavioral sense, strong and complete stabilizability will be described and defined in time-domain in Section 2. In the more present DAE literature these notions are sometimes mixed up and one must be careful which one is really used in a specific paper. We try to clarify this here and give several examples for which notions are used in the most important contributions. A comprehensive discussion of the introduced concepts as well as some first relations between them are also included in Section 2. In Section 3 we briefly revisit the solution theory of DAEs and then concentrate on normal forms under state space transformation and, further, under state space, input and feedback transformations. We introduce the concepts of system and feedback equivalence and state normal forms under these equivalences, which for instance generalize the Brunovsky form. It is also discussed when these forms are canonical and what properties (regarding controllability and stabilizability) the appearing subsystems have. The generalized Brunovsky form enables us to give short proofs of equivalent criteria, in particular generalizations of the Hautus test, for the controllability concepts in Section 4, the most of which are of course well-known - we discuss the relevant literature. In Section 5 we revisit the concept of feedback for DAE systems and proof new results concerning the equivalence of stabilizability of DAE control systems and the existence of a feedback which stabilizes the closed-loop system. In Section 6 we give a brief summary of some selected results of the geometric theory using invariant subspaces which lead to a representation of the reachability space and criteria for controllability at infinity, impulse controllability, controllability in the behavioral sense, complete and strong controllability. Finally, in Section 7 the results regarding the Kalman decomposition for DAE systems are stated and it is shown how the controllability concepts can be related to certain properties of the Kalman decomposition.

We close the introduction with the nomenclature used in this paper:

$\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$		set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, set of all integers, resp.
$\ell(\alpha)$, $\lvert\alpha\rvert$		length and absolute value of a multi-index $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^n$
$\mathbb{R}_{\geq 0}$ ($\mathbb{R}_{>0}$, $\mathbb{R}_{\leq 0}$, $\mathbb{R}_{<0}$)	$=$	$[0,\infty)$ ($(0,\infty)$, $(-\infty,0]$, $(-\infty,0)$), resp.
$\mathbb{C}_+$, $\mathbb{C}_-$ ($\overline{\mathbb{C}_+}$, $\overline{\mathbb{C}_-}$)		the open (closed) set of complex numbers with positive, negative real part, resp.
$\mathbf{Gl}_n(\mathbb{R})$		the set of invertible real $n \times n$ matrices
$\mathbb{R}[s]$		the ring of polynomials with coefficients in $\mathbb{R}$
$\mathbb{R}(s)$		the quotient field of $\mathbb{R}[s]$
$R^{n,m}$		the set of $n \times m$ matrices with entries in a ring R
$\sigma(A)$		spectrum of the matrix $A \in \mathbb{R}^{n,n}$
$f\vert_{\mathcal{I}}$		restriction of the function $f : \mathcal{T} \to \mathbb{R}^n$ to $\mathcal{I} \subseteq \mathcal{T}$,
$\mathcal{L}^1_{\text{loc}}(\mathcal{T};\mathbb{R}^n)$		locally Lebesgue integrable functions $f : \mathcal{T} \to \mathbb{R}^n$, see [1, Chap. 1]
$\dot{f}$ ($f^{(i)}$)		(i-th) distributional derivative of $f \in \mathcal{L}^1_{\text{loc}}(\mathcal{T};\mathbb{R}^n)$, $i \in \mathbb{N}_0$
$\mathcal{W}^{k,1}_{\text{loc}}(\mathcal{T};\mathbb{R}^n)$	$:=$	$\left\{\, x \in \mathcal{L}^1_{\text{loc}}(\mathcal{T};\mathbb{R}^n) \;\middle\vert\; x^{(i)} \in \mathcal{L}^1_{\text{loc}}(\mathcal{T};\mathbb{R}^n) \text{ for } i = 0,\ldots,k \,\right\}$, $k \in \mathbb{N}_0$
σ_τ		the τ-shift operator, i.e., for $f : \mathcal{T} \to \mathbb{R}^n$, $\mathcal{T} \subseteq \mathbb{R}$, $\sigma_\tau f : \mathcal{T} - \tau \to \mathbb{R}^n,\ t \mapsto f(t+\tau)$
ϱ		the reflection operator, i.e., for $f : \mathcal{T} \to \mathbb{R}^n$, $\mathcal{T} \subseteq \mathbb{R}$, $\varrho f : -\mathcal{T} \to \mathbb{R}^n,\ t \mapsto f(-t)$

2 Controllability concepts

We consider linear differential-algebraic control systems of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{2.1}$$

with $E, A \in \mathbb{R}^{k,n}$, $B \in \mathbb{R}^{k,m}$; the set of these systems is denoted by $\Sigma_{k,n,m}$, and we write $[E,A,B] \in \Sigma_{k,n,m}$.
We do not assume that the pencil $sE - A \in \mathbb{R}[s]^{k,n}$ is regular, that is $\operatorname{rk}_{\mathbb{R}(s)}(sE - A) = k = n$.
The function $u : \mathbb{R} \to \mathbb{R}^m$ is called *input*; $x : \mathbb{R} \to \mathbb{R}^n$ is called *(generalized) state*. Note that, strictly speaking, $x(t)$ is in general not a state in the sense that the free system (i.e., $u \equiv 0$) satisfies a semigroup property [83, Sec. 2.2]. We will, however, speak of the state $x(t)$ for sake of brevity, especially since $x(t)$ contains the full information about the system at time t.

A trajectory $(x,u) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m$ is said to be a *solution* of (2.1) if, and only if, it belongs to the *behavior* of (2.1):

$$\mathfrak{B}_{[E,A,B]} := \left\{ (x,u) \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \times \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^m) \;\middle|\; (x,u) \text{ satisfies (2.1) for almost all } t \in \mathbb{R} \right\}. \tag{2.2}$$

Note that any function $x \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$ is continuous. Moreover, by linearity of (2.1), $\mathfrak{B}_{[E,A,B]}$ is a vector space. Further, since the matrices in (2.1) do not depend on t, the behavior is *shift-invariant*, that is $(\sigma_\tau x, \sigma_\tau u) \in \mathfrak{B}_{[E,A,B]}$ for all $\tau \in \mathbb{R}$ and $(x,u) \in \mathfrak{B}_{[E,A,B]}$.
The following spaces play a fundamental role in this article:

(a) The *space of consistent initial states*

$$\mathcal{V}_{[E,A,B]} = \left\{ x(0) \;\middle|\; (x,u) \in \mathfrak{B}_{[E,A,B]} \right\}.$$

(b) The *space of consistent initial differential variables*

$$\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} = \left\{ x^0 \;\middle|\; \exists\, (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ with } Ex(0) = Ex^0 \right\}.$$

(c) The *reachability space at time* $t \in \mathbb{R}_{\geq 0}$

$$\mathcal{R}^t_{[E,A,B]} = \left\{ x(t) \;\middle|\; \exists\, (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ with } x(0) = 0 \right\}$$

and the *reachability space*

$$\mathcal{R}_{[E,A,B]} = \bigcup_{t \geq 0} \mathcal{R}^t_{[E,A,B]}.$$

(d) The *controllability space at time* $t \in \mathbb{R}_{\geq 0}$

$$\mathcal{C}^t_{[E,A,B]} = \left\{ x(0) \;\middle|\; \exists\, (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ with } x(t) = 0 \right\}$$

and the *controllability space*

$$\mathcal{C}_{[E,A,B]} = \bigcup_{t \geq 0} \mathcal{C}^t_{[E,A,B]}.$$

Note that, by linearity of the system, $\mathcal{V}_{[E,A,B]}$, $\mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$, $\mathcal{R}^t_{[E,A,B]}$ and $\mathcal{C}^t_{[E,A,B]}$ are linear subspaces of $\mathbb{R}^n$. We will show that $\mathcal{R}^{t_1}_{[E,A,B]} = \mathcal{R}^{t_2}_{[E,A,B]} = \mathcal{C}^{t_1}_{[E,A,B]} = \mathcal{C}^{t_2}_{[E,A,B]}$ for all $t_1, t_2 \in \mathbb{R}_{>0}$, see Lemma 2.3. This implies $\mathcal{R}_{[E,A,B]} = \mathcal{R}^t_{[E,A,B]} = \mathcal{C}^t_{[E,A,B]} = \mathcal{C}_{[E,A,B]}$ for all $t \in \mathbb{R}_{>0}$. Note further that, by shift-invariance, we have for all $t \in \mathbb{R}$ that

$$\mathcal{V}_{[E,A,B]} = \left\{ x(t) \;\middle|\; (x,u) \in \mathfrak{B}_{[E,A,B]},\ t \in \mathbb{R} \right\}, \tag{2.3}$$

$$\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} = \left\{ x^0 \;\middle|\; \exists\, (x,u) \in \mathfrak{B}_{[E,A,B]},\ t \in \mathbb{R} \text{ with } Ex(t) = Ex^0 \right\}. \tag{2.4}$$

In the following three lemmas we clarify some of the connections of the above defined spaces, before we state the controllability concepts.

Lemma 2.1 (Inclusions for reachability spaces)**.** *For* $[E,A,B] \in \Sigma_{k,n,m}$ *and* $t_1, t_2 \in \mathbb{R}_{>0}$ *with* $t_1 < t_2$, *the following holds true:*

(i)

$$\mathcal{R}^{t_1}_{[E,A,B]} \subseteq \mathcal{R}^{t_2}_{[E,A,B]}.$$

(ii) If

$$\mathcal{R}^{t_1}_{[E,A,B]} = \mathcal{R}^{t_2}_{[E,A,B]},$$

then

$$\mathcal{R}^{t_1}_{[E,A,B]} = \mathcal{R}^{t}_{[E,A,B]} \text{ for all } t \in \mathbb{R} \text{ with } t > t_1.$$

Proof: (i) Let $\bar{x} \in \mathcal{R}^{t_1}_{[E,A,B]}$. By definition, there exists some $(x,u) \in \mathfrak{B}_{[E,A,B]}$ with $x(0) = 0$ and $x(t_1) = \bar{x}$. Consider now $(x_1, u_1) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m$ with

$$(x_1(t), u_1(t)) = \begin{cases} (x(t - t_2 + t_1), u(t - t_2 + t_1)), & \text{if } t > t_2 - t_1 \\ (0, 0), & \text{if } t \le t_2 - t_1 \end{cases}$$

Then $x(0) = 0$ implies that x_1 is continuous at $t_2 - t_1$. Since, furthermore,

$$\begin{aligned} x_1|_{(-\infty, t_2 - t_1]} &\in \mathcal{W}^{1,1}_{\text{loc}}((-\infty, t_2 - t_1]; \mathbb{R}^n) \text{ and} \\ x_1|_{[t_2 - t_1, \infty)} &\in \mathcal{W}^{1,1}_{\text{loc}}([t_2 - t_1, \infty); \mathbb{R}^n), \end{aligned}$$

we have $(x_1, u_1) \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^n) \times \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^m)$. By shift-invariance, $E\dot{x}_1(t) = Ax_1(t) + Bu_1(t)$ holds true for almost all $t \in \mathbb{R}$, i.e., $(x_1, u_1) \in \mathfrak{B}_{[E,A,B]}$. Then, due to $x_1(0) = 0$ and $\bar{x} = x(t_1) = x_1(t_2)$, we obtain $\bar{x} \in \mathcal{R}^{t_2}_{[E,A,B]}$.

(ii) *Step 1:* We show that $\mathcal{R}^{t_1}_{[E,A,B]} = \mathcal{R}^{t_2}_{[E,A,B]}$ implies $\mathcal{R}^{t_1}_{[E,A,B]} = \mathcal{R}^{t_1 + 2(t_2 - t_1)}_{[E,A,B]}$: By (i), it suffices to show the inclusion "$\supseteq$". Assume that $\bar{x} \in \mathcal{R}^{t_1 + 2(t_2 - t_1)}_{[E,A,B]}$, i.e., there exists some $(x_1, u_1) \in \mathfrak{B}_{[E,A,B]}$ with $x_1(0) = 0$ and $x_1(t_1 + 2(t_2 - t_1)) = \bar{x}$. Since $x_1(t_2) \in \mathcal{R}^{t_2}_{[E,A,B]} = \mathcal{R}^{t_1}_{[E,A,B]}$, there exists some $(x_2, u_2) \in \mathfrak{B}_{[E,A,B]}$ with $x_2(0) = 0$ and $x_2(t_1) = x_1(t_2)$. Now consider the trajectory

$$(x(t), u(t)) = \begin{cases} (x_2(t), u_2(t)), & \text{if } t < t_1, \\ (x_1(t + (t_2 - t_1)), u_1(t + (t_2 - t_1))), & \text{if } t \ge t_1. \end{cases}$$

Since x is continuous at t_1, we can apply the same argumentation as in the proof of (i) to infer that $(x, u) \in \mathfrak{B}_{[E,A,B]}$. The result to be shown in this step is now a consequence of $x(0) = x_2(0) = 0$ and

$$\bar{x} = x_1(t_1 + 2(t_2 - t_1)) = x(t_2) \in \mathcal{R}^{t_2}_{[E,A,B]} = \mathcal{R}^{t_1}_{[E,A,B]}.$$

Step 2: We show (ii): From the result shown in the first step, we may inductively conclude that $\mathcal{R}^{t_1}_{[E,A,B]} = \mathcal{R}^{t_2}_{[E,A,B]}$ implies $\mathcal{R}^{t_1}_{[E,A,B]} = \mathcal{R}^{t_1 + l(t_2 - t_1)}_{[E,A,B]}$ for all $l \in \mathbb{N}$. Let $t \in \mathbb{R}$ with $t > t_1$. Then there exists some $l \in \mathbb{N}$ with $t \le t_1 + l(t_2 - t_1)$. Then statement (i) implies

$$\mathcal{R}^{t_1}_{[E,A,B]} \subseteq \mathcal{R}^{t}_{[E,A,B]} \subseteq \mathcal{R}^{t_1 + l(t_2 - t_1)}_{[E,A,B]},$$

and, by $\mathcal{R}^{t_1}_{[E,A,B]} = \mathcal{R}^{t_1 + l(t_2 - t_1)}_{[E,A,B]}$, we obtain the desired result. $\square$

Now we present some relations between controllability and reachability spaces of $[E, A, B] \in \Sigma_{k,n,m}$ and its *backward system*, which is simply defined by $[-E, A, B] \in \Sigma_{k,n,m}$. It can be easily verified that

$$\mathfrak{B}_{[-E,A,B]} = \left\{ \ (\varrho x, \varrho u) \ \middle| \ (x, u) \in \mathfrak{B}_{[E,A,B]} \ \right\}. \tag{2.5}$$

Lemma 2.2 (Reachability and controllability spaces of the backward system)**.** *For* $[E,A,B] \in \Sigma_{k,n,m}$ *and* $t \in \mathbb{R}_{>0}$*, there holds*

$$\mathcal{R}^t_{[E,A,B]} = \mathcal{C}^t_{[-E,A,B]}, \text{ and } \mathcal{C}^t_{[E,A,B]} = \mathcal{R}^t_{[-E,A,B]}.$$

Proof: Both assertions follow immediately from the fact that $(x,u) \in \mathfrak{B}_{[E,A,B]}$, if, and only if, $(\sigma_t(\varrho x), \sigma_t(\varrho u)) \in \mathfrak{B}_{[-E,A,B]}$. □

The previous lemma enables us to show that the controllability and reachability spaces of $[E,A,B] \in \Sigma_{k,n,m}$ are even equal. We further prove that both spaces do not depend on time $t \in \mathbb{R}_{>0}$.

Lemma 2.3 (Impulsive initial conditions and controllability spaces)**.** *For* $[E,A,B] \in \Sigma_{k,n,m}$*, the following holds true:*

(i)
$$\mathcal{R}^{t_1}_{[E,A,B]} = \mathcal{R}^{t_2}_{[E,A,B]} \text{ for all } t_1,t_2 \in \mathbb{R}_{>0}$$

(ii)
$$\mathcal{R}^{t}_{[E,A,B]} = \mathcal{C}^{t}_{[E,A,B]} \text{ for all } t \in \mathbb{R}_{>0}$$

(iii)
$$\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} = \mathcal{V}_{[E,A,B]} + \ker_{\mathbb{R}} E.$$

Proof: (i) By Lemma 2.3(i), there holds

$$\mathcal{R}^{\frac{t_1}{n+1}}_{[E,A,B]} \subseteq \mathcal{R}^{\frac{2t_1}{n+1}}_{[E,A,B]} \subseteq \cdots \subseteq \mathcal{R}^{\frac{nt_1}{n+1}}_{[E,A,B]} \subseteq \mathcal{R}^{t_1}_{[E,A,B]} \subseteq \mathbb{R}^n,$$

and thus

$$0 \le \dim \mathcal{R}^{\frac{t_1}{n+1}}_{[E,A,B]} \le \dim \mathcal{R}^{\frac{2t_1}{n+1}}_{[E,A,B]} \le \cdots \le \dim \mathcal{R}^{\frac{nt_1}{n+1}}_{[E,A,B]} \le \dim \mathcal{R}^{t_1}_{[E,A,B]} \le n.$$

As a consequence, there has to exist some $j \in \{1,\ldots,n+1\}$ with

$$\dim \mathcal{R}^{\frac{jt_1}{n+1}}_{[E,A,B]} = \dim \mathcal{R}^{\frac{(j+1)t_1}{n+1}}_{[E,A,B]}.$$

Together with the subset inclusion, this yields

$$\mathcal{R}^{\frac{jt_1}{n+1}}_{[E,A,B]} = \mathcal{R}^{\frac{(j+1)t_1}{n+1}}_{[E,A,B]}.$$

Lemma 2.3(ii) then implies the desired statement.

(ii) Let $\bar{x} \in \mathcal{R}^t_{[E,A,B]}$. Then there exists some $(x_1,u_1) \in \mathfrak{B}_{[E,A,B]}$ with $x_1(0)=0$ and $x_1(t)=\bar{x}$. Since, by (i), we have $x_1(2t) \in \mathcal{R}^t_{[E,A,B]}$, there also exists some $(x_2,u_2) \in \mathfrak{B}_{[E,A,B]}$ with $x_2(0)=0$ and $x_2(t)=x_1(2t)$. By linearity and shift-invariance, we have

$$(x,u) := (\sigma_t x_1 - x_2, \sigma_t u_1 - u_2) \in \mathfrak{B}_{[E,A,B]}.$$

The inclusion $\mathcal{R}^t_{[E,A,B]} \subseteq \mathcal{C}^t_{[E,A,B]}$ then follows by

$$x(0) = x_1(t) - x_2(0) = \bar{x}, \quad x(t) = x_1(2t) - x_2(t) = 0.$$

To prove the opposite inclusion, we make use of the previously shown subset relation and Lemma 2.2 to infer that

$$\mathcal{C}^t_{[E,A,B]} = \mathcal{R}^t_{[-E,A,B]} \subseteq \mathcal{C}^t_{[-E,A,B]} = \mathcal{R}^t_{[E,A,B]}.$$

(iii) We first show that $\mathcal{V}^{\text{diff}}_{[E,A,B]} \subseteq \mathcal{V}_{[E,A,B]} + \ker_{\mathbb{R}} E$: Assume that $x^0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$, i.e., $Ex^0 = Ex(0)$ for some $(x,u) \in \mathfrak{B}_{[E,A,B]}$. By $x(0) \in \mathcal{V}_{[E,A,B]}$, $x(0) - x^0 \in \ker_{\mathbb{R}} E$, we obtain

$$x^0 = x(0) + (x^0 - x(0)) \in \mathcal{V}_{[E,A,B]} + \ker_{\mathbb{R}} E.$$

To prove $\mathcal{V}_{[E,A,B]} + \ker_{\mathbb{R}} E \subseteq \mathcal{V}^{\text{diff}}_{[E,A,B]}$, assume that $x^0 = x(0) + \bar{x}$ for some $(x,u) \in \mathfrak{B}_{[E,A,B]}$ and $\bar{x} \in \ker_{\mathbb{R}} E$. Then $x^0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$ is a consequence of $Ex^0 = E(x(0) + \bar{x}) = Ex(0)$. □

By Lemma 2.3 it is sufficient to only consider the spaces $\mathcal{V}_{[E,A,B]}$ and $\mathcal{R}_{[E,A,B]}$ in the following.
We are now in the position to define the central notions of controllability, reachability and stabilizability considered in this article.

