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Abstract We consider a single-input-single output systems whose internal dynamics are described by the heat equation on some domain $\Omega \subset \mathbb{R}^d$ with sufficiently smooth boundary $\partial\Omega$. The input is formed by the Neumann boundary values; the output consists of the integral over the Dirichlet boundary values.
We show that the transfer function admits some partial fraction expansion with positive residuals. The location of transmission and invariant zeros of this system is furthermore investigated. We prove that the transmission zeros have an interlacing property in the sense that there is exactly one transmission zero between two poles of the transfer function. The set of transmission zeros is shown to be a subset of the invariant zeros.
Thereafter we consider the zero dynamics of this system. We prove that these are fully described by a self-adjoint and exponentially stable semigroup. The eigenvalues of the generator of this semigroup are proven to coincide with the set of invariant zeros.
Finally, we consider proportional output feedback. We show that any positive proportional gain results in an exponentially stable system. We further prove the root locus property: If the proportional gain tends to infinity, then the eigenvalues of the generator of the closed loop system will converge to the invariant zeros.

Keywords heat equation · infinite-dimensional linear systems · pole · transmission zero · invariant zero · zero dynamics · root locus · high-gain output feedback

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1 Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$. Consider the following heat equation with Neumann boundary control and observation formed by the spatial integral of the Dirichlet boundary values

$$\begin{aligned} \frac{\partial x}{\partial t}(\xi,t) &= \Delta_\xi x(\xi,t), && (\xi,t) \in \Omega \times \mathbb{R}_{\geq 0}, \\ u(t) &= \partial_\nu x(\xi,t), && (\xi,t) \in \partial\Omega \times \mathbb{R}_{\geq 0}, \\ y(t) &= \int_{\partial\Omega} x(\xi,t)\mathrm{d}\sigma_\xi && ,t \in \mathbb{R}_{\geq 0}, \\ x(\xi,0) &= x_0(\xi) && \xi \in \Omega. \end{aligned} \tag{1.1}$$

By setting $x(t) = x(\cdot,t)$, the boundary controlled heat equation (1.1) can be modeled as an infinite-dimensional linear system,

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \tag{1.2a}$$

$$y(t) = Cx(t). \tag{1.2b}$$

Due to the fact that control and observation are at the boundary, the operators B and C are now so-called *unbounded control and observation operators*.
This article is organized as follows: In Section 2, we recall the formulation of (1.1) as well-posed linear system from [6]. We also collect some known facts about the Neumann-Laplacian and its operator root. Section 3 deals with the transfer function of this system, i.e., we are dealing with the expression

$$G(s) = C(sI - A)^{-1}B.$$

We find a representation of the transfer function by a partial fraction expansion in which the poles are real and negative, and the corresponding residuals are all positive. This further gives rise to a detailed localization of the zeros of G. These will be called the *transmission zeros*, and they are shown to fulfill an interlacing property: All transmission zeros are negative and real; between two poles there exists exactly one transmission zero. The *invariant zeros* of the system are defined in Section 4. These are, loosely speaking, the numbers $\lambda \in \mathbb{C}$ for which the block operator

$$\begin{bmatrix} \lambda I - A & B \\ C & 0 \end{bmatrix}$$

has a non-trivial kernel. We will prove that the set of invariant zeros consists precisely of the unobservable eigenvalues and the transmission zeros. In Section 5 we define the *zero dynamics*. These are, again loosely speaking, the solutions of (1.1) with trivial output $y(\cdot) \equiv 0$. We show that the zero dynamics are fully described by an exponentially stable and self-adjoint semigroup. The spectrum of the generator of this semigroup coincides with the set of invariant zeros of the heat equation system. In the final Section 6, it is shown that proportional output feedback $u(t) = -k \cdot y(t)$ leads to the generator of an exponentially stable semigroup $T_k(\cdot)$ for all positive proportional gains k. For the generator A_k of this semigroup, we prove that with $k \to \infty$, the eigenvalues of A_k tend towards the invariant zeros of the heat equation system.

