Hamburger Beiträge
zur Angewandten Mathematik

An analytic approach to the Collatz $3n+1$ problem for negative start values with an appendix of tables

Lothar Berg and Gerhard Opfer

The first part of this paper will be published in a modified version with the title "An analytic approach to the Collatz 3n+1 problem for negative start values" in Computational Methods and Function Theory (CMFT), in 2013.

Nr. 2012-11
December 2012

AN ANALYTIC APPROACH TO THE COLLATZ $3N+1$ PROBLEM FOR NEGATIVE START VALUES WITH AN APPENDIX OF TABLES

LOTHAR BERG* AND GERHARD OPFER†

Abstract. In three papers, Meinardus, 1987, and Berg and Meinardus, 1994, and 1995, have shown, that the Collatz $3n+1$ problem for positive integers n as start values can be put into the theory of complex analysis. Here we investigate the Collatz $3n+1$ problem for negative start values. This problem is equivalent to the $3n-1$ problem for positive start values. It is known, that this problems differs from the $3n+1$ problem. One aspect is that all positive start values are tending, at least empirically, to either one, five or seventeen. We describe the corresponding analytic problem for this case, where one has to show that there are not more than three linearly independent, holomorphic solutions for this problem. However, this problem remains open.

Key words. Collatz $3n+1$ problem for negative start values, $3n-1$ problem, linear operators acting on holomorphic functions, natural boundary.

AMS subject classifications. 11B37, 11B83, 30D05, 39B32, 39B72.

1. Introduction: The Collatz $3n+1$ problem for arbitrary integer start values. Throughout the paper, let $\mathbb{N}$ be the set of positive integers and $\mathbb{Z}$ be the set of all integers. Let us define

$$c(n) := \begin{cases} \dfrac{n}{2} & \text{for even } n, \\ \dfrac{3n+1}{2} & \text{for odd } n, \end{cases} \qquad n \in \mathbb{Z}, \tag{1.1}$$

and call the mapping $c : \mathbb{Z} \to \mathbb{Z}$ the *Collatz function*. Since $3n+1$ is always even, the quotient $(3n+1)/2$ may be even or odd. Therefore, one could modify the quotient to $q(n) := (3n+1)/2^d$, where $d \geq 1$ is an integer chosen such that $q(n)$ is an odd integer. This idea was already introduced by Crandall, 1978, [5]. In our paper, it will turn out (see Theorem 3.6) that above all, only the even numbers of a Collatz sequence are important. For a fixed $n \in \mathbb{Z}$ we further define the iterates

$$c^{(k+1)}(n) := c^{(k)}(c(n)), \quad c^{(0)}(n) := n, \; k \in \mathbb{N} \cup \{0\}. \tag{1.2}$$

The sequence

$$\{c^{(0)}(n), c^{(1)}(n), c^{(2)}(n), \ldots, c^{(k)}(n), \ldots\}$$

which is defined for every $n \in \mathbb{Z}$ is called *Collatz sequence of n*. The number n is also called the *start value* of the Collatz sequence. For $n = 0$ the Collatz sequence is $\{0, 0, \ldots\}$, and thus, of no further interest. For $n \in \mathbb{N}$ the still open question whether there is always a $k \in \mathbb{N}$ such that $c^{(k)}(n) = 1$ is treated in [1, 2, 11, 12, 13] and in a comprehensive newer survey of 2010 by Lagarias, [9]. The only paper by Collatz on this subject, written 1986 is [4]. There is another paper by Guy, [8], who warns to put any energy into research of the Collatz and similar other problems. Here we

*University of Rostock, Institute for Mathematics, Ulmenstraße 8, 18057 Rostock, Germany, lothar.berg@uni-rostock.de

†University of Hamburg, Faculty for Mathematics, Informatics, and Natural Sciences [MIN], Bundestraße 55, 20146 Hamburg, Germany, opfer@math.uni-hamburg.de

will, nevertheless, study the Collatz sequence for negative start values. In this case the behavior differs from the Collatz sequence for positive start values.

In order to avoid negative integers we will use the following simple transformation

$$s(n) := -c(-n) = \begin{cases} \dfrac{n}{2} & \text{for even } n, \\ \dfrac{3n-1}{2} & \text{for odd } n, \end{cases} \quad n \in \mathbb{N}, \tag{1.3}$$

which produces the same numbers as c, but only with a positive sign. We will also call the function s the *Collatz function*, and this will be the main object of our study. For $n \in \mathbb{N}$ we define the iterates $s^{(k)}(n)$ analogously to (1.2). Empirically, the Collatz sequence defined by the function s, given in (1.3), will eventually produce a value which is located in one of the three following sets:

$$S_1 := \{1\}, \quad S_2 := \{5, 7, 10\}, \quad S_3 := \{17, 25, 37, 55, 82, 41, 61, 91, 136, 68, 34\}. \tag{1.4}$$

See Böhm and Sontacchi, 1978, [3], p. 261, where the corresponding sequence for S_1 bears a mistake. The three sets are the beginning of Collatz sequences, defined by s for the three start values $1, 5, 17$, respectively, which continue periodically, such that 1 is a fixed point. The essential part of the whole paper is the following conjecture. See Wirsching, p. 13, [12] for more details.

Conjecture 1.1. *Let $n \in \mathbb{N}$ be arbitrary. Then, there is always a smallest k such that $s^{(k)}(n) \in S_j$ for $j \in \{1, 2, 3\}$, where S_j is defined in (1.4).*

Definition 1.2. Let $n \in \mathbb{N}$ be given. If $s^{(k)}(n) \in S_j$ for $j \in \{1, 2, 3\}$, we will write, that *n belongs to case $j \in \{1, 2, 3\}$*. The first k for which $s^{(k)}(n) \in S_j$ will be called the *length of the Collatz sequence for n* and it is sufficient to stop the Collatz sequence at this k. In the theoretically possible case that a certain start value $n \in \mathbb{N}$ does not belong to one of the cases 1, 2, or 3, we will speak about *case* 0. We define four subsets $\mathbb{N}_j, j = 0, 1, 2, 3$ of $\mathbb{N}$:

$$\mathbb{N}_j := \{n \in \mathbb{N} : s^{(k)}(n) \in S_j\}, j \in \{1, 2, 3\}, \mathbb{N}_0 = \{n \in \mathbb{N} : n \notin \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N}_3\}. \tag{1.5}$$

Note, that $\mathbb{N}_j$, $j = 0, 1, 2, 3$ are pairwise distinct and $\mathbb{N}_j \neq \emptyset$, $j = 1, 2, 3$. With this definition the Collatz conjecture is equivalent to $\mathbb{N}_0 = \emptyset$ or to $\mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N}_3 = \mathbb{N}$.

For all $n \in [1, 10^4]$ we can determine the case j to which n belongs, and we can define a $10^2 \times 10^2$ matrix with the value j at the position n of that matrix, where we count columnwise, and color the point n red in case 1, green in case 2, and blue in case 3. By this we obtain the color graphic of Figure 1.4.

The graphic displays a (surprising) vertical structure. This means that there is a tendency that *neighbors* $n, n+1, \ldots, n+k$ for a variable $k \geq 1$ belong to the same case. In the interval $[1, 1600]$ we find 16 neighbors $1089, 1090, \ldots, 1104$ which all belong to case one. See Table 4.2. If we go to larger start values we find 222 neighbors 84 614 707 to 84 614 928 belonging all to case 1, 190 neighbors from 90 988 835 to 90 989 024 belonging to case 2, and 314 neighbors from 70 754 307 to 70 754 620 belonging to case 3. These are the longest series of neighbors up to 10^8.

We close this section with a little table on the frequencies of the start values $n \in [1, 10^k], k = 1, 2, \ldots, 8$ belonging to the case 1, 2, or 3. We see that there is an almost uniform distribution. The third case, where the start value n terminates in S_3

is a little more frequent than the other two cases.

TABLE 1.3. Frequencies of start values $n \in [1, 10^k]$, $k = 1, 2, \ldots, 8$ belonging to case 1, 2, or 3.

n	Case 1	Case 2	Case 3
$n \in [1, 10]$	6	4	0
$n \in [1, 10^2]$	38	31	31
$n \in [1, 10^3]$	349	306	345
$n \in [1, 10^4]$	3244	3213	3543
$n \in [1, 10^5]$	33030	32104	34866
$n \in [1, 10^6]$	327679	323351	348970
$n \in [1, 10^7]$	3273791	3244985	3481224
$n \in [1, 10^8]$	32697318	32470805	34831877

These numbers were computed by means of a MATLAB program, Version 7.12.0.635 (R2011a), on an Apple computer with Prozessor 1.8 GHz Intel Core i7.

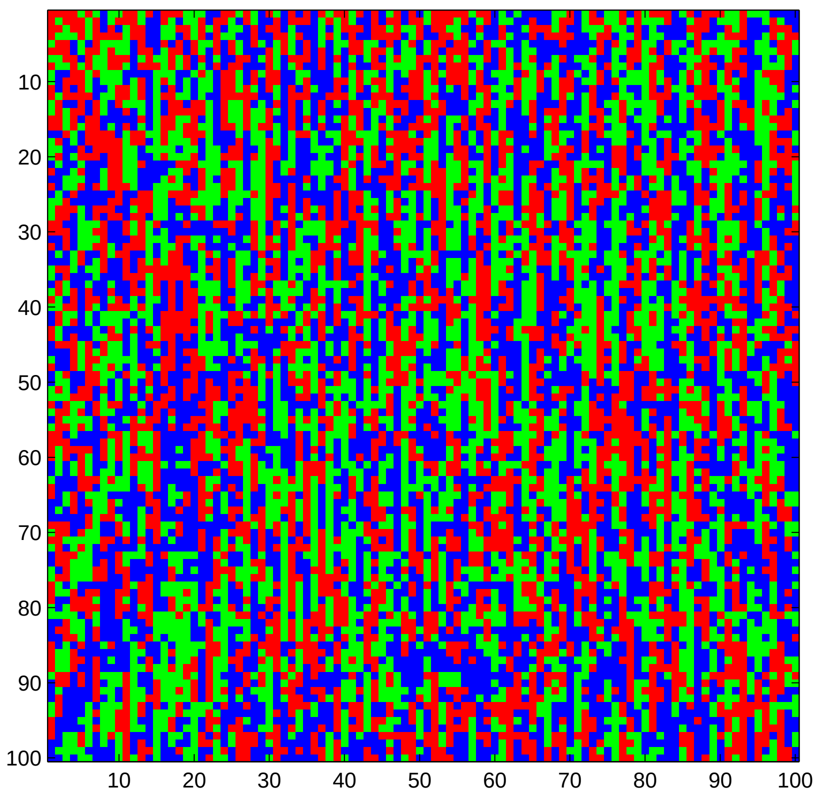

FIGURE 1.4. A graphical display of the three cases of the first 10^4 start values, case 1 is marked red, case 2 green, case 3 blue.