Definition 2.4. The system $[E,A,B] \in \Sigma_{k,n,m}$ is called

a) *controllable at infinity*, if for all $x^0 \in \mathbb{R}^n$, there exists some $(x,u) \in \mathcal{B}_{[E,A,B]}$ with $x^0 = x(0)$. In other words, if
$$\mathcal{V}_{[E,A,B]} = \mathbb{R}^n.$$

b) *impulse controllable*, if for all $x^0 \in \mathbb{R}^n$, there exists some $(x,u) \in \mathcal{B}_{[E,A,B]}$ with $Ex^0 = Ex(0)$. In other words, if
$$\mathcal{V}^{\text{diff}}_{[E,A,B]} = \mathbb{R}^n.$$

c) *controllable within the set of reachable states (R-controllable)*, if for all $x_0, x_f \in \mathcal{V}_{[E,A,B]}$ there exists some $t > 0$ and $(x,u) \in \mathcal{B}_{[E,A,B]}$ with $x(0) = x_0$ and $x(t) = x_f$.

d) *controllable in the behavioral sense*, if for all $(x_1,u_1), (x_2,u_2) \in \mathcal{B}_{[E,A,B]}$, there exist $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$, and $(x,u) \in \mathcal{B}_{[E,A,B]}$, such that
$$(x(t),u(t)) = \begin{cases} (x_1(t),u_1(t)), & \text{if } t < t_1, \\ (x_2(t),u_2(t)), & \text{if } t > t_2. \end{cases}$$

e) *stabilizable in the behavioral sense*, if for all $(x,u) \in \mathcal{B}_{[E,A,B]}$, there exists some $(x_0,u_0) \in \mathcal{B}_{[E,A,B]} \cap \Big(\mathcal{W}^{1,1}_{\text{loc}}(\mathcal{T};\mathbb{R}^n) \times \mathcal{W}^{1,1}_{\text{loc}}(\mathcal{T};\mathbb{R}^n)\Big)$, such that
$$(x(t),u(t)) = (x_0(t),u_0(t)) \quad \text{for all } t < 0,$$
and
$$\lim_{t\to\infty} (x_0(t),u_0(t)) = 0.$$

f) *completely reachable*, if there exists some $t \in \mathbb{R}_{>0}$ such that for all $x_f \in \mathbb{R}^n$ there exists some $(x,u) \in \mathcal{B}_{[E,A,B]}$ with $x(0) = 0$ and $x(t) = x_f$. In other words, if for some $t \in \mathbb{R}_{>0}$ it holds
$$\mathcal{R}^t_{[E,A,B]} = \mathbb{R}^n.$$

g) *completely controllable*, if there exists some $t \in \mathbb{R}_{>0}$ such that for all $x_0, x_f \in \mathbb{R}^n$, there exists some $(x,u) \in \mathcal{B}_{[E,A,B]}$ with $x(0) = x_0$ and $x(t) = x_f$.

h) *completely stabilizable*, if for all $x_0 \in \mathbb{R}^n$, there exists some $(x,u) \in \mathcal{B}_{[E,A,B]}$ with $x(0) = x_0$ and
$$\lim_{t\to\infty} x(t) = 0.$$

i) *strongly reachable*, if there exists some $t \in \mathbb{R}_{>0}$ such that for all $x_f \in \mathbb{R}^n$, there exists some $(x,u) \in \mathcal{B}_{[E,A,B]}$ with $Ex(0) = 0$ and $Ex(t) = Ex_f$.

j) *strongly controllable*, if there exists some $t \in \mathbb{R}_{>0}$ such that for all $x_0, x_f \in \mathbb{R}^n$, there exists some $(x,u) \in \mathcal{B}_{[E,A,B]}$ with $Ex(0) = Ex_0$ and $Ex(t) = Ex_f$.

k) *strongly stabilizable* (or merely *stabilizable*), if for all $x^0 \in \mathbb{R}^n$ there exists some $(x,u) \in \mathcal{B}_{[E,A,B]}$ with $Ex(0) = Ex^0$ and

$$\lim_{t\to\infty} Ex(t) = 0.$$

Some remarks on the definitions are warrant.

Remark 2.5. (i) The controllability concepts are sometimes mixed up in the literature. For instance, one has to pay attention if it is (tacitly) claimed that $[E,B] \in \mathbb{R}^{k,n+m}$ or $[E,A,B] \in \mathbb{R}^{k,2n+m}$ have full rank.

For regular systems we have the following: our notions of R- and complete controllability go along with the ones in [153], see also [39,46]; our notion of impulse controllability coincides with the one in [43] and controllability in [43] is our complete controllability; our strong controllability coincides with the respective notion in [143]; impulse, R- and strong controllability do also go along with the ones in [68, Rem. 2]; reachability at ∞ in [92] is our controllability at infinity and controllability at ∞ in [92] is our impulse controllability; controllability at infinity in [5,6,143] is our impulse controllability; impulse controllability in [58] is our strong controllability. Some of these aforementioned articles introduce the controllability by means of certain rank criteria for the matrix triple $[E,A,B]$. The connection of the concepts introduced in Definition 2.4 to linear algebraic properties of E, A and B will be highlighted in Section 4.

For general DAE systems we have: our notion of complete controllability coincides with controllability in [54] and complete controllability in [113], however controllability in [113] is our strong controllability; impulse controllability as defined in this article has also been studied in [57,66,69]; our behavioral controllability coincides with the framework which is introduced in [121, Definition 5.2.2] for so-called *differential behaviors*, which are general (possibly higher order) DAE systems with constant coefficients. Note that the concept of behavioral controllability does not require a distinction between input and state. The concepts of reachability and controllability in [11–14] coincide with our behavioral and complete controllability, resp. (see Sec. 4); full controllability of [155] is our complete controllability together with the additional assumption that solutions have to be unique.

(ii) Stabilizability in the behavioral sense is introduced in [121, Definition 5.2.2]. For regular systems stabilizability is usually defined either via linear algebraic properties of E, A and B, or by the existence of a stabilizing state feedback, see [31,32,53] and [46, Definition 3-1.2.]. Our concepts of behavioral stabilizability and stabilizability coincide with the notions of internal stability and complete stabilizability, resp., defined in [107] for the system $\mathcal{E}\dot{z}(t) = \mathcal{A}z(t)$ with $\mathcal{E} = [\,E\,,\,0\,]$, $\mathcal{A} = [\,A\,,\,B\,]$, $z(t) = [\,x^T(t)\,,\,u^T(t)\,]^T$.

(iii) Other concepts, not related to the ones considered in this article, are e.g. the instantaneous controllability (reachability) of order k in [113] or the impulsive mode controllability in [66].

(iv) The notion of consistent initial conditions is the most important one for DAE systems and therefore the consideration of the space $\mathcal{V}_{[E,A,B]}$ (for $B = 0$ when no control systems were considered) is as old as the theory of DAEs itself, see e.g. [56]. $\mathcal{V}_{[E,A,B]}$ is sometimes called viability kernel [27], see also [8,9]. The reachability and controllability space are some of the most important

notions for (DAE) control systems and have been considered in [92] for regular systems. They are the fundamental subspaces considered in the geometric theory, see Section 6. Further usage of these concepts can be found in the following: in [115] generalized reachability and controllability subspaces of regular systems are considered; Eliopoulou and Karcanias [52] consider reachability and almost reachability subspaces of general DAE systems; Frankowska [54] considers the reachability subspace in terms of differential inclusions.
A nice formula for the reachability space of a regular system has been derived by Yip et al. [153] (and later been adopted by Cobb [43], however called controllable subspace): Consider a regular system $[E, A, B] \in \Sigma_{n,n,m}$ in Weierstraß form [56], that is

$$E = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \quad A = \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

where N is nilpotent. Then [153, Thm. 2]

$$\mathcal{R}_{[E,A,B]} = \langle J|B_1 \rangle \times \langle N|B_2 \rangle,$$

where $\langle K|L \rangle := \operatorname{im}_{\mathbb{R}}[L, KL, \ldots, K^{n-1}L]$ for some matrices $K \in \mathbb{R}^{n \times n}, L \in \mathbb{R}^{n \times m}$. Furthermore, we have [153, Thm. 3]

$$\mathcal{V}_{[E,A,B]} = \mathbb{R}^{n_1} \times \langle N|B_2 \rangle.$$

This result has been improved later in [39] so that the Weierstraß form is no longer needed. Denoting by E^D the Drazin inverse of a given matrix $E \in \mathbb{R}^{n \times n}$ (see [37]), it is shown [39, Thm. 3.1] that, for $A = I$,

$$\mathcal{R}_{[E,A,B]} = E^D \langle E^D|B \rangle \oplus (I - EE^D)\langle E|B \rangle,$$

where the consideration of $A = I$ is justified by a certain (time-varying) transformation of the system [117]. We further have [39, Thm. 3.2]

$$\mathcal{V}_{[E,A,B]} = \operatorname{im}_{\mathbb{R}} E^D \oplus (I - EE^D)\langle E|B \rangle.$$

Yet another approach was followed by Cobb [40] who obtains that

$$\mathcal{R}_{[E,A,B]} = \langle (\alpha E - A)^{-1}E | (\alpha E - A)^{-1}B \rangle$$

for some $\alpha \in \mathbb{R}$ with $\det(\alpha E - A) \neq 0$. A simple proof of this result can also be found in [154].

(v) The notion $\mathcal{V}^{\text{diff}}_{[E,A,B]}$ comes from the possible impulsive behavior of solutions of (2.1), i.e., x may have jumps, when distributional solutions are permitted, see e.g. [43] as a very early contribution in this regard. Since these jumps have no effect on the solutions if they occur at the initial time and within the kernel of E this leads to the definition of $\mathcal{V}^{\text{diff}}_{[E,A,B]}$. See also the definition of impulse controllability.

(vi) Impulse controllability and controllability at infinity are usually defined by considering distributional solutions of (2.1), see e.g. [43,57,69], sometimes called impulsive modes, see e.g. [18,66,143]. For regular systems, impulse controllability has been introduced by Verghese et al. [143] (called controllability at infinity in this work) as controllability of the impulsive modes of the system, and later made more precise by Cobb [43], see also Armentano [5,6] (who also calls it controllability at infinity) for a more geometric point of view. In [143] the authors do also develop the notion of strong controllability as impulse controllability with, additionally, controllability in the regular sense. Cobb [41] showed that under the condition of impulse controllability, the infinite

eigenvalues of regular $sE - A$ can be assigned via a state feedback $u = Fx$ to arbitrary finite positions. ARMENTANO [5] later showed how to calculate F. This topic has been further pursued in [88] in the form of invariant polynomial assignment.
Controllability at infinity has been introduced by ROSENBROCK [127] - although he does not use this phrase - as controllability of the infinite frequency zeros. Later COBB [43] did a comparison of the concepts of impulse controllability and controllability at infinity, see [43, Thm. 5].
These concepts have later been generalized by GEERTS [57] (see [57, Thm. 4.5 & Rem. 4.9], however he does not use the name "controllability at infinity"). Controllability at infinity of (2.1) is equivalent to the strictness of the corresponding differential inclusion [54, Prop. 2.6]. The concept of impulsive mode controllability in [66] is even weaker than impulse controllability.

(vii) Controllability concepts with a distributional solution setup have been considered in [57,113,123] for instance, see also [43]. A typical argumentation in these works is that inconsistent initial values cause distributional solutions in a way that the state trajectory is composed of a continuous function and a linear combination of Dirac's delta impulse and some of its derivatives. However, some frequency domain considerations in [109] refute this approach. This justifies that we do only consider weakly differentiable solutions as defined in the behavior $\mathcal{B}_{[E,A,B]}$.

For a mathematically rigorous approach to distributional solution theory of linear DAEs, we refer to [57] by GEERTS, and [133, 134] by TRENN. The latter works introduce the notions of impulse controllability and jump controllability which coincide with our impulse controllability and behavioral controllability, resp.

(viii) R-controllability has been first defined in [153] for regular DAEs. Roughly speaking, R-controllability is the property that any consistent initial state x_0 can be steered to any reachable state x_f, where here x_f is reachable if, and only if, there exist $t > 0$ and $(x, u) \in \mathcal{B}_{[E,A,B]}$ such that $x(t) = x_f$; by (2.3) the latter is equivalent to $x_f \in \mathcal{V}_{[E,A,B]}$, as stated in Definition 2.4.

(ix) The concept of behavioral controllability has been introduced by WILLEMS [146], see also [121]. This concept is very suitable for generalizations in various directions, see e.g. [33, 38, 67, 91, 125, 148, 152]. Having found the behavior of the considered control system one can take over the definition of behavioral controllability without the need for any further changes. From this point of view this appears to be the most natural of the controllability concepts. However, this concept also seems to be the least regarded in the DAE literature.

(x) The controllability theory of DAE systems can also be treated with the theory of differential inclusions [8, 9] as showed by FRANKOWSKA [54].

(xi) KARCANIAS and HAYTON [79] pursued a special ansatz to simplify the system (2.1): provided that B has full column rank, we take a left annihilator N and a pseudoinverse $B^\dagger$ of B (i.e., $NB = 0$ and $B^\dagger B = I$) such that $W = \begin{bmatrix} N \\ B^\dagger \end{bmatrix}$ is invertible and then pre-multiply (2.1) by W, thus obtaining the equivalent system

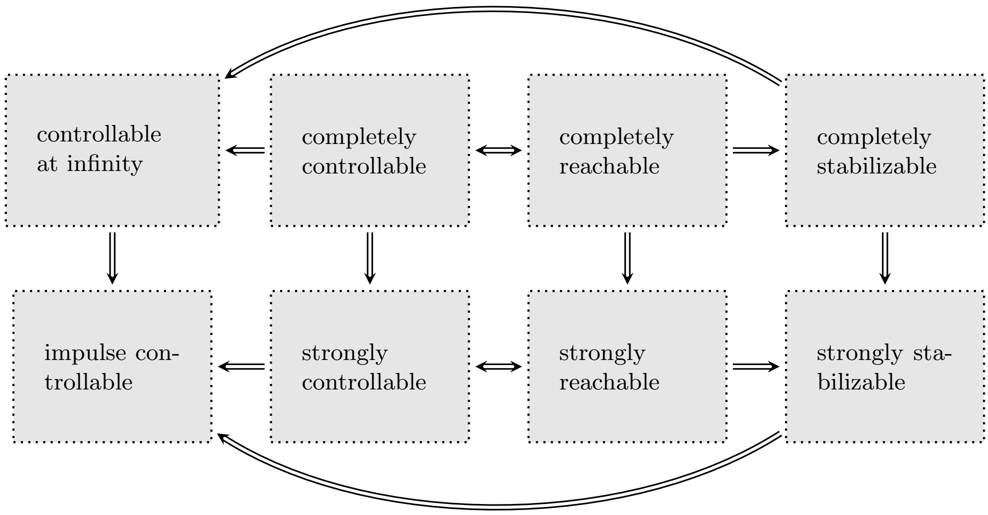
$$\begin{aligned} NE\dot{x} &= NAx, \\ u &= B^\dagger(E\dot{x} - Ax). \end{aligned}$$

The reachability (controllability) properties of (2.1) may now be studied in terms of the pencil $sNE - NA$, which is called the restriction pencil [72], first introduced as zero pencil for the investigation of system zeros of ODEs in [85, 86], see also [82]. For a comprehensive study of the properties of the pencil $sNE - NA$ see e.g. [78–81].

(xii) Banaszuk and Przyłuski [14] have considered perturbations of DAE control systems and obtained conditions under which the sets of all completely controllable systems (systems controllable in the behavioral sense) within the set of all systems $\Sigma_{k,n,m}$ contain an open and dense subset, or its complement contains an open and dense subset. $\diamond$

The following dependencies hold true between the concepts from Definition 2.4. Some further relations will be derived in Section 4.

Proposition 2.6. *For any* $[E, A, B] \in \Sigma_{k,n,m}$ *the following implications hold true:*



Proof: The following implications are immediate consequences of Definition 2.4:

completely controllable ⇒ controllable at infinity ⇒ impulse controllable,

completely controllable ⇒ strongly controllable ⇒ impulse controllable,

completely controllable ⇒ completely reachable ⇒ strongly reachable,

strongly controllable ⇒ strongly reachable,

completely stabilizable ⇒ controllable at infinity,

strongly stabilizable ⇒ impulse controllable,

completely stabilizable ⇒ strongly stabilizable.

It remains to prove the following assertions:

(i) completely reachable ⇒ completely controllable,

(ii) strongly reachable ⇒ strongly controllable,

(iii) completely reachable ⇒ completely stabilizable,

(iv) strongly reachable ⇒ strongly stabilizable.

(i) Let $x_0, x_f \in \mathbb{R}^n$. Then, by complete reachability of $[E, A, B]$, there exist $t > 0$ and some $(x_1, u_1) \in \mathcal{B}_{[E,A,B]}$ with $x_1(0) = 0$ and $x_1(t) = x_0$. Further, there exists $(x_2, u_2) \in \mathcal{B}_{[E,A,B]}$ with $x_2(0) = 0$ and $x_2(t) = x_f - x_1(2t)$. By linearity and shift-invariance, we have

$$(x, u) := (\sigma_t x_1 + x_2, \sigma_t u_1 + u_2) \in \mathcal{B}_{[E,A,B]}.$$

On the other hand, this trajectory fulfills $x(0) = x_1(t) + x_2(0) = x_0$ and $x(t) = x_1(2t) + x_2(t) = x_f$.

(ii) The proof of this statement is analogous to (i).

(iii) By (i) it follows that the system is completely controllable. Complete controllability implies that there exists some $t > 0$, such that for all $x_0 \in \mathbb{R}^n$ there exists $(x_1, u_1) \in \mathcal{B}_{[E,A,B]}$ with $x_1(0) = x_0$ and $x_1(t) = 0$. Then, since (x, u) with

$$(x(\tau), u(\tau)) = \begin{cases} (x_1(\tau), u_1(\tau)), & \text{if } \tau \le t \\ (0, 0), & \text{if } \tau \ge t \end{cases}$$

satisfies $(x, u) \in \mathcal{B}_{[E,A,B]}$ (cf. the proof of Lemma 2.1(i)) , the system $[E, A, B]$ is completely stabilizable.

(iv) The proof of this statement is analogous to (iii). $\square$

3 Solutions, relations and normal forms

In this section we give the definitions for system and feedback equivalence of DAE control systems (see [58, 127, 143]), revisit the solution theory of DAEs (see [90, 145] and also [24]), and state a normal form under system and feedback equivalence (see [98]).

3.1 System and feedback equivalence

We define the essential concepts of system and feedback equivalence. System equivalence was first studied by Rosenbrock [127] (called restricted system equivalence in his work, see also [143]) and later became a crucial concept in the control theory of DAEs [21, 22, 58, 59, 64]. Feedback equivalence for DAEs seems to have been first considered in [58] to derive a feedback canonical form for regular systems, little later also in [98] (for general DAEs) where additionally also derivative feedback was investigated and respective canonical forms derived, see also Section 3.3.

Definition 3.1 (System and feedback equivalence)**.**
Two systems $[E_i, A_i, B_i] \in \Sigma_{k,n,m}$, $i = 1, 2$, are called

- *system equivalent* if, and only if,

$$\begin{aligned} &\exists\, W \in \mathbf{Gl}_k(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R}) : \\ &\begin{bmatrix} sE_1 - A_1 & B_1 \end{bmatrix} = W \begin{bmatrix} sE_2 - A_2 & B_2 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix} ; \end{aligned} \tag{3.1}$$

  we write

$$[\, E_1\,,\, A_1\,,\, B_1\,] \overset{W,T}{\sim_s} [\, E_2\,,\, A_2\,,\, B_2\,]\,.$$

- *feedback equivalent* if, and only if,

$$\begin{aligned}&\exists\, W \in \mathbf{Gl}_k(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R}), V \in \mathbf{Gl}_m(\mathbb{R}), F \in \mathbb{R}^{m,n} :\\ &\begin{bmatrix} sE_1 - A_1 & B_1 \end{bmatrix} = W \begin{bmatrix} sE_2 - A_2 & B_2 \end{bmatrix} \begin{bmatrix} T & 0 \\ -F & V \end{bmatrix} ;\end{aligned} \tag{3.2}$$

  we write

$$[\, E_1\,,\, A_1\,,\, B_1\,] \overset{W,T,V,F}{\sim_f} [\, E_2\,,\, A_2\,,\, B_2\,].$$

◇

It is easy to observe that both system and feedback equivalence are equivalence relations on $\Sigma_{k,n,m}$. To see the latter, note that if $[\, E_1\,,\, A_1\,,\, B_1\,] \overset{W,T,V,F}{\sim_f} [\, E_2\,,\, A_2\,,\, B_2\,]$, then

$$[\, E_2\,,\, A_2\,,\, B_2\,] \overset{W^{-1},T^{-1},V^{-1},-V^{-1}FT^{-1}}{\sim_f} [\, E_1\,,\, A_1\,,\, B_1\,].$$

The behaviors of system and feedback equivalent systems are connected via

$$\begin{aligned}&\text{If } [\, E_1\,,\, A_1\,,\, B_1\,] \overset{W,T}{\sim_s} [\, E_2\,,\, A_2\,,\, B_2\,],\ \text{then}\\ &(x,u) \in \mathfrak{B}_{[E_1,A_1,B_1]} \Leftrightarrow (Tx,u) \in \mathfrak{B}_{[E_2,A_2,B_2]}\end{aligned}$$

$$\begin{aligned}&\text{If } [\, E_1\,,\, A_1\,,\, B_1\,] \overset{W,T,V,F}{\sim_f} [\, E_2\,,\, A_2\,,\, B_2\,],\ \text{then}\\ &(x,u) \in \mathfrak{B}_{[E_1,A_1,B_1]} \Leftrightarrow (Tx, Fx+Vu) \in \mathfrak{B}_{[E_2,A_2,B_2]}.\end{aligned}$$

In particular, if $[\, E_1\,,\, A_1\,,\, B_1\,] \overset{W,T}{\sim_s} [\, E_2\,,\, A_2\,,\, B_2\,]$, then

$$\mathcal{V}_{[E_1,A_1,B_1]} = T^{-1} \cdot \mathcal{V}_{[E_2,A_2,B_2]}, \quad \mathcal{R}^t_{[E_1,A_1,B_1]} = T^{-1} \cdot \mathcal{R}^t_{[E_2,A_2,B_2]}.$$

Further, if $[\, E_1\,,\, A_1\,,\, B_1\,] \overset{W,T,V,F}{\sim_f} [\, E_2\,,\, A_2\,,\, B_2\,]$, then

$$\mathcal{V}_{[E_1,A_1,B_1]} = T^{-1} \cdot \mathcal{V}_{[E_2,A_2,B_2]}, \quad \mathcal{R}^t_{[E_1,A_1,B_1]} = T^{-1} \cdot \mathcal{R}^t_{[E_2,A_2,B_2]},$$

and properties of controllability at infinity, impulse controllability, R-controllability, behavioral controllability, behavioral stabilizability, complete controllability, complete stabilizability, strong controllability and strong stabilizability are invariant under system and feedback equivalence.
In order to study normal forms under system and feedback equivalence we introduce the following notation: For $k \in \mathbb{N}$ we introduce the matrices $N_k \in \mathbb{R}^{k,k}$, $K_k, L_k \in \mathbb{R}^{k-1,k}$ with

$$N_k = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \quad K_k = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \quad L_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}.$$

Further, let $e_i^{[k]} \in \mathbb{R}^k$ be the i-th canonical unit vector, and, for some multi-index $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathbb{N}^l$, we define

$$\begin{aligned} N_\alpha &= \operatorname{diag}(N_{\alpha_1}, \dots, N_{\alpha_l}) \in \mathbb{R}^{|\alpha|,|\alpha|},\\ K_\alpha &= \operatorname{diag}(K_{\alpha_1}, \dots, K_{\alpha_l}) \in \mathbb{R}^{|\alpha|-l,|\alpha|},\\ L_\alpha &= \operatorname{diag}(L_{\alpha_1}, \dots, L_{\alpha_l}) \in \mathbb{R}^{|\alpha|-l,|\alpha|},\\ E_\alpha &= \operatorname{diag}(e^{[\alpha_1]}_{\alpha_1}, \dots, e^{[\alpha_l]}_{\alpha_l}) \in \mathbb{R}^{|\alpha|,l}. \end{aligned} \tag{3.3}$$

KRONECKER proved [87] that any matrix pencil $sE - A \in \mathbb{R}[s]^{k,n}$ can be put into a certain canonical form, called Kronecker canonical form nowadays, of which a more comprehensive proof has been provided by GANTMACHER [56]. In the following we may use the quasi-Kronecker form derived in [24], since in general the Kronecker canonical form is complex-valued even though the given pencil $sE - A$ is real-valued, what we need to avoid. The obtained form then is not canonical anymore, but it is a normal form (see Remark 3.5).