Let us give some relations to existing literature: The zero dynamics for a boundary controlled heat equation is considered in [4]. There it is presented that the zero dynamics are represented by a certain semigroup on $L^2(\Omega)$. A detailed analysis of the semigroup property and, in particular, the correspondence between eigenvalues of this generator and the invariant zeros has not been established. The article [3] treats root-locus for parabolic problems of type $\dot{x}(\xi,t) = Lx(\xi,t)$, where $\xi \in [0,1]$ and L is some differential operator of even order. Input and output are formed by boundary values which, in the case where L is a second-order differential operator, coincide with our assumptions in the special case where $\Omega = [0,1]$. Further note that, in [27], invariant zeros of infinite-dimensional systems with bounded input and output operators are considered: The interlacing property has been shown for the case where A is self-adjoint and $B = C^*$. Zero dynamics of infinite-dimensional linear systems with bounded control and observation operators are treated in [12]. There it is shown that the zero dynamics are, under certain additional boundedness assumptions on B and C which are intimately connected to the relative degree of the transfer function, the zero dynamics are described by a semigroup which is defined on some closed proper subspace of the state space.

1.1 Nomenclature and fundamentals

$\mathbb{N}$, $\mathbb{N}_0$		set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, set of all integers, resp.
$\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$	=	$[0,\infty)$, $(0,\infty)$, resp.
$\operatorname{Re}\lambda$, $\operatorname{Im}\lambda$, $\overline{\lambda}$		real part, imaginary part, complex conjugate, resp. of a complex number $\lambda \in \mathbb{C}$
$\ker A$, $\operatorname{im} A$		kernel and range of a linear operator A
$A\|_Y$		restriction of a mapping $A : X \to H$ to the subset $Y \subset X$
I		identity mapping
$\mathscr{B}(X,Y)$		the set of bounded linear operator from X to Y
$\rho(A)$, $\sigma(A)$		the resolvent set and spectrum of a linear operator A
$\ell^p(\mathbb{N})$, $\ell^p(\mathbb{N}_0)$		$p \in [1,\infty]$, the space of p-summable sequences $(a_k)_{k\in\mathbb{N}}$, resp. $(a_k)_{k\in\mathbb{N}_0}$
$L^p(\Omega;X)$		$p \in [1,\infty]$, the Lebesgue space of measurable functions $x : \Omega \to X$, see [9, Chapter IV]
$L^p_{\mathrm{loc}}(\Omega;X)$		space of measurable functions from Ω to X that are locally in L^p
$L^p(\Omega)$, $L^p_{\mathrm{loc}}(\Omega)$	=	$L^p(\Omega;\mathbb{C})$, $L^p_{\mathrm{loc}}(\Omega;\mathbb{C})$, resp.

$\mathscr{C}^k(\Omega;X)$		$k\in\mathbb{N}_0$, the set of k-times continuously differentiable functions from Ω to X
$\mathscr{C}(\Omega;X)$, $\mathscr{C}^k(\Omega)$, $\mathscr{C}(\Omega)$	=	$\mathscr{C}^0(\Omega;X)$, $\mathscr{C}^k(\Omega;\mathbb{C})$, $\mathscr{C}^0(\Omega;\mathbb{C})$, resp.
$\mathscr{C}_c^\infty(\Omega)$		the set of infinite times differentiable functions from Ω to $\mathbb{C}$ with compact support in Ω
$H^k(\Omega)$		$k\geq 0$ (fractional) Sobolev space of functions $x:\Omega\to\mathbb{C}$, see [10, Chapter 4]

