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ON THE DIFFERENTIABILITY OF STATIONARY FLUID-STRUCTURE INTERACTION PROBLEMS WITH RESPECT TO THE PROBLEM DATA

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Abstract. In this work, a coupled system of stationary fluid-structure equations in an arbitrary Lagrangian-Eulerian framework is considered. Existence of solutions is shown under a small data assumption. Finally, differentiability of the solutions with respect to the given data, volume forces and boundary values, is shown.

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Introduction

In this article, we consider a stationary fluid-structure interaction (FSI) problem. The main contribution of this paper will be the proof that there is at least one locally unique solution that is Fréchet-differentiable with respect to the boundary data and volume forces.

Fluid-structure interactions are of great importance in many real-life applications, such as industrial processes, aero-elasticity, and biomechanics. More specifically, fluid-structure interactions (FSI) are important to describe flows around elastic structures as for instance in the flutter analysis of airplanes [23], parachute FSI [29], or blood flow in the cardiovascular system and heart valve dynamics, see, e.g, [18, 21, 27].

Derivatives of the solution to FSI problems with respect to given data such as volume forces or boundary values have been used excessively in numerical papers concerned with sensitivity calculations, derivative based minimization problems, see, e.g., [1,19,26,28,31]. In addition, sensitivity calculations are required for a posteriori estimation. Studies with a particular emphasize on FSI problems are carried out, for instance in [11,15,25,32,33]. Optimization-related problems subject to FSI has not yet been tackled. Some first steps have been done in [4] where the case of a nonstationary problem with fixed interface was considered. Further, stabilization of such problems with a lower dimensional structure equations are considered [8, 24].

In the last years, several attempts have been made to prove existence of fully unsteady fluid-structure interaction problems in three dimensions. The first results were derived for structures that were modeled as a rigid body, e.g., [9], or it was modeled by a finite number of modal functions [10]. Existence results for three dimensional fluid-structure interaction, where the structure was described as an elastic material, was shown in [12–14]. The extension to nonstationary problems is far from being trivial. The well-known problem, in the nonstationary case, is the regularity gap of the fluid and the structure velocity on the interface. First

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proofs of well-posedness and existence of nonstationary fluid-structure interaction have been derived by [6, 7]. Moreover, regularity properties of the FSI solution have been studied in [2, 3].

However, to the best of the authors knowledge no result on the differentiability of solutions to FSI problems with respect to the data has been shown. Some work in this direction is contained in [20] where differentiability of the solution-map for a lower-dimensional structure was considered.

The purpose of this work is to extend the existence theory for a stationary FSI problem given in [13] to include local uniqueness of at least one solution under a small data assumption. Then, the main contribution of this article, namely differentiability of the solution with respect to the problem data is shown.

Finally, we emphasize that the dilemma of non-matching coordinate systems of fluids and structures has to be overcome. Consequently, in this work, the well-known arbitrary Lagrangian-Eulerian (ALE) frame of reference (see, e.g., [16, 17, 22]) is used. Using this method, the fluid equations are reformulated on a fixed (but arbitrary) reference configuration.

The paper is organized as follows. In Section 1, we will describe the considered setting for the FSI problem under consideration. Then, some results on the fluid and structure problem will be collected in Section 2. Here, special emphasis is placed on the fluid problem which is posed in the ALE framework, while the structure equation is just standard linear elasticity. The paper concludes with Section 2.1; here the main result, namely differentiability of solutions to the FSI problem, will be shown. This will be done using a fixed point argument in combination with several applications of the implicit function theorem.

1. Problem Setting

In the sequel, let $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ be a given domain with a $C^{1,1}$ boundary. We decompose Ω into a fluid part Ω_f and a structure part Ω_s. In Ω_f the incompressible Stokes equations and on Ω_s a linear elastic structure equation is valid. We denote the interface between the two subdomains by $\Gamma_i := \overline{\Omega_f} \cap \overline{\Omega_s}$. Boundary parts where we prescribe Dirichlet conditions, are denoted by Γ_f and Γ_s (for fluid or structure, respectively). These boundary parts are chosen such that $\partial\Omega = \Gamma_f \cup \Gamma_s$. In order to avoid problems due to singularities arising from the change of the boundary conditions, we assume that the interface Γ_i has positive distance to the boundary $\partial\Omega$ of the domain.

We use standard notation for the usual Lebesgue and Sobolev spaces. On the Hilbert space $L^2(\Omega)$, we denote the scalar product by $(\cdot,\cdot)$ and the corresponding norm by $||\cdot||$. The space $W^{m,p}(\Omega)$ contains those functions whose weak derivatives up to order m are in $L^p(\Omega)$. We use $H^m(\Omega) := W^{m,2}(\Omega)$. For any $d-1$ dimensional set $\Gamma \subset \overline{\Omega}$ we define $W^{m,p}_{0,\Gamma}(\Omega) \subset W^{m,p}(\Omega)$ by zero Dirichlet conditions on Γ, assuming $|\Gamma| \neq 0$ in the $d-1$ dimensional sense. For vector valued function spaces, we indicate this by adding the image space to the definition, e.g., $H^m(\Omega;\mathbb{R}^d)$ for H^m functions with values in $\mathbb{R}^d$. Throughout, we assume $p > d$ and hence $W^{1,\infty}(\Omega) \subset W^{2,p}(\Omega)$ by standard embeddings.

Due to the coupling of fluid and structure equations, the domains Ω_f and Ω_s are unknown a priori. Under the assumption that the overall domain Ω is fixed we will reformulate the coupled system on a fixed domain $\widehat{\Omega} = \Omega$. To do so, we introduce a *reference configuration* by denoting $\widehat{\Omega}_f \subset \Omega$ and $\widehat{\Omega}_s = \Omega \setminus \overline{\widehat{\Omega}_f} \subset \Omega$. The interface $\widehat{\Gamma}_i = \partial\widehat{\Omega}_f \cap \partial\widehat{\Omega}_s$ is assumed to be of class $C^{1,1}$. Then, the problem of finding the domains $\Omega_{f,s}$ is equivalent to finding a transformation $\widehat{\mathcal{A}}\colon \widehat{\Omega} \to \widehat{\Omega}$ such that $\widehat{\mathcal{A}}(\widehat{\Omega}_f) = \Omega_f$, $\widehat{\mathcal{A}}(\widehat{\Omega}_s) = \Omega_s$, and $\widehat{\mathcal{A}}(\widehat{\Gamma}_i) = \Gamma_i$. This transformation is called the ALE map [16, 17, 22]. A natural choice for $\widehat{\Omega}_s$ is the domain initially occupied by the structure, and hence $\widehat{\mathcal{A}}\colon \widehat{\Omega}_s \to \Omega_s \subset \widehat{\Omega}$ is given by the displacement $\hat{u}_s$, e.g., $\widehat{\mathcal{A}}\Big|_{\widehat{\Omega}_s} = I + \hat{u}_s$, where I denotes the identity on $\mathbb{R}^d$. To obtain its values on the fluid domain, one has to choose an arbitrary continuation, e.g., of harmonic type [17], to obtain a displacement $\hat{u}_f$ on the fluid reference domain $\widehat{\Omega}_f$ that satisfies the following geometrical coupling condition:

$$\widehat{\mathcal{A}} := \widehat{\mathcal{A}}(\hat{x}) = \hat{u}_s(\hat{x}) \quad \text{on } \widehat{\Gamma}_i.$$

Let $\mathcal{R} : W^{2-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d) \to W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \cap H^1_{0,\widehat{\Gamma}_f}(\widehat{\Omega}_f;\mathbb{R}^d)$ be a continuous (linear) extension operator. Then, the ALE map on $\widehat{\Omega}_f$ is determined by

$$\widehat{\mathcal{A}}\Big|_{\widehat{\Omega}_f} = I + \mathcal{R}(\gamma_i(\hat{u}_s)),$$

where γ_i denotes the trace operator over $\widehat{\Gamma}_i$. This leads to $\hat{u}_f = \hat{u}_s$ on $\widehat{\Gamma}_i$. For reasons that will become clear later, it is desirable to choose the continuation $\mathcal{R}$ such that $\|\widehat{\mathcal{A}}\|_{W^{1,\infty}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})}$ is small.