Proposition 3.2 (Quasi-Kronecker form [24, 56])**.**
For any matrix pencil $sE - A \in \mathbb{R}[s]^{k,n}$, there exist $W \in \mathbf{Gl}_k(\mathbb{R})$, $T \in \mathbf{Gl}_n(\mathbb{R})$ such that

$$W(sE - A)T = \begin{bmatrix} sI_{n_s} - A_s & 0 & 0 & 0 \\ 0 & sN_\alpha - I_{|\alpha|} & 0 & 0 \\ 0 & 0 & sK_\beta - L_\beta & 0 \\ 0 & 0 & 0 & sK_\gamma^\top - L_\gamma^\top \end{bmatrix} \tag{3.4}$$

for some $A_s \in \mathbb{R}^{n_s,n_s}$ and multi-indices $\alpha \in \mathbb{N}^{n_\alpha}$, $\beta \in \mathbb{N}^{n_\gamma}$, $\gamma \in \mathbb{N}^{n_\gamma}$. The multi-indices α, β, γ are uniquely determined by $sE - A$. Further, the matrix A_s is unique up to similarity. $\diamond$

The (components of the) multi-indices α, β, γ are often called minimal indices and elementary divisors and play an important role in the analysis of matrix pencils, see e.g. [56, 97, 98, 106], where the components of α are the orders of the infinite elementary divisors, the components of β are the column minimal indices and the components of γ are the row minimal indices. In fact, the number of column (row) minimal indices equal to one corresponds to the dimension of $\ker_\mathbb{R} E \cap \ker_\mathbb{R} A$ ($\ker_\mathbb{R} E^\top \cap \ker_\mathbb{R} A^\top$), or, equivalently, the number of zero columns (rows) in a quasi-Kronecker form of $sE - A$. Further, note that $sI_{n_s} - A_s$ may be further transformed into Jordan canonical form to obtain the finite elementary divisors.
Since the multi-indices $\alpha \in \mathbb{N}^{n_\alpha}$, $\beta \in \mathbb{N}^{n_\gamma}$, $\gamma \in \mathbb{N}^{n_\gamma}$ are well-defined by means of the pencil $sE - A$ and, furthermore, the matrix A_s is unique up to similarity, this justifies the introduction of the following quantities.

Definition 3.3 (Index of $sE - A$)**.** Let the matrix pencil $sE - A \in \mathbb{R}[s]^{k,n}$ be given with quasi-Kronecker form (3.4). Then the *index* $\nu \in \mathbb{N}_0$ of $sE - A$ is defined as

$$\nu = \max\{\alpha_1, \ldots, \alpha_{\ell(\alpha)}, \gamma_1, \ldots, \gamma_{\ell(\gamma)}\}.$$

$\diamond$

The index is larger or equal to the index of nilpotency ζ of N_α, i.e., $\zeta \le \nu$, $N_\alpha^\zeta = 0$ and $N_\alpha^{\zeta-1} \neq 0$. By means of the quasi-Kronecker form (3.4) it can be seen that the index of $sE - A$ does not exceed one if, and only if,

$$\operatorname{im}_\mathbb{R} A \subseteq \operatorname{im}_\mathbb{R} E + A \cdot \ker_\mathbb{R} E. \tag{3.5}$$

This is moreover equivalent to the fact that for some (and hance any) real matrix Z with $\operatorname{im}_\mathbb{R} Z = \ker_\mathbb{R} E$, there holds

$$\operatorname{im}_\mathbb{R}[E, AZ] = \operatorname{im}_\mathbb{R}[E, A]. \tag{3.6}$$

Since each block in $sK_\beta - L_\beta$ ($sK_\gamma^\top - L_\gamma^\top$) causes a single drop of the column (row) rank of $sE - A$, we have

$$\ell(\beta) = n - \operatorname{rk}_{\mathbb{R}(s)}(sE - A), \quad \ell(\gamma) = k - \operatorname{rk}_{\mathbb{R}(s)}(sE - A). \tag{3.7}$$

Further, $\lambda \in \mathbb{C}$ is a generalized eigenvalue of $sE - A$ if, and only if,

$$\operatorname{rk}_\mathbb{C}(\lambda E - A) < \operatorname{rk}_{\mathbb{R}(s)}(sE - A).$$

3.2 A normal form under system equivalence

Using Proposition 3.2 it is easy to determine a normal form under system equivalence. For regular systems this normal form was first discovered by ROSENBROCK [127].

Corollary 3.4 (Decoupled DAE)**.**
Let $[E, A, B] \in \Sigma_{k,n,m}$. Then there exist $W \in \mathbf{Gl}_k(\mathbb{R})$, $T \in \mathbf{Gl}_n(\mathbb{R})$ such that

$$[E, A, B] \overset{W,T}{\sim_s} \left[\begin{bmatrix} I_{n_s} & 0 & 0 & 0 \\ 0 & N_\alpha & 0 & 0 \\ 0 & 0 & K_\beta & 0 \\ 0 & 0 & 0 & K_\gamma^\top \end{bmatrix}, \begin{bmatrix} A_s & 0 & 0 & 0 \\ 0 & I_{|\alpha|} & 0 & 0 \\ 0 & 0 & L_\beta & 0 \\ 0 & 0 & 0 & L_\gamma^\top \end{bmatrix}, \begin{bmatrix} B_s \\ B_f \\ B_u \\ B_o \end{bmatrix} \right], \tag{3.8}$$

for some $B_s \in \mathbb{R}^{n_s,m}$, $B_f \in \mathbb{R}^{|\alpha|,m}$, $B_o \in \mathbb{R}^{|\beta|-\ell(\beta),m}$, $B_u \in \mathbb{R}^{|\gamma|,m}$, $A_s \in \mathbb{R}^{n_s,n_s}$ and multi-indices $\alpha \in \mathbb{N}^{n_\alpha}$, $\beta \in \mathbb{N}^{n_\beta}$, $\gamma \in \mathbb{N}^{n_\gamma}$. This is interpreted, in terms of the DAE (2.1), as follows: $(x, u) \in \mathfrak{B}_{[E,A,B]}$ if, and only if,

$$\left(x_s(\cdot)^\top, x_f(\cdot)^\top, x_u(\cdot)^\top, x_o(\cdot)^\top\right)^\top := Tx(\cdot)$$

with

$$x_f(\cdot) = \begin{pmatrix} x_{f[1]}(\cdot) \\ \vdots \\ x_{f[\ell(\alpha)]}(\cdot) \end{pmatrix}, \quad x_u(\cdot) = \begin{pmatrix} x_{u[1]}(\cdot) \\ \vdots \\ x_{u[\ell(\beta)]}(\cdot) \end{pmatrix}, \quad x_o(\cdot) = \begin{pmatrix} x_{o[1]}(\cdot) \\ \vdots \\ x_{o[\ell(\gamma)]}(\cdot) \end{pmatrix}$$

solves the decoupled DAEs

$$\dot{x}_s(t) = A_s x_s(t) + B_s u(t), \tag{3.9a}$$

$$N_{\alpha_i} \dot{x}_{f[i]}(t) = x_{f[i]}(t) + B_{f[i]} u(t) \qquad \text{for } i = 1, \ldots, \ell(\alpha), \tag{3.9b}$$

$$K_{\beta_i} \dot{x}_{u[i]}(t) = L_{\beta_i} x_{u[i]}(t) + B_{u[i]} u(t) \qquad \text{for } i = 1, \ldots, \ell(\beta), \tag{3.9c}$$

$$K_{\gamma_i}^\top \dot{x}_{o[i]}(t) = L_{\gamma_i}^\top x_{o[i]}(t) + B_{o[i]} u(t) \qquad \text{for } i = 1, \ldots, \ell(\gamma) \tag{3.9d}$$

with suitably labeled partitions of B_f, B_u and B_o. ◇

Remark 3.5 (Canonical and normal form)**.** Recall the definition of a canonical form: given a group G, a set $\mathcal{S}$, and a group action $\alpha : G \times \mathcal{S} \to \mathcal{S}$ which defines an equivalence relation $s \overset{\alpha}{\sim} s'$ if, and only if, $\exists\, U \in G : \alpha(U, s) = s'$. Then a map $\gamma : \mathcal{S} \to \mathcal{S}$ is called a *canonical form for* α [26] if, and only if,

$$\forall\, s, s' \in \mathcal{S} \;:\; \gamma(s) \overset{\alpha}{\sim} s \quad \wedge \quad \left[s \overset{\alpha}{\sim} s' \Leftrightarrow \gamma(s) = \gamma(s') \right].$$

Therefore, the set $\mathcal{S}$ is divided into disjoint orbits (i.e., equivalence classes) and the mapping γ picks a unique representative in each equivalence class. In the setup of system equivalence, the group is $G = \mathbf{Gl}_n(\mathbb{R}) \times \mathbf{Gl}_n(\mathbb{R})$, the considered set is $\mathcal{S} = \Sigma_{k,n,m}$ and the group action $\alpha\big((W, T), [E, A, B]\big) = [WET, WAT, WB]$ corresponds to $\overset{W^{-1},T^{-1}}{\sim}$. However, Corollary 3.4 does not provide a mapping γ. That means that the form (3.8) is not a unique representative within the equivalence class and hence it is not a canonical form. Nevertheless, we may call it a *normal form*, since every entry is (at least) unique up to similarity. ◇

Remark 3.6 (Canonical forms for regular systems)**.** For regular systems which are completely controllable two actual canonical forms of $[E, A, B] \in \Sigma_{n,n,m}$ under system equivalence have been obtained: the Jordan control canonical form in [59] and, later, the more simple canonical form in [64] based on the Hermite canonical form for controllable ODEs $[I, A, B]$. ◇

Remark 3.7 (DAEs corresponding to the blocks in the quasi-Kronecker form)**.** Corollary 3.4 leads to the separate consideration of the differential-algebraic equations (3.9a)-(3.9c):

a) (3.9a) is an ordinary differential equation whose solution is determined by variations of constants, that is

$$x_s(t) = e^{A_s t} x_s(0) + \int_0^t e^{A_s(t-\tau)} B_s u(\tau) d\tau, \quad t \in \mathbb{R}.$$

In particular, solvability is guaranteed by $u \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^m)$. The initial value $x_s(0) \in \mathbb{R}^n$ can be chosen arbitrarily; the prescription of $u \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^m)$ and $x_s(0) \in \mathbb{R}^n$ guarantees uniqueness of the solution.

b) The solutions of (3.9b) can be calculated by successive differentiation and pre-multiplication with N_{α_i}, hence we have

$$0 = N_{\alpha_i}^{\alpha_i} x_{f[i]}^{(\alpha_i)}(t) \overset{(3.9b)}{=} N_{\alpha_i}^{\alpha_i - 1} x_{f[i]}(t)^{(\alpha_i - 1)} + N_{\alpha_i}^{\alpha_i - 1} B_{f[i]} u^{(\alpha_i - 1)}(t) = \ldots = x_{f[i]}(t) + \sum_{j=0}^{\alpha_i - 1} N_{\alpha_i}^j B_{f[i]} u^{(j)}(t),$$

where $u^{(j)}$ denotes the j-th distributional derivative of u. As a consequence, the solution requires a certain smoothness of the input, expressed by

$$\sum_{j=0}^{\alpha_i - 1} N_{\alpha_i}^j B_{f[i]} u^{(j)} \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^{\alpha_i}).$$

In particular, condition $u \in \mathcal{W}^{\alpha_i,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^{\alpha_i})$ guarantees solvability of the DAE (3.9b). Note that the initial value $x_{f[i]}(0)$ cannot be chosen at all: It is fixed by u via the relation

$$x_{f[i]}(0) = -\left(\sum_{j=0}^{\alpha_i - 1} N_{\alpha_i}^j B_{f[i]} u^{(j)} \right)(0).$$

On the other hand, for any (sufficiently smooth) input there exists a unique solution for appropriately chosen initial value.

c) Denoting

$$x_{u[i]-} = \begin{bmatrix} x_{u[i],1} \\ \vdots \\ x_{u[i],\beta_i - 1} \end{bmatrix}$$

then (3.9c) is equivalent to

$$\dot{x}_{u[i]-} = N_{\beta_i - 1}^\top x_{u[i]-} + e_{\beta_i - 1}^{[\beta_i - 1]} x_{u[i],\beta_i} + B_{u[i]} u(t).$$

Hence, a solution exists for all inputs $u \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^m)$ and all $x_{u[i],\beta_i} \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R})$ as well as $x_{u[i],1}(0) \ldots, x_{u[i],\beta_i - 1}(0)$. This system is therefore under-determined in the sense that one component as well as all initial values can be freely chosen. Hence any existing solution for fixed input u and fixed initial value $x_{u[i]}(0)$ is far from being unique.

d) Denoting

$$x_{o[i]+} = \begin{bmatrix} 0_{1,1} \\ x_{o[i]} \end{bmatrix}$$

then (3.9d) can be rewritten as

$$N_{\gamma_i}^\top \dot{x}_{o[i]+} = x_{o[i]+} + B_{o[i]} u(t).$$

Hence we obtain $x_{o[i]+}(t) = -\sum_{j=0}^{\gamma_i-1} (N_{\gamma_i}^\top)^j B_{o[i]} u^{(j)}(t)$, which gives

$$x_{o[i]}(t) = -[0_{(\gamma_i-1),1}, I_{\gamma_i-1}] \sum_{j=0}^{\gamma_i-1} (N_{\gamma_i}^\top)^j B_{o[i]} u^{(j)}(t)$$

together with the consistency condition on the input:

$$\left(e_1^{[\gamma_i]}\right)^\top \sum_{j=0}^{\gamma_i-1} (N_{\gamma_i}^\top)^j B_{o[i]} u^{(j)}(t) = 0. \tag{3.10}$$

The smoothness condition

$$\sum_{j=0}^{\gamma_i-1} (N_{\gamma_i}^\top)^j B_{o[i]} u^{(j)} \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^{\gamma_i})$$

is therefore not enough to guarantee existence of a solution; the additional constraint formed by (3.10) has to be satisfied, too. Furthermore, as in b), the initial value $x_{o[i]}(0)$ is fixed by the input u. Hence, a solution does only exist if the consistency conditions on the input and initial value are satisfied, but then the solution is unique. ◇

Remark 3.8 (Solutions on (finite) time intervals)**.** The solution of a DAE $[E, A, B] \in \Sigma_{k,n,m}$ on some time interval $I \subsetneq \mathbb{R}$ can be defined in a straightforward manner (compare (2.2)). By the considerations in Remark 3.7, we can infer that any solution (x, u) on some finite time interval $I \subsetneq \mathbb{R}$ can be extended to a solution on the whole real axis. Consequently, all concepts which have been defined in Sec. 2 could be also made based on solutions on intervals I including zero.

3.3 A normal form under feedback equivalence

A normal form under feedback transformation (3.2) was first studied for systems governed by ordinary differential equations by Brunovsky [30]. In this section we present a generalization of the Brunovsky form for general DAE systems $[E, A, B] \in \Sigma_{k,n,m}$ from [98]. For more details on the feedback form and a more geometric point of view on feedback invariants and feedback canonical forms see [81, 98].

Remark 3.9 (Feedback for regular systems)**.** It is known [11, 58] that the class of regular DAE systems is not closed under the action of state feedback. Therefore, in [130] the class of regular systems is divided into the families

$$\Sigma_\theta := \{ \ (E, A, B) \in \Sigma_{n,n,m} \ | \ \det(\cos\theta \ E - \sin\theta \ A) \neq 0 \ \}, \quad \theta \in [0, \pi),$$

and it is shown that any of these families is dense in the set of regular systems and the union of these families is exactly the set of regular systems. The authors of [130] then introduce the "constant-ratio proportional and derivative" feedback on Σ_θ, i.e.

$$u = F(\cos\theta \ x - \sin\theta \ \dot{x}) + v.$$

This feedback leads to a group action and enables them to obtain a generalization of Brunovsky's theorem [30] on each of the subsets of completely controllable systems in Σ_θ, see [130, Thm. 6]. Glüsing-Lüerssen [58] derived a canonical form under the unchanged feedback equivalence (3.2) on the set of strongly controllable (called impulse controllability in [58]) regular systems, see [58, Thm. 4.7]. In particular it was shown that this set is closed under the action of a feedback group. ◇

Theorem 3.10 (Normal form under feedback equivalence [98]).
Let $[E, A, B] \in \Sigma_{k,n,m}$. Then there exist $W \in \mathbf{Gl}_k(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R}), V \in \mathbf{Gl}_m(\mathbb{R}), F \in \mathbb{R}^{m,n}$ such that

$$[E, A, B] \overset{W,T,V,F}{\sim_f}$$
$$\left[\begin{bmatrix} I_{|\alpha|} & 0 & 0 & 0 & 0 & 0 \\ 0 & K_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & L_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & K_\delta^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & N_\kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_{\overline{c}}} \end{bmatrix}, \begin{bmatrix} N_\alpha^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & L_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & K_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & L_\delta^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{\overline{c}} \end{bmatrix}, \begin{bmatrix} E_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E_\gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right], \tag{3.11}$$

for some multi-indices $\alpha, \beta, \gamma, \delta, \kappa$ and a matrix $A_{\overline{c}} \in \mathbb{R}^{n_{\overline{c}}, n_{\overline{c}}}$. This is interpreted, in terms of the DAE (2.1), as follows: $(x, u) \in \mathfrak{B}_{[E,A,B]}$ if, and only if,

$$\left(x_c(\cdot)^\top, x_u(\cdot)^\top, x_{ob}(\cdot)^\top, x_o(\cdot)^\top, x_f(\cdot)^\top, x_{\overline{c}}(\cdot)^\top\right)^\top := Tx(\cdot), \quad \left(u_c(\cdot)^\top, u_{ob}(\cdot)^\top, u_s(\cdot)^\top\right)^\top := V(u(\cdot) - Fx(\cdot)),$$

with

$$x_c(\cdot) = \begin{pmatrix} x_{c[1]}(\cdot) \\ \vdots \\ x_{c[\ell(\alpha)]}(\cdot) \end{pmatrix}, \quad u_c(\cdot) = \begin{pmatrix} u_{c[1]}(\cdot) \\ \vdots \\ x_{c[\ell(\alpha)]}(\cdot) \end{pmatrix}, \quad x_u(\cdot) = \begin{pmatrix} x_{u[1]}(\cdot) \\ \vdots \\ x_{u[\ell(\beta)]}(\cdot) \end{pmatrix},$$
$$x_{ob}(\cdot) = \begin{pmatrix} x_{ob[1]}(\cdot) \\ \vdots \\ x_{ob[\ell(\gamma)]}(\cdot) \end{pmatrix}, \quad u_{ob}(\cdot) = \begin{pmatrix} u_{ob[1]}(\cdot) \\ \vdots \\ u_{ob[\ell(\gamma)]}(\cdot) \end{pmatrix}, \quad x_o(\cdot) = \begin{pmatrix} x_{o[1]}(\cdot) \\ \vdots \\ x_{o[\ell(\delta)]}(\cdot) \end{pmatrix}, \quad x_f(\cdot) = \begin{pmatrix} x_{f[1]}(\cdot) \\ \vdots \\ x_{f[\ell(\kappa)]}(\cdot) \end{pmatrix}$$

solves the decoupled DAEs

$$\dot{x}_{c[i]}(t) = N_{\alpha_i}^\top x_c(t) + e_{\alpha_i}^{[\alpha_i]} u_{c[i]}(t) \qquad \text{for } i = 1, \ldots, \ell(\alpha) \tag{3.12a}$$

$$K_{\beta_i} \dot{x}_{u[i]}(t) = L_{\beta_i} x_{u[i]}(t) \qquad \text{for } i = 1, \ldots, \ell(\beta), \tag{3.12b}$$

$$L_{\gamma_i}^\top \dot{x}_{ob[i]}(t) = K_{\gamma_i}^\top x_{ob[i]}(t) + e_{\gamma_i}^{[\gamma_i]} u_{ob[i]} \qquad \text{for } i = 1, \ldots, \ell(\gamma), \tag{3.12c}$$

$$K_{\delta_i}^\top \dot{x}_{o[i]}(t) = L_{\delta_i}^\top x_{o[i]}(t) \qquad \text{for } i = 1, \ldots, \ell(\delta), \tag{3.12d}$$

$$N_{\kappa_i} \dot{x}_{f[i]}(t) = x_c(t) \qquad \text{for } i = 1, \ldots, \ell(\alpha) \tag{3.12e}$$

$$\dot{x}_{\overline{c}}(t) = A_{\overline{c}} x_{\overline{c}}(t). \tag{3.12f}$$

◇

Note that by Remark 3.5 the form (3.11) is a normal form. However, if we apply an additional state space transformation to the block $[I_{n_{\overline{c}}}, A_{\overline{c}}, 0]$ which puts $A_{\overline{c}}$ into Jordan canonical form, and then prescribe the order of the blocks of each type, e.g. from largest dimension to lowest (what would mean $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{\ell(\alpha)}$ for α for instance), then (3.11) becomes a canonical form.