2. The analytic approach. Let $\mathbb{C}$ be the notation for the set of complex numbers and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ for the open, unit disk in $\mathbb{C}$. For $n \in \mathbb{N}$ and for the Collatz function c defined in (1.1) the analytic approach was introduced by Meinardus, 1987, [10] and by Berg and Meinardus, 1994, 1995, [1, 2]. For this reason, we will be brief in this section. In order to study the Collatz sequence generated by s we introduce holomorphic functions $h:\mathbb{D}\to\mathbb{C}$ defined by the power series

$$h(z) := \sum_{n=1}^{\infty} h_n z^n \tag{2.1}$$

with coefficients h_n satisfying

$$h_n = h_{s(n)}, \text{ for all } n\in\mathbb{N},\ h_n\in\{0,1\}, \tag{2.2}$$

where s is the Collatz function defined in (1.3). According to (2.2), the power series defined in (2.1) will always converge in $\mathbb{D}$ and $|h(z)|\le |z|/(1-|z|)$ for all $z\in\mathbb{D}$.

EXAMPLE 2.1. *Let $h_n = 1$ for all $n\in\mathbb{N}$. Then, the properties mentioned in (2.2) are satisfied and $h(z)=\frac{z}{1-z}$.*

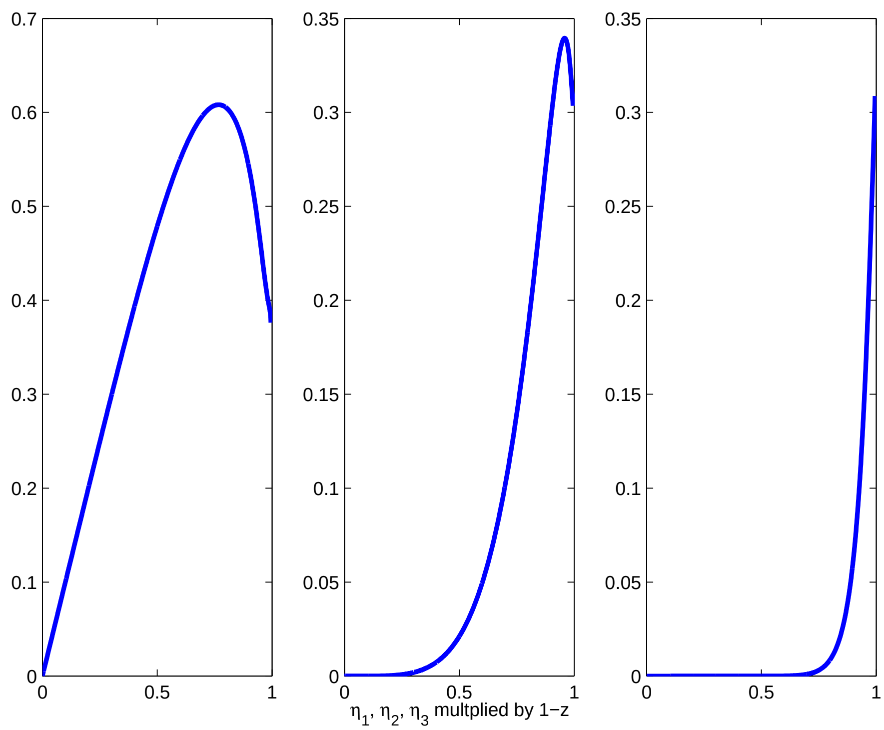

FIGURE 2.2. The three holomorphic functions η_j corresponding to the three cases $j=1,2,3$, multiplied by $1-z$.

THEOREM 2.3. *The Collatz conjecture for the Collatz function s, defined in (1.3) is true if and only if there exist exactly three linearly independent holomorphic functions defined on the open unit disk $\mathbb{D}$ with Taylor coefficients $h_n\in\{0,1\}$, with properties given in (2.2).*

Proof. Let

$$\eta_j(z) := \sum_{n\in\mathbb{N}_j} z^n,\quad j=0,1,2,3,$$

where $\mathbb{N}_j$ are defined in (1.5). Since $\mathbb{N}_j$, $j = 0, 1, 2, 3$ are pairwise distinct and $\mathbb{N}_j \neq \emptyset$, $j = 1, 2, 3$, the linear space $\langle \eta_0, \eta_1, \eta_2, \eta_3 \rangle$ has dimension three if and only if $\mathbb{N}_0 = \emptyset$ and, therefore, $\eta_0 = 0$. □

The graphs of the three functions η_j, $j = 1, 2, 3$ are all very much alike the graph of $f(z) = z/(1-z)$, therefore, we do not show these graphs. However, the graphs of $\eta_j(z)(1-z), j = 1, 2, 3$ are more interesting, and are represented in Figure 2.2. Since $\eta_1(z)+\eta_2(z)+\eta_3(z) = z/(1-z)$, at least empirically, $(\eta_1+\eta_2+\eta_3)(1-z) = z$ follows. In order to draw these graphs we have reduced η_j to the corresponding Taylor polynomial t_j of degree 1000. Since all coefficients are in $\{0, 1\}$ there is no danger of cancelation. The interval $[0, 0.995]$ was discretized in the form $x = 0, 0.001, 0.002, \ldots, 0.994, 0.995$. We have reduced the interval $[0, 1]$ to $[0, 0.995]$ in order to avoid the influence of the canceled remainder. From Figure 2.2 we can see that

$$\eta_j(z) \leq c_j \frac{1}{1-z}, \quad c_1 := 0.61,\ c_2 := 0.35,\ c_3 := 0.35, z \in [0, 1[.$$

The functions h will satisfy some functional equations which we will develop now.

Theorem 2.4. *Let h be given as in (2.1), (2.2) and let*

$$\lambda := \exp(2\pi \mathbf{i}/3).$$

Then, h satisfies

$$h(z) + h(-z) = 2h(z^2), \tag{2.3}$$

$$h(z^3) - h(-z^3) = \frac{2}{3} z \sum_{\nu=0}^{2} \lambda^{-\nu} h(\lambda^\nu z^2). \tag{2.4}$$

Proof. Obviously, by using (2.2), we have,

$$h(z) = \sum_{n=1}^{\infty} h_{2n} z^{2n} + \sum_{n=1}^{\infty} h_{2n-1} z^{2n-1} = h(z^2) + \sum_{n=1}^{\infty} h_{3n-2} z^{2n-1} \Rightarrow$$

$$h(z^3) = h(z^6) + \sum_{n=1}^{\infty} h_{3n-2} z^{6n-3}. \tag{2.5}$$

A further calculation by using $\lambda^3 = 1$ and $1 + \lambda + \lambda^2 = 0$ yields

$$h(z^2) + \frac{1}{\lambda} h(\lambda z^2) + \frac{1}{\lambda^2} h(\lambda^2 z^2) = 3 \sum_{n=1}^{\infty} h_{3n-2} z^{6n-4},$$

and a comparison with (2.5) shows that

$$h(z^3) = h(z^6) + \frac{z}{3} \sum_{\nu=0}^{2} \lambda^{-\nu} h(\lambda^\nu z^2), \tag{2.6}$$

which is equivalent to the system (2.3), (2.4). □

Remark 2.5. The three linearly independent functions η_j, $j = 1, 2, 3$ are three linearly independent solutions of (2.4) in $\mathbb{D}$. We conjecture, that they have $\partial\mathbb{D}$ (boundary of $\mathbb{D}$) as the natural boundary, i. e. they cannot be continued analytically to $|z| > 1$. The decisive tool for proving this statement is a theorem by Fabry, [7] on power series with gaps. We quote from Erdős, p. 102, [6]: "The gap theorem of Fabry states that if $f(z) = \sum a_k z^{n_k}$ is a power series whose circle of convergence is the unit circle and $\lim n_k/k = \infty$ then the unit circle is the natural boundary of $f(z)$." According to the vertical structure of the graph in Figure 1.4, mentioned at the end of Section 1, we expect that the condition of Fabry is satisfied, since neighbors in one of the three η functions imply gaps in the other two. On the other hand, (2.6) has the two linearly independent solutions $h = 1$ and $h = \frac{z}{1-z}$ in the whole plane $\mathbb{C}$. Note that equation (2.6) is closely connected with formula (29), $h(z^3) = h(z^6) + \frac{1}{3z}\sum_{\nu=0}^{2} \lambda^\nu h(\lambda^\nu z^2)$, of [1] by the changes $z \to 1/z$ and $h(z) \to h(1/z)$.

Lemma 2.6. *Equation (2.3) is equivalent to*

$$h_{2n} = h_n \text{ for all } n \in \mathbb{N}, \tag{2.7}$$

and equation (2.4) is equivalent to

$$h_{2n-1} = h_{3n-2} \text{ for all } n \in \mathbb{N}. \tag{2.8}$$

Proof. Equation (2.7) follows immediately from (2.3) and vice versa. Let us now consider equation (2.8). Multiplying (2.8) by $2z^{6n-3}$ and summing over n yields, by means of (2.1), (2.2), and (2.5), the equivalence of (2.8) and (2.4). □

Lemma 2.7. *Let $n > 1$ be odd. Then n can be represented as $n = 2^k m + 1$ with $k \geq 1, m = 2p+1, p \geq 0$. By repeated application of (2.2) we obtain*

$$h_{2^k m+1} = h_{3\cdot 2^{k-1}m+1} = \cdots = h_{3^j\cdot 2^{k-j}m+1} = \cdots = h_{3^k m+1}, \tag{2.9}$$

where the first k indices are odd, and the last one, the $k+1$st is even.

Proof. Straightforward using (2.8). □

A solution of (2.7) was already given in [1, 2]. But we do not need it here.

Theorem 2.8. *The general solution of (2.4) can be expressed in the form*

$$h(z) = \sum_{n=1}^{\infty} h_n p_n(z), \quad h_n \in \{0, 1\}, \tag{2.10}$$

where $p_1(z) = z$ and for $n > 1$ we have

$$p_n(z) := \begin{cases} 0 & \text{for } n \equiv 1 \bmod 6 \text{ and } n \equiv 4 \bmod 6, \\ z^n & \text{for } n \equiv 0 \bmod 6 \text{ and } n \equiv 2 \bmod 6, \\ q_n(z) & \text{for } n \equiv 3 \bmod 6 \text{ and } n \equiv 5 \bmod 6, \end{cases} \tag{2.11}$$

where

$$q_n(z) := z^{r_{n0}} + z^{r_{n1}} + \cdots + z^{r_{nk}} \tag{2.12}$$

with

$$r_{nj} = 3^j 2^{k-j} m + 1, \quad j = 0, 1, \ldots, k, k \in \mathbb{N} \cup \{0\}, \tag{2.13}$$

where $n = 2^k m + 1 = r_{n0}$ and $m \equiv \pm 1 \bmod 6$.