For later purposes we define two terms the deformation gradient $\widehat{F}$ and its determinant $\hat{J}$, related to the ALE map, that we will use quite frequently, by

$$\widehat{F} := \widehat{F}(\hat{u}) = \widehat{\nabla}\widehat{\mathcal{A}}, \quad \hat{J} := \hat{J}(\hat{u}) = \det(\widehat{F}). \tag{1}$$

The existence proof is based on a fixed point argument. Hence we begin by stating some properties of the fluid and structure problem.

Remark 1.1. Note that, by Sobolev embeddings and the assumptions on $\mathcal{R}$, we have

$$\|\hat{u}_f\|_{W^{1,\infty}(\widehat{\Omega}_f;\mathbb{R}^d)} \leq c\|\hat{u}_f\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)} \leq c\|\hat{u}_s\|_{W^{2-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)} \leq c\|\hat{u}_s\|_{W^{2,p}(\widehat{\Omega}_s;\mathbb{R}^d)}.$$

In particular, $\hat{J} > 0$ if $\|\hat{u}_s\|_{W^{2,p}(\widehat{\Omega}_s;\mathbb{R}^d)}$ is sufficiently small and we see that this implies $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{A}}^{-1} \in W^{1,\infty}(\widehat{\Omega};\mathbb{R}^{d\times d})$.

2. Properties of the subproblems

The fluid problem reads (in Eulerian coordinates):

Problem 2.1. Given a domain Ω_f with a $C^{1,1}$ boundary, and $g_f \in W^{2-1/p,p}(\Gamma_f;\mathbb{R}^d)$ with the compatibility condition, see, e.g, [30, Theorem 2.4],

$$\int_{\Gamma_f} g_f \cdot n_f \, ds = 0. \tag{2}$$

Denote, again by g_f the continuation of g_f in $W^{2,p}(\Omega_f;\mathbb{R}^d) \cap H^1_{0,\Gamma_i}(\Omega_f;\mathbb{R}^d)$. Then we need to find $(v_f, p_f) \in H^1_{0,\Gamma_f\cup\Gamma_i}(\Omega_f;\mathbb{R}^d) + (g_f, 0) \times L^2(\Omega_f)/\mathbb{R}$ such that

$$\begin{aligned} -\widehat{\operatorname{div}}\,(\sigma_f) &= 0 \quad \text{in } \Omega_f,\\ \widehat{\operatorname{div}}\, v_f &= 0 \quad \text{in } \Omega_f,\\ v_f &= 0 \quad \text{on } \Gamma_i,\\ v_f &= g_f \quad \text{on } \Gamma_f. \end{aligned}$$

Here, the constitutive tensor σ_f is given by

$$\sigma_f := \sigma_f(v_f, p_f) = -p_f I + \nu_f \nabla v_f.$$

where ν_f describes the kinematic viscosity of the fluid.

The existence and regularity of solutions to Problem 2.1 is studied in [30] under the assumption of a regular domain Ω_f. Unfortunately, since the domain Ω_f is given by the unknown mapping $\hat{u}_s$, i.e., $\Omega_f = \widehat{\mathcal{A}}(\widehat{\Omega}_f)$, this can not be asserted a priori.

In order to proceed in the reference domain, we transform the fluid equations to a fixed arbitrary reference configuration:

Problem 2.2. Given a displacement $\hat{u}_s \in W^{2-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)$, and $g_f \in W^{2-1/p,p}(\Gamma_f;\mathbb{R}^d)$ satisfying (2). Define $\widehat{F}$ and $\hat{J}$ by (1), and the transformed constitutive tensor as

$$\widehat{\sigma}_f := \widehat{\sigma}_f(\hat{v}_f,\hat{p}_f) = -\hat{p}_f I + \nu_f \widehat{\nabla}\hat{v}_f \widehat{F}^{-1}.$$

Find $(\hat{v}_f,\hat{p}_f) \in H^1_{0,\Gamma_f\cup\Gamma_i}(\widehat{\Omega}_f;\mathbb{R}^d) + (g_f,0) \times L^2(\widehat{\Omega}_f)/\mathbb{R}$ such that

$$\begin{aligned} -\widehat{\operatorname{div}}\,(\widehat{\sigma}_f \hat{J}\widehat{F}^{-T}) &= 0 \quad \text{in } \widehat{\Omega}_f,\\ \widehat{\operatorname{div}}\,(\hat{J}\widehat{F}^{-1}\hat{v}_f) &= 0 \quad \text{in } \widehat{\Omega}_f,\\ \hat{v}_f &= 0 \quad \text{on } \widehat{\Gamma}_i,\\ \hat{v}_f &= g_f \quad \text{on } \Gamma_f. \end{aligned} \tag{3}$$

Under the assumption of $W^{2,p}$ regularity for $\hat{u}_f = \mathcal{R}(\hat{u}_s)$, with sufficiently small norm, the solutions of the Problems 2.1 and 2.2 coincide by setting $\hat{v}_f = v_f \circ \widehat{\mathcal{A}}^{-1}$ and $\hat{p}_f = p_f \circ \widehat{\mathcal{A}}^{-1}$. Since $\Gamma_f = \widehat{\Gamma}_f$ the boundary data g_f do not need to be transformed..

The structure problem (in Lagrangian coordinates) is defined by:

Problem 2.3. Given forces $\hat{f}_s \in L^p(\widehat{\Omega}_s;\mathbb{R}^d)$ and $\hat{g}_s \in W^{1-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)$. Find $\hat{u}_s \in W^{2,p}(\widehat{\Omega}_s;\mathbb{R}^d)$, such that

$$\begin{aligned} -\widehat{\operatorname{div}}\,(\widehat{\Sigma}_s) &= \hat{f}_s \quad \text{in } \widehat{\Omega}_s,\\ \hat{u}_f &= 0 \quad \text{on } \widehat{\Gamma}_s,\\ \widehat{\Sigma}_s \hat{n}_s &= \hat{g}_s \quad \text{on } \widehat{\Gamma}_i. \end{aligned}$$

with the Piola-Kirchhoff stress tensor $\widehat{\Sigma}_s$ and the (linear) Green-Lagrange tensor $\widehat{E}$:

$$\widehat{\Sigma}_s := \widehat{\Sigma}_s(\hat{u}_s) = \lambda \operatorname{tr}(\widehat{E})I + 2\mu\widehat{E}, \quad \widehat{E} := \widehat{E}(\hat{u}_s) = \frac{1}{2}(\widehat{\nabla}\hat{u}_s + \widehat{\nabla}\hat{u}_s^T),$$

where the constants λ and μ denote the Lamé parameters.