Remark 3.11 (DAEs corresponding to the blocks in the feedback form)**.** The form in Theorem 3.10 again leads to the separate consideration of the differential-algebraic equations (3.12a)-(3.12f):

a) (3.12a) is given by $[I_{\alpha_i}, N_{\alpha_i}^\top, e_{\alpha_i}^{[\alpha_i]}]$, and is completely controllable by the classical results for ODE systems (see e.g. [136, Sec. 3.2]). This system has furthermore the properties of being R-controllable, and both controllable and stabilizable in the behavioral sense.

b) (3.12b) corresponds to an under-determined system with zero dimensional input space. Since $x_{u[i]}$ satisfies (3.12b) if, and only if, there exists some $v_i \in \mathcal{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R})$ with

$$\dot{x}_{u[i]}(t) = \quad N_{\beta_i}^\top x_{u[i]}(t) + e_{\beta_i}^{[\beta_i]}\, v_i(t),$$

this system has the same properties as (3.12a).

c) Denoting

$$z_{ob[i]} = \begin{bmatrix} x_{ob[i]} \\ u_{ob[i]} \end{bmatrix},$$

then (3.12c) can be rewritten as

$$N_{\gamma_i}\dot{z}_{ob[i]}(t) = z_{ob[i]}(t),$$

which has, by b) in Remark 3.7, the unique solution $z_{ob[i]} = 0$. Hence,

$$\mathfrak{B}_{[L_{\gamma_i}^\top, K_{\gamma_i}^\top, e_{\gamma_i}^{[\gamma_i]}]} = \{0\}.$$

The system $[L_{\gamma_i}^\top, K_{\gamma_i}^\top, e_{\gamma_i}^{[\gamma_i]}]$ is therefore completely controllable if, and only if, $\gamma_i = 1$. In the case where $\gamma_i > 1$, this system is not even impulse controllable. However, independent of γ_i, $[L_{\gamma_i}^\top, K_{\gamma_i}^\top, e_{\gamma_i}^{[\gamma_i]}]$ is R-controllable, and both controllable and stabilizable in the behavioral sense.

d) Again, there holds

$$\mathfrak{B}_{[K_{\delta_i}^\top, L_{\delta_i}^\top, 0_{\delta_i,0}]} = \{0\},$$

whence, in dependence on δ_i, we can infer the same properties as in c).

e) Due to

$$\mathfrak{B}_{[N_{\kappa_i}, I_{\kappa_i}, 0_{\kappa_i,0}]} = \{0\},$$

the system $[N_{\kappa_i}, I_{\kappa_i}, 0_{\kappa_i,0}]$ is never controllable at infinity, but always R-controllable and both controllable and stabilizable in the behavioral sense. $[N_{\kappa_i}, I_{\kappa_i}, 0_{\kappa_i,0}]$ is strongly controllable if, and only if, $\kappa_i = 1$.

f) The system $[I_{n_{\overline{c}}}, A_{\overline{c}}, 0_{\overline{c},0}]$ satisfies

$$\mathfrak{B}_{[I_{n_{\overline{c}}}, A_{\overline{c}}, 0_{n_{\overline{c}},0}]} = \left\{ \; e^{A_{\overline{c}}\cdot} x^0 \;\middle|\; x^0 \in \mathbb{R}^{n_{\overline{c}}} \; \right\},$$

whence it is controllable at infinity, but neither strongly controllable nor controllable in the behavioral sense nor R-controllable. The properties of being completely stabilizable and stabilizable in the behavioral sense are attained if, and only if, $\sigma(A_{\overline{c}}) \subseteq \mathbb{C}_-$.

By using the implications shown in Proposition 2.6, we can deduce the following for the systems arising in the feedback form:

	$[I_{\alpha_i}, N_{\alpha_i}^\top, e_{\alpha_i}^{[\alpha_i]}]$	$[K_{\beta_i}, L_{\beta_i}, 0_{\alpha_i,0}]$	$[L_{\gamma_i}^\top, K_{\gamma_i}^\top, e_{\gamma_i}^{[\gamma_i]}]$	$[K_{\delta_i}^\top, L_{\delta_i}^\top, 0_{\delta_i,0}]$	$[N_{\kappa_i}, I_{\kappa_i}, 0_{\kappa_i,0}]$	$[I_{n_{\overline{c}}}, A_{\overline{c}}, 0_{\overline{c},0}]$
controllable at infinity	✓	✓	$\Leftrightarrow \gamma_i = 1$	$\Leftrightarrow \delta_i = 1$	✗	✓
impulse controllable	✓	✓	$\Leftrightarrow \gamma_i = 1$	$\Leftrightarrow \delta_i = 1$	$\Leftrightarrow \kappa_i = 1$	✓
completely controllable	✓	✓	$\Leftrightarrow \gamma_i = 1$	$\Leftrightarrow \delta_i = 1$	✗	✗
completely reachable	✓	✓	$\Leftrightarrow \gamma_i = 1$	$\Leftrightarrow \delta_i = 1$	✗	✗
strongly controllable	✓	✓	$\Leftrightarrow \gamma_i = 1$	$\Leftrightarrow \delta_i = 1$	$\Leftrightarrow \kappa_i = 1$	✗
strongly reachable	✓	✓	$\Leftrightarrow \gamma_i = 1$	$\Leftrightarrow \delta_i = 1$	$\Leftrightarrow \kappa_i = 1$	✗
completely stabilizable	✓	✓	$\Leftrightarrow \gamma_i = 1$	$\Leftrightarrow \delta_i = 1$	✗	$\Leftrightarrow \sigma(A_{\overline{c}}) \subseteq \mathbb{C}_-$
strongly stabilizable	✓	✓	$\Leftrightarrow \gamma_i = 1$	$\Leftrightarrow \delta_i = 1$	$\Leftrightarrow \kappa_i = 1$	$\Leftrightarrow \sigma(A_{\overline{c}}) \subseteq \mathbb{C}_-$
R-controllable	✓	✓	✓	✓	✓	✗
controllable in the behavioral sense	✓	✓	✓	✓	✓	✗
stabilizable in the behavioral sense	✓	✓	✓	✓	✓	$\Leftrightarrow \sigma(A_{\overline{c}}) \subseteq \mathbb{C}_-$

Using the characterizations of the subsystems we may deduce some statements about the overall system.

Corollary 3.12. *A system $[E, A, B] \in \Sigma_{k,n,m}$ with feedback form* (3.11) *is*

(i) *controllable at infinity if, and only if, $\gamma = (1, \dots, 1)$, $\delta = (1, \dots, 1)$ and $\ell(\kappa) = 0$;*

(ii) *impulse controllable if, and only if, $\gamma = (1, \dots, 1)$, $\delta = (1, \dots, 1)$ and $\kappa = (1, \dots, 1)$;*

(iii) *strongly controllable (and thus also strongly reachable) if, and only if, $\gamma = (1, \dots, 1)$, $\delta = (1, \dots, 1)$, $\kappa = (1, \dots, 1)$ and $n_{\overline{c}} = 0$;*

(iv) *completely controllable (and thus also completely reachable) if, and only if, $\gamma = (1, \dots, 1)$, $\delta = (1, \dots, 1)$ and $\ell(\kappa) = n_{\overline{c}} = 0$;*

(v) *R-controllable if, and only if, $n_{\overline{c}} = 0$;*

(vi) *controllable in the behavioral sense if, and only if, $n_{\overline{c}} = 0$;*

(vii) *strongly stabilizable if, and only if,* $\gamma = (1, \dots, 1)$, $\delta = (1, \dots, 1)$, $\ell(\kappa) = 0$, *and* $\sigma(A_{\overline{c}}) \subseteq \mathbb{C}_-$*;*

(viii) *completely stabilizable if and only if,* $\gamma = (1, \dots, 1)$, $\delta = (1, \dots, 1)$, $\kappa = (1, \dots, 1)$, *and* $\sigma(A_{\overline{c}}) \subseteq \mathbb{C}_-$*;*

(ix) *stabilizable in the behavioral sense if, and only if,* $\sigma(A_{\overline{c}}) \subseteq \mathbb{C}_-$*.*

Remark 3.13 (Derivative feedback)**.** A canonical form under proportional and derivative feedback (PD feedback) was derived in [98] as well (note that PD feedback defines an equivalence relation on $\Sigma_{k,n,m}$). The main tool for doing this is the restriction pencil (see Remark 2.5(xi)): Clearly, the system

$$\begin{aligned} NE\dot{x} &= NAx, \\ u &= B^{\dagger}(E\dot{x} - Ax) \end{aligned}$$

is equivalent, via PD feedback, to the system

$$\begin{aligned} NE\dot{x} &= NAx, \\ u &= 0. \end{aligned}$$

Then putting $sNE - NA$ into Kronecker canonical form yields a PD canonical form for the DAE system with a 5×4-block structure.

We may, however, directly derive this PD canonical form from the normal form (3.11). To this end we may observe that the system $[I_{\alpha_i}, N_{\alpha_i}^\top, e_{\alpha_i}^{[\alpha_i]}]$ can be written as

$$K_{\alpha_i}\dot{x}_{c[i]}(t) = L_{\alpha_i}x_{c[i]}(t), \quad \dot{x}_{c[i],\alpha_i}(t) = u_{c[i]}(t),$$

and hence is, via PD feedback, equivalent to the system $\left[\begin{bmatrix} K_{\alpha_i} \\ 0 \end{bmatrix}, \begin{bmatrix} L_{\alpha_i} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$. On the other hand, the system $[L_{\gamma_i}^\top, K_{\gamma_i}^\top, e_{\gamma_i}^{[\gamma_i]}]$ can be written as

$$N_{\gamma_i-1}\dot{x}_{ob[i]}(t) = x_{ob[i]}(t), \quad \dot{x}_{ob[i],\gamma_{i-1}}(t) = u_{ob[i]}(t),$$

and hence is, via PD feedback, equivalent to the system $\left[\begin{bmatrix} N_{\gamma_i-1} \\ 0 \end{bmatrix}, \begin{bmatrix} I_{\gamma_i-1} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$. A canonical form for $[E, A, B] \in \Sigma_{k,n,m}$ under PD feedback is therefore given by

$$[E, A, B] \sim_{PD} \left[\begin{bmatrix} K_\beta & 0 & 0 & 0 \\ 0 & K_\delta^\top & 0 & 0 \\ 0 & 0 & N_\kappa & 0 \\ 0 & 0 & 0 & I_{n_{\overline{c}}} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} L_\beta & 0 & 0 & 0 \\ 0 & L_\delta^\top & 0 & 0 \\ 0 & 0 & I_{|\kappa|} & 0 \\ 0 & 0 & 0 & A_{\overline{c}} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I_\zeta & 0 \end{bmatrix}\right],$$

where $A_{\overline{c}}$ is in Jordan canonical form, and the blocks of each type are ordered from largest dimension to lowest.

Note that the properties of complete controllability, controllability at infinity and controllability in the behavioral sense are invariant under PD feedback. However, since derivative feedback changes the set of differential variables, the properties of strong controllability as well as impulse controllability may be lost/gained after PD feedback. $\diamond$

Remark 3.14 (Connection to Kronecker form)**.** We may observe from (3.2) that feedback transformation may be alternatively considered as a transformation of the extended pencil

$$s\mathcal{E} - \mathcal{A} = \begin{bmatrix} sE - A, & -B \end{bmatrix}, \tag{3.13}$$

that is based on a multiplication from the left by $\mathcal{W} = W \in \mathbf{Gl}_k(\mathbb{R})$, and from the right by

$$\mathcal{T} = \begin{bmatrix} T & 0 \\ F & V \end{bmatrix} \in \mathbf{Gl}_{n+m}(\mathbb{R}).$$

This equivalence is therefore a subclass of the class which is induced by the pre- and post-multiplication of $s\mathcal{E} - \mathcal{A}$ by arbitrary invertible matrices. Loosely speaking, one can hence expect a normal form under feedback equivalence which specializes the quasi-Kronecker form of $s\mathcal{E} - \mathcal{A}$. Indeed, the latter form may be obtained from the feedback form of $[E, A, B]$ by several simple row transformations $s\mathcal{E} - \mathcal{A}$ which are not interpretable as feedback group actions anymore. More precise, simple permutations of columns lead to the separate consideration of the extended pencils corresponding to the systems (3.12a)-(3.12f): The extended pencils corresponding to $[I_{\alpha_i}, N_{\alpha_i}^\top, e_{\alpha_i}^{[\alpha_i]}]$ and $[K_{\beta_i}, L_{\beta_i}, 0_{\alpha_i,0}]$ are $sK_{\alpha_i} - L_{\alpha_i}$ and $sK_{\beta_i} - L_{\beta_i}$, resp. The extended matrix pencil corresponding to the system $[L_{\gamma_i}^\top, K_{\gamma_i}^\top, e_{\gamma_i}^{[\gamma_i]}]$ is given by $sN_{\gamma_i} - I_{\gamma_i}$. The extended matrix pencils corresponding to the systems $[K_{\delta_i}^\top, L_{\delta_i}^\top, 0_{\delta_i,0}]$, $[N_{\kappa_i}, I_{\kappa_i}, 0_{\kappa_i,0}]$ and $[I_{n_{\overline{c}}}, A_{\overline{c}}, 0_{\overline{c},0}]$ are obviously given by $sK_{\delta_i}^\top - L_{\delta_i}^\top$, $sN_{\kappa_i} - I_{\kappa_i}$ and $sI_{n_{\overline{c}}} - A_{\overline{c}}$, respectively. In particular, $\lambda \in \mathbb{C}$ is a generalized eigenvalue of $s\mathcal{E} - \mathcal{A}$, if, and only if, $\lambda \in \sigma(A_{\overline{c}})$. $\diamond$

4 Algebraic criteria

In this section we derive equivalent criteria on the matrices $E, A \in \mathbb{R}^{k,n}$, $B \in \mathbb{R}^{k,m}$ for the controllability and stabilizability concepts of Definition 2.4. The criteria are generalizations of the Hautus test (also called Popov-Belevitch-Hautus test, since independently developed by Popov [122], Belevitch [16] and Hautus [63]) in terms of rank criteria on the involved matrices. Note that these conditions are not new - we refer to the relevant literature. However, we provide new proofs using only the feedback normal form (3.11).
First we show that certain rank criteria on the matrices involved in control systems are invariant under feedback equivalence. After that, we relate these rank criteria to the feedback form (3.11).

Lemma 4.1. *Let $[E_1, A_1, B_1], [E_2, A_2, B_2] \in \Sigma_{k,n,m}$ be given such that for $W \in \mathbf{Gl}_k(\mathbb{R})$, $T \in \mathbf{Gl}_n(\mathbb{R})$, $V \in \mathbf{Gl}_m(\mathbb{R})$ and $F \in \mathbb{R}^{m,n}$, there holds*

$$[\, E_1\, ,\, A_1\, ,\, B_1\,] \overset{W,T,V,F}{\sim_f} [\, E_2\, ,\, A_2\, ,\, B_2\,].$$

Then

$$\begin{aligned}
\operatorname{im}_{\mathbb{R}} E_1 + \operatorname{im}_{\mathbb{R}} A_1 + \operatorname{im}_{\mathbb{R}} B_1 &= W \cdot (\operatorname{im}_{\mathbb{R}} E_2 + \operatorname{im}_{\mathbb{R}} A_2 + \operatorname{im}_{\mathbb{R}} B_2)\,, \\
\operatorname{im}_{\mathbb{R}} E_1 + A_1 \cdot \ker_{\mathbb{R}} E_1 + \operatorname{im}_{\mathbb{R}} B_1 &= W \cdot (\operatorname{im}_{\mathbb{R}} E_2 + A_2 \cdot \ker_{\mathbb{R}} E_2 + \operatorname{im}_{\mathbb{R}} B_2)\,, \\
\operatorname{im}_{\mathbb{R}} E_1 + \operatorname{im}_{\mathbb{R}} B_1 &= W \cdot (\operatorname{im}_{\mathbb{R}} E_2 + \operatorname{im}_{\mathbb{R}} B_2)\,, \\
\operatorname{im}_{\mathbb{C}} (\lambda E_1 - A_1) + \operatorname{im}_{\mathbb{C}} B_1 &= W \cdot (\operatorname{im}_{\mathbb{C}} (\lambda E_2 - A_2) + \operatorname{im}_{\mathbb{C}} B_2) \quad \textit{for all } \lambda \in \mathbb{C}, \\
\operatorname{im}_{\mathbb{R}(s)} (sE_1 - A_1) + \operatorname{im}_{\mathbb{R}(s)} B_1 &= W \cdot \big(\operatorname{im}_{\mathbb{R}(s)} (sE_2 - A_2) + \operatorname{im}_{\mathbb{R}(s)} B_2\big)\,.
\end{aligned}$$

Proof: Immediate from (3.2). $\square$

Lemma 4.2 (Algebraic criteria via feedback form)**.** *For a system $[E, A, B] \in \Sigma_{k,n,m}$ with feedback form* (3.11) *the following statements hold true:*

a)

$$\begin{aligned} &\operatorname{im}_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} A + \operatorname{im}_{\mathbb{R}} B = \operatorname{im}_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} B \\ \iff\ &\gamma = (1,\ldots,1),\ \delta = (1,\ldots,1),\ \ell(\kappa) = 0. \end{aligned}$$

b)

$$\begin{aligned} &\operatorname{im}_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} A + \operatorname{im}_{\mathbb{R}} B = \operatorname{im}_{\mathbb{R}} E + A \cdot \ker_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} B \\ \iff\ &\gamma = (1,\ldots,1),\ \delta = (1,\ldots,1),\ \kappa = (1,\ldots,1). \end{aligned}$$

c)

$$\begin{aligned} &\operatorname{im}_{\mathbb{C}} E + \operatorname{im}_{\mathbb{C}} A + \operatorname{im}_{\mathbb{R}} B = \operatorname{im}_{\mathbb{C}}(\lambda E - A) + \operatorname{im}_{\mathbb{C}} B \\ \iff\ &\delta = (1,\ldots,1),\ \lambda \notin \sigma(A_{\overline{c}}). \end{aligned}$$

d) For $\lambda \in \mathbb{C}$ we have

$$\begin{aligned} &\dim\big(\operatorname{im}_{\mathbb{R}(s)}(sE - A) + \operatorname{im}_{\mathbb{R}(s)} B\big) = \dim(\operatorname{im}_{\mathbb{C}}(\lambda E - A) + \operatorname{im}_{\mathbb{C}} B \\ \iff\ &\lambda \notin \sigma(A_{\overline{c}}). \end{aligned}$$

Proof: It is, by Lemma 4.1, no loss of generality to assume that $[E, A, B]$ is already in feedback normal form. The results then follow by a simple verification of the above statements by means of the feedback form. □

Combining Lemmas 4.1 and 4.2 with Corollary 3.12, we may deduce the following criteria for the controllability and stabilizability concepts introduced in Definition 2.4.