Proof. The index of h on the right-hand side of (2.8) is $\equiv 1 \bmod 3$ (i.e. $\equiv 1 \bmod 6$ or $\equiv 4 \bmod 6$), hence all h with such indices equal h with a smaller index according to (2.9) and are collected with this h in (2.10). An even index, which is $\equiv 0 \bmod 6$ or $\equiv 2 \bmod 6$, is not $\equiv 1 \bmod 3$, and cannot appear in (2.8). Hence, a term $h_n z^n$ with such an index cannot be collected with other terms. Finally, the numbers n, which are $\equiv 3 \bmod 6$ or $\equiv 5 \bmod 6$, can be written as $n = 2^k m + 1$ with $m \equiv \pm 1 \bmod 6$, so that (2.12) follows from (2.9). □

Remark 2.9. The case $k = 0$ in the above formulas (2.12) to (2.13) implies $q_n(z) = z^n$, such that the middle case in (2.11) could be included in the third case.

Example 2.10. The first terms of the general solution of (2.4) are

$$\begin{aligned} h(z) = {} & h_1 z + h_2 z^2 + h_3(z^3 + z^4) + h_5(z^5 + z^7 + z^{10}) + h_6 z^6 + h_8 z^8 + \\ & h_9(z^9 + z^{13} + z^{19} + z^{28}) + h_{11}(z^{11} + z^{16}) + h_{12} z^{12} + h_{14} z^{14} + \\ & h_{15}(z^{15} + z^{22}) + h_{17}(z^{17} + z^{25} + z^{37} + z^{55} + z^{82}) + h_{18} z^{18} + h_{20} z^{20} + \\ & h_{21}(z^{21} + z^{31} + z^{46}) + h_{23}(z^{23} + z^{34}) + h_{24} z^{24} + h_{26} z^{26} + \cdots \end{aligned}$$

Lemma 2.11. *In the expansion for h, given in (2.10) all exponents $k \in \mathbb{N}$ appear exactly once.*

Proof. If we put $h_n = 1$ for all $n \in \mathbb{N}$ we have $h(z) = \frac{z}{1-z}$ which proves the theorem. See Example 2.1. ◻

3. A solution technique. We follow here an idea of [11]. Instead of looking for the solutions of the functional equation (2.3) we will be looking at the zero solutions of the following linear operator:

$$U[h](z) := \frac{1}{2}\left(h(z) + h(-z)\right) - h(z^2). \tag{3.1}$$

By another investigation, [11], we know, that this operator is continuous. Thus, the application to a power series can be executed term by term. If we apply this operator to h, defined in (2.10), then the solutions of $U[h] = 0$ (0 = zero function) are the solutions of the system of the two functional equations (2.3), (2.4). Let p_n be defined as in (2.10), (2.11). Then, we have $U[p_1](z) = -z^2$ and for $n > 1$ we have

$$U[p_n](z) = \begin{cases} 0 & \text{for } n \equiv 1 \bmod 6 \text{ and } n \equiv 4 \bmod 6, \\ z^n - z^{2n} & \text{for } n \equiv 0 \bmod 6 \text{ and } n \equiv 2 \bmod 6, \\ z^{r_{nk}} - q_n(z^2) & \text{for } n \equiv 3 \bmod 6 \text{ and } n \equiv 5 \bmod 6, \end{cases} \tag{3.2}$$

where r_{nk} is the highest exponent of q_n. See (2.12), (2.13). Note, that all powers occurring in (3.2) are even. Following the Remark 2.9, formula (3.2) implies

$$U[h](z) = -h_1 z^2 + \sum_{n \not\equiv 1 \bmod 3} h_n(z^{r_{nk}} - q_n(z^2)), \tag{3.3}$$

where q_n and r_{nk} are defined in (2.12), (2.13), respectively. We will use the notation

$$\pi_n := U[p_n]$$

and present some examples in Table 4.6. The same table, only ordered with respect to the exponents of the positive term can be found in Table 4.7. Using Lemma 2.11 and (3.1), it is clear, that the exponents of the positive terms of π_n, resulting from $\frac{1}{2}(h(z) + h(-z))$, exactly cover the even numbers from 2 on, and the same applies for the exponents of the negative terms, stemming from $h(z^2)$ in the definition of $U[h]$. Thus, we can write

$$U[h](z) = \sum_{\ell=1}^{\infty} (h_{\ell'} - h_{\ell''}) z^{2\ell} \tag{3.4}$$

where ℓ' is the position n of the exponent ℓ in the set of exponents of the positive terms in the expansion (3.3), and ℓ'' is the position n of ℓ in the set of exponents of the negative terms, i. e. $\pi_{\ell'}(z) = z^{2\ell} - \cdots$, $\pi_{\ell''}(z) = +\cdots - z^{2\ell} - \cdots$. There are some examples in Table 3.1.

TABLE 3.1. The numbers ℓ', ℓ'' in (3.4) as functions of 2ℓ.

2ℓ	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36
ℓ'	2	3	6	8	5	12	14	11	18	20	15	24	26	9	30	32	23	36
ℓ''	1	2	3	3	5	6	5	8	9	5	11	12	9	14	15	11	17	18

2ℓ	38	40	42	44	46	48	50	52	54	56	58	60	62	64	66	68	70	72
ℓ'	38	27	42	44	21	48	50	35	54	56	39	60	62	29	66	68	47	72
ℓ''	9	20	21	15	23	24	17	26	27	9	29	30	21	32	33	23	35	36

2ℓ	74	76	78	80	82	84	86	88	90	92	94	96	98	100	102	104	106	108
ℓ'	74	51	78	80	17	84	86	59	90	92	63	96	98	45	102	104	71	108
ℓ''	17	38	39	27	41	42	29	44	45	21	47	48	33	50	51	35	53	54

There is one interesting case. Because of $5' = 5'' = 5$ we have $\pi_5(z) = z^{10} - z^{10} - z^{14} - z^{20}$ which means that the 10th power cancels.

LEMMA 3.2. *The linear operator $U[h]$, applied to h defined in (2.1) will vanish if and only if*

$$h_{\ell'} - h_{\ell''} = 0 \text{ for all } \ell = 1, 2, \ldots, \tag{3.5}$$

where the meaning of $h_{\ell'}, h_{\ell''}$ is explained in connection with formula (3.4).

Proof. The Taylor expansion of $U[h]$ is given in (3.4) and the statement is elementary. □

Let us have a look at the following three tables.

TABLE 3.3. Case 1.

2ℓ	ℓ'	ℓ''
30	30	15
22	15	11
16	11	8
8	8	3
4	3	2
2	2	1

TABLE 3.4. Case 2.

2ℓ	ℓ'	ℓ''
224	224	75
112	75	56
56	56	9
28	9	14
14	14	5
10	5	5

TABLE 3.5. Case 3

2ℓ	ℓ'	ℓ''
100	45	50
50	50	17
82	17	41
136	41	68
68	68	23
34	23	17

The Tables 3.3 to 3.5 are all constructed in the same manner. We will call them *2ℓ tables*. There is a 2ℓ column and an ℓ' and an ℓ'' column and ℓ', ℓ'' are defined in (3.4) depending on 2ℓ. Let $2\ell_1, \ell_1', \ell_1''$ be one row of such a table, then the next row $2\ell_2, \ell_2', \ell_2''$ is chosen in such a way that $\ell_2' = \ell_1''$. This determines the next row uniquely. There is always a continuation to the next row with one exception. If a row happens to be $2, 2, 1$, then a continuation is impossible, because there is no $\ell' = 1$. See Tables 3.1 and 4.6. Since the inverse s^{-1} of s cannot be uniquely defined, e.g. $s^{-1}(4) = 8$ or $s^{-1}(4) = 3$, the 2ℓ tables cannot be uniquely continued in the direction of the top of that table.

We note that the set of all integers ℓ' will cover $\mathbb{N}' := \{n \in \mathbb{N} : n \not\equiv 1 \bmod 3\}$ and that the mapping $2\mathbb{N} \to \mathbb{N}'$ defined by

$$2\ell \to \ell'$$

is invertible on $\mathbb{N}'$. By having a look at Table 3.1 and some finite extension, we can deduce from (3.5) the following set of three explicit solution h_j with $j \leq 50$:

$$h_1 = h_j, \quad j = 2, 3, 6, 8, 11, 12, 15, 24, 29, 30, 32, 39, 44, 48,$$
$$h_5 = h_j, \quad j = 9, 14, 18, 20, 26, 27, 35, 36, 38, 47,$$
$$h_{17} = h_j, \quad j = 21, 23, 33, 41, 42, 45, 50.$$

Thus, h_j are exactly the coefficients of the corresponding η functions, disregarding the coefficients h_j with $j \equiv 1 \bmod 3$. However, the main problem is not to compute the three solutions, but to show that there is no room for a fourth solution. The pairs (ℓ', ℓ'') have the property that every number from $\mathbb{N}'$ will appear exactly once in the first component of (ℓ', ℓ''), and all numbers from $\mathbb{N}' \cup \{1\}$ will appear in the second component, possibly several, but finitely many times, with the exception of $\ell'' \equiv 0 \bmod 6$ and $\ell'' \equiv 2 \bmod 6$ which appear also exactly once. We can define the following equivalence relation: The pairs $(\ell'_1, \ell''_1), (\ell'_2, \ell''_2)$ will be called *equivalent* if they have a component in common. Let $\ell'_1 \neq \ell'_2$, then the two pairs are equivalent if and only if $\ell'_1 = \ell''_2$ or $\ell'_2 = \ell''_1$ or $\ell''_1 = \ell''_2$. In this setting, the Collatz problem reduces to showing that there are exactly three equivalence classes. The first 250 triples $(2\ell, \ell', \ell'')$ in the order of $2\ell, \ell', \ell''$, respectively, are given in Table 4.8.

Information about the 2ℓ tables is collected in the following theorem and its proof.