Remark 2.4. We should note that in the above setting other boundary conditions can be considered as well. The only strong requirement we have to enforce is that the solutions of the problems are sufficiently regular on $\widehat{\Gamma}_i$, i.e., $\hat{u}_f$ and $\hat{v}_f$ need to be in $W^{2-1/p,p}(\widehat{\Gamma}_i)$. The choice of the particular boundary values here is guided by the aim to have as little technicalities as possible. For the same reason it is possible to replace the equations with appropriate nonlinear versions, e.g., Navier-Stokes, nonlinear elasticity, etc. However, we refrain from such generalizations as we feel the proofs are already involved enough and not much value is added by an additional fixed point argument to obtain solvability of the (non-coupled) fluid and structure equations.

Similar arguments can be made to incorporate additional volume forces or boundary values.

We will now derive some properties of solutions to the Problems 2.2 and 2.3. Before these are stated, we recall two assumptions from [13, Page 83]. They are dealing with the regularity of the transformation $\widehat{F}$ and its determinant $\hat{J}$.

Remark 2.5. We simplify notation by setting $\widehat{A} := \widehat{F}^{-1}\hat{J}\widehat{F}^{-T}$ and $\widehat{B} := \hat{J}\widehat{F}^{-T}$ in the following section. Further, let $\widehat{I}$ be the identity matrix in $\mathbb{R}^{d\times d}$. The following assumptions are made on the $\widehat{A}$ and $\widehat{B}$ to proof existence of the transformed Stokes problem. Assume:

a) $\widehat{A}$ is a symmetric positive definite matrix whose coefficients are elements of $W^{1,p}(\widehat{\Omega}_f)$. There exists some positive constant α such that $\widehat{A} \geq \alpha\widehat{I}$.
b) $\widehat{B}$ is invertible, with both $\widehat{B}$ and $\widehat{B}^{-1}$ in $W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})$.

c) There is $C_u > 0$ such that $\widehat{A}$ and $\widehat{B}$ verify

$$\|\widehat{I} - \widehat{A}\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \le C_u, \quad \|\widehat{I} - \widehat{B}\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \le C_u, \quad \|\widehat{I} - \widehat{B}^T\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \le C_u.$$

As both $\widehat{A}$ and $\widehat{B}$ are given by $\widehat{F}$, we remark that the assumptions hold if $\widehat{\mathcal{A}}$ is regular enough, i.e., for small deformations of the fluid mesh ($\|\hat{u}_f\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)} \ll 1$), see also our Remark 1.1. Further, we note that the mappings $\widehat{A}, \widehat{B}, \widehat{B}^{-1}\colon W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \to W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})$ are Fréchet-differentiable with

$$\begin{aligned}
\|D_u\widehat{A}(u)\|_{\mathcal{L}(W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d),W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d}))} &\le C_u,\\
\|D_u\widehat{B}(u)\|_{\mathcal{L}(W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d),W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d}))} &\le C_u,\\
\|D_U\widehat{B}^{-1}(u)\|_{\mathcal{L}(W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d),W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} &\le C_u,
\end{aligned}$$

with a constant C_u depending on $\|u\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)}$. We note that, in fact, the constant satisfies $C_u \le C_{\rho,M} < \infty$ as long as $0 < \rho \le \hat{J}$ and $\|u\|_{W^{2,p}(\widehat{\Omega}_f);\mathbb{R}^d} \le M$.

Before we come to the existence and regularity of solutions to Problem 2.2 we need to introduce a slightly generalized problem that we will need frequently in the following proofs.

Problem 2.6. Given a displacement $\hat{u}_s \in W^{2-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)$, $\hat{f}_f \in L^p(\widehat{\Omega}_f;\mathbb{R}^d)$, and $g_f \in W^{2-1/p,p}(\Gamma_f;\mathbb{R}^d)$ satisfying (2) and $\widehat{P} \in W^{1,p}(\widehat{\Omega}_f)$ with compatibility condition

$$\int_{\widehat{\Omega}_f} \widehat{P}\,dx = 0,$$

or $\widehat{P} \in W^{1,p}(\widehat{\Omega}_f)/\mathbb{R}$. Define $\widehat{F}$ and $\hat{J}$ by (1).
Find $(\hat{v}_f, \hat{p}_f) \in W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \cap H^1_{0,\Gamma_f\cup\widehat{\Gamma}_i}(\widehat{\Omega}_f;\mathbb{R}^d) + (g_f, 0) \times W^{1,p}(\widehat{\Omega}_f)/\mathbb{R}$ such that

$$\begin{aligned}
-\widehat{\operatorname{div}}\,(\hat{\sigma}_f\widehat{B}) &= \hat{f}_f \quad \text{in } \widehat{\Omega}_f,\\
\widehat{\operatorname{div}}\,(\widehat{B}^T\hat{v}_f) &= \widehat{P} \quad \text{in } \widehat{\Omega}_f,\\
\hat{v}_f &= 0 \quad \text{on } \widehat{\Gamma}_i,\\
\hat{v}_f &= g_f \quad \text{on } \Gamma_f,
\end{aligned} \tag{4}$$

with $\widehat{B}$ given by Remark 2.5.

It is immediately clear that, Problem 2.2 can be obtained from Problem 2.6, by setting $\hat{f}_f = 0$ and $\widehat{P} = 0$.

Theorem 2.7 (Existence of Stokes in the ALE framework)**.** *Let $\widehat{\Omega}_f \subset \mathbb{R}^d$ be any open domain, with a $C^{1,1}$ boundary. Let $\hat{u}_s \in W^{2-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)$, $\hat{f}_f \in L^p(\widehat{\Omega}_f;\mathbb{R}^d)$ and $g_f \in W^{2-1/p,p}(\Gamma_f;\mathbb{R}^d)$ be given. Define $\widehat{F}$ and $\hat{J}$ by (1). In addition, let the assumptions a) - c) hold true, i.e., $\hat{u}_s \in W^{2-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)$ is small enough. Then, there exists a unique solution to Problem 2.6 and it holds the estimate*

$$\|\hat{v}_f\|_{W^{2,p}(\widehat{\Omega}_f)} + \|\hat{p}_f\|_{W^{1,p}(\widehat{\Omega}_f)/\mathbb{R}} \le C\Big(\|\hat{f}_f\|_{L^p(\widehat{\Omega}_f;\mathbb{R}^d)} + \|g_f\|_{W^{2-1/p,p}(\Gamma_f;\mathbb{R}^d)} + \|\widehat{P}\|_{W^{1,p}(\widehat{\Omega}_f)/\mathbb{R}}\Big). \tag{5}$$

The constant C depends on the assumptions a) $-$ c) and hence on $\|\hat{u}_s\|_{W^{2-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)}$; it remains bounded by $C_{\rho,M}$ as long as $0 < \rho \le \hat{J}$ and $\|\hat{u}_s\|_{W^{2-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)} \le M$. In particular, it holds

$$\|\hat{\sigma}_f\widehat{B}\hat{n}_f\|_{W^{1-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)} \le C\Big(\|\hat{f}_f\|_{L^p(\widehat{\Omega}_f;\mathbb{R}^d)} + \|g_f\|_{W^{2-1/p,p}(\Gamma_f;\mathbb{R}^d)} + \|\widehat{P}\|_{W^{1,p}(\widehat{\Omega}_f)/\mathbb{R}}\Big).$$

Further, the mapping $\mathcal{G} : W^{2-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d) \times L^p(\widehat{\Omega}_f;\mathbb{R}^d) \times W^{2-1/p,p}(\Gamma_f;\mathbb{R}^d) \times W^{1,p}(\widehat{\Omega}_f)/\mathbb{R} \to W^{1-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)$ *defined by*

$$(\hat{u}_s, \hat{f}_f, g_f, \widehat{P}) \mapsto \widehat{\sigma}_f \widehat{B} \hat{n}_f$$

is continuously differentiable.