Corollary 4.3 (Algebraic criteria for controllability/stabilizability)**.** *Let a system $[E, A, B] \in \Sigma_{k,n,m}$ be given.*

(i) $[E, A, B]$ *is controllable at infinity if, and only if,*

$$\operatorname{im}_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} A + \operatorname{im}_{\mathbb{R}} B = \operatorname{im}_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} B.$$

(ii) $[E, A, B]$ *is impulse controllable, if, and only if,*

$$\operatorname{im}_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} A + \operatorname{im}_{\mathbb{R}} B = \operatorname{im}_{\mathbb{R}} E + A \cdot \ker_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} B.$$

(iii) $[E, A, B]$ *is completely controllable if, and only if,*

$$\begin{aligned} &\operatorname{im}_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} A + \operatorname{im}_{\mathbb{R}} B = \operatorname{im}_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} B \\ \wedge\ &\operatorname{im}_{\mathbb{C}} E + \operatorname{im}_{\mathbb{C}} A + \operatorname{im}_{\mathbb{C}} B = \operatorname{im}_{\mathbb{C}}(\lambda E - A) + \operatorname{im}_{\mathbb{C}} B \quad \forall\, \lambda \in \mathbb{C}. \end{aligned}$$

(iv) $[E, A, B]$ *is strongly controllable if, and only if,*

$$\begin{aligned} &\operatorname{im}_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} A + \operatorname{im}_{\mathbb{R}} B = A \cdot \ker_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} B \\ \wedge\ &\operatorname{im}_{\mathbb{C}} E + \operatorname{im}_{\mathbb{C}} A + \operatorname{im}_{\mathbb{C}} B = \operatorname{im}_{\mathbb{C}}(\lambda E - A) + \operatorname{im}_{\mathbb{C}} B \quad \forall\, \lambda \in \mathbb{C}. \end{aligned}$$

(v) $[E, A, B]$ *is completely stabilizable if, and only if,*

$$\begin{aligned} &\operatorname{im}_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} A + \operatorname{im}_{\mathbb{R}} B = \operatorname{im}_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} B \\ \wedge\ &\operatorname{im}_{\mathbb{C}} E + \operatorname{im}_{\mathbb{C}} A + \operatorname{im}_{\mathbb{C}} B = \operatorname{im}_{\mathbb{C}}(\lambda E - A) + \operatorname{im}_{\mathbb{C}} B \quad \forall\, \lambda \in \overline{\mathbb{C}_+}. \end{aligned}$$

(vi) $[E, A, B]$ *is strongly stabilizable if, and only if,*

$$\operatorname{im}_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} A + \operatorname{im}_{\mathbb{R}} B = \operatorname{im}_{\mathbb{R}} E + A \cdot \ker_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} B$$
$$\wedge\ \operatorname{im}_{\mathbb{C}} E + \operatorname{im}_{\mathbb{C}} A + \operatorname{im}_{\mathbb{C}} B = \operatorname{im}_{\mathbb{C}}(\lambda E - A) + \operatorname{im}_{\mathbb{C}} B \quad \forall\, \lambda \in \overline{\mathbb{C}_+}.$$

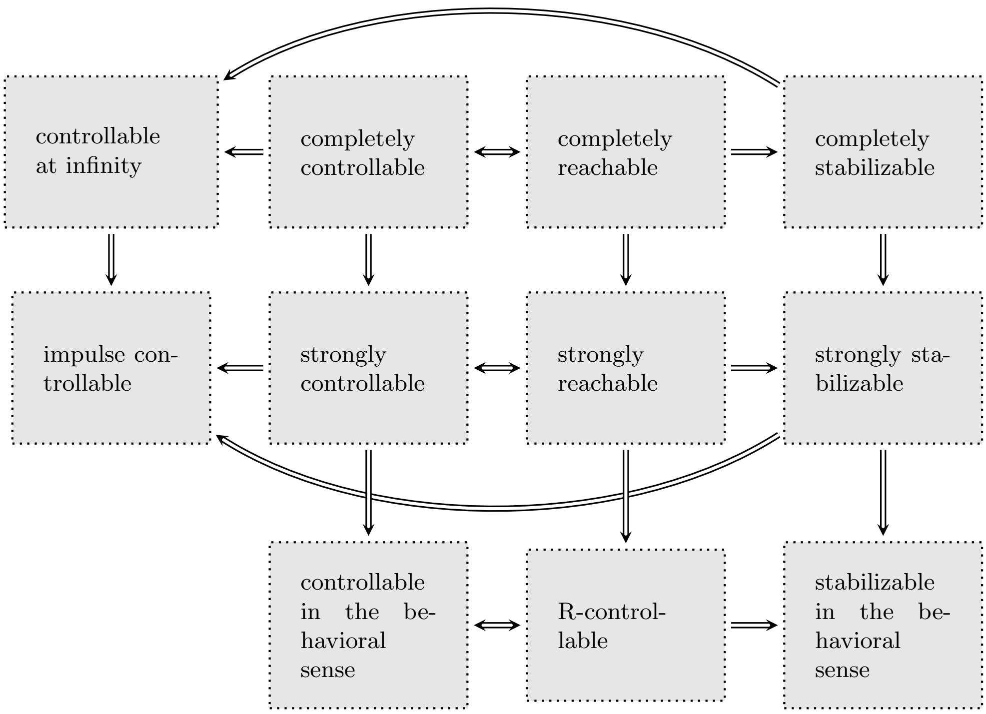
(vii) $[E, A, B]$ *is controllable in the behavioral sense if, and only if,*

$$\operatorname{rk}_{\mathbb{R}(s)}[sE - A, B] = \operatorname{rk}_{\mathbb{C}}[\lambda E - A, B] \quad \forall \lambda \in \mathbb{C}.$$

(viii) $[E, A, B]$ *is stabilizable in the behavioral sense if, and only if,*

$$\operatorname{rk}_{\mathbb{R}(s)}[sE - A, B] = \operatorname{rk}_{\mathbb{C}}[\lambda E - A, B] \quad \forall \lambda \in \overline{\mathbb{C}_+}.$$

The above result leads to the following extension of the diagram in Proposition 2.6. Note that the equivalence of R-controllability and controllability in the behavioral sense was already shown in Corollary 3.12.



In the following we will consider further criteria for the concepts introduced in Definition 2.4.

Remark 4.4 (Controllability at infinity)**.** Corollary 4.3(i) immediately implies that controllability at infinity is equivalent to

$$\operatorname{im}_{\mathbb{R}} A \subseteq \operatorname{im}_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} B.$$

In terms of a rank criterion, this is the same as

$$\operatorname{rk}_{\mathbb{R}}[E, A, B] = \operatorname{rk}_{\mathbb{R}}[E, B]. \tag{4.1}$$

Criterion (4.1) has first been derived by Geerts [57, Thm. 4.5] for the case $\operatorname{rk}[E, A, B] = k$, although he does not use the name “controllability at infinity”.

In the case of regular $sE - A \in \mathbb{R}[s]^{n,n}$, condition (4.1) reduces to

$$\operatorname{rk}_{\mathbb{R}}[E, B] = n. \tag{4.2}$$

$\diamond$

Remark 4.5 (Impulse controllability)**.** By Corollary 4.3(ii), impulse controllability of $[E, A, B] \in \Sigma_{k,n,m}$ is equivalent to

$$\operatorname{im}_{\mathbb{R}} A \subseteq \operatorname{im}_{\mathbb{R}} E + A \cdot \ker_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} B.$$

Another equivalent characterization is that, for one (and hence any) matrix Z with $\operatorname{im}_{\mathbb{R}}(Z) = \ker_{\mathbb{R}}(E)$, there holds

$$\operatorname{rk}_{\mathbb{R}}[E, A, B] = \operatorname{rk}_{\mathbb{R}}[E, AZ, B]. \tag{4.3}$$

This has first been derived by Geerts [57, Rem. 4.9], again for the case $\operatorname{rk}[E, A, B] = k$. In [69, Thm. 3] and [66] it has been obtained that impulse controllability is equivalent to

$$\operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = \operatorname{rk}_{\mathbb{R}}[E, A, B] + \operatorname{rk}_{\mathbb{R}} E,$$

which is in fact equivalent to (4.3). It has also been shown in [69, p. 1] that impulse controllability is equivalent to

$$\operatorname{rk}_{\mathbb{R}(s)}(s\mathcal{E} - \mathcal{A}) = \operatorname{rk}_{\mathbb{R}}[E, A, B].$$

This criterion can be alternatively shown by using the feedback form (3.11). Using condition (3.6) we may also infer that this is equivalent to the index of the extended pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{R}[s]^{k,n+m}$ being at most one.
If the pencil $sE - A$ is regular, then condition (4.3) reduces to

$$\operatorname{rk}_{\mathbb{R}}[E, AZ, B] = n. \tag{4.4}$$

This condition can be also inferred from [46, Th. 2-2.3]. $\diamond$

Remark 4.6 (Controllability in the behavioral sense and R-controllability)**.** The concepts of controllability in the behavioral sense and R-controllability are equivalent by Corollary 3.12. The algebraic criterion in Corollary 4.3(vii) is equivalent to the extended matrix pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{R}[s]^{k,n+m}$ having no generalized eigenvalues, or, equivalently, in the feedback form (3.11) it holds $n_{\overline{c}} = 0$.
The criterion in (vii) is shown in [121, Thm. 5.2.10] for the larger class of linear differential behaviors. R-controllability for systems with regular $sE-A$ was considered in [46, Thm. 2-2.2], where the condition

$$\operatorname{rk}_{\mathbb{R}}[\lambda E - A, B] = n \ \ \forall \lambda \in \mathbb{C} \tag{4.5}$$

was derived. This is, for regular $sE - A$, in fact equivalent to Corollary 4.3(viii) $\diamond$

Remark 4.7 (Complete controllability and strong controllability)**.** By Corollary 4.3, complete controllability of $[E, A, B] \in \Sigma_{k,n,m}$ is equivalent to $[E, A, B]$ being R-controllable and controllable at infinity, whereas strong controllability of $[E, A, B] \in \Sigma_{k,n,m}$ is equivalent to $[E, A, B]$ being R-controllable and impulse controllable.
Banaszuk et al. [11] already obtained the condition in Corollary 4.3(vi) for complete controllability considering discrete systems. Complete controllability is called $\mathcal{H}$-controllability in [11]. Recently, Zubova [155] considered full controllability, which is just complete controllability with the additional assumption that solutions have to be unique, and obtained three equivalent criteria [155, Sec. 7], where the first one characterizes the uniqueness and the other two are equivalent to the condition in Corollary 4.3(vi).
For regular systems, the conditions in Corollary 4.3 for complete and strong controllability are also derived in [46, Thm. 2-2.1 & Thm. 2-2.3]. $\diamond$

Remark 4.8 (Stabilizability)**.** By Corollary 4.3, complete stabilizability of $[E, A, B] \in \Sigma_{k,n,m}$ is equivalent to $[E, A, B]$ being stabilizable in the behavioral sense and controllable at infinity, whereas strong stabilizability of $[E, A, B] \in \Sigma_{k,n,m}$ is equivalent to $[E, A, B]$ being stabilizable in the behavioral sense and impulse controllable.
The criterion in (viii) for stabilizability in the behavioral sense is shown in [121, Thm. 5.2.30] for the class of linear differential behaviors. ◇

Remark 4.9 (Kalman criterion for regular systems)**.** For regular systems $[E, A, B] \in \Sigma_{n,n,m}$ with $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$ the usual Hautus and Kalman criteria can be found in a summarized form e.g. in [46]. Other approaches to derive controllability criteria rely on the expansion of $(sE - A)^{-1}$ as a power series in s, which is only feasible in the regular case. For instance, in [108] the numerator matrices of this expansion, i.e., the coefficients of the polynomial $\operatorname{adj}(sE - A)$, are used to derive a rank criterion for complete controllability. Then again, in [84] Kalman rank criteria for complete controllability, R-controllability and controllability at infinity are derived in terms of the coefficients of the power series expansion of $(sE - A)^{-1}$. The advantage of these criteria, especially the last one, is that no transformation of the system needs to be performed as it is usually necessary in order to derive Kalman rank criteria for DAEs, see e.g. [46].
However, simple criteria can be obtained using only a left transformation of little impact: if $\alpha \in \mathbb{R}$ is chosen such that $\det(\alpha E - A) \neq 0$ then the system is complete controllable if, and only if, [154, Cor. 1]

$$\operatorname{rk}_{\mathbb{R}} \Big[(\alpha E - A)^{-1}B, \big((\alpha E - A)^{-1}E\big)(\alpha E - A)^{-1}B, \ldots, \big((\alpha E - A)^{-1}E\big)^{n-1}(\alpha E - A)^{-1}B\Big] = n,$$

and it is impulse controllable if, and only if, [154, Thm. 2]

$$\operatorname{im}_{\mathbb{R}}(\alpha E - A)^{-1}E + \ker(\alpha E - A)^{-1}E + \operatorname{im}_{\mathbb{R}}(\alpha E - A)^{-1}B = \mathbb{R}^n.$$

The result concerning complete controllability has also been obtained in [39, Thm. 4.1] for the case $A = I$ and $\alpha = 0$.
Yet another approach was followed by Kučera and Zagalak [88] who introduced controllability indices and characterized strong controllability in terms of an equation for these indices. ◇

5 Feedback, stability and autonomous systems

State feedback is, roughly speaking, the special choice of the input being a function of the state. Due to the mutual dependence of state and input in a feedback system, this is often referred to as *closed-loop control.* In the linear case, feedback is the imposition of the additional relation $u(t) = Fx(t)$ for some $F \in \mathbb{R}^{m,n}$. This results in the system

$$E\dot{x}(t) = (A + BF)x(t). \tag{5.1}$$

Feedback for linear ODE systems was first studied by Wonham [150], where it is shown that controllability of $[I, A, B] \in \Sigma_{n,n,m}$ is equivalent to any set $\Lambda \subseteq \mathbb{C}$ which has at most n elements and is symmetric with respect to the imaginary axis (that is, $\lambda \in \Lambda \Leftrightarrow \overline{\lambda} \in \Lambda$) being achievable by a suitable feedback, i.e., there exists some $F \in \mathbb{R}^{m,n}$ with the property that $\sigma(A + BF) = \Gamma$. In particular, the input may be chosen in a way that the closed-loop system is stable, i.e., any state trajectory tends to zero. Using the *Kalman decomposition* [76] (see also Section 7), it can be shown for ODE systems that stabilizability is equivalent to the existence of a feedback such that the resulting system is stable.
These results have been generalized to regular DAE systems by Cobb [41], see also [46, 53, 95, 96, 114, 116]. Note that, for DAE systems, not only the problem of assignment of eigenvalues occurs, but also the index may be changed by imposing feedback.
The crucial ingredient for the treatment of DAE systems with non-regular pencil $sE - A$ will be the feedback form by Loiseau et al. [98] (see Thm. 3.10).

5.1 Autonomy and stability

The feedback law $u(t) = Fx(t)$ results in a DAE in which the input is completely eliminated. We now focus on DAEs without input, and we introduce several properties and concepts. For matrices $E, A \in \mathbb{R}^{k,n}$, consider a DAE

$$E\dot{x}(t) = Ax(t). \tag{5.2}$$

Its *behavior* is given by

$$\mathfrak{B}_{[E,A]} := \left\{ x \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \;\middle|\; x \text{ satisfies (5.2) for almost all } t \in \mathbb{R} \right\}.$$

Definition 5.1 (Stability concepts for DAEs, autonomous DAEs)**.** A linear time-invariant DAE $[E, A] \in \Sigma_{k,n}$ is called

a) *completely stable*, if for all $x^0 \in \mathbb{R}^n$, there exists some $x \in \mathfrak{B}_{[E,A]}$ with $x(0) = x^0$ and

$$\lim_{t\to\infty} x(t) = 0.$$

b) *strongly stable*, if for all $x^0 \in \mathbb{R}^n$, there exists some $x \in \mathfrak{B}_{[E,A]}$ with $Ex(0) = Ex^0$ and

$$\lim_{t\to\infty} x(t) = 0.$$

c) *stable in the behavioral sense*, if for all $x \in \mathcal{B}_{[E,A]}$, there exists some $x_0 \in \mathcal{B}_{[E,A]}$, such that

$$x(t) = x_0(t) \text{ for all } t < 0,$$

and

$$\lim_{t\to\infty} x_0(t) = 0.$$

d) *autonomous*, if for all $x_1, x_2 \in \mathfrak{B}_{[E,A]}$ with

$$x_1(t) = x_2(t) \quad \text{for all } t < 0,$$

there holds

$$x_1(t) = x_2(t) \quad \text{for all } t \in \mathbb{R}.$$

$\diamond$

Remark 5.2 (Stable and autonomous DAEs)**.** In Definition 5.1 we have defined complete stability as the property of existence of a solution which tends to zero for any given initial values. Whereas the term "complete" rather suggests that any possible solution should converge to zero, it was our practice throughout this article to relate this term to the consistency of any initial value, as it is the case in Definition 5.1. Of course, due to possible underdetermined β-blocks, a completely stable DAE may have solutions which grow unboundedly as well.
The notion of autonomy is introduced by Willems and Polderman in [121, Sec. 3.2] for general behaviors. For DAE systems $E\dot{x}(t) = Ax(t)$ we can further conclude that autonomy is equivalent to any $x \in \mathfrak{B}_{[E,A]}$ being uniquely determined by $x(0)$. This gives also rise to the fact that autonomy is equivalent to $\dim_{\mathbb{R}} \mathfrak{B}_{[E,A]} \le n$ which is, on the other hand, equivalent to $\dim_{\mathbb{R}} \mathfrak{B}_{[E,A]} < \infty$. Autonomy indeed means that the DAE is not underdetermined. $\diamond$

In regard of Remark 3.7 we can infer the following:

Corollary 5.3 (Stability criteria and quasi-Kronecker form)**.** *Let* $[E, A] \in \Sigma_{k,n}$ *be a linear time-invariant DAE and assume that the quasi-Kronecker form of* $sE - A$ *is given by* (3.4). *Then the following statements hold true:*

a) $[E, A]$ *is completely stable if, and only if,* $\ell(\alpha) = 0$, $\gamma = (1, \dots, 1)$ *and* $\sigma(A_s) \subseteq \mathbb{C}_-$.

b) $[E, A]$ *is strongly stable if, and only if,* $\alpha = (1, \dots, 1)$, $\gamma = (1, \dots, 1)$ *and* $\sigma(A_s) \subseteq \mathbb{C}_-$.

c) $[E, A]$ *is stable in the behavioral sense if, and only if,* $\sigma(A_s) \subseteq \mathbb{C}_-$.

d) $[E, A]$ *is autonomous if, and only if,* $\ell(\beta) = 0$.

The subsequent algebraic criteria for the previously defined notions of stability and autonomy can be inferred from Corollary 5.3 by using further arguments similar to the ones of Section 4.

Corollary 5.4 (Algebraic criteria for stability)**.** *Let* $[E, A] \in \Sigma_{k,n}$. *Then the following statements hold true:*

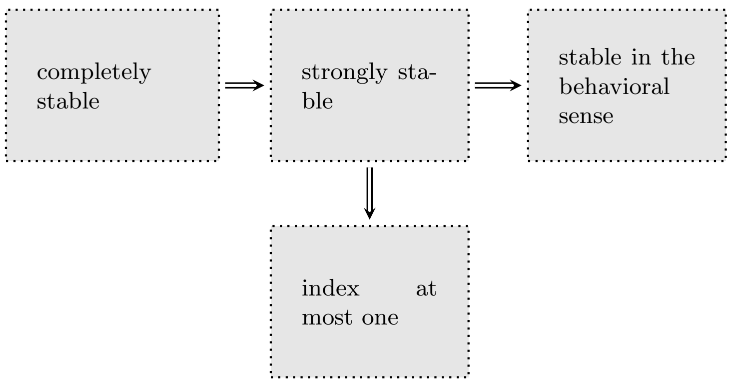
a) $[E, A]$ *is completely stable if, and only if,* $\operatorname{im}_{\mathbb{R}} A \subseteq \operatorname{im}_{\mathbb{R}} E$ *and* $\operatorname{rk}_{\mathbb{R}(s)}(sE - A) = \operatorname{rk}_{\mathbb{C}}(\lambda E - A)$ *for all* $\lambda \in \overline{\mathbb{C}_+}$.

b) $[E, A]$ *is strongly stable if, and only if,* $\operatorname{im}_{\mathbb{R}} A \subseteq \operatorname{im}_{\mathbb{R}} E + A \cdot \ker_{\mathbb{R}} E$ *and* $\operatorname{rk}_{\mathbb{R}(s)}(sE - A) = \operatorname{rk}_{\mathbb{C}}(\lambda E - A)$ *for all* $\lambda \in \overline{\mathbb{C}_+}$.

c) $[E, A]$ *is stable in the behavioral sense if, and only if,* $\operatorname{rk}_{\mathbb{R}(s)}(sE - A) = \operatorname{rk}_{\mathbb{C}}(\lambda E - A)$ *for all* $\lambda \in \overline{\mathbb{C}_+}$.

d) $[E, A]$ *is autonomous if, and only if,* $\ker_{\mathbb{R}(s)}(sE - A) = \{0\}$.

Corollary 5.4 leads to the following implications:



Remark 5.5. a) Strong stability implies that the index of $sE - A$ is at most one. In the case where the matrix $[E, A] \in \mathbb{R}^{k,2n}$ has full row rank, complete stability is sufficient for the index of $sE - A$ being zero.

On the other hand, behavioral stability of $[E, A]$ together with the index of $sE - A$ being not greater than one implies strong stability of $[E, A]$. Furthermore, for systems $[E, A] \in \Sigma_{k,n}$ with $\operatorname{rk}_{\mathbb{R}}[E, A] = k$, complete stability is equivalent to behavioral stability together with the property that the index of $sE - A$ is zero.