THEOREM 3.6. *Let $2\ell \in 2\mathbb{N}$ be an even but otherwise arbitrary integer ≥ 4. Then, the 2ℓ column of the corresponding 2ℓ table consists of the even entries of the Collatz sequence for the start value 2ℓ.*

Proof. From (2.12), (3.3), and (3.4) it follows that $\ell' = n$ belongs to $2\ell = r_{nk}$ and that $\ell'' = n$ belongs to $2\ell = r_{nj}$ using the notation of Theorem 2.8. The clou of the further proof is to recognize that an arbitrary even integer $2\ell > 2$ can be written uniquely not only as $2\ell = 2r_{nj}$ but also as $2\ell = 2r_{n'k'} = 3^{k'}m' + 1$ with $n' = 2^{k'} + 1$ and $m' \equiv \pm 1 \bmod 6$. Hence, $(2\ell, \ell', \ell'') = (2\ell, n', n)$ follows. On the other hand, the next even number in the Collatz sequence of $2\ell = 2r_{nj}$ is $2\ell_1 = r_{nk} = 3^k m + 1$ and, as we already know, $\ell'_1 = n = 2^k m + 1$ belongs to this number. This implies, that the pair $(2\ell_1, \ell'_1)$ contains the first two components of the next row $(2\ell_1, \ell'_1, \ell''_1)$ in the 2ℓ table. ◻

COROLLARY 3.7.
For $2\ell \equiv 0 \bmod 6$ we have $\ell' = 2\ell, \ell'' = \ell$.
For $2\ell \equiv 2 \bmod 6$ we have $\ell' = 2\ell, \ell'' < \ell$.
For $2\ell \equiv 4 \bmod 6$ we have $\ell' < 2\ell, \ell'' = \ell$.
For all cases we have $\ell' \leq 2\ell, \ell'' \leq \ell$.

THEOREM 3.8. *Let every 2ℓ table $(2l \geq 4)$ have the property that the 2ℓ column contains an entry which is smaller than the first entry 2ℓ. Then, the Collatz conjecture is true.*

Proof. Let $n_\aleph$ be a large positive integer such that the Collatz conjecture is true for all $n \leq n_\aleph$. According to Theorem 3.6 the first column of the 2ℓ table is a Collatz sequence for the start value 2ℓ, with the odd entries missing. Our assumptions imply that the Collatz sequence contains an element $2\ell_1 < 2\ell$, thus, $2\ell_1 \leq 2\ell - 2$. If $2\ell_1 > n_\aleph$, this argument can be repeated and, eventually we have $2\ell_j \leq 2\ell - 2j$ and, for sufficiently large j, we have $2\ell_j \leq n_\aleph$ and the Collatz conjecture is true. ◻

Note, that the one row 2ℓ table $(2, 2, 1)$ is excluded in the above theorem, since it has no continuation. But, nevertheless, the Collatz conjecture is true for $n = 2$. In the proof of Theorem 3.8 we have introduced a quantity $n_\aleph$. According to Table 1.3 we have $n_\aleph \geq 10^8$.

Acknowledgment. The research of the second mentioned author was supported by the German Science Foundation, DFG, GZ: OP 33/19-1. The same author would like to thank the librarian, Ruth Ellebracht, of the Department of Mathematics of the University of Hamburg for her permanent help in searching the relevant literature.

REFERENCES

[1] L. Berg and G. Meinardus, *Functional equations connected with the Collatz problem*, Results Math., **25** (1994), pp. 1–12.

[2] L. Berg and G. Meinardus, *The* $3n+1$ *Collatz problem and functional equations*, Rostock. Math. Kolloq. **48** (1995), pp. 11–18.

[3] C. Böhm and G. Sontacchi, *On the existence of cycles of given length in integer sequences like* $x_{n+1} = x_n/2$ *if* x_n *even, and* $x_{n+1} = 3x_n + 1$ *otherwise*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (8) Mat. Appl. Vol. LXIV (1978), pp. 260–264.

[4] L. Collatz, *Über die Entstehung des (3n+1) Problems*, J. Qufu Norm. Univ. Nat. Sci. Ed., **3** (1986), pp. 9–11 (in Chinese).[1]

[5] R. E. Crandall, *On the "*$3x + 1$*" problem*, Math. Comp., **32** (1978), pp. 1281–1292.

[6] P. Erdős, Note on the converse of Fabry's gap theorem, Trans. AMS, **57** (1945), pp. 102–104.

[7] E. Fabry, *Sur les points singuliers d'une fonction donée par son développement en série et l'impossibilité du prolongement analytique dans des cas très généraux*, Ann. Sci. Ecole Norm. Sup., 3^e série, **13** (1896), pp. 367–399.

[8] R. K Guy, *Don't try to solve these problems*, Amer. Math. Monthly, **90** (1983), pp. 35–41.

[9] J. C. Lagarias, ed., The Ultimate Challenge: The $3x + 1$ Problem, Amer. Math. Soc., 2010, 344 pp.

[10] G. Meinardus, *Über das Syracuse-Problem*, Preprint Nr. 67, Universität Mannheim, 1987, 32 pp.

[11] G. Opfer, *An analytic approach to the Collatz* $3n + 1$ *problem.* Hamburger Beiträge zur Angewandten Mathematik, Preprint Nr. 2011-09, May 2011. http://preprint.math.uni-hamburg.de/public/papers/hbam/hbam2011-09.pdf.

[12] G. J. Wirsching, The dynamical system generated by the $3n + 1$ function, Lecture Notes in Mathematics, 1681, Springer, Berlin, 1998, 159 pp.

[13] G. J. Wirsching, *Über das* $3n + 1$ *Problem*, Elem. Math. **55** (2000), pp. 142–155.

[1] The German title as used here was handwritten by Collatz on a copy of the Chinese paper.

4. Appendix: Tables related to the $3n-1$ problem.

Contents:

TABLE 4.1. The case numbers of the odd start values $1, 3, \ldots, 999$

		100	200	300	400	500	600	700	800	900
1	1	1	1	1	2	1	2	3	2	2
3	1	1	2	1	1	2	1	3	1	3
5	2	1	1	3	2	3	1	2	2	2
7	2	2	1	1	3	1	2	2	1	1
9	2	3	3	1	1	2	1	2	2	2
11	1	3	1	3	1	2	3	3	2	1
13	2	1	3	3	1	3	1	2	3	1
15	1	1	2	1	3	1	1	3	3	3
17	3	3	1	2	3	1	1	1	1	3
19	2	2	3	3	1	3	1	1	1	1
21	3	2	3	3	2	3	3	3	1	1
23	3	3	2	2	2	3	3	2	1	1
25	3	2	3	1	3	3	3	3	3	1
27	2	1	1	1	2	3	1	1	1	3
29	1	1	3	3	2	2	2	1	3	3
31	3	3	1	3	2	2	2	1	3	3
33	3	2	3	2	1	3	3	1	2	1
35	2	1	1	2	3	2	2	1	3	3
37	3	1	2	3	3	2	3	3	2	3
39	1	2	3	1	3	1	1	3	2	1
41	3	1	2	2	3	2	3	2	2	2
43	1	2	3	3	3	3	2	3	1	2
45	3	1	1	1	2	1	3	1	2	2
47	2	3	3	1	2	1	1	2	2	2
49	3	2	1	3	3	1	1	2	2	3
51	2	1	2	3	1	1	3	1	3	2
53	1	3	1	2	2	3	1	2	1	1
55	3	1	2	2	2	2	3	1	3	3
57	1	1	3	2	3	3	3	3	1	1
59	1	2	1	1	1	1	3	1	2	1
61	3	2	3	2	1	2	3	3	2	3
63	2	3	3	3	1	1	1	2	1	2
65	1	3	2	1	3	2	2	2	3	3
67	3	2	3	1	3	3	2	2	1	3
69	1	1	1	3	3	1	2	3	3	1
71	1	1	2	2	2	2	2	1	1	3
73	3	1	1	1	2	2	3	3	3	1
75	2	3	1	2	2	2	1	1	3	1
77	1	2	3	2	3	3	3	3	3	3
79	1	3	3	1	1	1	2	3	3	1
81	2	2	2	2	3	1	2	3	3	1
83	3	1	1	2	2	3	1	3	3	3
85	1	3	2	3	1	3	3	2	1	3
87	1	2	2	1	1	3	1	3	2	3
89	2	1	1	3	1	3	1	2	2	2
91	3	2	3	3	3	3	1	2	2	3
93	2	1	3	3	3	2	3	2	2	2
95	1	3	3	3	2	2	3	2	2	1
97	1	3	2	2	1	2	3	2	1	2
99	3	2	2	2	2	3	3	3	3	2

TABLE 4.2. The first 500 start values with case = 1.

		50	100	150	200	250	300	350	400	450
1	1	129	258	403	548	690	844	1002	1156	1307
2	2	130	259	404	549	691	848	1007	1157	1308
3	3	135	260	409	550	692	853	1011	1158	1325
4	4	137	269	410	551	694	857	1012	1160	1326
5	6	138	270	411	552	696	863	1014	1161	1327
6	8	141	273	412	559	704	866	1016	1162	1335
7	11	142	274	413	563	717	867	1024	1203	1343
8	12	145	275	414	564	718	868	1029	1204	1345
9	15	151	276	419	566	719	871	1030	1205	1349
10	16	154	282	420	568	727	885	1032	1206	1350
11	22	155	283	422	569	729	897	1033	1208	1352
12	24	156	284	424	578	730	902	1034	1210	1355
13	29	157	289	433	579	731	904	1035	1211	1356
14	30	158	290	434	580	732	907	1036	1212	1360
15	32	169	301	451	581	733	908	1040	1215	1365
16	39	170	302	452	602	734	911	1049	1217	1366
17	43	171	303	454	603	735	912	1071	1218	1368
18	44	172	307	456	604	745	913	1075	1221	1369
19	48	173	308	459	605	746	918	1076	1225	1373
20	53	174	309	460	606	751	919	1077	1226	1374
21	57	176	310	461	609	755	920	1078	1227	1376
22	58	183	312	462	613	756	921	1080	1228	1377
23	59	189	314	463	614	758	922	1089	1229	1378
24	60	190	315	464	615	759	923	1090	1230	1379
25	64	192	316	470	616	760	924	1091	1231	1380
26	65	193	325	472	617	768	925	1092	1232	1381
27	69	194	327	479	618	771	926	1093	1234	1382
28	71	201	338	480	619	772	928	1094	1235	1384
29	77	202	339	485	620	774	933	1095	1236	1387
30	78	205	340	487	624	775	939	1096	1238	1388
31	79	206	342	489	627	776	940	1097	1240	1392
32	85	207	344	490	628	803	944	1098	1248	1399
33	86	210	345	497	630	804	953	1099	1254	1408
34	87	211	346	498	632	806	957	1100	1256	1429
35	88	212	347	501	639	807	958	1101	1259	1434
36	95	217	348	506	647	808	959	1102	1260	1435
37	96	226	352	507	649	817	960	1103	1264	1436
38	97	227	359	508	650	818	969	1104	1271	1437
39	101	228	365	512	653	819	970	1117	1277	1438
40	103	230	366	515	654	820	973	1118	1278	1445
41	105	231	367	516	663	821	974	1126	1279	1453
42	106	232	373	517	675	822	975	1128	1281	1454
43	113	235	378	518	676	823	978	1131	1285	1455
44	114	236	379	520	678	824	979	1132	1293	1457
45	115	240	380	538	680	826	980	1136	1294	1458
46	116	245	384	539	683	827	981	1137	1298	1459
47	118	249	386	540	684	828	994	1138	1299	1460
48	120	253	387	545	687	838	995	1143	1300	1461
49	127	254	388	546	688	840	996	1151	1301	1462
50	128	256	402	547	689	843	1001	1155	1306	1463

TABLE 4.3. The first 500 start values with case = 2.