Proof. The existence of a unique solution $(\hat{v}_f, \hat{p}_f) \in H^1_0(\widehat{\Omega}_f;\mathbb{R}^d) + (g_f, 0) \times L^2(\widehat{\Omega}_f)/\mathbb{R}$ to Problem 2.6 can be seen by transformation back to the physical domain Ω_f, noting that the mean value of the back transformed $\widehat{P}$ is preserved, see for instance [30, Prop. 2.2] (compare Problem 2.1).

The regularity follows as in [13, Lemma 4] with the obvious modifications for non-homogeneous Dirichlet data and inhomogeneity $\widehat{P}$. To show differentiability it is sufficient to consider the map $(\hat{u}_s, \hat{f}_f, g_f, \widehat{P}) \mapsto \widehat{\sigma}_f \widehat{B} \hat{n}_f$.

In order to show differentiability, we employ an implicit function type argument. To this end, for given values $\hat{u}_s, \hat{f}_f, g_f, \widehat{P}$ the solution $(\hat{v}_f, \hat{p}_f)$ is given by the equation (4), which we abbreviate as

$$a(\hat{u}_s, \hat{f}_f, g_f, \widehat{P}, \hat{v}_f, \hat{p}_f) = 0,$$

where

$$\begin{aligned} a \colon W^{2-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d) \times L^p(\widehat{\Omega}_f;\mathbb{R}^d) \times W^{2-1/p,p}(\Gamma_f;\mathbb{R}^d) \times W^{1,p}(\widehat{\Omega}_f)/\mathbb{R} \\ \times W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \cap H^1_{0,\Gamma_f\cup\Gamma_i}(\widehat{\Omega}_f;\mathbb{R}^d) \times W^{1,p}(\widehat{\Omega}_f)/\mathbb{R} \\ \to L^p(\widehat{\Omega}_f;\mathbb{R}^d) \times W^{1,p}(\widehat{\Omega}_f)/\mathbb{R}. \end{aligned}$$

To show differentiability of $(\hat{v}_f, \hat{p}_f)$ with respect to the given data, we employ the implicit function theorem. To do this end, we note that

$$D_{v,p} a(\hat{u}_s, \hat{f}_f, g_f, \widehat{P}, \hat{v}_f, \hat{p}_f) \colon W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \cap H^1_{0,\Gamma_f\cup\Gamma_i}(\widehat{\Omega}_f;\mathbb{R}^d) \times W^{1,p}(\widehat{\Omega}_f)/\mathbb{R} \to L^p(\widehat{\Omega}_f;\mathbb{R}^d) \times W^{1,p}(\widehat{\Omega}_f)/\mathbb{R},$$

corresponds to the transformed Stokes Problem 2.6, since it is linear in $\hat{v}_f, \hat{p}_f$. Thus, by what we have seen above, the operator
$D_{v,p} a(\hat{u}_s, \hat{f}_f, g_f, \hat{v}_f, \hat{p}_f)$ is invertible. This shows the assertion. □

For later purposes, we intend to derive some properties of the derivative $D_u\mathcal{G}$. To do so, we start with a Theorem calculating the derivative of $(\hat{v}_f, \hat{p}_f)$ with respect to the domain transformation $\hat{u}_f = \mathcal{R}\hat{u}_s$.

Theorem 2.8. *The mapping*

$$\begin{aligned} S \colon W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \cap H^1_{0,\Gamma_f}(\widehat{\Omega}_f;\mathbb{R}^d) \to W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \cap H^1_{0,\Gamma_f\cup\Gamma_i}(\widehat{\Omega}_f;\mathbb{R}^d) + (g_f, 0) \times W^{1,p}(\widehat{\Omega}_f)/\mathbb{R} \\ \hat{u}_f \mapsto (\hat{v}_f, \hat{p}_f) \end{aligned}$$

given by Problem 2.6 with fixed $(\hat{f}_f, g_f, \widehat{P})$ *is Fréchet-differentiable and the derivative*

$$S'(\hat{u}_f) \colon W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \cap H^1_{0,\Gamma_f}(\widehat{\Omega}_f;\mathbb{R}^d) \to W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \cap H^1_{0,\Gamma_f\cup\Gamma_i}(\widehat{\Omega}_f;\mathbb{R}^d) \times W^{1,p}(\widehat{\Omega}_f)/\mathbb{R},$$

is given as follows. For any given $\delta u \in W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)$, *the pair*

$$(\delta v, \delta p) = S'(\hat{u}_f)\delta u \in W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \cap H^1_{0,\Gamma_f\cup\Gamma_i}(\widehat{\Omega}_f;\mathbb{R}^d) \times W^{1,p}(\widehat{\Omega}_f)/\mathbb{R}$$

is given as the unique solution to the problem

$$\begin{aligned} -\widehat{div}((-\delta p I + \nu_f \nabla \delta v \widehat{F}^{-1})\widehat{B}) &= \widehat{div}\left(-\hat{p}_f D_u \widehat{B}\delta u + \nu_f \nabla \hat{v}_f D_u \widehat{A}\delta u\right) && \text{in } \widehat{\Omega}_f, \\ \widehat{div}(\widehat{B}^T \delta v) &= -\widehat{div}((D_u \widehat{B}^T \delta u)\hat{v}_f) && \text{in } \widehat{\Omega}_f, \\ \delta v &= 0 && \text{on } \widehat{\Gamma}_i \cup \Gamma_f, \end{aligned} \tag{6}$$

where $(\hat{v}_f, \hat{p}_f) = S(\hat{u}_f)$ *and* $\widehat{A}$ *and* $\widehat{B}$ *are given in Remark 2.5. In particular, if* $\|\hat{u}_f\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)}$ *is sufficiently small (compare Remark 2.5) then*

$$\|S'(\hat{u}_f)\|_{\mathcal{L}(W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d), W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)\cap H^1_{0,\Gamma_f\cup\Gamma_i}(\widehat{\Omega}_f;\mathbb{R}^d)\times W^{1,p}(\widehat{\Omega}_f)/\mathbb{R})} \le C(\hat{u}_f),$$

where $C(\hat{u}_f)$ *depends on* $\|\hat{u}_f\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^2)}$*, and the data* $(\hat{f}_f, g_f)$ *only. Further, it holds* $C(\hat{u}_f) \to 0$ *if* $(\hat{u}_f, \hat{f}_f, g_f) \to 0$.