The scalar product $\langle\cdot,\cdot\rangle$ of a Hilbert H space is defined to be linear in the first and anti-linear in the second component. On the dual space H' we define multiplication such that $(\lambda y)(x) := \overline{\lambda} y(x)$ for $y\in H'$ and $x\in H$. With this definition the dual pairing $\langle y,x\rangle := \overline{y(x)}$ for $y\in H'$ and $x\in H$ becomes linear in the first and anti-linear in the second component.
In this article $\Omega\subset\mathbb{R}^d$ is always a bounded open set with a uniformly $\mathscr{C}^2$-boundary $\partial\Omega$ [1, Chapter 4]. Integration on the surface of this manifold is indicated by σ_ξ and $|\partial\Omega| := \int 1 \mathrm{d}\sigma_\xi$ is the surface area of the boundary. For $\xi\in\partial\Omega$ we denote by $\nu(\xi)$ the outward normal of $\partial\Omega$ and by $\partial_\nu x(\xi)$ the directional derivative of some function $x\in L^2(\mathbb{R}^d)$ along ν at the point ξ, whenever it is well-defined. By ∇x, Δx we denote the (distributional) gradient, respectively Laplacian of x.
For the notion of (strongly continuous, contractive, analytic, bounded, exponentially stable) semigroup we refer to [24]. A definition of sesquilinear forms can be found in [13].
For the readers convenience, a known but crucial result on the connection between semigroups and sesquilinear forms is depicted. This fundamental theorem will be used several times.

Theorem 1.1 *Let H be a Hilbert space, which is continuously and densely embedded into the Hilbert space X and let $a: H\times H\to\mathbb{C}$ be a continuous, symmetric sesquilinear form. If, for some $\alpha>0$, the form fulfills*

$$\operatorname{Re} a(x,x) = a(x,x) \geq \alpha\|x\|_X \qquad \forall x\in X,$$

then the following holds

(i) The operator

$$\begin{aligned} D(A) &:= \left\{x\in H \;\middle|\; \exists z(x)\in X : a(x,\varphi) = \langle z(x),\varphi\rangle_X \quad \forall\varphi\in H\right\}, \\ Ax &:= -z(x) \qquad \forall x\in D(A) \end{aligned}$$

is well-defined, self-adjoint, non-positive and generates a contractive, analytic semigroup in X.

(ii) $D(A)$ is dense in H with respect to $\|\cdot\|_H$.

(iii) For any $\lambda \in \mathbb{R}_{\geq 0}$, the operator root (in the sense of [13]) of $\lambda I - A$ fulfills

$$D((\lambda I - A)^{\frac{1}{2}}) = H,$$
$$\langle (\lambda I - A)^{\frac{1}{2}}, (\lambda I - A)^{\frac{1}{2}} \rangle_X = \lambda \langle x, y \rangle_X + a(x,y) \qquad \forall x \in D(A).$$

We call A the operator associated to the sesqulinear form $a(\cdot,\cdot)$.

Proof The first part of this is [2, Theorem 4.2]. Assertions (ii) and (iii) are contained in Kato's First and Second Representation Theorem [13, Section VI.2]. □

2 Heat equation as infinite-dimensional linear system

In [6] the partial differential equation (1.1) was put into the framework of infinite-dimensional well-posed linear systems. This is the framework within which we will analyse and solve the equation. Therefore, we recollect several facts from [6]: By taking $x(t) := x(\cdot,t) \in L^2(\Omega)$, the heat equation (1.1) can be interpreted as an infinite-dimensional linear system (1.2) on the state space $X := L^2(\Omega)$ with A, B and C as presented in the following:

(a) $A : D(A) \subset X \to X$ with

$$Ax = \Delta x \qquad \forall x \in D(A) = \left\{ \, x \in H^2(\Omega) \;\middle|\; \partial_\nu x|_{\partial\Omega} = 0 \, \right\}; \tag{2.1a}$$

(b) $B \in \mathscr{B}(\mathbb{C}, \left(H^1(\Omega)\right)')$ with

$$\langle Bu, \varphi \rangle = u \cdot \int_{\partial\Omega} \overline{\varphi(\xi)} \mathrm{d}\sigma_\xi \qquad \forall \varphi \in H^1(\Omega); \tag{2.1b}$$

(c) $C : D(C) \to \mathbb{C}$ with

$$Cx = \int_{\partial\Omega} x(\xi) \mathrm{d}\sigma_\xi \qquad \forall x \in D(C) \supset H^1(\Omega). \tag{2.1c}$$

Note that B and C are well-defined, because there exists a continuous linear trace operator mapping $H^1(\Omega)$ onto $L^2(\partial\Omega)$. In fact, C is well defined on $H^{\frac{1}{2}+\varepsilon}$ for any $\varepsilon > 0$, according to [10, Theorem 4.24 (i)]. The actual domain $D(C)$ is defined precisely in [6, Equation (6.9)]. For our purposes it suffices to know that $H^1(\Omega)$ is contained in $D(C)$.