		50	100	150	200	250	300	350	400	450
1	5	161	335	500	667	811	990	1130	1286	1446
2	7	162	341	502	668	812	993	1139	1288	1448
3	9	167	353	503	669	816	997	1140	1291	1481
4	10	177	354	504	670	832	998	1142	1292	1482
5	13	178	355	509	671	833	999	1144	1296	1483
6	14	181	356	510	679	837	1000	1145	1317	1484
7	18	186	357	511	681	839	1003	1146	1319	1488
8	19	187	361	529	682	841	1004	1147	1329	1489
9	20	188	362	530	705	842	1005	1148	1330	1494
10	26	191	371	531	706	845	1006	1149	1331	1495
11	27	199	372	532	707	846	1008	1150	1332	1496
12	28	203	374	535	708	847	1013	1152	1333	1497
13	35	204	375	537	709	849	1017	1173	1334	1498
14	36	208	376	541	710	854	1018	1177	1336	1499
15	38	214	377	542	712	856	1019	1183	1337	1500
16	40	215	381	555	713	858	1020	1185	1338	1503
17	47	216	382	556	714	859	1021	1186	1339	1504
18	51	223	383	560	722	860	1022	1187	1340	1506
19	52	224	397	561	723	861	1023	1188	1341	1507
20	54	237	398	562	724	862	1041	1189	1342	1508
21	56	238	399	565	741	864	1047	1190	1351	1509
22	63	241	401	570	742	887	1055	1191	1357	1519
23	70	242	405	571	744	889	1057	1192	1358	1523
24	72	250	406	572	747	890	1058	1193	1359	1524
25	75	251	408	573	748	891	1059	1194	1361	1525
26	76	252	416	574	749	892	1060	1195	1362	1526
27	80	255	421	575	750	893	1061	1196	1363	1527
28	81	265	423	576	752	894	1062	1200	1364	1528
29	89	266	427	593	753	895	1063	1201	1407	1530
30	93	271	428	594	754	896	1064	1202	1409	1531
31	94	278	429	595	762	901	1069	1207	1410	1532
32	102	280	430	596	763	905	1070	1213	1411	1533
33	104	281	431	597	764	906	1073	1214	1412	1534
34	107	285	432	598	765	909	1074	1216	1413	1561
35	108	286	445	600	766	910	1079	1241	1414	1569
36	112	287	446	601	767	941	1082	1249	1415	1570
37	119	288	447	607	785	942	1083	1253	1416	1577
38	121	297	448	608	789	943	1084	1255	1417	1578
39	125	298	453	629	791	945	1110	1257	1418	1581
40	126	299	455	631	793	946	1111	1258	1419	1582
41	133	300	471	634	794	947	1112	1261	1420	1583
42	139	304	473	635	795	948	1113	1262	1424	1585
43	140	317	474	636	796	950	1119	1263	1425	1586
44	143	318	475	640	797	951	1120	1267	1426	1587
45	144	320	476	643	798	952	1121	1268	1427	1588
46	149	322	482	644	801	963	1122	1269	1428	1589
47	150	323	483	646	802	964	1123	1270	1431	1590
48	152	324	484	648	805	966	1124	1272	1439	1591
49	159	333	495	665	809	968	1127	1273	1443	1592
50	160	334	499	666	810	989	1129	1280	1444	1594

TABLE 4.4. The first 500 start values with case = 3.

		50	100	150	200	250	300	350	400	450
1	17	168	321	465	610	738	881	1027	1169	1305
2	21	175	326	466	611	739	882	1028	1170	1309
3	23	179	328	467	612	740	883	1031	1171	1310
4	25	180	329	468	621	743	884	1037	1172	1311
5	31	182	330	469	622	757	886	1038	1174	1312
6	33	184	331	477	623	761	888	1039	1175	1313
7	34	185	332	478	625	769	898	1042	1176	1314
8	37	195	336	481	626	770	899	1043	1178	1315
9	41	196	337	486	633	773	900	1044	1179	1316
10	42	197	343	488	637	777	903	1045	1180	1318
11	45	198	349	491	638	778	914	1046	1181	1320
12	46	200	350	492	641	779	915	1048	1182	1321
13	49	209	351	493	642	780	916	1050	1184	1322
14	50	213	358	494	645	781	917	1051	1197	1323
15	55	218	360	496	651	782	927	1052	1198	1324
16	61	219	363	505	652	783	929	1053	1199	1328
17	62	220	364	513	655	784	930	1054	1209	1344
18	66	221	368	514	656	786	931	1056	1219	1346
19	67	222	369	519	657	787	932	1065	1220	1347
20	68	225	370	521	658	788	934	1066	1222	1348
21	73	229	385	522	659	790	935	1067	1223	1353
22	74	233	389	523	660	792	936	1068	1224	1354
23	82	234	390	524	661	799	937	1072	1233	1367
24	83	239	391	525	662	800	938	1081	1237	1370
25	84	243	392	526	664	813	949	1085	1239	1371
26	90	244	393	527	672	814	954	1086	1242	1372
27	91	246	394	528	673	815	955	1087	1243	1375
28	92	247	395	533	674	825	956	1088	1244	1383
29	98	248	396	534	677	829	961	1105	1245	1385
30	99	257	400	536	685	830	962	1106	1246	1386
31	100	261	407	543	686	831	965	1107	1247	1389
32	109	262	415	544	693	834	967	1108	1250	1390
33	110	263	417	553	695	835	971	1109	1251	1391
34	111	264	418	554	697	836	972	1114	1252	1393
35	117	267	425	557	698	850	976	1115	1265	1394
36	122	268	426	558	699	851	977	1116	1266	1395
37	123	272	435	567	700	852	982	1125	1274	1396
38	124	277	436	577	701	855	983	1133	1275	1397
39	131	279	437	582	702	865	984	1134	1276	1398
40	132	291	438	583	703	869	985	1135	1282	1400
41	134	292	439	584	711	870	986	1141	1283	1401
42	136	293	440	585	715	872	987	1153	1284	1402
43	146	294	441	586	716	873	988	1154	1287	1403
44	147	295	442	587	720	874	991	1159	1289	1404
45	148	296	443	588	721	875	992	1163	1290	1405
46	153	305	444	589	725	876	1009	1164	1295	1406
47	163	306	449	590	726	877	1010	1165	1297	1421
48	164	311	450	591	728	878	1015	1166	1302	1422
49	165	313	457	592	736	879	1025	1167	1303	1423
50	166	319	458	599	737	880	1026	1168	1304	1430

TABLE 4.5. The distribution of the first 500 primes $2, 3, \ldots, 3571$ to the case numbers 1,2 or 3.

Case 1	Case 2	Case 3
167	158	175

TABLE 4.6. Table of exponents of polynomials π_n ordered with respect to n. The numbers in the second column represent the exponents belonging to the positive term of π_n, the numbers in the third and later columns represent the exponents of the negative terms of π_n. Entries x with $x-1=3k$ appear in red. Example: $\pi_9 = x^{28} - (x^{18} + x^{26} + x^{38} + x^{56})$.

$n=1$	-	2					
2	2	4					
3	4	6	8				
5	10	10	14	20			
6	6	12					
8	8	16					
9	28	18	26	38	56		
11	16	22	32				
12	12	24					
14	14	28					
15	22	30	44				
17	82	34	50	74	110	164	
18	18	36					
20	20	40					
21	46	42	62	92			
23	34	46	68				
24	24	48					
26	26	52					
27	40	54	80				
29	64	58	86	128			
30	30	60					
32	32	64					
33	244	66	98	146	218	326	488
35	52	70	104				
36	36	72					
38	38	76					
39	58	78	116				
41	136	82	122	182	272		
42	42	84					
44	44	88					
45	100	90	134	200			
47	70	94	140				
48	48	96					
50	50	100					
51	76	102	152				
53	118	106	158	236			
54	54	108					
56	56	112					
57	190	114	170	254	380		
59	88	118	176				

60	60	120								
62	62	124								
63	94	126	188							
65	730	130	194	290	434	650	974	1460		
66	66	132								
68	68	136								
69	154	138	206	308						
71	106	142	212							
72	72	144								
74	74	148								
75	112	150	224							
77	172	154	230	344						
78	78	156								
80	80	160								
81	406	162	242	362	542	812				
83	124	166	248							
84	84	168								
86	86	172								
87	130	174	260							
89	298	178	266	398	596					
90	90	180								
92	92	184								
93	208	186	278	416						
95	142	190	284							
96	96	192								
98	98	196								
99	148	198	296							
101	226	202	302	452						
102	102	204								
104	104	208								
105	352	210	314	470	704					
107	160	214	320							
108	108	216								
110	110	220								
111	166	222	332							
113	568	226	338	506	758	1136				
114	114	228								
116	116	232								
117	262	234	350	524						
119	178	238	356							
120	120	240								
122	122	244								
123	184	246	368							
125	280	250	374	560						
126	126	252								
128	128	256								
129	2188	258	386	578	866	1298	1946	2918	4376	
131	196	262	392							
132	132	264								
134	134	268								
135	202	270	404							
137	460	274	410	614	920					
138	138	276								
140	140	280								

141	316	282	422	632			
143	214	286	428				
144	144	288					
146	146	292					
147	220	294	440				
149	334	298	446	668			
150	150	300					
152	152	304					
153	514	306	458	686	1028		
155	232	310	464				
156	156	312					
158	158	316					
159	238	318	476				
161	1216	322	482	722	1082	1622	2432
162	162	324					
164	164	328					
165	370	330	494	740			
167	250	334	500				
168	168	336					
170	170	340					
171	256	342	512				
173	388	346	518	776			
174	174	348					
176	176	352					
177	892	354	530	794	1190	1784	
179	268	358	536				
180	180	360					
182	182	364					
183	274	366	548				
185	622	370	554	830	1244		
186	186	372					
188	188	376					
189	424	378	566	848			
191	286	382	572				
192	192	384					
194	194	388					
195	292	390	584				
197	442	394	590	884			
198	198	396					
200	200	400					
201	676	402	602	902	1352		
203	304	406	608				
204	204	408					
206	206	412					
207	310	414	620				
209	1054	418	626	938	1406	2108	
210	210	420					
212	212	424					
213	478	426	638	956			
215	322	430	644				
216	216	432					
218	218	436					
219	328	438	656				
221	496	442	662	992			