Proof. To see that problem (6) has a unique solution, we show that the prerequisites of Problem 2.6 are fulfilled, which shows the claimed existence using Theorem 2.7. The regularity requirements for the right hand side are given by definition of $\widehat{F}$ and $(\hat{v}_f, \hat{p}_f)$ (using that $W^{1,p}$ and $W^{2,p}$ are algebras). Further to see that $\widehat{\mathrm{div}}\,((D_u \widehat{B}^T \delta u)\hat{v}_f) \in W^{1,p}(\widehat{\Omega}_f)$ we note that due to the Piola identity, see, e.g., [5, Chapter I, p 39], it holds $\widehat{\mathrm{div}}\,(\hat{J}(u)\widehat{F}^{-T}(u)) = 0$ for all u and thus also $\widehat{\mathrm{div}}\,((D_u \widehat{B}^T \delta u)) = 0$. This implies

$$\widehat{\mathrm{div}}\,((D_u \widehat{B}^T \delta u)\hat{v}_f) = D_u \widehat{B}^T \delta u : \nabla \hat{v}_f \in W^{1,p}(\widehat{\Omega}_f).$$

Thus, the only question remaining is the compatibility condition

$$\int_{\widehat{\Omega}_f} \widehat{\mathrm{div}}\,((D_u \widehat{B}^T \delta u)\hat{v}_f)\, dx = 0.$$

To this end, one applies the Gauss divergence theorem, which shows the compatibility condition using $\hat{v}_f = 0$ on $\widehat{\Gamma}_i$ and $D_u \widehat{B}^T \delta u = 0$ on Γ_f; by definition that $\widehat{\mathcal{A}} = I$ on Γ_f for all $u \in W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \cap H^1_{0,\Gamma_f}(\widehat{\Omega}_f;\mathbb{R}^d)$.

The claimed bound on $\|S'(\hat{u}_f)\|$ follows from the stability estimate (5) in Theorem 2.7 and the bounds on $D_u \widehat{F}^{-1}$, $D_u \hat{J}$, $\hat{v}_f$, and $\hat{f}_f$. For the bounds on $D_u \widehat{F}^{-1}$ and $D_u \hat{J}$ the smallness assumption in Remark 2.5 is required. Moreover, because $(\hat{v}_f, \hat{p}_f)$ in the right hand side of the defining equations of $(\delta v, \delta p)$ tends to zero if $(\hat{u}_f, \hat{f}_f, g_f) \to 0$ it holds $C(\hat{u}_f) \to 0$ if all data tends to zero. Note that C_u in Remark 2.5 c), and thus C in (5), is non increasing as $\hat{u}_f \to 0$.

Now, to show differentiability we define $(\hat{v}_f, \hat{p}_f) = S(\hat{u}_f)$ and $(v, p) = S(\hat{u}_f + \delta u)$ where δu is sufficiently small to assert the smallness assumption for $\hat{u}_f + \delta u$ following Remark 2.5.

Then, from strong convergence of the coefficients in Problem 2.6, i.e.,

$$\widehat{F}(\hat{u}_f + \delta u)^{-1} \to \widehat{F}(\hat{u}_f) \quad \text{as} \quad \|\delta u\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)} \to 0,$$

it follows easily, by comparing the defining equations and application of the stability estimate in Theorem 2.7, that $(v, p) \to (\hat{v}_f, \hat{v}_p)$ in $W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \times W^{1,p}(\widehat{\Omega}_f)$ as $\|\delta u\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)} \to 0$.

Now, we show that in fact $(\delta v, \delta p)$ given in the statement of the Theorem is the claimed derivative $S'(\hat{u}_f)\delta u$. We define the linearization error

$$(e_v, e_p) = S(\hat{u}_f + \delta u) - S(\hat{u}_f) - S'(\hat{u}_f)\delta u = (v - \hat{v}_f - \delta v, p - \hat{p}_f - \delta p).$$

By the following Lemma 2.9 we obtain

$$\|e_v\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)} + \|e_p\|_{W^{1,p}(\widehat{\Omega}_f)/\mathbb{R}} \leq o(\|\delta u\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)}).$$

This shows the asserted differentiability. □

Lemma 2.9. *Let $\hat{u}_f \in W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \cap H^1_{0,\Gamma_f}(\widehat{\Omega}_f;\mathbb{R}^d)$ be small enough such that the mapping S from Theorem 2.8 is well defined in $\hat{u}_f$. Further, let ϵ be such that S is well defined for all $\hat{u}_f + \delta u$ where $\|\delta u\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)} \leq \epsilon$.*

Let $(\hat{v}_f, \hat{p}_f) = S(\hat{u}_f)$ and $(v,p) = S(\hat{u}_f + \delta u)$ and $(\delta v, \delta p)$ be given by (6).

Then the linearization error $(e_v, e_p) = (v - \hat{v}_f - \delta v, p - \hat{p}_f - \delta p)$ satisfies:

$$\begin{aligned}
&\|e_v\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)} + \|e_p\|_{W^{1,p}(\widehat{\Omega}_f)/\mathbb{R}} \\
&\qquad \leq C\Big(\|v - \hat{v}_f\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)} + \|p - \hat{p}_f\|_{W^{1,p}(\widehat{\Omega}_f)/\mathbb{R}}\Big)\|\delta u\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)} + o\Big(\|\delta u\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)}\Big) \\
&\qquad = o\Big(\|\delta u\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)}\Big).
\end{aligned}$$

Proof. We compare the equation (4) for $(\hat{v}_f, \hat{p}_f)$

$$\begin{aligned}
-\widehat{\operatorname{div}}\,((-\hat{p}_f I + \nu_f \widehat{\nabla}\hat{v}_f \widehat{F}^{-1})\widehat{B}) &= \hat{f}_f \quad \text{in } \widehat{\Omega}_f, \\
\widehat{\operatorname{div}}\,(\widehat{B}^T \hat{v}_f) &= \widehat{P} \quad \text{in } \widehat{\Omega}_f, \\
\hat{v}_f &= 0 \quad \text{on } \widehat{\Gamma}_i, \\
\hat{v}_f &= g_f \quad \text{on } \Gamma_f,
\end{aligned}$$

where $\widehat{B} = \hat{J}\widehat{F}^{-T}$, and (v,p) where $F = \widehat{\nabla}\widehat{\mathcal{A}}(\hat{u}_f + \delta u)$, $J = \det F$ and $B = JF^{-T}$

$$\begin{aligned}
-\widehat{\operatorname{div}}\,((-pI + \nu_f \widehat{\nabla} v F^{-1})B) &= \hat{f}_f \quad \text{in } \widehat{\Omega}_f, \\
\widehat{\operatorname{div}}\,(B^T v) &= \widehat{P} \quad \text{in } \widehat{\Omega}_f, \\
v &= 0 \quad \text{on } \widehat{\Gamma}_i, \\
v &= g_f \quad \text{on } \Gamma_f,
\end{aligned}$$

with the equation (6) for $(\delta v, \delta p)$

$$\begin{aligned}
-\widehat{\operatorname{div}}\,((-\delta p I + \nu_f \nabla\delta v \widehat{F}^{-1})\widehat{B}) &= \widehat{\operatorname{div}}\,\big(-\hat{p}_f D_u \widehat{B}\delta u + \nu_f \nabla \hat{v}_f D_u \widehat{A}\delta u\big) && \text{in } \widehat{\Omega}_f, \\
\widehat{\operatorname{div}}\,(\widehat{B}^T \delta v) &= -\widehat{\operatorname{div}}\,((D_u \widehat{B}^T \delta u)\hat{v}_f) && \text{in } \widehat{\Omega}_f, \\
\delta v &= 0 && \text{on } \widehat{\Gamma}_i \cup \Gamma_f.
\end{aligned}$$