2.1 The Neumann Laplacian

The Laplacian with Neumann boundary condition is the main operator of our system. Since it plays such an important role, we collect several facts about it that are mostly known, see e.g. [10, 6].

First there is a deep result, which states that weak solutions of the Neumann problem are in fact $H^2(\Omega)$-functions.

Lemma 2.1 [10, Proposition 5.26 (ii)] *Let $x \in H^1(\Omega)$ and $f \in L^2(\Omega)$, satisfy*

$$\int_\Omega \nabla x(\xi)\overline{\nabla\varphi(\xi)}\mathrm{d}\xi = \int_\Omega f(\xi)\overline{\varphi(\xi)}\mathrm{d}\xi$$

for all $\varphi \in C^\infty(\Omega)$ with $\partial_\nu\varphi|_{\partial\Omega} \equiv 0$. Then $x \in H^2(\Omega)$ and $\partial_\nu x|_{\partial\Omega} \equiv 0$.

Lemma 2.2

(i) The sesquilinear form

$$D(a): H^1(\Omega)\times H^1(\Omega), \qquad a(x,z) := \int_\Omega \nabla x(\xi)\overline{\nabla z(\xi)}d\xi, \tag{2.2}$$

is symmetric and continuous on the complex Hilbert-space $H^1(\Omega)$. The operator associated to $a(\cdot,\cdot)$ (cf. Theorem 1.1) is A as in (2.1a)*.*

(ii) A is self-adjoint and non-positive and it generates a contractive semigroup $T(\cdot)$ on $X = L^2(\Omega)$.

(iii) The space $D(A)$ is densely and continuously embedded into the space $H^1(\Omega)$ and there holds

$$\|x\|^2_{H^1(\Omega)} = \|x\|^2_{L^2(\Omega)} - \langle x, Ax\rangle_{L^2(\Omega)} \qquad \forall x \in D(A), \tag{2.3}$$

Proof Statement (i) is proven in [10, Theorem 5.31 (ii)], and statement (iii) in [10, Proposition 5.28 (i)]. Assertion (ii) can be deduced from Theorem 1.1, or, alternatively, from [6, Statement 1]. □

For $k,s \in \mathbb{R}_{>0}$ we define the Banach spaces

$$X_k := (sI-A)^{-k}X, \qquad \|x\|_{X_k} := \left\|(sI-A)^k x\right\|_X,$$

and X_{-k} as the dual space of X_k with respect to the pivot space X, see [24, Section 2.9]. The space X_k is independent of the choice of $s > 0$ in this definition and X_{-k} equals the completion of X with respect to the norm $\|(sI-A)^{-k}\cdot\|_X$, [24, Proposition 2.10.2]. For $k \in (0,1]$ the semigroup T extends to a strongly continuous semigroup on X_{-k}, also denoted by T, see [19, Theorem 3.10.11]. The generator of this semigroup is the continuous extension of $A \in \mathscr{B}(X_1;X)$ to $A \in \mathscr{B}(X_{-k+1};X_{-k})$.

Lemma 2.3 *Writing $\sim$ for norm equivalence, the following holds true:*

(i) $X_1 = D(A)$ and $\|\cdot\|_{X_1} \sim \|\cdot\|_X + \|A\cdot\|_X \sim \|\cdot\|_{H^2(\Omega)}$.

(ii) $X_{\frac12} = D(A^{\frac12}) = H^1(\Omega)$ and $\|\cdot\|_{X_{\frac12}} \sim \|\cdot\|_{H^1(\Omega)}$. Consequently, $X_{-\frac12} = \left(H^1(\Omega)\right)'$.