For ODEs the notions of complete stability, strong stability and stability in the behavioural sense are equivalent.

b) The behaviour of an autonomous system $[E, A]$ satisfies $\dim_{\mathbb{R}} \mathfrak{B}_{[E,A]} = n_s$, where n_s denotes the number of rows of the matrix A_s in the quasi-Kronecker form (3.4) of $sE - A$. Note that regularity of $sE - A$ is sufficient for autonomy of $[E, A]$.

c) Autonomy has been algebraically characterized for linear differential behaviours in [121, Sec. 3.2]. Statement d) in Corollary 5.4 can indeed be generalized to a larger class of linear differential equations. $\diamond$

5.2 Stabilization by feedback

A system $[E, A, B] \in \Sigma_{k,n,m}$ can, via state feedback with some $F \in \mathbb{R}^{m,n}$, be turned into a DAE $[E, A + BF] \in \Sigma_{k,n}$. We now present some properties of $[E, A + BF] \in \Sigma_{k,n}$ that can be achieved by a suitable feedback matrix $F \in \mathbb{R}^{m,n}$.

Theorem 5.6 (Stabilizing feedback)**.** *For a system $[E, A, B] \in \Sigma_{k,n,m}$ the following holds true:*

a) $[E, A, B]$ is impulse controllable if, and only if, there exists $F \in \mathbb{R}^{m,n}$ such that the index of $sE - (A + BF)$ is at most one.

b) $[E, A, B]$ is completely stabilizable if, and only if, there exists $F \in \mathbb{R}^{m,n}$ such that $[E, A + BF]$ is completely stable.

c) $[E, A, B]$ is strongly stabilizable if, and only if, there exists $F \in \mathbb{R}^{m,n}$ such that $[E, A + BF]$ is strongly stable.

Proof: a) Let $[E, A, B]$ be impulse controllable. Then $[E, A, B]$ can be put into feedback form (3.11), i.e., there exist $W \in \mathbf{Gl}_k(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R})$ and $\tilde{F} \in \mathbb{R}^{m,n}$ such that

$$
\begin{aligned}
&W(sE - (A + B\tilde{F}T^{-1})T \\
&= \begin{bmatrix}
sI_{|\alpha|} - N_\alpha^\top & 0 & 0 & 0 & 0 & 0 \\
0 & sK_\beta - L_\beta & 0 & 0 & 0 & 0 \\
0 & 0 & sL_\gamma^\top - K_\gamma^\top & 0 & 0 & 0 \\
0 & 0 & 0 & sK_\delta^\top - L_\delta^\top & 0 & 0 \\
0 & 0 & 0 & 0 & sN_\kappa - I_{|\kappa|} & 0 \\
0 & 0 & 0 & 0 & 0 & sI_{n_{\overline{c}}} - A_{\overline{c}}
\end{bmatrix}.
\end{aligned}
\tag{5.3}
$$

By Corollary 3.12(ii) the impulse controllability of $[E, A, B]$ implies that $\gamma = (1, \ldots, 1)$, $\delta = (1, \ldots, 1)$ and $\kappa = (1, \ldots, 1)$. Therefore, we have that, with $F = \tilde{F}T^{-1}$, the pencil $sE - (A + BF)$ has index at most one as the index is preserved under system equivalence.
Conversely, assume that $[E, A, B]$ is not impulse controllable. We show that for all $F \in \mathbb{R}^{m,n}$ the index of $sE-(A+BF)$ is greater than one. To this end, let $F \in \mathbb{R}^{m,n}$ and choose $W \in \mathbf{Gl}_k(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R})$ and $\tilde{F} \in \mathbb{R}^{m,n}$ such that (3.11) holds. Then, partitioning $V^{-1}FT = [F_{ij}]_{i=1,\ldots,3,j=1,\ldots,6}$ accordingly, we obtain

$$
\begin{aligned}
&s\tilde{E} - \tilde{A} := W(sE - (A + BF + B\tilde{F}T^{-1}))T = W(sE - (A + B\tilde{F}T^{-1}))T - WBVV^{-1}FT \\
&= \begin{bmatrix}
sI_{|\alpha|} - (N_\alpha^\top + E_\alpha F_{11}) & -E_\alpha F_{12} & -E_\alpha F_{13} & -E_\alpha F_{14} & -E_\alpha F_{15} & -E_\alpha F_{16} \\
0 & sK_\beta - L_\beta & 0 & 0 & 0 & 0 \\
-E_\gamma F_{21} & -E_\gamma F_{22} & sL_\gamma^\top - (K_\gamma^\top + E_\gamma F_{23}) & -E_\gamma F_{24} & -E_\gamma F_{25} & -E_\gamma F_{26} \\
0 & 0 & 0 & sK_\delta^\top - L_\delta^\top & 0 & 0 \\
0 & 0 & 0 & 0 & sN_\kappa - I_{|\kappa|} & 0 \\
0 & 0 & 0 & 0 & 0 & sI_{n_{\overline{c}}} - A_{\overline{c}}
\end{bmatrix}.
\end{aligned}
\tag{5.4}
$$

Now the assumption that $[E, A, B]$ is not impulse controllable leads to $\gamma \neq (1, \ldots, 1)$, $\delta \neq (1, \ldots, 1)$ or $\kappa \neq (1, \ldots, 1)$. We will now show that the index of $sE - (A + BF + B\tilde{F}T^{-1})$ is greater than one by showing this for the equivalent pencil in (5.4) via applying the condition in (3.6): Let Z be a real matrix with $\operatorname{im}_{\mathbb{R}} Z = \ker_{\mathbb{R}} \tilde{E}$. Then

$$Z = \begin{bmatrix} 0 & Z_1^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Z_2^\top & 0 \end{bmatrix}^\top,$$

where $\operatorname{im} Z_1 = \ker K_\beta = \operatorname{im} E_\beta$ and $\operatorname{im} Z_2 = \ker N_\kappa = \operatorname{im} E_\kappa$. Taking into account that $\operatorname{im}_{\mathbb{R}} E_\gamma \subseteq \operatorname{im}_{\mathbb{R}} L_\gamma^\top$, we obtain that

$$\operatorname{im}_{\mathbb{R}} \begin{bmatrix} 0_{|\alpha|-\ell(\alpha)+|\beta|-\ell(\beta),k} & I_{|\gamma|+|\delta|+|\kappa|} & 0_{k,n_{\overline{c}}} \end{bmatrix} \begin{bmatrix} \tilde{E} & \tilde{A}Z \end{bmatrix} = \operatorname{im}_{\mathbb{R}} \begin{bmatrix} L_\gamma^\top & 0 & 0 & E_\gamma F_{25} Z_2 \\ 0 & K_\delta^\top & 0 & 0 \\ 0 & 0 & N_\kappa & Z_2 \end{bmatrix}.$$

On the other hand, there holds

$$\begin{aligned} &\operatorname{im}_{\mathbb{R}} \begin{bmatrix} 0_{|\alpha|-\ell(\alpha)+|\beta|-\ell(\beta),k} & I_{|\gamma|+|\delta|+|\kappa|} & 0_{k,n_{\overline{c}}} \end{bmatrix} \begin{bmatrix} \tilde{E} & \tilde{A} \end{bmatrix} \\ &\qquad = \operatorname{im}_{\mathbb{R}} \begin{bmatrix} L_\gamma^\top & 0 & 0 & K_\gamma^\top + E_\gamma F_{23} & E_\gamma F_{24} & E_\gamma F_{25} \\ 0 & K_\delta^\top & 0 & 0 & L_\delta^\top & 0 \\ 0 & 0 & N_\kappa & 0 & 0 & I_{|\kappa|} \end{bmatrix}. \end{aligned}$$

Since the assumption that at least one of the multi-indices satisfies $\gamma \neq (1, \ldots, 1)$, $\delta \neq (1, \ldots, 1)$ or $\kappa \neq (1, \ldots, 1)$ and that $\operatorname{im} Z_2 = \operatorname{im} E_\kappa$ leads to

$$\operatorname{im}_{\mathbb{R}} \begin{bmatrix} L_\gamma^\top & 0 & 0 & E_\gamma F_{25} Z_2 \\ 0 & K_\delta^\top & 0 & 0 \\ 0 & 0 & N_\kappa & Z_2 \end{bmatrix} \subsetneq \operatorname{im}_{\mathbb{R}} \begin{bmatrix} L_\gamma^\top & 0 & 0 & K_\gamma^\top + E_\gamma F_{23} & E_\gamma F_{24} & E_\gamma F_{25} \\ 0 & K_\delta^\top & 0 & 0 & L_\delta^\top & 0 \\ 0 & 0 & N_\kappa & 0 & 0 & I_{|\kappa|} \end{bmatrix},$$

and thus

$$\operatorname{im}_{\mathbb{R}} \begin{bmatrix} \tilde{E} & \tilde{A}Z \end{bmatrix} \subsetneq \operatorname{im}_{\mathbb{R}} \begin{bmatrix} \tilde{E} & \tilde{A} \end{bmatrix},$$

whence, by condition (3.6), the index of $sE - (A + BF + B\tilde{F}T^{-1})$ has to be greater than one. Since F was chosen arbitrarily we may conclude that $sE - (A + BF)$ has index greater than one for all $F \in \mathbb{R}^{m,n}$, which completes the proof of a).

b) If $[E, A, B]$ is completely stabilizable, then we may transform the system into feedback form and obtain (5.3). Then Corollary 3.12(viii) implies $\gamma = (1, \ldots, 1)$, $\delta = (1, \ldots, 1)$, $\ell(\kappa) = 0$, and $\sigma(A_{\overline{c}}) \subseteq \mathbb{C}_-$. Further, by [136, Thm. 4.20], there exists some $F_{11} \in \mathbb{R}^{|\alpha|,\ell(\alpha)}$ such that $\sigma(N_\alpha + E_\alpha F_{11}) \subseteq \mathbb{C}_-$. Setting $\hat{F} := [F_{ij}]_{i=1,\ldots,3,j=1,\ldots,6}$ with $F_{ij} = 0$ for $i \neq 1$ or $j \neq 1$ we obtain that with $F = \tilde{F}T^{-1} + V\hat{F}T^{-1}$ the system $[E, A + BF]$ is completely stable by Corollary 5.3 a) as complete stability is preserved under system equivalence.
On the other hand, assume that $[E, A, B]$ is not completely stabilizable. We show that for all $F \in \mathbb{R}^{m,n}$ the system $[E, A + BF]$ is not completely stable. To this end, let $F \in \mathbb{R}^{m,n}$ and observe that we may do a transformation as in (5.4). Then the assumption that $[E, A, B]$ is not completely stabilizable yields $\gamma \neq (1, \ldots, 1)$, $\delta \neq (1, \ldots, 1)$, $\ell(\kappa) > 0$ or $\sigma(A_{\overline{c}}) \not\subseteq \mathbb{C}_-$. If $\gamma \neq (1, \ldots, 1)$, $\delta \neq (1, \ldots, 1)$ or $\ell(\kappa) > 0$, then $\operatorname{im}_{\mathbb{R}} \tilde{A} \not\subseteq \operatorname{im}_{\mathbb{R}} \tilde{E}$, and by Corollary 5.4 a) the system $[\tilde{E}, \tilde{A}]$ is not completely stable. On the other hand, if $\gamma = (1, \ldots, 1)$, $\delta = (1, \ldots, 1)$, $\ell(\kappa) = 0$ and $\lambda \in \sigma(A_{\overline{c}}) \cap \overline{\mathbb{C}_+}$, we find $\operatorname{im}_{\mathbb{C}} \left(\lambda\tilde{E} - \tilde{A}\right) \subsetneq \operatorname{im}_{\mathbb{C}} \tilde{E}$, which implies

$$\operatorname{rk}_{\mathbb{C}}(\lambda\tilde{E} - \tilde{A}) < \operatorname{rk}_{\mathbb{C}} \tilde{E} = n - \ell(\beta) - \ell(\kappa) = n - \ell(\beta) \overset{(3.7)}{=} \operatorname{rk}_{\mathbb{R}(s)}(s\tilde{E} - \tilde{A}).$$

Hence, applying Corollary 5.4 a) again, the system $[\tilde{E}, \tilde{A}]$ is not completely stable. As complete stability is invariant under system equivalence it follows that $[E, A+BF+B\tilde{F}T^{-1}]$ is not completely stable. Since F was chosen arbitrarily we may conclude that $[E, A+BF]$ is not completely stable for all $F \in \mathbb{R}^{m,n}$, which completes the proof of b).

c) The proof is analogous to b). $\square$

Remark 5.7 (State feedback)**.** (i) If the pencil $sE - A$ is regular and $[E, A, B]$ is impulse controllable, then a feedback $F \in \mathbb{R}^{m,n}$ can be constructed such that the pencil $sE - (A + BF)$ is regular and its index does not exceed one: First we choose $W, T, \tilde{F}$ such that we can put the system into the form (5.3). Now, impulse controllability implies that $\gamma = (1, \ldots, 1)$, $\delta = (1, \ldots, 1)$ and $\kappa = (1, \ldots, 1)$. Assuming $\ell(\delta) > 0$ implies that any quasi-Kronecker form of the pencil $sE - (A + B\tilde{F}T^{-1} + B\hat{F})$ contains an overdetermined γ-block (w.r.t. the form (3.4)) for all $\hat{F} \in \mathbb{R}^{m,n}$ as the feedback cannot act on this block, which contradicts regularity of $sE - A$. Hence it holds $\ell(\delta) = 0$ and from $k = n$ we further obtain that $\ell(\gamma) = \ell(\beta)$. Now applying another feedback as in (5.4), where we choose $F_{22} = E_\beta^\top \in \mathbb{R}^{\ell(\beta),|\beta|}$ and $F_{ij} = 0$ otherwise, we obtain, taking into account that $E_\gamma = I_{\ell(\gamma)}$ and that the pencil $\begin{bmatrix} sK_\beta - L_\beta \\ -E_\beta^\top \end{bmatrix}$ is regular, that $sE - (A + BF)$ is indeed regular with index at most one.

(ii) The matrix F_{11} in the proof of Theorem 5.6 b) can be constructed as follows: For $j = 1, \ldots, \ell(\alpha)$, consider vectors

$$a_j = -[a_{j\alpha_j-1}, \ldots, a_{j0}] \in \mathbb{R}^{1,\alpha_j}.$$

Then, for

$$F_{11} = \operatorname{diag}(a_1, \ldots, a_{\ell(\alpha)}) \in \mathbb{R}^{\ell(\alpha),|\alpha|}$$

the matrix $N_\alpha + E_\alpha F_{11}$ is diagonally composed of companion matrices, whence, for

$$p_j(s) = s^{\alpha_j} + a_{j\alpha_j-1}s^{\alpha_j-1} + \ldots + a_{j0} \in \mathbb{R}[s]$$

the characteristic polynomial of $N_\alpha + E_\alpha F_{11}$ is given by

$$\det(sI_{|\alpha|} - (N_\alpha + E_\alpha F_{11})) = \prod_{j=1}^{\ell(\alpha)} p_j(s).$$

Hence, choosing the coefficients a_{ji}, $j = 1, \ldots, \ell(\alpha)$, $i = 0, \ldots, \alpha_j$ such that the polynomials $p_1(s), \ldots, p_{\ell(\alpha)}(s) \in \mathbb{R}[s]$ are all Hurwitz, i.e., all roots of $p_1(s), \ldots, p_{\ell(\alpha)}(s)$ are in $\mathbb{C}_-$, we obtain stability. $\diamond$

5.3 Feedback in the behavioral sense

The hitherto presented feedback concept consists of the additional application of the relation $u(t) = Fx(t)$ to the system $E\dot{x}(t) = Ax(t) + Bu(t)$. Feedback can therefore be seen as an additional algebraic constraint that can be resolved for the input. Feedback in the behavioral sense, or, also called, *feedback via interconnection* [148] generalizes this approach by also allowing further algebraic relations in which the state not necessarily uniquely determines the input. That is, for given (or to be determined) $K = [K_x, K_u]$ with $K_x \in \mathbb{R}^{l,n}$, $K_u \in \mathbb{R}^{l,m}$, we consider

$$\mathfrak{B}^K_{[E,A,B]} := \left\{ (x, u) \in \mathfrak{B}_{[E,A,B]} \;\middle|\; \forall\, t \in \mathbb{R} : \; (x(t)^\top, u(t)^\top)^\top \in \ker_{\mathbb{R}}(K) \right\} = \mathfrak{B}_{[E,A,B]} \cap \mathfrak{B}_{[0_{l,n}, K_x, K_u]}.$$

We can alternatively write

$$\mathfrak{B}^K_{[E,A,B]} = \mathfrak{B}_{[E^K,A^K]},$$

where

$$[E^K, A^K] = \left[\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ K_x & K_u \end{bmatrix}\right]. \tag{5.5}$$

The concept of feedback in the behavioral sense has its origin in the works by Willems, Polderman and Trentelman [17,121,135,147,148], where differential behaviors and their stabilization via *feedback via interconnection* is considered. The latter means a systematic addition of some further (differential) equations in a way that a desired behavior is achieved. In contrast to these works we only add equations which are purely algebraic. This justifies to speak of *static feedback via interconnection*. We will give equivalent conditions for this type of feedback stabilizing the system. Note that, in principle, one could make the rigorous choice $K = I_{n+m}$ to end up with a behavior $\mathfrak{B}^K_{[E,A,B]} = \{0\}$ which is obviously autonomous and stable. This, however, is not suitable from a practical point of view, since in this interconnection, the space of consistent initial differential variables is a proper subset of the initial differential variables which are consistent with the original system $[E, A, B]$. Consequently, the interconnected system does not have the causality property - that is, the implementation of the controller at a certain time $t \in \mathbb{R}$ is not possible, since this causes jumps in the differential variables. To avoid this, we introduce the concept of *compatibility*.

Definition 5.8 (Compatible and stabilizing feedback)**.** The static feedback $K = [K_x, K_u]$, defined by $K_x \in \mathbb{R}^{l,n}$, $K_u \in \mathbb{R}^{l,m}$, is called

a) *compatible*, if for any $x^0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$, there exists some $(x, u) \in \mathfrak{B}^K_{[E,A,B]}$ with $Ex(0) = Ex^0$.

b) *stabilizing*, if $[E^K, A^K] \in \Sigma_{k+l,n}$ is stable in the behavioral sense. $\diamond$

Remark 5.9 (Compatible feedback)**.** Our definition of compatible feedback is a slight modification of the concept introduced by Julius and van der Schaft in [73], where an interconnection is called compatible, if any trajectory of the system without feedback can be concatenated with a trajectory of the interconnected system. This certainly implies that the space of initial differential variables of the interconnected system cannot be smaller than the corresponding set for the system without feedback. $\diamond$

Theorem 5.10 (Stabilizing feedback in the behavioral sense)**.** *Let $[E, A, B] \in \Sigma_{k,n,m}$ be given. Then there exists a compatible and stabilizing feedback $K = [K_x, K_u]$ with $K_x \in \mathbb{R}^{l,n}$, $K_u \in \mathbb{R}^{l,m}$, if, and only if, $[E, A, B]$ is stabilizable in the behavioral sense. In case of $[E, A, B]$ being stabilizable in the behavioral, the compatible and stabilizing feedback K can moreover be chosen in a way that $[E^K, A^K]$ is autonomous.*

Proof: Since, by definition, $[E, A, B] \in \Sigma_{k,n,m}$ is stabilizable in the behavioral sense if, and only if, for $s\mathcal{E} - \mathcal{A} = [sE - A, -B]$ the DAE $[\mathcal{E}, \mathcal{A}] \in \Sigma_{k,n+m}$ is stable in the behavioral sense, necessity follows from setting $l = 0$.
In order to show sufficiency, let $K = [K_x, K_u]$ with $K_x \in \mathbb{R}^{l,n}$, $K_u \in \mathbb{R}^{l,m}$, be a compatible and stabilizing feedback for $[E, A, B]$. Now the system can be put into feedback form, i.e., there exist $W \in \mathbf{Gl}_k(\mathbb{R})$, $T \in \mathbf{Gl}_n(\mathbb{R})$, $V \in \mathbf{Gl}_m(\mathbb{R})$ and $F \in \mathbb{R}^{m,n}$ such that

$$\begin{bmatrix} s\tilde{E} - \tilde{A} & \tilde{B} \\ -\tilde{K}_x & \tilde{K}_u \end{bmatrix} = \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sE - A & B \\ -K_x & K_u \end{bmatrix} \begin{bmatrix} T & 0 \\ -F & V \end{bmatrix},$$

where $[\tilde{E}, \tilde{A}, \tilde{B}]$ is in the form (3.11). Now the behavioral stability of $[E^K, A^K]$ implies that the system $[\tilde{E}^K, \tilde{A}^K] := \left[\begin{bmatrix} \tilde{E} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{K}_x & \tilde{K}_u \end{bmatrix}\right]$ is stable in the behavioral sense as well. Assume that $[E, A, B]$ is not stabilizable in the behavioral sense, that is, by Corollary 3.12(ix), there exists $\lambda \in \sigma(A_{\overline{c}}) \cap \overline{\mathbb{C}_+}$. Hence we find $x_6^0 \in \mathbb{R}^{n_{\overline{c}}} \setminus \{0\}$ such that $A_{\overline{c}} x_6^0 = \lambda x_6^0$. Then, with $x(\cdot) := \left(0, \ldots, 0, (e^{\lambda \cdot} x_6^0)^\top\right)^\top$, we have that $(x, 0) \in \mathcal{B}_{[\tilde{E}, \tilde{A}, \tilde{B}]}$. As $x(0) \in \mathcal{V}^{\mathrm{diff}}_{[\tilde{E}, \tilde{A}, \tilde{B}]} = T^{-1} \cdot \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$, the compatibility of the feedback K implies that there exists $(\tilde{x}, \tilde{u}) \in \mathfrak{B}^K_{[E,A,B]}$ with $E\tilde{x}(0) = ETx(0)$. This gives $(WET)T^{-1}\tilde{x}(0) = WETx(0)$ and writing $T^{-1}\tilde{x}(t) = (\tilde{x}_1(t)^\top, \ldots, \tilde{x}_6(t)^\top)^\top$ with vectors of appropriate size, we obtain $\tilde{x}_6(0) = x_6^0$. Since the solution of the initial value problem $\dot{y} = A_{\overline{c}} y$, $y(0) = x_6^0$, is unique, we find $\tilde{x}_6(t) = e^{\lambda t} x_6^0$ for all $t \in \mathbb{R}$. Now $(T^{-1}\tilde{x}, -V^{-1}FT^{-1}\tilde{x} + V^{-1}\tilde{u}) \in \mathcal{B}_{[\tilde{E}^K, \tilde{A}^K]}$ and as for all $(\hat{x}, \hat{u}) \in \mathcal{B}_{[\tilde{E}^K, \tilde{A}^K]}$ with $(\hat{x}(t), \hat{u}(t)) = (T^{-1}\tilde{x}(t), -V^{-1}FT^{-1}\tilde{x} + V^{-1}\tilde{u}(t))$ for all $t < 0$ we have $\hat{x}_6(t) = \tilde{x}_6(t)$ for all $t \in \mathbb{R}$, and $\tilde{x}_6(t) \not\to_{t\to\infty} 0$ since $\lambda \in \overline{\mathbb{C}_+}$, this contradicts that $[\tilde{E}^K, \tilde{A}^K]$ is stable in the behavioral sense.