Comparing the first of each of the three equations yields, using $\widehat{A} = \widehat{F}^{-1}\widehat{B}$ and $A = F^{-1}B$,

$$\begin{aligned}
-\widehat{\operatorname{div}}\,((-e_pI + \nabla e_v\widehat{F}^{-1})\widehat{B}) &= -\widehat{\operatorname{div}}\,((-pI + \nu_f\widehat{\nabla}v\widehat{F}^{-1})\widehat{B}) - \hat{f}_f \\
&\quad - \widehat{\operatorname{div}}\left(-\hat{p}_fD_u\widehat{B}\delta u + \nu_f\nabla\hat{v}_fD_u\widehat{A}\delta u\right) \\
&= -\widehat{\operatorname{div}}\,((-pI + \nu_f\widehat{\nabla}v\widehat{F}^{-1})\widehat{B}) + \widehat{\operatorname{div}}\,((-pI + \nu_f\widehat{\nabla}vF^{-1})B) \\
&\quad - \widehat{\operatorname{div}}\,((-pI + \nu_f\widehat{\nabla}vF^{-1})B) - \hat{f}_f \\
&\quad - \widehat{\operatorname{div}}\left(-\hat{p}_fD_u\widehat{B}\delta u + \nu_f\nabla\hat{v}_fD_u\widehat{A}\delta u\right) \\
&= -\widehat{\operatorname{div}}\,(-p(\widehat{B} - B) - \hat{p}_fD_u\widehat{B}\delta u) \\
&\quad - \nu_f\widehat{\operatorname{div}}\,(\widehat{\nabla}v(\widehat{A} - A) + \nabla\hat{v}_fD_u\widehat{A}\delta u) \\
&=: R_1.
\end{aligned} \tag{7}$$

Noting that $W^{1,p}$ is an algebra, we see that $R_1 \in L^p(\widehat{\Omega}_f;\mathbb{R}^d)$ and by Frechét-differentiability of

$$\begin{aligned}
\widehat{A}, \widehat{B}\colon W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) &\to W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d}), \\
\hat{J}\colon W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) &\to W^{1,p}(\widehat{\Omega}_f),
\end{aligned}$$

in a neighborhood of $\hat{u}_f$; we can estimate the L^p-norm of R_1 as follows, using the constant C_u given in Remark 2.5,

$$\begin{aligned}
\|R_1\|_{L^p(\widehat{\Omega}_f;\mathbb{R}^d)} &\le \|-p(\widehat{B} - B) - \hat{p}_fD_u\widehat{B}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\
&\quad + \nu_f\|\widehat{\nabla}v(\widehat{A} - A) + \nabla\hat{v}_fD_u\widehat{A}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\
&\le \|p\|_{W^{1,p}(\widehat{\Omega}_f)/\mathbb{R}}\|B - \widehat{B} - D_u\widehat{B}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\
&\quad + \|p - \hat{p}_f\|_{W^{1,p}(\widehat{\Omega}_f)/\mathbb{R}}\|D_u\widehat{B}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\
&\quad + \nu_f\|\widehat{\nabla}v\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})}\|A - \widehat{A} - D_u\widehat{A}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\
&\quad + \nu_f\|\widehat{\nabla}(v - \hat{v}_f)\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})}\|D_u\widehat{A}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\
&\le C_u\|p - \hat{p}_f\|_{W^{1,p}(\widehat{\Omega}_f)/\mathbb{R}}\|\delta u\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)} \\
&\quad + \nu_fC_u\|\widehat{\nabla}(v - \hat{v}_f)\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})}\|\delta u\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)} + o(\|\delta u\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)}).
\end{aligned} \tag{8}$$

Now, we compare the second of each of the three equations; with the Piola identity, as in the proof of Theorem 2.8, we obtain

$$\begin{aligned}
\widehat{\operatorname{div}}\,(\widehat{B}^Te_v) &= \widehat{\operatorname{div}}\,(\widehat{B}^Tv) - \widehat{\operatorname{div}}\,(B^Tv) + \widehat{\operatorname{div}}\,(B^Tv) - \widehat{P} + \widehat{\operatorname{div}}\,((D_u\widehat{B}^T\delta u)\hat{v}_f) \\
&= \widehat{\operatorname{div}}\,((\widehat{B}^T - B^T)v + (D_u\widehat{B}^T\delta u)\hat{v}_f) \\
&= (\widehat{B}^T - B^T) : \widehat{\nabla}v + D_u\widehat{B}^T\delta u : \widehat{\nabla}\hat{v}_f \\
&=: R_2.
\end{aligned} \tag{9}$$

Further, we have the compatibility condition

$$\begin{aligned}\int_{\widehat{\Omega}_f} R_2\,dx &= \int_{\widehat{\Omega}_f} \widehat{\operatorname{div}}\,((\widehat{B}^T - B^T)v + (D_u\widehat{B}^T\delta u)\hat{v}_f)\,dx \\ &= \int_{\widehat{\Gamma}_i} (\widehat{B}^T - B^T)v\hat{n}_f + (D_u\widehat{B}^T\delta u)\hat{v}_f\hat{n}_f\,ds \\ &\quad + \int_{\Gamma_f} (\widehat{B}^T - B^T)v\hat{n}_f + (D_u\widehat{B}^T\delta u)\hat{v}_f\hat{n}_f\,ds = 0\end{aligned}$$

since $v, \hat{v}_f = 0$ on $\widehat{\Gamma}_i$ and $\widehat{B}^T = B^T$ on Γ_f and thus $D_u\widehat{B}^T\delta u = 0$ on Γ_f.

Again, using the algebra properties of $W^{1,p}$ and the properties of $\widehat{B}$, we assert $R_2 \in W^{1,p}(\widehat{\Omega}_f)$ and we calculate

$$\begin{aligned}\|R_2\|_{W^{1,p}(\widehat{\Omega}_f)} &\leq \|B^T - \widehat{B}^T D_u\widehat{B}^T\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})}\|v\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)} \\ &\quad + \|v - \hat{v}_f\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)}\|D_u\widehat{B}^T\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\ &\leq C_u\|v - \hat{v}_f\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)}\|\delta u\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)} + o(\|\delta u\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)}).\end{aligned} \tag{10}$$

Combining the above calculations asserts that (e_v, e_p) solves the problem

$$\begin{aligned}-\widehat{\operatorname{div}}\,((-e_pI + \nabla e_v\widehat{F}^{-1})\widehat{B}) &= R_1 && \text{in } \widehat{\Omega}_f, \\ \widehat{\operatorname{div}}\,(\widehat{B}^T e_v) &= R_2 && \text{in } \widehat{\Omega}_f, \\ e_v &= 0 && \text{on } \widehat{\Gamma}_i \cup \Gamma_f,\end{aligned}$$

with R_1 and R_2 given by (7) and (9). From the estimates (8) and (10) the assertion follows by Theorem 2.7. □

Now, we can state the next preliminary result concerning the derivative of the 'Dirichlet-to-Neumann' map given in Theorem 2.7:

Theorem 2.10. *For the mapping $\mathcal{G}$ given by Theorem 2.7 it holds; for any given constant $\varepsilon > 0$, one can assert that*

$$\|D_u\mathcal{G}(\hat{u}_s, \hat{f}_f, g_f)\| \leq \varepsilon$$

where the norm is taken with respect to

$$D_u\mathcal{G}(\hat{u}_s, \hat{f}_f, g_f)\colon W^{2-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d) \to W^{1-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)$$

provided that $\hat{f}_f \in L^p(\widehat{\Omega}_f;\mathbb{R}^d)$, $g_f \in W^{2-1/p,p}(\Gamma_f;\mathbb{R}^d)$, satisfying (2), and $\hat{u}_s \in W^{2-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)$ are sufficiently small.