222	222	444								
224	224	448								
225	1702	450	674	1010	1514	2270	3404			
227	340	454	680							
228	228	456								
230	230	460								
231	346	462	692							
233	784	466	698	1046	1568					
234	234	468								
236	236	472								
237	532	474	710	1064						
239	358	478	716							
240	240	480								
242	242	484								
243	364	486	728							
245	550	490	734	1100						
246	246	492								
248	248	496								
249	838	498	746	1118	1676					
251	376	502	752							
252	252	504								
254	254	508								
255	382	510	764							
257	6562	514	770	1154	1730	2594	3890	5834	8750	13124
258	258	516								
260	260	520								
261	586	522	782	1172						
263	394	526	788							
264	264	528								
266	266	532								
267	400	534	800							
269	604	538	806	1208						
270	270	540								
272	272	544								
273	1378	546	818	1226	1838	2756				
275	412	550	824							
276	276	552								
278	278	556								
279	418	558	836							
281	946	562	842	1262	1892					
282	282	564								
284	284	568								
285	640	570	854	1280						
287	430	574	860							
288	288	576								
290	290	580								
291	436	582	872							
293	658	586	878	1316						
294	294	588								
296	296	592								
297	1000	594	890	1334	2000					
299	448	598	896							
300	300	600								
302	302	604								

303	454	606	908						
305	1540	610	914	1370	2054	3080			
306	306	612							
308	308	616							
309	694	618	926	1388					
311	466	622	932						
312	312	624							
314	314	628							
315	472	630	944						
317	712	634	950	1424					
318	318	636							
320	320	640							
321	3646	642	962	1442	2162	3242	4862	7292	
323	484	646	968						
324	324	648							
326	326	652							
327	490	654	980						
329	1108	658	986	1478	2216				
330	330	660							
332	332	664							
333	748	666	998	1496					
335	502	670	1004						
336	336	672							
338	338	676							
339	508	678	1016						
341	766	682	1022	1532					
342	342	684							
344	344	688							
345	1162	690	1034	1550	2324				
347	520	694	1040						
348	348	696							
350	350	700							
351	526	702	1052						
353	2674	706	1058	1586	2378	3566	5348		
354	354	708							
356	356	712							
357	802	714	1070	1604					
359	538	718	1076						
360	360	720							
362	362	724							
363	544	726	1088						
365	820	730	1094	1640					
366	366	732							
368	368	736							
369	1864	738	1106	1658	2486	3728			
371	556	742	1112						
372	372	744							
374	374	748							
375	562	750	1124						
377	1270	754	1130	1694	2540				
378	378	756							
380	380	760							
381	856	762	1142	1712					
383	574	766	1148						

384	384	768						
386	386	772						
387	580	774	1160					
389	874	778	1166	1748				
390	390	780						
392	392	784						
393	1324	786	1178	1766	2648			
395	592	790	1184					
396	396	792						
398	398	796						
399	598	798	1196					
401	2026	802	1202	1802	2702	4052		
402	402	804						
404	404	808						
405	910	810	1214	1820				
407	610	814	1220					
408	408	816						
410	410	820						
411	616	822	1232					
413	928	826	1238	1856				
414	414	828						
416	416	832						
417	3160	834	1250	1874	2810	4214	6320	
419	628	838	1256					
420	420	840						
422	422	844						
423	634	846	1268					
425	1432	850	1274	1910	2864			
426	426	852						
428	428	856						
429	964	858	1286	1928				
431	646	862	1292					
432	432	864						
434	434	868						
435	652	870	1304					
437	982	874	1310	1964				
438	438	876						
440	440	880						
441	1486	882	1322	1982	2972			
443	664	886	1328					
444	444	888						
446	446	892						
447	670	894	1340					
449	5104	898	1346	2018	3026	4538	6806	10208
450	450	900						
452	452	904						
453	1018	906	1358	2036				
455	682	910	1364					
456	456	912						
458	458	916						
459	688	918	1376					
461	1036	922	1382	2072				
462	462	924						
464	464	928						

465	2350	930	1394	2090	3134	4700					
467	700	934	1400								
468	468	936									
470	470	940									
471	706	942	1412								
473	1594	946	1418	2126	3188						
474	474	948									
476	476	952									
477	1072	954	1430	2144							
479	718	958	1436								
480	480	960									
482	482	964									
483	724	966	1448								
485	1090	970	1454	2180							
486	486	972									
488	488	976									
489	1648	978	1466	2198	3296						
491	736	982	1472								
492	492	984									
494	494	988									
495	742	990	1484								
497	2512	994	1490	2234	3350	5024					
498	498	996									
500	500	1000									
501	1126	1002	1502	2252							
503	754	1006	1508								
504	504	1008									
506	506	1012									
507	760	1014	1520								
509	1144	1018	1526	2288							
510	510	1020									
512	512	1024									
513	19684	1026	1538	2306	3458	5186	7778	11666	17498	26246	39368
515	772	1030	1544								
516	516	1032									
518	518	1036									
519	778	1038	1556								
521	1756	1042	1562	2342	3512						
522	522	1044									
524	524	1048									
525	1180	1050	1574	2360							
527	790	1054	1580								
528	528	1056									
530	530	1060									
531	796	1062	1592								
533	1198	1066	1598	2396							
534	534	1068									
536	536	1072									
537	1810	1074	1610	2414	3620						
539	808	1078	1616								
540	540	1080									
542	542	1084									
543	814	1086	1628								
545	4132	1090	1634	2450	3674	5510	8264				

546	546	1092					
548	548	1096					
549	1234	1098	1646	2468			
551	826	1102	1652				
552	552	1104					
554	554	1108					
555	832	1110	1664				
557	1252	1114	1670	2504			
558	558	1116					
560	560	1120					
561	2836	1122	1682	2522	3782	5672	
563	844	1126	1688				
564	564	1128					
566	566	1132					
567	850	1134	1700				
569	1918	1138	1706	2558	3836		
570	570	1140					
572	572	1144					
573	1288	1146	1718	2576			
575	862	1150	1724				
576	576	1152					
578	578	1156					
579	868	1158	1736				
581	1306	1162	1742	2612			
582	582	1164					
584	584	1168					
585	1972	1170	1754	2630	3944		
587	880	1174	1760				
588	588	1176					
590	590	1180					
591	886	1182	1772				
593	2998	1186	1778	2666	3998	5996	
594	594	1188					
596	596	1192					
597	1342	1194	1790	2684			
599	898	1198	1796				
600	600	1200					
602	602	1204					
603	904	1206	1808				
605	1360	1210	1814	2720			
606	606	1212					
608	608	1216					
609	4618	1218	1826	2738	4106	6158	9236
611	916	1222	1832				
612	612	1224					
614	614	1228					
615	922	1230	1844				
617	2080	1234	1850	2774	4160		
618	618	1236					
620	620	1240					
621	1396	1242	1862	2792			
623	934	1246	1868				
624	624	1248					
626	626	1252					

627	940	1254	1880						
629	1414	1258	1886	2828					
630	630	1260							
632	632	1264							
633	2134	1266	1898	2846	4268				
635	952	1270	1904						
636	636	1272							
638	638	1276							
639	958	1278	1916						
641	10936	1282	1922	2882	4322	6482	9722	14582	21872
642	642	1284							
644	644	1288							
645	1450	1290	1934	2900					
647	970	1294	1940						
648	648	1296							
650	650	1300							
651	976	1302	1952						
653	1468	1306	1958	2936					
654	654	1308							
656	656	1312							
657	3322	1314	1970	2954	4430	6644			
659	988	1318	1976						
660	660	1320							
662	662	1324							
663	994	1326	1988						
665	2242	1330	1994	2990	4484				
666	666	1332							
668	668	1336							
669	1504	1338	2006	3008					
671	1006	1342	2012						
672	672	1344							
674	674	1348							
675	1012	1350	2024						
677	1522	1354	2030	3044					
678	678	1356							
680	680	1360							
681	2296	1362	2042	3062	4592				
683	1024	1366	2048						
684	684	1368							
686	686	1372							
687	1030	1374	2060						
689	3484	1378	2066	3098	4646	6968			
690	690	1380							
692	692	1384							
693	1558	1386	2078	3116					
695	1042	1390	2084						
696	696	1392							
698	698	1396							
699	1048	1398	2096						
701	1576	1402	2102	3152					
702	702	1404							
704	704	1408							
705	8020	1410	2114	3170	4754	7130	10694	16040	
707	1060	1414	2120						

708	708	1416					
710	710	1420					
711	1066	1422	2132				
713	2404	1426	2138	3206	4808		
714	714	1428					
716	716	1432					
717	1612	1434	2150	3224			
719	1078	1438	2156				
720	720	1440					
722	722	1444					
723	1084	1446	2168				
725	1630	1450	2174	3260			
726	726	1452					
728	728	1456					
729	2458	1458	2186	3278	4916		
731	1096	1462	2192				
732	732	1464					
734	734	1468					
735	1102	1470	2204				
737	5590	1474	2210	3314	4970	7454	11180
738	738	1476					
740	740	1480					
741	1666	1482	2222	3332			
743	1114	1486	2228				
744	744	1488					
746	746	1492					
747	1120	1494	2240				
749	1684	1498	2246	3368			

TABLE 4.7. Table of exponents of polynomials π_n ordered with respect to the exponent of the positive term. The numbers in the second column represent the exponents belonging to the positive term of π_n, the numbers in the third and later columns represent the exponents of the negative terms of π_j. Entries x with $x-1=3k$ appear in red.