(iii) The extension of the semigroup T to $X_{-\frac12}$ is generated by the mapping

$$A_{-\frac12}: H^1(\Omega) \to \left(H^1(\Omega)\right)', \qquad \langle A_{-\frac12}x,\varphi\rangle := a(x,\varphi), \qquad \varphi \in H^1(\Omega). \tag{2.4}$$

Proof Regarding (i), it is clear that $X_1 = (sI-A)^{-1}X = D(A)$ and easy to show that $\|x\|_{X_1} := \|(sI-A)x\|_X \sim \|x\|_{L^2(\Omega)} + \|\Delta x\|_{L^2(\Omega)}$ for $x \in D(A)$. Theorem 5.11 of [10] states that the latter norm is equivalent to $H^2(\Omega)$ norm. Statements (ii) and (iii) are consequences of (2.2) and the fact that the domain of the form a equals $D((sI-A)^{\frac12})$ by KATO's Second Representation Theorem [13, Section VI.2]. □

Lemma 2.4 *Let A be defined as in* (2.1a). *The resolvent of A is compact and there is a real valued sequence $(\lambda_k)_{k\in\mathbb{N}_0}$ such that*

a) (λ_k) *is nondecreasing,* $\lambda_0 = 0$, $\lambda_1 > 0$, *and* $\lim_{k\to\infty}\lambda_k = \infty$;
b) $\sigma(A) = \{-\lambda_k \mid k \in \mathbb{N}_0\}$;

and there is an orthonormal basis $(v_k)_{k\in\mathbb{N}_0}$ of $L^2(\Omega)$ with $v_k \in D(A)$ for all $k \in \mathbb{N}_0$, and

$$Ax = -\sum_{k=0}^{\infty} \lambda_k \langle x, v_k\rangle_{L^2(\Omega)} \cdot v_k \qquad \forall x \in D(A). \tag{2.5}$$

The domain of A can be represented by

$$D(A) = \left\{ \sum_{k=0}^{\infty} c_k v_k \,\middle|\, (c_k), (\lambda_k c_k) \in \ell^2(\mathbb{N}_0) \right\}, \tag{2.6}$$

$$\left\| \sum_{k=0}^{\infty} c_k v_k \right\|_{D(A)}^2 = \|(c_k)\|_{\ell^2(\mathbb{N}_0)}^2 + \|(\lambda_k c_k)\|_{\ell^2(\mathbb{N}_0)}^2. \tag{2.7}$$

Moreover,

$$H^1(\Omega) = \left\{ \sum_{k=0}^{\infty} a_k v_k \,\middle|\, (a_k), (\sqrt{\lambda_k} a_k) \in \ell^2(\mathbb{N}_0) \right\}, \tag{2.8}$$

$$\left\| \sum_{k=0}^{\infty} a_k v_k \right\|_{H^1(\Omega)}^2 = \|(a_k)\|_{\ell^2(\mathbb{N}_0)}^2 + \left\|(\sqrt{\lambda_k} a_k)\right\|_{\ell^2(\mathbb{N}_0)}^2. \tag{2.9}$$

Proof Since $(\lambda I - A)^{-1}L^2(\Omega) = D(A) \subset H^2(\Omega)$ and $H^2(\Omega)$ is compactly embedded into $L^2(\Omega)$ by the Rellich-Kondrachov Theorem [10, Theorem 4.17 (i)], the resolvent of A is compact. The part about the spectrum and the representation (2.5) follow with the spectral theorem for compact operators and can for example be found in [10, Theorem 7.13 (ii)].