Proof. We note that $\mathcal{G}$ is given as a composition of the following three maps:

$$\begin{aligned}\mathcal{R}\colon &\quad W^{2-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d) \to W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \\ &\quad \hat{u}_s \mapsto \mathcal{R}(\hat{u}_s) = \hat{u}_f, \\ S\colon &\quad W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \times L^p(\widehat{\Omega}_f;\mathbb{R}^d) \times W^{2-1/p,p}(\Gamma_f;\mathbb{R}^d) \to W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \times W^{1,p}(\widehat{\Omega}_f)/\mathbb{R}, \\ &\quad (\hat{u}_f, \hat{f}_f, g_f) \mapsto (\hat{v}_f, \hat{p}_f) \text{ solving Problem 2.2}, \\ \tau\colon &\quad W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \times W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \times W^{1,p}(\widehat{\Omega}_f)/\mathbb{R} \to W^{1-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d), \\ &\quad (\hat{u}_f, \hat{v}_f, \hat{p}_f) \mapsto \hat{J}\hat{\sigma}_f\widehat{F}^{-T}\hat{n}_f.\end{aligned}$$

Here, for fixed $\hat{f}_f$ and g_f the operator S is the same as defined in Theorem 2.8.

This means

$$\mathcal{G}(\hat{u}_s, \hat{f}_f, g_f) = \tau\Big(\mathcal{R}(\hat{u}_s), S(\mathcal{R}(\hat{u}_s), \hat{f}_f, g_f)\Big).$$

Noting that $\mathcal{R}$ is linear and bounded, and thus has no influence on the differentiability, we neglect the influence of $\mathcal{R}$ and write $\hat{u}_f$ instead of $\mathcal{R}(\hat{u}_s)$. By the chain rule, we get for any direction δu that

$$\begin{aligned} D_u\mathcal{G}(\hat{u}_f, \hat{f}_f, g_f)\delta u = {} & D_u\tau(\hat{u}_f, S(\hat{u}_f, \hat{f}_f, g_f))\delta u \\ & + D_{v,p}\tau(\hat{u}_f, S(\hat{u}_f, \hat{f}_f, g_f))D_u S(\hat{u}_f, \hat{f}_f, g_f)\delta u. \end{aligned}$$

Because τ is linear in v and p, it follows

$$\begin{aligned} D_u\mathcal{G}(\hat{u}_f, \hat{f}_f, g_f)\delta u = {} & D_u\tau(\hat{u}_f, S(\hat{u}_f, \hat{f}_f, g_f))\delta u \\ & + \tau(\hat{u}_f, D_u S(\hat{u}_f, \hat{f}_f, g_f)\delta u). \end{aligned}$$

We take a closer look on the summands on the right hand side.

For the first term $D_u\tau$, we note

$$\tau(\hat{u}_f, S(\hat{u}_f, \hat{f}_f, g_f)) = \tau(\hat{u}_f, \hat{v}_f, \hat{p}_f) = (-\hat{p}_f I + \nu_f \widehat{\nabla}\hat{v}_f \widehat{F}^{-1})\widehat{B}\hat{n}_f.$$

Then, for the derivative, we obtain

$$D_u\tau(\hat{u}_f, S(\hat{u}_f, \hat{f}_f, g_f))\delta u = (-\hat{p}_f I + \nu_f \widehat{\nabla}\hat{v}_f \widehat{F}^{-1})(D_u\widehat{B}\delta u)\hat{n}_f + (\nu_f \widehat{\nabla}\hat{v}_f D_u\widehat{F}^{-1}\delta u)\widehat{B}\hat{n}_f.$$

From this, using that $W^{1-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)$ is the trace space of $W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^d)$, and an algebra for $p > d$, we calculate:

$$\begin{aligned} \|D_u\tau(\hat{u}_f, S(\hat{u}_f, \hat{f}_f, g_f))\delta u\|_{W^{1-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)} \le {} & \|\widehat{\sigma}_f(D_u\widehat{B}\delta u)\hat{n}_f\|_{W^{1-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)} \\ & + \|(\nu_f \widehat{\nabla}\hat{v}_f D_u\widehat{F}^{-1}\delta u)\widehat{B}\hat{n}_f\|_{W^{1-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)} \\ \le {} & C\|\widehat{\sigma}_f\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})}\|D_u\widehat{B}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \\ & + C\|\hat{v}_f\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)}\|\widehat{B}\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})}\|D_u\widehat{F}^{-1}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})}. \end{aligned}$$

Using the definitions of $\hat{J}$ and $\widehat{F}^{-T}$, we get that for $\|\hat{u}_f\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)}$ sufficiently small, there is a constant, depending on the size of the norm only, such that

$$\|D_u\widehat{B}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} + \|D_u\widehat{F}^{-1}\delta u\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^{d\times d})} \le C\|\delta u\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)}.$$

Hence,

$$\|D_u\tau(\hat{u}_f, S(\hat{u}_f, \hat{f}_f, g_f))\|_{\mathcal{L}(W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d), W^{1-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d))} \to 0 \quad ((\hat{u}_f, \hat{f}_f, g_f) \to 0)$$

uniformly in $\|\delta u\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)} \le 1$ since $\|\widehat{\sigma}_f\|_{W^{1,p}(\widehat{\Omega}_f;\mathbb{R}^d)}, \|\hat{v}_f\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)} \to 0$.

Now, we come to the second term

$$\tau(\hat{u}_f, D_u S(\hat{u}_f, \hat{f}_f, g_f)\delta u).$$

We note that by definition $D_u S(\hat{u}_f, \hat{f}_f, g_f)\delta u = S'(\hat{u}_f)\delta u$ is given in Theorem 2.8. By the definition of τ it holds with $(\delta v, \delta p) = D_u S(\hat{u}_f, \hat{f}_f, g_f)\delta u$

$$\|\tau(\hat{u}_f, \delta v, \delta p)\|_{W^{1-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)} \le C\|S'(\hat{u}_f)\|\|\delta u\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)}.$$

Thus, by Theorem 2.8 we get

$$\|\tau(\hat{u}_f, D_u S(\hat{u}_f, \hat{f}_f, g_f)\delta u)\|_{W^{1-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)} \to 0 \quad ((\hat{u}_f, \hat{f}_f, g_f) \to 0)$$

uniformly in $\|\delta u\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)} \leq 1$. This yields the assertion. □

Theorem 2.11 (Existence of the structure problem)**.** *Let $\hat{f}_s \in L^p(\widehat{\Omega}_s;\mathbb{R}^d)$ and $\hat{g}_s \in W^{1-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)$. Then, there exists a unique solution $\hat{u}_s \in W^{2,p}(\widehat{\Omega}_s;\mathbb{R}^d) \cap W^{1,p}_{0,\widehat{\Gamma}_s}(\widehat{\Omega}_s;\mathbb{R}^d)$ to Problem 2.3. Further, it holds the estimate:*

$$\|\hat{u}_s\|_{W^{2,p}(\widehat{\Omega}_s;\mathbb{R}^d)} \leq C\left(\|\hat{f}_s\|_{L^p(\widehat{\Omega}_s;\mathbb{R}^d)} + \|\hat{g}_s\|_{W^{1-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)}\right).$$

In addition mapping $\mathcal{H} : L^p(\widehat{\Omega}_s;\mathbb{R}^d) \times W^{1-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d) \to W^{2,p}(\widehat{\Omega}_s;\mathbb{R}^d)$ defined by

$$(\hat{f}_s, \hat{g}_s) \mapsto \hat{u}_s$$

is continuously differentiable.