It remains to show the second assertion, that is, for a system $[E, A, B] \in \Sigma_{k,n,m}$ that is stabilizable in the behavioral sense, there exists some compatible and stabilizing feedback K such that $[E^K, A^K]$ is autonomous:
Since, for $[E_1, A_1, B_1], [E_2, A_2, B_2] \in \Sigma_{k,n,m}$ with

$$[\,E_1\,,\,A_1\,,\,B_1\,] \overset{W,T,V,F}{\sim_f} [\,E_2\,,\,A_2\,,\,B_2\,],$$

$K_2 \in \mathbb{R}^{l,n+m}$ and

$$K_1 = K_2 \begin{bmatrix} T & 0 \\ F & V \end{bmatrix},$$

the behaviors of the feedback systems are related by

$$\begin{bmatrix} T & 0 \\ F & V \end{bmatrix} \mathfrak{B}^{K_1}_{[E_1,A_1,B_1]} = \mathfrak{B}^{K_2}_{[E_2,A_2,B_2]},$$

it is no loss of generality to assume that $[E, A, B]$ is in feedback form (3.11), i.e.,

$$sE - A$$
$$= \begin{bmatrix} sI_{|\alpha|} - N_\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & sK_\beta - L_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & sK_\gamma^\top - L_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & sK_\delta^\top - L_\delta^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & sN_\kappa - I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & sI_{n_{\overline{c}}} - A_{\overline{c}} \end{bmatrix}, \quad B = \begin{bmatrix} E_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E_\gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $F_{11} \in \mathbb{R}^{\ell(\alpha),|\alpha|}$ such that $\det(sI_{|\alpha|} - (N_\alpha + E_\alpha F_{11}))$ is Hurwitz. Then the DAE

$$\begin{bmatrix} I_{|\alpha|} & 0 \\ 0 & 0 \end{bmatrix} \dot{z}(t) = \begin{bmatrix} N_\alpha & E_\alpha \\ F_{11} & -I_{\ell(\alpha)} \end{bmatrix} z(t)$$

is autonomous and stable in the behavioral sense. Furthermore, by an argumentation as in Remark 5.7(ii), for

$$a_j = [a_{j\beta_j-2}, \ldots, a_{j0}, 1] \in \mathbb{R}^{1,\beta_j}$$

with the property that the polynomials

$$p_j(s) = s^{\beta_j} + a_{j\beta_j-1}s^{\beta_j-1} + \ldots + a_{j0} \in \mathbb{R}[s]$$

are Hurwitz for $j = 1, \ldots, \ell(\alpha)$, the choice

$$K_u = \mathrm{diag}(a_1, \ldots, a_{\ell(\beta)}) \in \mathbb{R}^{\ell(\beta),|\beta|}$$

leads to an autonomous system

$$\begin{bmatrix} K_\beta \\ 0 \end{bmatrix} \dot{z}(t) = \begin{bmatrix} L_\beta \\ K_u \end{bmatrix} z(t),$$

which is also stable in the behavioral sense. Since, moreover, by Corollary 3.12(ix), there holds $\sigma(A_{\overline{c}}) \subseteq \mathbb{C}_-$, the choice

$$K = \begin{bmatrix} F_{11} & 0 & 0 & 0 & 0 & 0 & -I_{\ell(\alpha)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_u & 0 \end{bmatrix}$$

leads to a behavioral stable and autonomous system. Since the differential variables can be arbitrarily initialized in any of the previously discussed subsystems, the constructed feedback is also compatible. □

6 Invariant subspaces

This section is dedicated to some selected results of the geometric theory of differential-algebraic control systems. Geometric theory plays a fundamental role in standard ODE system theory, see the work by Basile and Marro [15] and also the famous book by Wonham [151]. In [93] Lewis gave an overview of the to date geometric theory of DAEs. As we will do here he put special emphasis on the two fundamental sequences of subspaces $\mathcal{V}_i$ and $\mathcal{W}_i$ defined as follows:

$$\begin{aligned} \mathcal{V}_0 &:= \mathbb{R}^n, & \mathcal{V}_{i+1} &:= A^{-1}(E\mathcal{V}_i + \operatorname{im}_{\mathbb{R}} B) \subseteq \mathbb{R}^n, \\ \mathcal{W}_0 &:= \{0\}, & \mathcal{W}_{i+1} &:= E^{-1}(A\mathcal{W}_i + \operatorname{im}_{\mathbb{R}} B) \subseteq \mathbb{R}^n. \end{aligned}$$

Let $\mathcal{V}^* := \bigcap_{i\in\mathbb{N}_0} \mathcal{V}_i$ and $\mathcal{W}^* := \bigcup_{i\in\mathbb{N}_0} \mathcal{W}_i$ be the limits of the sequences, which we may call Wong sequences [23, 24], since Wong [149] was the first one who used both sequences (with $B = 0$) for the analysis of matrix pencils. However, the Wong sequences can be traced back to Dieudonné [49], who focused on the first of the two Wong sequences. Bernhard [25] and Armentano [6] used the Wong sequences to carry out a geometric analysis of matrix pencils. They appear also in [3, 4, 89, 138].
In control theory, that is when $B \neq 0$, the Wong sequences have been extensively studied by several authors, see e.g. [92, 105, 106, 111, 112, 114, 115, 140] for regular systems and [3, 10, 12, 13, 27, 28, 52, 93, 98, 113, 123] for general DAE systems. Frankowska [54] did a nice investigation of systems (2.1) in terms of differential inclusions [8, 9], however requiring controllability at infinity (see [54, Prop. 2.6]). Nevertheless, she is the first to derive a formula for the reachability space [54, Thm. 3.1], which was later generalized by Przyłuski and Sosnowski [123, Sec. 4] (in fact, the same generalization has been announced in [98, p. 296], [93, Sec. 5] and [10, p. 1510], however without proof); it also occurred in [52, Thm. 2.5].

Proposition 6.1 (Reachability space [123, Sec. 4]). *For $[E, A, B] \in \Sigma_{k,n,m}$ and limits $\mathcal{V}^*$ and $\mathcal{W}^*$ of the Wong sequences we have*

$$\mathcal{R}_{[E,A,B]} = \mathcal{V}^* \cap \mathcal{W}^*.$$

It has been shown in [12] (for discrete systems), see also [10, 13, 27, 113], that the first Wong sequence is the space of consistent initial conditions. For regular systems this was proved in [92].

Proposition 6.2 (Consistent initial conditions [12]). *For $[E, A, B] \in \Sigma_{k,n,m}$ and limit $\mathcal{V}^*$ of the first Wong sequence we have*

$$\mathcal{V}_{[E,A,B]} = \mathcal{V}^*.$$

Various other properties of $\mathcal{V}^*$ and $\mathcal{W}^*$ have been derived in [12] in the context of discrete systems. A characterization of the spaces $\mathcal{V}^*$ and $\mathcal{W}^*$ in terms of distributions is also given in [123]: $\mathcal{V}^* + \ker_{\mathbb{R}} E$ is the set of all initial values such that the distributional initial value problem [123, (3)] has a smooth solution (x, u); $\mathcal{W}^*$ is the set of all initial values such that [123, (3)] has an impulsive solution (x, u); $\mathcal{V}^* + \mathcal{W}^*$ is the set of all initial values such that [123, (3)] has an impulsive-smooth solution (x, u).

For regular systems Özçaldiran [112] showed that $\mathcal{V}^*$ is the supremal $(A, E, \mathrm{im}_{\mathbb{R}} B)$-invariant subspace of $\mathbb{R}^n$ and $\mathcal{W}^*$ is the infimal restricted $(E, A, \mathrm{im}_{\mathbb{R}} B)$-invariant subspace of $\mathbb{R}^n$. These concepts, which have also been used in [3, 12, 92, 106] are defined as follows.

Definition 6.3 ($(A, E, \mathrm{im}_{\mathbb{R}} B)$- and $(E, A, \mathrm{im}_{\mathbb{R}} B)$-invariance [112])**.** Let $E, A \in \mathbb{R}^{k,n}$ and $B \in \mathbb{R}^{n,m}$. A subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is called *$(A, E, \mathrm{im}_{\mathbb{R}} B)$-invariant* if

$$A\mathcal{V} \subseteq E\mathcal{V} + \mathrm{im}_{\mathbb{R}} B.$$

A subspace $\mathcal{W} \subseteq \mathbb{R}^n$ is called *restricted $(E, A, \mathrm{im}_{\mathbb{R}} B)$-invariant* if

$$\mathcal{W} = E^{-1}(A\mathcal{W} + \mathrm{im}_{\mathbb{R}} B).$$

$\diamond$

It is easy to verify that the proofs given in [112, Lems. 2.1 & 2.2] remain the same for general $E, A \in \mathbb{R}^{k,n}$ and $B \in \mathbb{R}^{n,m}$ - this was shown in [12] as well. For $\mathcal{V}^*$ this can be found in [3], see also [106]. So we have the following proposition.

Proposition 6.4 (Wong sequences as invariant subspaces)**.** *Consider $[E, A, B] \in \Sigma_{k,n,m}$ and the limits $\mathcal{V}^*$ and $\mathcal{W}^*$ of the Wong sequences. Then the following statements hold true.*

(i) *$\mathcal{V}^*$ is $(A, E, \mathrm{im}_{\mathbb{R}} B)$-invariant and for any $\mathcal{V} \subseteq \mathbb{R}^n$ which is $(A, E, \mathrm{im}_{\mathbb{R}} B)$-invariant it holds $\mathcal{V} \subseteq \mathcal{V}^*$;*

(ii) *$\mathcal{W}^*$ is restricted $(E, A, \mathrm{im}_{\mathbb{R}} B)$-invariant and for any $\mathcal{W} \subseteq \mathbb{R}^n$ which is restricted $(E, A, \mathrm{im}_{\mathbb{R}} B)$-invariant it holds $\mathcal{W}^* \subseteq \mathcal{W}$.*

It is now clear how the controllability concepts can be characterized in terms of the invariant subspaces $\mathcal{V}^*$ and $\mathcal{W}^*$. However, the statement about R-controllability/behavioral controllability seems to be new. The only other appearance of a subspace inclusion as a characterization of R-controllability that the authors are aware of occurs in [39] for regular systems: if $A = I$, then the system is R-controllable if, and only if, $\mathrm{im}_{\mathbb{R}} E^D \subseteq \langle E^D | B \rangle$, where E^D is the Drazin inverse of E, see Remark 2.5(iv).

Theorem 6.5 (Geometric criteria for controllability)**.** *Consider $[E, A, B] \in \Sigma_{k,n,m}$ and the limits $\mathcal{V}^*$ and $\mathcal{W}^*$ of the Wong sequences. Then $[E, A, B]$ is*

(i) *controllable at infinity if, and only if, $\mathcal{V}^* = \mathbb{R}^n$;*

(ii) *impulse controllable if, and only if, $\mathcal{V}^* + \ker_{\mathbb{R}} E = \mathbb{R}^n$ or, equivalently, $E\mathcal{V}^* = \mathrm{im}_{\mathbb{R}} E$;*

(iii) *controllable in the behavioral sense if, and only if, $\mathcal{V}^* \subseteq \mathcal{W}^*$;*

(iv) *completely controllable if, and only if, $\mathcal{V}^* \cap \mathcal{W}^* = \mathbb{R}^n$;*

(v) *strongly controllable if, and only if, $(\mathcal{V}^* \cap \mathcal{W}^*) + \ker_{\mathbb{R}} E = \mathbb{R}^n$ or, equivalently, $E(\mathcal{V}^* \cap \mathcal{W}^*) = \mathrm{im}_{\mathbb{R}} E$.*

Proof: By Propositions 6.1 and 6.2 it is clear that it only remains to prove (iii). We proceed in several steps.

Step 1: Let $[E_1, A_1, B_1], [E_2, A_2, B_2] \in \Sigma_{k,n,m}$ such that for some $W \in \mathbf{Gl}_k(\mathbb{R})$, $T \in \mathbf{Gl}_n(\mathbb{R})$, $V \in \mathbf{Gl}_m(\mathbb{R})$ and $F \in \mathbb{R}^{m,n}$ it holds

$$[\, E_1 \,,\, A_1 \,,\, B_1 \,] \overset{W,T,V,F}{\sim_f} [\, E_2 \,,\, A_2 \,,\, B_2 \,].$$

We show that the Wong sequences $\mathcal{V}_i^1, \mathcal{W}_i^1$ of $[E_1, A_1, B_1]$ and the Wong sequences $\mathcal{V}_i^2, \mathcal{W}_i^2$ of $[E_2, A_2, B_2]$ are related by

$$\forall\, i \in \mathbb{N}_0 : \ \mathcal{V}_i^1 = T^{-1}\mathcal{V}_i^2 \ \wedge \ \mathcal{W}_i^1 = T^{-1}\mathcal{W}_i^2.$$

We proof the statement by induction. It is clear that $\mathcal{V}_0^1 = T^{-1}\mathcal{V}_0^2$. Assuming that $\mathcal{V}_i^1 = T^{-1}\mathcal{V}_i^2$ for some $i \geq 0$ we find that, by (3.2),

$$\begin{aligned}
\mathcal{V}_{i+1}^1 &= A_1^{-1}(E_1\mathcal{V}_i^1 + \operatorname{im}_{\mathbb{R}} B_1) \\
&= \left\{ \, x \in \mathbb{R}^n \;\middle|\; \exists\, y \in \mathcal{V}_i^1 \ \exists\, u \in \mathbb{R}^m : \ W(A_2T + B_2T)x = WE_2Ty + WB_2Vu \, \right\} \\
&= \left\{ \, x \in \mathbb{R}^n \;\middle|\; \exists\, z \in \mathcal{V}_i^2 \ \exists\, v \in \mathbb{R}^m : \ A_2Tx = E_2z + B_2v \, \right\} \\
&= T^{-1}\left(A_2^{-1}(E_2\mathcal{V}_i^1 + \operatorname{im}_{\mathbb{R}} B_2)\right) = T^{-1}\mathcal{V}_{i+1}^2.
\end{aligned}$$

The statement about $\mathcal{W}_i^1$ and $\mathcal{W}_i^2$ can be proved analogous.

Step 2: By Step 1 we may without loss of generality assume that $[E, A, B]$ is given in feedback form (3.11). We make the convention that if $\alpha \in \mathbb{N}^l$ is some multi-index, then $\alpha - 1 := (\alpha_1 - 1, \ldots, \alpha_l - 1)$. It not follows that

$$\forall\, i \in \mathbb{N}_0 : \ \mathcal{V}_i = \mathbb{R}^{|\alpha|} \times \mathbb{R}^{|\beta|} \times \operatorname{im}_{\mathbb{R}} N_{\gamma-1}^i \times \operatorname{im}_{\mathbb{R}} (N_{\delta-1}^\top)^i \times \operatorname{im}_{\mathbb{R}} N_\kappa^i \times \mathbb{R}^{n_{\overline{c}}}, \tag{6.1}$$

which is immediate from observing that $K_\gamma^\top x = L_\gamma^\top y + E_\gamma u$ for some x, y, u of appropriate dimension yields $x = N_{\gamma-1}y$ and $L_\delta^\top x = K_\delta^\top y$ for some x, y yields $x = N_{\delta-1}^\top y$. Note that in the case $\gamma_i = 1$ or $\delta_i = 1$, i.e., we have a 1×0 block, we find that N_{γ_i-1} and N_{δ_i-1} are absent, so these relations are consistent.
On the other hand we find that

$$\forall\, i \in \mathbb{N}_0 : \ \mathcal{W}_i = \ker_{\mathbb{R}} N_\alpha^i \times \ker_{\mathbb{R}} N_\beta^i \times \ker_{\mathbb{R}} N_{\gamma-1}^i \times \{0\}^{|\delta| - \ell(\delta)} \times \ker_{\mathbb{R}} N_\kappa^i \times \{0\}^{n_{\overline{c}}}, \tag{6.2}$$

which indeed needs some more rigorous proof. First observe that $\operatorname{im}_{\mathbb{R}} E_\alpha = \ker_{\mathbb{R}} N_\alpha$, $\ker_{\mathbb{R}} K_\beta = \ker_{\mathbb{R}} N_\beta$ and $(L_\gamma^\top)^{-1}(\operatorname{im}_{\mathbb{R}} E_\gamma) = \operatorname{im}_{\mathbb{R}} E_{\gamma-1} = \ker_{\mathbb{R}} N_{\gamma-1}$. Therefore we have

$$\mathcal{W}_1 = E^{-1}(\operatorname{im}_{\mathbb{R}} B) = \ker_{\mathbb{R}} N_\alpha \times \ker_{\mathbb{R}} N_\beta \times \ker_{\mathbb{R}} N_{\gamma-1} \times \{0\}^{|\delta| - \ell(\delta)} \times \ker_{\mathbb{R}} N_\kappa \times \{0\}^{n_{\overline{c}}}.$$

Further observe that $N_\alpha^i N_\alpha^\top = N_\alpha N_\alpha^\top N_\alpha^{i-1}$ for all $i \in \mathbb{N}$ and, hence, if $x = N_\alpha^\top y + E_\alpha u$ for some x, u and $y \in \ker_{\mathbb{R}} N_\alpha^{i-1}$ it follows $x \in \ker_{\mathbb{R}} N_\alpha^i$. Likewise, if $L_\gamma^\top x = K_\gamma^\top y + E_\gamma u$ for some x, u and $y \in \ker_{\mathbb{R}} N_{\gamma-1}^{i-1}$ we find $x = N_{\gamma-1}^\top y + E_{\gamma-1}^\top u$ and hence $x \in \ker_{\mathbb{R}} N_{\gamma-1}^i$. Finally, if $K_\beta x = L_\beta y$ for some x and some $y \in \ker_{\mathbb{R}} N_\beta^{i-1}$ it follows that by adding some zero rows we obtain $N_\beta x = N_\beta N_\beta^\top y$ and hence, as above, $x \in \ker_{\mathbb{R}} N_\beta^i$. This proves (6.2).

Step 3: From (6.1) and (6.2) it follows that

$$\mathcal{V}^* = \mathbb{R}^{|\alpha|} \times \mathbb{R}^{|\beta|} \times \operatorname{im}_{\mathbb{R}}\{0\}^{|\gamma| - \ell(\gamma)} \times \{0\}^{|\delta| - \ell(\delta)} \times \{0\}^{|\kappa|} \times \mathbb{R}^{n_{\overline{c}}},$$

$$\mathcal{W}^* = \mathbb{R}^{|\alpha|} \times \mathbb{R}^{|\beta|} \times \operatorname{im}_{\mathbb{R}} \mathbb{R}^{|\gamma|-\ell(\gamma)} \times \{0\}^{|\delta|-\ell(\delta)} \times \mathbb{R}^{|\kappa|} \times \{0\}^{n_{\overline{c}}}.$$

As by Corollary 3.12(vi) the system $[E, A, B]$ is controllable in the behavioral sense if, and only if, $n_{\overline{c}} = 0$ we may immediately deduce that this is the case if, and only if, $\mathcal{V}^* \subseteq \mathcal{W}^*$. This proves the theorem. □

Remark 6.6 (Representation of the reachability space)**.** From Proposition 6.1 and the proof of Theorem 6.5 we may immediately observe that, using the notation from Theorem 3.10, we have

$$\mathcal{R}_{[E,A,B]} = T^{-1}\left(\mathbb{R}^{|\alpha|} \times \mathbb{R}^{|\beta|} \times \operatorname{im}_{\mathbb{R}}\{0\}^{|\gamma|-\ell(\gamma)} \times \{0\}^{|\delta|-\ell(\delta)} \times \{0\}^{|\kappa|} \times \{0\}^{n_{\overline{c}}}\right).$$

◇

7 Kalman decomposition

Nearly fifty years ago Kalman [76] obtained his famous decomposition of linear ODE control systems. This decomposition has later been generalized to regular DAEs by Verghese at al. [143], see also [46]. A Kalman decomposition of general discrete-time DAE systems has been provided by Banaszuk et al. [13] (later generalized to systems with output equation in [10]) in a very nice way using the Wong sequences (cf. Section 6). They derive a system

$$\left[\begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix}\right], \tag{7.1}$$

which is system equivalent to given $[E, A, B] \in \Sigma_{k,n,m}$ with the properties that the system $[E_{11}, A_{11}, B_1]$ is completely controllable and the matrix $[E_{11}, A_{11}, B_1]$ has full row rank (strongly $\mathcal{H}$-controllable in the notation of [13]) and, furthermore, $\mathcal{R}_{[E_{22},A_{22},0]} = \{0\}$.