$n=1$	-	2			
2	2	4			
3	4	6	8		
6	6	12			
8	8	16			
5	10	10	14	20	
12	12	24			
14	14	28			
11	16	22	32		
18	18	36			
20	20	40			
15	22	30	44		
24	24	48			
26	26	52			
9	28	18	26	38	56
30	30	60			
32	32	64			
23	34	46	68		
36	36	72			

38	38	76				
27	40	54	80			
42	42	84				
44	44	88				
21	46	42	62	92		
48	48	96				
50	50	100				
35	52	70	104			
54	54	108				
56	56	112				
39	58	78	116			
60	60	120				
62	62	124				
29	64	58	86	128		
66	66	132				
68	68	136				
47	70	94	140			
72	72	144				
74	74	148				
51	76	102	152			
78	78	156				
80	80	160				
17	82	34	50	74	110	164
84	84	168				
86	86	172				
59	88	118	176			
90	90	180				
92	92	184				
63	94	126	188			
96	96	192				
98	98	196				
45	100	90	134	200		
102	102	204				
104	104	208				
71	106	142	212			
108	108	216				
110	110	220				
75	112	150	224			
114	114	228				
116	116	232				
53	118	106	158	236		
120	120	240				
122	122	244				
83	124	166	248			
126	126	252				
128	128	256				
87	130	174	260			
132	132	264				
134	134	268				
41	136	82	122	182	272	
138	138	276				
140	140	280				
95	142	190	284			
144	144	288				

146	146	292						
99	148	198	296					
150	150	300						
152	152	304						
69	154	138	206	308				
156	156	312						
158	158	316						
107	160	214	320					
162	162	324						
164	164	328						
111	166	222	332					
168	168	336						
170	170	340						
77	172	154	230	344				
174	174	348						
176	176	352						
119	178	238	356					
180	180	360						
182	182	364						
123	184	246	368					
186	186	372						
188	188	376						
57	190	114	170	254	380			
192	192	384						
194	194	388						
131	196	262	392					
198	198	396						
200	200	400						
135	202	270	404					
204	204	408						
206	206	412						
93	208	186	278	416				
210	210	420						
212	212	424						
143	214	286	428					
216	216	432						
218	218	436						
147	220	294	440					
222	222	444						
224	224	448						
101	226	202	302	452				
228	228	456						
230	230	460						
155	232	310	464					
234	234	468						
236	236	472						
159	238	318	476					
240	240	480						
242	242	484						
33	244	66	98	146	218	326	488	
246	246	492						
248	248	496						
167	250	334	500					
252	252	504						

254	254	508			
171	256	342	512		
258	258	516			
260	260	520			
117	262	234	350	524	
264	264	528			
266	266	532			
179	268	358	536		
270	270	540			
272	272	544			
183	274	366	548		
276	276	552			
278	278	556			
125	280	250	374	560	
282	282	564			
284	284	568			
191	286	382	572		
288	288	576			
290	290	580			
195	292	390	584		
294	294	588			
296	296	592			
89	298	178	266	398	596
300	300	600			
302	302	604			
203	304	406	608		
306	306	612			
308	308	616			
207	310	414	620		
312	312	624			
314	314	628			
141	316	282	422	632	
318	318	636			
320	320	640			
215	322	430	644		
324	324	648			
326	326	652			
219	328	438	656		
330	330	660			
332	332	664			
149	334	298	446	668	
336	336	672			
338	338	676			
227	340	454	680		
342	342	684			
344	344	688			
231	346	462	692		
348	348	696			
350	350	700			
105	352	210	314	470	704
354	354	708			
356	356	712			
239	358	478	716		
360	360	720			

362	362	724				
243	364	486	728			
366	366	732				
368	368	736				
165	370	330	494	740		
372	372	744				
374	374	748				
251	376	502	752			
378	378	756				
380	380	760				
255	382	510	764			
384	384	768				
386	386	772				
173	388	346	518	776		
390	390	780				
392	392	784				
263	394	526	788			
396	396	792				
398	398	796				
267	400	534	800			
402	402	804				
404	404	808				
81	406	162	242	362	542	812
408	408	816				
410	410	820				
275	412	550	824			
414	414	828				
416	416	832				
279	418	558	836			
420	420	840				
422	422	844				
189	424	378	566	848		
426	426	852				
428	428	856				
287	430	574	860			
432	432	864				
434	434	868				
291	436	582	872			
438	438	876				
440	440	880				
197	442	394	590	884		
444	444	888				
446	446	892				
299	448	598	896			
450	450	900				
452	452	904				
303	454	606	908			
456	456	912				
458	458	916				
137	460	274	410	614	920	
462	462	924				
464	464	928				
311	466	622	932			
468	468	936				

470	470	940				
315	472	630	944			
474	474	948				
476	476	952				
213	478	426	638	956		
480	480	960				
482	482	964				
323	484	646	968			
486	486	972				
488	488	976				
327	490	654	980			
492	492	984				
494	494	988				
221	496	442	662	992		
498	498	996				
500	500	1000				
335	502	670	1004			
504	504	1008				
506	506	1012				
339	508	678	1016			
510	510	1020				
512	512	1024				
153	514	306	458	686	1028	
516	516	1032				
518	518	1036				
347	520	694	1040			
522	522	1044				
524	524	1048				
351	526	702	1052			
528	528	1056				
530	530	1060				
237	532	474	710	1064		
534	534	1068				
536	536	1072				
359	538	718	1076			
540	540	1080				
542	542	1084				
363	544	726	1088			
546	546	1092				
548	548	1096				
245	550	490	734	1100		
552	552	1104				
554	554	1108				
371	556	742	1112			
558	558	1116				
560	560	1120				
375	562	750	1124			
564	564	1128				
566	566	1132				
113	568	226	338	506	758	1136
570	570	1140				
572	572	1144				
383	574	766	1148			
576	576	1152				

578	578	1156				
387	580	774	1160			
582	582	1164				
584	584	1168				
261	586	522	782	1172		
588	588	1176				
590	590	1180				
395	592	790	1184			
594	594	1188				
596	596	1192				
399	598	798	1196			
600	600	1200				
602	602	1204				
269	604	538	806	1208		
606	606	1212				
608	608	1216				
407	610	814	1220			
612	612	1224				
614	614	1228				
411	616	822	1232			
618	618	1236				
620	620	1240				
185	622	370	554	830	1244	
624	624	1248				
626	626	1252				
419	628	838	1256			
630	630	1260				
632	632	1264				
423	634	846	1268			
636	636	1272				
638	638	1276				
285	640	570	854	1280		
642	642	1284				
644	644	1288				
431	646	862	1292			
648	648	1296				
650	650	1300				
435	652	870	1304			
654	654	1308				
656	656	1312				
293	658	586	878	1316		
660	660	1320				
662	662	1324				
443	664	886	1328			
666	666	1332				
668	668	1336				
447	670	894	1340			
672	672	1344				
674	674	1348				
201	676	402	602	902	1352	
678	678	1356				
680	680	1360				
455	682	910	1364			
684	684	1368				

686	686	1372						
459	688	918	1376					
690	690	1380						
692	692	1384						
309	694	618	926	1388				
696	696	1392						
698	698	1396						
467	700	934	1400					
702	702	1404						
704	704	1408						
471	706	942	1412					
708	708	1416						
710	710	1420						
317	712	634	950	1424				
714	714	1428						
716	716	1432						
479	718	958	1436					
720	720	1440						
722	722	1444						
483	724	966	1448					
726	726	1452						
728	728	1456						
65	730	130	194	290	434	650	974	1460
732	732	1464						
734	734	1468						
491	736	982	1472					
738	738	1476						
740	740	1480						
495	742	990	1484					
744	744	1488						
746	746	1492						
333	748	666	998	1496				
750	750	1500						
752	752	1504						
503	754	1006	1508					
756	756	1512						
758	758	1516						
507	760	1014	1520					
762	762	1524						
764	764	1528						
341	766	682	1022	1532				
768	768	1536						
770	770	1540						
515	772	1030	1544					
774	774	1548						
776	776	1552						
519	778	1038	1556					
780	780	1560						
782	782	1564						
233	784	466	698	1046	1568			
786	786	1572						
788	788	1576						
527	790	1054	1580					
792	792	1584						

794	794	1588				
531	796	1062	1592			
798	798	1596				
800	800	1600				
357	802	714	1070	1604		
804	804	1608				
806	806	1612				
539	808	1078	1616			
810	810	1620				
812	812	1624				
543	814	1086	1628			
816	816	1632				
818	818	1636				
365	820	730	1094	1640		
822	822	1644				
824	824	1648				
551	826	1102	1652			
828	828	1656				
830	830	1660				
555	832	1110	1664			
834	834	1668				
836	836	1672				
249	838	498	746	1118	1676	
840	840	1680				
842	842	1684				
563	844	1126	1688			
846	846	1692				
848	848	1696				
567	850	1134	1700			
852	852	1704				
854	854	1708				
381	856	762	1142	1712		
858	858	1716				
860	860	1720				
575	862	1150	1724			
864	864	1728				
866	866	1732				
579	868	1158	1736			
870	870	1740				
872	872	1744				
389	874	778	1166	1748		
876	876	1752				
878	878	1756				
587	880	1174	1760			
882	882	1764				
884	884	1768				
591	886	1182	1772			
888	888	1776				
890	890	1780				
177	892	354	530	794	1190	1784
894	894	1788				
896	896	1792				
599	898	1198	1796			
900	900	1800				

902	902	1804			
603	904	1206	1808		
906	906	1812			
908	908	1816			
405	910	810	1214	1820	
912	912	1824			
914	914	1828			
611	916	1222	1832		
918	918	1836			
920	920	1840			
615	922	1230	1844		
924	924	1848			
926	926	1852			
413	928	826	1238	1856	
930	930	1860			
932	932	1864			
623	934	1246	1868		
936	936	1872			
938	938	1876			
627	940	1254	1880		
942	942	1884			
944	944	1888			
281	946	562	842	1262	1892
948	948	1896			
950	950	1900			
635	952	1270	1904		
954	954	1908			
956	956	1912			
639	958	1278	1916		
960	960	1920			
962	962	1924			
429	964	858	1286	1928	
966	966	1932			
968	968	1936			
647	970	1294	1940		
972	972	1944			
974	974	1948			
651	976	1302	1952		
978	978	1956			
980	980	1960			
437	982	874	1310	1964	
984	984	1968			
986	986	1972			
659	988	1318	1976		
990	990	1980			
992	992	1984			
663	994	1326	1988		
996	996	1992			
998	998	1996			
297	1000	594	890	1334	2000

Table 4.8. Table of the first two hundred and fifty triples $(2\ell, \ell', \ell'')$, in the order of 2ℓ, and ordered with respect to the ℓ' column and to the ℓ'' column.