Proof. The proof can be found in Ciarlet [5, Theorem 6.3.6] noting that $\widehat{\Gamma}_i$ has positive distance to $\widehat{\Gamma}_s$; and thus no singularities from non-matching boundary conditions may arise. □

2.1. Existence of a solution to the fluid-structure problem

In this section, we state the forward problem under consideration, and give some results on existence of a locally unique solution. We consider the stationary Stokes equations involving a nonlinear coupling with linear elasticity. Due to the employed fixed-point arguments it is not surprising that we will obtain existence, local uniqueness and regularity of a solution under a small data assumption only.

We are now prepared to show that there exists a (unique) solution to the coupling of the Problems 2.2 and 2.3. Here the principal unknowns are the fluid velocity $\hat{v}_f$, the fluid pressure $\hat{p}_f$, the structure displacement $\hat{u}_s$, and the fluid domain displacement (mesh motion) $\hat{u}_f$. For the convenience to the reader, we recall the coupling conditions on $\widehat{\Gamma}_i$. The first one is the geometry coupling condition (to specify the ALE map $\widehat{\mathcal{A}}$) such that the unknown fluid domain follows the interface. The next is a velocity condition for the fluid problem and finally, the stress balance on the interface. Thus, we have

$$\begin{aligned} \hat{u}_f &= \mathcal{R}(\gamma_i(\hat{u}_s)) && \text{on } \widehat{\Gamma}_i, \\ \hat{v}_f &= 0 && \text{on } \widehat{\Gamma}_i, \\ \widehat{\Sigma}_s\hat{n}_s - \hat{J}\hat{\sigma}_f\widehat{F}^{-T}\hat{n}_f &= 0 && \text{on } \widehat{\Gamma}_i. \end{aligned} \tag{11}$$

It can be inferred from the third condition that $\hat{g}_s = \hat{J}\hat{\sigma}_f\widehat{F}^{-T}\hat{n}_f$ in Problem 2.3.

Then, the coupled problem reads:

Problem 2.12. Find $(\hat{v}_f, \hat{p}_f) \in W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \times W^{1,p}(\widehat{\Omega})/\mathbb{R}$ such that

$$\begin{aligned} -\widehat{\operatorname{div}}\,(\hat{\sigma}_f\widehat{B}) &= 0 \quad \text{in } \widehat{\Omega}_f, \\ \widehat{\operatorname{div}}\,(\widehat{B}^T\hat{v}_f) &= 0 \quad \text{in } \widehat{\Omega}_f, \\ \hat{v}_f &= g_f \quad \text{on } \Gamma_f, \end{aligned}$$

and find $\hat{u}_s \in W^{2,p}(\widehat{\Omega}_s;\mathbb{R}^d)$, such that

$$
\begin{aligned}
-\widehat{\operatorname{div}}\,(\widehat{\Sigma}_s) &= \hat{f}_s \quad \text{in } \widehat{\Omega}_s,\\
\hat{u}_f &= 0 \quad \text{on } \widehat{\Gamma}_s,
\end{aligned}
$$

using the coupling conditions (11).

The following proof of existence for the FSI problem is similar to [13]. However, a major extension is the differentiability of the solution map.

Theorem 2.13. *Let $g_f \in W^{2-1/p,p}(\Gamma_f;\mathbb{R}^d)$, and $\hat{f}_s \in L^p(\widehat{\Omega}_s;\mathbb{R}^d)$ be given with $3 < p < \infty$. Assuming that g_f, and $\hat{f}_s$ are small enough, meaning that for a certain constant $M > 0$,*

$$\|g_f\|_{W^{2-1/p,p}(\Gamma_f)} + \|\hat{f}_s\|_{L^p(\widehat{\Omega}_s)} \le M.$$

Then there exists a solution $\widehat{U} = (\hat{v}_f, \hat{p}_f, \hat{u}_s) \in \widehat{W} + (g_f, 0, 0)$ with

$$\widehat{W} = W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d) \cap W^{1,p}_{0,\Gamma_f\cup\widehat{\Gamma}_i}(\widehat{\Omega}_f;\mathbb{R}^d) \times W^{1,p}(\widehat{\Omega}_f)/\mathbb{R} \times W^{2,p}(\widehat{\Omega};\mathbb{R}^d) \cap W^{1,p}_{0,\partial\widehat{\Omega}}(\widehat{\Omega};\mathbb{R}^d)$$

to the coupled Problem 2.12 with $\widehat{F}$ and $\hat{J}$ are defined by (1).

Assuming that M is sufficiently small, the solution is uniquely determined under the additional condition:

$$\|\hat{v}_f\|_{W^{2,p}(\widehat{\Omega}_f;\mathbb{R}^d)} + \|\hat{p}_f\|_{W^{1,p}(\widehat{\Omega}_f)/\mathbb{R}} + \|\hat{u}_s\|_{W^{2,p}(\widehat{\Omega}_s;\mathbb{R}^d)} \le M. \tag{12}$$

In addition, the herewith defined mapping

$$
\begin{aligned}
\mathcal{F}\colon W^{2-1/p,p}(\Gamma_f;\mathbb{R}^d) \times L^p(\widehat{\Omega}_s;\mathbb{R}^d) &\to \widehat{W} + (g_f, 0, 0)\\
(g_f, \hat{f}_s) &\mapsto (\hat{v}_f, \hat{p}_f, \hat{u}_s)
\end{aligned}
$$

is continuously differentiable in a neighborhood of $(0,0)$.

Proof. The existence of a solution can be obtained by the arguments in [13], in particular Theorem 1. Hence all we need to show is differentiability of the thus defined mapping $\mathcal{F}$.

To do this, we note that

$$(\hat{v}_f, \hat{p}_f, \hat{u}_s) = \mathcal{F}(g_f, \hat{f}_s)$$

iff the displacement $\hat{u}_s$ satisfies

$$\hat{u}_s = \mathcal{H}(\hat{f}_s, \mathcal{G}(\hat{u}_s, 0, g_f, 0)). \tag{13}$$

Then, the velocity and pressure $(\hat{v}_f, \hat{p}_f)$ depend continuously differentiable on $(\hat{u}_s, g_f)$ by Theorem 2.7. Thus, it is sufficient to show differentiability of the mapping

$$(g_f, \hat{f}_s) \mapsto \hat{u}_s$$

given by the above fix point relation (13). To see that this defines a differentiable mapping, we employ the implicit function theorem. We note that

$$D_g\mathcal{H}(\hat{f}_s, \mathcal{G}(\hat{u}_s, 0, g_f, 0))\colon W^{1-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d) \to W^{2,p}(\widehat{\Omega}_s;\mathbb{R}^d)$$

corresponds to the solution operator to the linear elasticity problem 2.3 and is thus bounded, see Theorem 2.11. The second part

$$D_u\mathcal{G}(\hat{u}_s, 0, g_f, 0))\colon W^{2,p}(\widehat{\Omega}_s;\mathbb{R}^d) \to W^{1-1/p,p}(\widehat{\Gamma}_i;\mathbb{R}^d)$$

corresponds to the shape derivative of the flow and from Theorem 2.10, we know that $\|D_u \mathcal{G}\|$ can be made arbitrarily small by choosing M possibly smaller. Thus, $I + D_g \mathcal{H} \circ D_u \mathcal{G}$ is invertible. The implicit function theorem yields the asserted local uniqueness of $(\hat{v}_f, \hat{p}_f, \hat{u}_s)$ and differentiability of $\mathcal{F}$. □

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