This last condition is very reasonable, as one should wonder what properties a Kalman decomposition of a DAE system should have. In the case of ODEs the decomposition simply took the form

$$\left[\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix}\right],$$

where $[A_{11}, B_1]$ is controllable. Therefore, an ODE system is decomposed into a controllable and an uncontrollable part, since clearly $[A_{22}, B_2]$ is not controllable at all. For DAEs however, the situation is more subtle, since in a decomposition (7.1) with $[E_{11}, A_{11}, B_1]$ completely controllable (and $[E_{11}, A_{11}, B_1]$ full row rank) the conjectural "uncontrollable" part $[E_{22}, A_{22}, 0]$ may still have a controllable subsystem, since systems of the type $[K_\beta, L_\beta, 0]$ are always controllable. To exclude this and ensure that all controllable parts are included $[E_{11}, A_{11}, B_1]$ we may state the additional condition (as in [13]) that

$$\mathcal{R}_{[E_{22},A_{22},0]} = \{0\}.$$

This then also guarantees certain uniqueness properties of the Kalman decomposition. Hence, any system (7.1) with the above properties which is system equivalent to $[E, A, B]$ we may call a Kalman decomposition of $[E, A, B]$. We cite the result of [13], but also give some remarks on how the decomposition may be easily derived.

Theorem 7.1 (Kalman decomposition [13])**.** *For $[E, A, B] \in \Sigma_{k,n,m}$, there exist $W \in \mathbf{Gl}_k(\mathbb{R})$, $T \in \mathbf{Gl}_n(\mathbb{R})$ such that*

$$[E, A, B] \overset{W,T}{\sim_s} \left[\begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix}\right], \tag{7.2}$$

with $E_{11}, A_{11} \in \mathbb{R}^{k_1,n_1}$, $E_{12}, A_{12} \in \mathbb{R}^{k_1,n_2}$, $E_{22}, A_{22} \in \mathbb{R}^{k_2,n_2}$ and $B_1 \in \mathbb{R}^{k_1,m}$, such that $[E_{11}, A_{11}, B_1] \in \Sigma_{k_1,n_1,m}$ is completely controllable, $\operatorname{rk}_{\mathbb{R}}[E_{11}, A_{11}, B_1] = k_1$ and $\mathcal{R}_{[E_{22},A_{22},0_{k_2,m}]} = \{0\}$.

Remark 7.2 (Derivation of the Kalman decomposition)**.** Let $[E, A, B] \in \Sigma_{k,n,m}$ be given. The Kalman decomposition (7.2) can be derived using the limits $\mathcal{V}^*$ and $\mathcal{W}^*$ of the Wong sequences presented in Section 6. It is clear that these spaces satisfy the following subspace relations:

$$\begin{aligned} E(\mathcal{V}^* \cap \mathcal{W}^*) &\subseteq (E\mathcal{V}^* + \operatorname{im}_{\mathbb{R}} B) \cap (A\mathcal{W}^* + \operatorname{im}_{\mathbb{R}} B), \\ A(\mathcal{V}^* \cap \mathcal{W}^*) &\subseteq (E\mathcal{V}^* + \operatorname{im}_{\mathbb{R}} B) \cap (A\mathcal{W}^* + \operatorname{im}_{\mathbb{R}} B). \end{aligned}$$

Therefore, if we choose any full rank matrices $R_1 \in \mathbb{R}^{n,n_1}, P_1 \in \mathbb{R}^{n,n_2}, R_2 \in \mathbb{R}^{k,k_1}, P_2 \in \mathbb{R}^{k,k_2}$ such that

$$\begin{aligned} \operatorname{im}_{\mathbb{R}} R_1 &= \mathcal{V}^* \cap \mathcal{W}^*, & \operatorname{im}_{\mathbb{R}} R_2 &= (E\mathcal{V}^* + \operatorname{im}_{\mathbb{R}} B) \cap (A\mathcal{W}^* + \operatorname{im}_{\mathbb{R}} B), \\ \operatorname{im}_{\mathbb{R}} R_1 \oplus \operatorname{im}_{\mathbb{R}} P_1 &= \mathbb{R}^n, & \operatorname{im}_{\mathbb{R}} R_2 \oplus \operatorname{im}_{\mathbb{R}} P_2 &= \mathbb{R}^k, \end{aligned}$$

then $[R_1, P_1] \in \mathbf{Gl}_n(\mathbb{R})$ and $[R_2, P_2] \in \mathbf{Gl}_k(\mathbb{R})$, and, furthermore, there exists matrices $E_{11}, A_{11} \in \mathbb{R}^{k_1,n_1}$, $E_{12}, A_{12} \in \mathbb{R}^{k_1,n_2}$, $E_{22}, A_{22} \in \mathbb{R}^{k_2,n_2}$ such that

$$\begin{aligned} ER_1 &= R_2E_{11}, & AR_1 &= R_2A_{11}, \\ EP_1 &= R_2E_{12} + P_2E_{22}, & AP_1 &= R_2A_{12} + P_2A_{22}. \end{aligned}$$

As, moreover, $\operatorname{im}_{\mathbb{R}} B \subseteq (E\mathcal{V}^* + \operatorname{im}_{\mathbb{R}} B) \cap (A\mathcal{W}^* + \operatorname{im}_{\mathbb{R}} B) = \operatorname{im}_{\mathbb{R}} R_2$ there exists $B_1 \in \mathbb{R}^{k_1,m}$ such that $B = R_2B_1$. All these relations together yield the decomposition (7.2) with $W = [R_2, P_2]$ and $T = [R_1, P_1]^{-1}$. The properties of the subsystems essentially rely on the observation that by Proposition 6.1

$$\mathcal{R}_{[E,A,B]} = \mathcal{V}^* \cap \mathcal{W}^* = \operatorname{im}_{\mathbb{R}} R_1 = T^{-1}(\mathbb{R}^{n_1} \times \{0\}^{n_2}).$$

$\diamond$

Remark 7.3 (Kalman decomposition)**.** It is important to note that a trivial reachability space does not necessarily imply that $B = 0$. An intriguing example which illustrates this is the system

$$[E, A, B] = \left[\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right]. \tag{7.3}$$

Another important fact we like to stress by means of this example is that $B \neq 0$ does no necessarily imply $n_1 \neq 0$ in the Kalman decomposition (7.2). In fact, the above system $[E, A, B]$ is already in Kalman decomposition with $k_1 = k_2 = 1, n_1 = 0, n_2 = 1, m = 1$ and $E_{12} = 1$, $A_{12} = 0$, $B_1 = 1$ as well as $E_{22} = 0$, $A_{22} = 1$. Then all the required properties are obtained, in particular $\operatorname{rk}_{\mathbb{R}}[E_{11}, A_{11}, B_1] = \operatorname{rk}_{\mathbb{R}}[1] = 1$ and the system $[E_{11}, A_{11}, B_1]$ is completely controllable as it is in feedback form (3.11) with exactly one γ-block with $\gamma = 1$; complete controllability then follows from Corollary 3.12. However, $[E_{11}, A_{11}, B_1]$ is hard to view as a control system as no equation can be written down. Nevertheless, the space $\mathcal{R}_{[E_{11},A_{11},B_1]}$ has dimension zero and obviously every state can be steered to every other state. $\diamond$

We now analyze how two forms of type (7.2) of one system $[E, A, B] \in \Sigma_{k,n,m}$ differ.

Proposition 7.4 (Uniqueness of the Kalman decomposition)**.** *Let $[E, A, B] \in \Sigma_{k,n,m}$ be given and assume that, for all $i \in \{1, 2\}$, the systems $[E_i, A_i, B_i] \overset{W_i,T_i}{\sim_s} [E, A, B]$ with*

$$sE_i - A_i = \begin{bmatrix} sE_{11,i} - A_{11,i} & sE_{12,i} - A_{12,i} \\ 0 & sE_{22,i} - A_{22,i} \end{bmatrix}, \quad B_i = \begin{bmatrix} B_{1,i} \\ 0 \end{bmatrix}$$

where $E_{11,i}, A_{11,i} \in \mathbb{R}^{k_{1,i},n_{1,i}}$, $E_{12,i}, A_{12,i} \in \mathbb{R}^{k_{1,i},n_{2,i}}$, $E_{22,i}, A_{22,i} \in \mathbb{R}^{k_{2,i},n_{2,i}}$, $B_{1,i} \in \mathbb{R}^{k_{1,i},m}$ satisfy

$$\operatorname{rk}_{\mathbb{R}} \begin{bmatrix} E_{11,i} & A_{11,i} & B_{1,i} \end{bmatrix} = k_{1,i}$$

and, in addition, $[E_{11,i}, A_{11,i}, B_{c,i}] \in \Sigma_{k_{1,i},n_{1,i},m}$ *is completely controllable and* $\mathcal{R}_{[E_{22,i},A_{22,i},0_{k_{2,i},m}]} = \{0\}$. *Then* $k_{1,1} = k_{1,2}$, $k_{2,1} = k_{2,2}$, $n_{1,1} = n_{1,2}$, $n_{2,1} = n_{2,2}$. *Moreover, for some* $W_{11} \in \mathbf{Gl}_{k_{1,1}}(\mathbb{R})$, $W_{12} \in \mathbb{R}^{k_{1,1},k_{2,1}}$, $W_{22} \in \mathbf{Gl}_{k_{2,1}}(\mathbb{R})$, $T_{11} \in \mathbf{Gl}_{n_{1,1}}(\mathbb{R})$, $T_{12} \in \mathbb{R}^{n_{1,1},n_{2,1}}$, $T_{22} \in \mathbf{Gl}_{n_{2,1}}(\mathbb{R})$, *there holds*

$$W_2 W_1^{-1} = \begin{bmatrix} W_{11} & W_{12} \\ 0 & W_{22} \end{bmatrix}, \quad T_1^{-1} T_2 = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}.$$

In particular, the systems $[E_{11,1}, A_{11,1}, B_{1,1}]$, $[E_{11,2}, A_{11,2}, B_{1,2}]$ *and, respectively,* $[E_{22,1}, A_{22,1}, 0]$, $[E_{22,2}, A_{22,2}, 0]$ *are system equivalent.*

Proof: It is no loss of generality to assume that $W_1 = I_k$, $T_1 = I_n$. Then we obtain

$$\mathbb{R}^{n_{1,1}} \times \{0\} = \mathcal{R}_{[E_1,A_1,B_1]} = T_2 \mathcal{R}_{[E_2,A_2,B_2]} = T_2\big(\mathbb{R}^{n_{1,2}} \times \{0\}\big).$$

This implies $n_{1,1} = n_{1,2}$ and

$$T_2 = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \text{ for some } T_{11} \in \mathbf{Gl}_{n_{1,1}},\ T_{12} \in \mathbb{R}^{n_{1,1},n_{2,1}},\ T_{22} \in \mathbf{Gl}_{n_{2,1}}.$$

Now partitioning

$$W_2 = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, \quad W_{11} \in \mathbb{R}^{k_{1,1},k_{1,2}},\ W_{12} \in \mathbb{R}^{k_{1,1},k_{2,2}},\ W_{21} \in \mathbb{R}^{k_{2,1},k_{c,2}},\ W_{22} \in \mathbb{R}^{k_{2,1},k_{2,2}},$$

the block $(2,1)$ of the equations $W_1E_1T_1 = E_2$, $W_1A_1T_1 = A_2$ and $W_1B_1 = B_2$ give rise to

$$0 = W_{21} \begin{bmatrix} E_{11,2} & A_{11,2} & B_{1,2} \end{bmatrix}.$$

Since the latter matrix is supposed to have full row rank, we obtain $W_{21} = 0$. The assumption of W_2 being invertible then leads to $k_{1,1} \le k_{1,2}$. Reversing the roles of $[E_1, A_1, B_1]$ and $[E_2, A_2, B_2]$, we further obtain $k_{1,2} \le k_{1,1}$, whence $k_{1,2} = k_{1,1}$. Using again the invertibility of W, we obtain that both W_{11} and W_{22} are invertible. □

It is immediate from the form (7.2) that $[E, A, B]$ is completely controllable if, and only if, $n_1 = n$. The following result characterizes the further controllability and stabilizability notions in terms of properties of the submatrices in (7.2).

Corollary 7.5 (Properties induced from the Kalman decomposition)**.** *Consider* $[E, A, B] \in \Sigma_{k,n,m}$ *with*

$$[E, A, B] \overset{W,T}{\sim_s} \left[\begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \right]$$

such that $[E_{11}, A_{11}, B_1] \in \Sigma_{k_1,n_1,m}$ *is completely controllable,* $\operatorname{rk}_{\mathbb{R}}[E_{11}, A_{11}, B_1] = k_1$ *and* $\mathcal{R}_{[E_{22},A_{22},0_{k_2,m}]} = \{0\}$. *Then the following statements hold true:*

(i) $\operatorname{rk}_{\mathbb{R}(s)}(sE_{22} - A_{22}) = n_2$.

(ii) If $sE - A$ *is regular, then both pencils* $sE_{11} - A_{11}$ *and* $sE_{22} - A_{22}$ *are regular. In particular, it holds* $k_1 = n_1$ *and* $k_2 = n_2$.

(iii) If $[E, A, B]$ *is impulse controllable, then the index of the pencil* $sE_{22} - A_{22}$ *is at most one.*

(iv) $[E, A, B]$ *is controllable at infinity if, and only if,* $\operatorname{im}_{\mathbb{R}} A_{22} \subseteq \operatorname{im}_{\mathbb{R}} E_{22}$.

(v) $[E, A, B]$ *is controllable in the behavioral sense if, and only if,* $\mathrm{rk}_{\mathbb{R}(s)}(sE_{22} - A_{22}) = \mathrm{rk}_{\mathbb{C}}(\lambda E_{22} - A_{22})$ *for all* $\lambda \in \mathbb{C}$*.*

(vi) $[E, A, B]$ *is stabilizable in the behavioral sense if, and only if,* $\mathrm{rk}_{\mathbb{R}(s)}(sE_{22} - A_{22}) = \mathrm{rk}_{\mathbb{C}}(\lambda E_{22} - A_{22})$ *for all* $\lambda \in \overline{\mathbb{C}_+}$*.*

Proof: (i) Assuming that $\mathrm{rk}_{\mathbb{R}(s)}(sE_{22} - A_{22}) < n_2$, then, in a quasi-Kronecker (3.4) form of $sE_{22} - A_{22}$, it holds $\ell(\beta) > 0$ by (3.7). By the findings of Remark 3.11 b), we can conclude $\mathcal{R}_{[E_{22},A_{22},0_{k_2,m}]} \neq \{0\}$, a contradiction.

(ii) We can infer from (i) that $n_2 \leq k_2$. We can further infer from the regularity of $sE - A$ that $n_2 \geq k_2$. The regularity of $sE_{11} - A_{11}$ and $sE_{22} - A_{22}$ then follows immediately from $\det(sE - A) = c \cdot \det(sE_{11} - A_{11}) \cdot \det(sE_{22} - A_{22})$, where $c = \det(W \cdot T)$.

(iii) Assume that $[E, A, B]$ is impulse controllable. By Corollary 4.3(ii) and the invariance of impulse controllability under system equivalence this implies that

$$\mathrm{im}_{\mathbb{R}} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \subseteq \mathrm{im}_{\mathbb{R}} \begin{bmatrix} E_{11} & E_{12} & B_1 & A_{11}Z_1 + A_{12}Z_2 \\ 0 & E_{22} & 0 & A_{22}Z_2 \end{bmatrix},$$

where $Z = [Z_1^\top, Z_2^\top]^\top$ is a real matrix such that $\mathrm{im}_{\mathbb{R}} Z = \ker_{\mathbb{R}} \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix}$. The last condition in particular implies that $\mathrm{im}_{\mathbb{R}} Z_2 \subseteq \ker_{\mathbb{R}} E_{22}$ and therefore we obtain

$$\mathrm{im}_{\mathbb{R}} A_{22} \subseteq \mathrm{im}_{\mathbb{R}} E_{22} + A_{22} \cdot \ker_{\mathbb{R}} E_{22},$$

which is, by (3.5), equivalent to the index of $sE_{22} - A_{22}$ being at most one.

(iv) Since $\mathrm{rk}_{\mathbb{R}}[E_{11}, A_{11}, B_1] = k_1$ and the system $[E_{11}, A_{11}, B_1]$ is controllable at infinity, Corollary 4.3(i) leads to $\mathrm{rk}_{\mathbb{R}}[E_{11}, B_1] = k_1$. Therefore, we have

$$\mathrm{im}_{\mathbb{R}} \begin{bmatrix} E_{11} & E_{12} & B_1 \\ 0 & E_{22} & 0 \end{bmatrix} = \mathbb{R}^{k_1} \times \mathrm{im}_{\mathbb{R}} E_{22}.$$

Analogously, we obtain

$$\mathrm{im}_{\mathbb{R}} \begin{bmatrix} E_{11} & E_{12} & A_{11} & A_{12} & B_1 \\ 0 & E_{22} & 0 & A_{22} & 0 \end{bmatrix} = \mathbb{R}^{k_1} \times (\mathrm{im}_{\mathbb{R}} E_{22} + \mathrm{im}_{\mathbb{R}} A_{22}).$$

Again using Corollary 4.3(i) and the invariance of controllability at infinity under system equivalence, we see that $[E, A, B]$ is controllable at infinity if, and only if,

$$\mathbb{R}^{k_1} \times (\mathrm{im}_{\mathbb{R}} E_{22} + \mathrm{im}_{\mathbb{R}} A_{22}) = \mathbb{R}^{k_1} \times \mathrm{im}_{\mathbb{R}} E_{22},$$

which is equivalent to $\mathrm{im}_{\mathbb{R}} A_{22} \subseteq \mathrm{im}_{\mathbb{R}} E_{22}$.

(v) Since $\mathrm{rk}_{\mathbb{R}}[E_{11}, A_{11}, B_1] = k_1$ and $[E_{11}, A_{11}, B_1] \in \Sigma_{k_1,n_1,m}$ is completely controllable it holds

$$\mathrm{rk}_{\mathbb{C}}[\lambda E_{11} - A_{11}, B_1] = k_1 \quad \text{for all } \lambda \in \mathbb{C}.$$

Therefore, we have

$$\mathrm{rk}_{\mathbb{C}}[\lambda E - A, B] = \mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E_{11} - A_{11} & \lambda E_{12} - A_{12} & B_1 \\ 0 & \lambda E_{22} - A_{22} & 0 \end{bmatrix} = k_1 + \mathrm{rk}_{\mathbb{C}}(\lambda E_{22} - A_{22}),$$

and, analogously, $\mathrm{rk}_{\mathbb{R}(s)}[sE - A, B] = k_1 + \mathrm{rk}_{\mathbb{R}(s)}(sE_{22} - A_{22})$. Now applying Corollary 4.3 (vii) we find that $[E, A, B]$ is controllable in the behavioral sense if, and only if, $\mathrm{rk}_{\mathbb{R}(s)}(sE_{22} - A_{22}) = \mathrm{rk}_{\mathbb{C}}(\lambda E_{22} - A_{22})$ for all $\lambda \in \mathbb{C}$.

(vi) The proof of this statement is analogous to d). $\square$

Remark 7.6 (Kalman decomposition and controllability)**.** Note that the condition of the index of $sE_{22} - A_{22}$ being at most one in Corollary 7.5(iii) is equivalent to the system $[E_{22}, A_{22}, 0_{k_2,m}]$ being impulse controllable. Likewise the condition $\operatorname{im}_{\mathbb{R}} A_{22} \subseteq \operatorname{im}_{\mathbb{R}} E_{22}$ in (iv) is equivalent to $[E_{22}, A_{22}, 0_{k_2,m}]$ being controllable at infinity. Obviously, the conditions in (v) and (vi) are equivalent to behavioral controllability and stabilizability of $[E_{22}, A_{22}, 0_{k_2,m}]$, resp.
Furthermore, the converse statement to (ii) does not hold true. That is, the index of $sE_{22} - A_{22}$ being at most one is in general not sufficient for $[E, A, B]$ being impulse controllable. For instance, reconsider the system (7.3) which is not impulse controllable, but $sE_{22} - A_{22} = -1$ is of index one. Even in the case where $sE - A$ is regular, the property of the index of $sE_{22} - A_{22}$ being zero or one is not enough to infer impulse controllability of $sE - A$. As a counterexample, consider

$$[E, A, B] = \left[\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right].$$

$\diamond$

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