We have $X_{\frac{1}{2}} = H^1(\Omega)$ by Lemma 2.3, so it remains to prove (2.8): Using (2.3) and the spectral decomposition (2.5), we see that for all $v = \sum_{k=0}^{\infty} a_k v_k \in D(A)$ holds

$$\begin{aligned}
\|v\|_{H^1(\Omega)}^2 &= \|v\|_{L^2(\Omega)}^2 - \langle v, Av\rangle_{L^2(\Omega)} = \|(a_k)\|_{\ell_2(\mathbb{N}_0)}^2 + \left\langle v, \sum_{k=0}^{\infty} \lambda_k \langle v, v_k\rangle v_k \right\rangle \\
&= \|(a_k)\|_{\ell_2(\mathbb{N}_0)} + \sum_{k=0}^{\infty} \lambda_k a_k \underbrace{|\langle v, v_k\rangle|^2}_{=|a_k|^2} = \|(a_k)\|_{\ell_2(\mathbb{N}_0)}^2 + \sum_{k=0}^{\infty} \lambda_k |a_k|^2 \\
&= \|(a_k)\|_{\ell_2(\mathbb{N}_0)}^2 + \left\| \left(\sqrt{\lambda_k} a_k \right) \right\|_{\ell_2(\mathbb{N}_0)}^2.
\end{aligned}$$

The representation (2.6) implies that linear combinations of $(v_k)_{k\in\mathbb{N}_0}$ are dense in $D(A)$. Since Lemma 2.2 (iii) states that $D(A)$ is dense in $H^1(\Omega)$, we can infer from the above computations that $H^1(\Omega)$ is equal to the completion of span$\{\ v_k \mid k \in \mathbb{N}_0\ \}$ with respect to the the norm

$$\left\| \sum_{k=0}^{\infty} a_k v_k \right\|^2 := \|(a_k)\|^2_{\ell_2(\mathbb{N}_0)} + \left\| \left(\sqrt{\lambda_k} a_k \right) \right\|^2_{\ell_2(\mathbb{N}_0)},$$

whence (2.8) holds true. □

2.2 Inputs and outputs

In the sequel we collect some special properties of the system given by A, B and C in (2.1): As mentioned earlier, the operator $C : D(C) \supset H^1(\Omega) \to \mathbb{C}$, is well defined on $H^1(\Omega)$. The operator $B : \mathbb{C} \to \left(H^1(\Omega)\right)'$ is the adjoint operator of $C|_{H^1(\Omega)}$ in the sense that

$$\langle Bu, \varphi \rangle = \langle u, C\varphi \rangle \qquad \forall u \in \mathbb{C}, \varphi \in H^1(\Omega). \tag{2.10}$$

We will often identify the mapping B with the element $B \in \left(H^1(\Omega)\right)'$.
For $x_0 \in L^2(\Omega)$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$ the variation of constants formula

$$x(t) := T(t)x_0 + \int_0^t T(t-\tau)Bu(\tau)\mathrm{d}\tau, \qquad t \in \mathbb{R}_{\geq 0}, \tag{2.11}$$

is well defined as B maps into $X_{-\frac{1}{2}} \subset X_{-1}$. The function $x(\cdot) : \mathbb{R}_{\geq 0} \to X_{-1}$ defined by (2.11) is called *mild solution* of (1.2a). A *strong solution* of (1.2a) in X_s is a function $x \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; X_{s+1}) \cap C(\mathbb{R}_{\geq 0}; X_s)$, that satisfies

$$x(t) = x_0 + \int_0^t Ax(\tau) + Bu(\tau)\mathrm{d}\tau \quad \text{ in } X_s.$$

The following result shows that the mild solution (2.11) is even pointwisely in X and moreover, $x(t) \in D(C)$ for almost all $t \in \mathbb{R}_{\geq 0}$.

Theorem 2.5 *Let $X = L^2(\Omega)$ and the operators A, B and C as in* (2.1) *be given. Then the following holds true:*

(i) For all $u \in L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$, $x_0 \in X$, the function defined in (2.11) *fulfills*
  a) $x(t) \in X$ for all $t \in \mathbb{R}_{\geq 0}$;
  b) $x(t) \in D(C)$ for almost all $t \in \mathbb{R}_{\geq 0}$.