2ℓ	ℓ'	ℓ''	2ℓ	ℓ'	ℓ''	2ℓ	ℓ'	ℓ''
2	2	1	2	2	1	2	2	1
4	3	2	4	3	2	4	3	2
6	6	3	10	5	5	6	6	3
8	8	3	6	6	3	8	8	3
10	5	5	8	8	3	10	5	5
12	12	6	28	9	14	14	14	5
14	14	5	16	11	8	20	20	5
16	11	8	12	12	6	12	12	6
18	18	9	14	14	5	16	11	8
20	20	5	22	15	11	18	18	9
22	15	11	82	17	41	26	26	9
24	24	12	18	18	9	38	38	9
26	26	9	20	20	5	56	56	9
28	9	14	46	21	23	22	15	11
30	30	15	34	23	17	32	32	11
32	32	11	24	24	12	24	24	12
34	23	17	26	26	9	28	9	14
36	36	18	40	27	20	30	30	15
38	38	9	64	29	32	44	44	15
40	27	20	30	30	15	34	23	17
42	42	21	32	32	11	50	50	17
44	44	15	244	33	122	74	74	17
46	21	23	52	35	26	110	110	17
48	48	24	36	36	18	164	164	17
50	50	17	38	38	9	36	36	18
52	35	26	58	39	29	40	27	20
54	54	27	136	41	68	42	42	21
56	56	9	42	42	21	62	62	21
58	39	29	44	44	15	92	92	21
60	60	30	100	45	50	46	21	23
62	62	21	70	47	35	68	68	23
64	29	32	48	48	24	48	48	24
66	66	33	50	50	17	52	35	26
68	68	23	76	51	38	54	54	27
70	47	35	118	53	59	80	80	27
72	72	36	54	54	27	58	39	29
74	74	17	56	56	9	86	86	29
76	51	38	190	57	95	128	128	29
78	78	39	88	59	44	60	60	30
80	80	27	60	60	30	64	29	32
82	17	41	62	62	21	66	66	33
84	84	42	94	63	47	98	98	33
86	86	29	730	65	365	146	146	33
88	59	44	66	66	33	218	218	33
90	90	45	68	68	23	326	326	33
92	92	21	154	69	77	488	488	33

94	63	47	106	71	53	70	47	35
96	96	48	72	72	36	104	104	35
98	98	33	74	74	17	72	72	36
100	45	50	112	75	56	76	51	38
102	102	51	172	77	86	78	78	39
104	104	35	78	78	39	116	116	39
106	71	53	80	80	27	82	17	41
108	108	54	406	81	203	122	122	41
110	110	17	124	83	62	182	182	41
112	75	56	84	84	42	272	272	41
114	114	57	86	86	29	84	84	42
116	116	39	130	87	65	88	59	44
118	53	59	298	89	149	90	90	45
120	120	60	90	90	45	134	134	45
122	122	41	92	92	21	200	200	45
124	83	62	208	93	104	94	63	47
126	126	63	142	95	71	140	140	47
128	128	29	96	96	48	96	96	48
130	87	65	98	98	33	100	45	50
132	132	66	148	99	74	102	102	51
134	134	45	226	101	113	152	152	51
136	41	68	102	102	51	106	71	53
138	138	69	104	104	35	158	158	53
140	140	47	352	105	176	236	236	53
142	95	71	160	107	80	108	108	54
144	144	72	108	108	54	112	75	56
146	146	33	110	110	17	114	114	57
148	99	74	166	111	83	170	170	57
150	150	75	568	113	284	254	254	57
152	152	51	114	114	57	380	380	57
154	69	77	116	116	39	118	53	59
156	156	78	262	117	131	176	176	59
158	158	53	178	119	89	120	120	60
160	107	80	120	120	60	124	83	62
162	162	81	122	122	41	126	126	63
164	164	17	184	123	92	188	188	63
166	111	83	280	125	140	130	87	65
168	168	84	126	126	63	194	194	65
170	170	57	128	128	29	290	290	65
172	77	86	2188	129	1094	434	434	65
174	174	87	196	131	98	650	650	65
176	176	59	132	132	66	974	974	65
178	119	89	134	134	45	1460	1460	65
180	180	90	202	135	101	132	132	66
182	182	41	460	137	230	136	41	68
184	123	92	138	138	69	138	138	69
186	186	93	140	140	47	206	206	69
188	188	63	316	141	158	308	308	69
190	57	95	214	143	107	142	95	71

192	192	96	144	144	72	212	212	71
194	194	65	146	146	33	144	144	72
196	131	98	220	147	110	148	99	74
198	198	99	334	149	167	150	150	75
200	200	45	150	150	75	224	224	75
202	135	101	152	152	51	154	69	77
204	204	102	514	153	257	230	230	77
206	206	69	232	155	116	344	344	77
208	93	104	156	156	78	156	156	78
210	210	105	158	158	53	160	107	80
212	212	71	238	159	119	162	162	81
214	143	107	1216	161	608	242	242	81
216	216	108	162	162	81	362	362	81
218	218	33	164	164	17	542	542	81
220	147	110	370	165	185	812	812	81
222	222	111	250	167	125	166	111	83
224	224	75	168	168	84	248	248	83
226	101	113	170	170	57	168	168	84
228	228	114	256	171	128	172	77	86
230	230	77	388	173	194	174	174	87
232	155	116	174	174	87	260	260	87
234	234	117	176	176	59	178	119	89
236	236	53	892	177	446	266	266	89
238	159	119	268	179	134	398	398	89
240	240	120	180	180	90	596	596	89
242	242	81	182	182	41	180	180	90
244	33	122	274	183	137	184	123	92
246	246	123	622	185	311	186	186	93
248	248	83	186	186	93	278	278	93
250	167	125	188	188	63	416	416	93
252	252	126	424	189	212	190	57	95
254	254	57	286	191	143	284	284	95
256	171	128	192	192	96	192	192	96
258	258	129	194	194	65	196	131	98
260	260	87	292	195	146	198	198	99
262	117	131	442	197	221	296	296	99
264	264	132	198	198	99	202	135	101
266	266	89	200	200	45	302	302	101
268	179	134	676	201	338	452	452	101
270	270	135	304	203	152	204	204	102
272	272	41	204	204	102	208	93	104
274	183	137	206	206	69	210	210	105
276	276	138	310	207	155	314	314	105
278	278	93	1054	209	527	470	470	105
280	125	140	210	210	105	704	704	105
282	282	141	212	212	71	214	143	107
284	284	95	478	213	239	320	320	107
286	191	143	322	215	161	216	216	108
288	288	144	216	216	108	220	147	110

290	290	65	218	218	33	222	222	111
292	195	146	328	219	164	332	332	111
294	294	147	496	221	248	226	101	113
296	296	99	222	222	111	338	338	113
298	89	149	224	224	75	506	506	113
300	300	150	1702	225	851	758	758	113
302	302	101	340	227	170	1136	1136	113
304	203	152	228	228	114	228	228	114
306	306	153	230	230	77	232	155	116
308	308	69	346	231	173	234	234	117
310	207	155	784	233	392	350	350	117
312	312	156	234	234	117	524	524	117
314	314	105	236	236	53	238	159	119
316	141	158	532	237	266	356	356	119
318	318	159	358	239	179	240	240	120
320	320	107	240	240	120	244	33	122
322	215	161	242	242	81	246	246	123
324	324	162	364	243	182	368	368	123
326	326	33	550	245	275	250	167	125
328	219	164	246	246	123	374	374	125
330	330	165	248	248	83	560	560	125
332	332	111	838	249	419	252	252	126
334	149	167	376	251	188	256	171	128
336	336	168	252	252	126	258	258	129
338	338	113	254	254	57	386	386	129
340	227	170	382	255	191	578	578	129
342	342	171	258	258	129	866	866	129
344	344	77	260	260	87	1298	1298	129
346	231	173	586	261	293	1946	1946	129
348	348	174	394	263	197	2918	2918	129
350	350	117	264	264	132	262	117	131
352	105	176	266	266	89	392	392	131
354	354	177	400	267	200	264	264	132
356	356	119	604	269	302	268	179	134
358	239	179	270	270	135	270	270	135
360	360	180	272	272	41	404	404	135
362	362	81	1378	273	689	274	183	137
364	243	182	412	275	206	410	410	137
366	366	183	276	276	138	614	614	137
368	368	123	278	278	93	920	920	137
370	165	185	418	279	209	276	276	138
372	372	186	946	281	473	280	125	140
374	374	125	282	282	141	282	282	141
376	251	188	284	284	95	422	422	141
378	378	189	640	285	320	632	632	141
380	380	57	430	287	215	286	191	143
382	255	191	288	288	144	428	428	143
384	384	192	290	290	65	288	288	144
386	386	129	436	291	218	292	195	146

388	173	194	658	293	329	294	294	147
390	390	195	294	294	147	440	440	147
392	392	131	296	296	99	298	89	149
394	263	197	1000	297	500	446	446	149
396	396	198	448	299	224	668	668	149
398	398	89	300	300	150	300	300	150
400	267	200	302	302	101	304	203	152
402	402	201	454	303	227	306	306	153
404	404	135	1540	305	770	458	458	153
406	81	203	306	306	153	686	686	153
408	408	204	308	308	69	1028	1028	153
410	410	137	694	309	347	310	207	155
412	275	206	466	311	233	464	464	155
414	414	207	312	312	156	312	312	156
416	416	93	314	314	105	316	141	158
418	279	209	472	315	236	318	318	159
420	420	210	712	317	356	476	476	159
422	422	141	318	318	159	322	215	161
424	189	212	320	320	107	482	482	161
426	426	213	3646	321	1823	722	722	161
428	428	143	484	323	242	1082	1082	161
430	287	215	324	324	162	1622	1622	161
432	432	216	326	326	33	2432	2432	161
434	434	65	490	327	245	324	324	162
436	291	218	1108	329	554	328	219	164
438	438	219	330	330	165	330	330	165
440	440	147	332	332	111	494	494	165
442	197	221	748	333	374	740	740	165
444	444	222	502	335	251	334	149	167
446	446	149	336	336	168	500	500	167
448	299	224	338	338	113	336	336	168
450	450	225	508	339	254	340	227	170
452	452	101	766	341	383	342	342	171
454	303	227	342	342	171	512	512	171
456	456	228	344	344	77	346	231	173
458	458	153	1162	345	581	518	518	173
460	137	230	520	347	260	776	776	173
462	462	231	348	348	174	348	348	174
464	464	155	350	350	117	352	105	176
466	311	233	526	351	263	354	354	177
468	468	234	2674	353	1337	530	530	177
470	470	105	354	354	177	794	794	177
472	315	236	356	356	119	1190	1190	177
474	474	237	802	357	401	1784	1784	177
476	476	159	538	359	269	358	239	179
478	213	239	360	360	180	536	536	179
480	480	240	362	362	81	360	360	180
482	482	161	544	363	272	364	243	182
484	323	242	820	365	410	366	366	183

486	486	243	366	366	183	548	548	183
488	488	33	368	368	123	370	165	185
490	327	245	1864	369	932	554	554	185
492	492	246	556	371	278	830	830	185
494	494	165	372	372	186	1244	1244	185
496	221	248	374	374	125	372	372	186
498	498	249	562	375	281	376	251	188
500	500	167	1270	377	635	378	378	189