(ii) For all $t \in \mathbb{R}_{\geq 0}$, there exists some $c_t \in \mathbb{R}_{\geq 0}$, such that for all $u \in L^2([0,t])$, $x_0 \in X$, the solutions of (1.2) *fulfill*

$$\|y(\cdot)\|_{L^2([0,t])} + \|x(t)\|_X \leq c_t \cdot \left(\|u(\cdot)\|_{L^2([0,t])} + \|x_0\|_X \right). \tag{2.12}$$

Proof All of this is contained in in [6, Corollary 1]. □

The above statement means that the system (1.2) is *well-posed*. This basically comprises four properties, namely the boundedness of semigroup $T(\cdot)$ on each compact interval $[0,t]$ (which is guaranteed anyway by its strong continuity), as well as the boundedness of the input-to-state maps $\mathfrak{B}_t$, state-to-output maps $\mathfrak{C}_t$, and the input-output maps $\mathfrak{D}_t$, which are defined as follows:

$$\begin{aligned}
\mathfrak{B}_t: \quad & L^2([0,t]) \to X, & \mathfrak{C}_t: \quad & X \to L^2([0,t]),\\
& u(\cdot) \mapsto \int_0^t T(t-\tau)Bu(\tau)\mathrm{d}\tau, & & x \mapsto CT(\cdot)x,\\
\mathfrak{D}_t: \quad & L^2([0,t]) \to L^2([0,t]),\\
& u(\cdot) \mapsto C\int_0^{\cdot} T(\cdot-\tau)Bu(\tau)\mathrm{d}\tau.
\end{aligned} \tag{2.13}$$

Moreover, we can define the infinite-time state-to-output and input-to-output mappings

$$\begin{aligned}
\mathfrak{C}: \quad & X \to L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0}), & \mathfrak{D}: \quad & L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0}) \to L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0}),\\
& x \mapsto CT(\cdot)x, & & u(\cdot) \mapsto C\int_0^{\cdot} T(\cdot-\tau)Bu(\tau)\mathrm{d}\tau.
\end{aligned} \tag{2.14}$$

For any input function $u \in L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$ and initial value $x_0 \in X$ the state and output of the system (1.2) are defined by

$$x(t) = T(t)x_0 + \mathfrak{B}_t u|_{[0,t]}, \qquad t \in \mathbb{R}_{\geq 0}, \tag{2.15a}$$

$$y = \mathfrak{C}x_0 + \mathfrak{D}u. \tag{2.15b}$$

3 The transfer function

We collect properties of the transfer function $C(sI-A)^{-1}B$ with A, B and C as in (2.1). We show that it admits a partial fraction expansion with positive and real residuals and nonpositive real poles. This representation will be the basis for further investigations which, in particular, comprise an analysis of the location of the zeros.
We first define transfer functions of infinite-dimensional systems (cf. [26, 23] and the bibliographies therein).

Definition 3.1 Let X be a Hilbert space and let A, B and C be operators with the following properties:

(a) $A: D(A) \subset X \to X$ is the generator of a strongly continuous semigroup on X;
(b) $B \in \mathscr{B}(\mathbb{C}, D(A^*)')$;
(c) $C: D(C) \subset X \to \mathbb{C}$ for some dense subspace $D(C) \subset X$;
(d) For the space

$$V = \{\, x \in X \mid Ax \in X + \mathrm{im}B \,\}$$

with norm

$$\|x\|_V^2 = \inf\left\{\, \|x\|_X^2 + \|Ax+Bu\|_X^2 \;\middle|\; u \in \mathbb{C} \text{ with } Ax+Bu \in X \,\right\}$$

holds that $V \subset D(C)$ and C restricts to an element of $\mathscr{B}(V, \mathbb{C})$.

Let $r(A,B,C) \subset \sigma(A)$ be the set of removable singularities of the function

$$\rho(A) \to \mathbb{C}, \qquad s \mapsto C(sI-A)^{-1}B$$

Let $D(G) = \rho(A) \cup r(A,B,C)$. We define the *transfer function* $G : D(G) \to \mathbb{C}$ of (A,B,C) by the analytic extension of $C(sI-A)^{-1}B$. A complex number $s \in \mathbb{C}$ is called *transmission zero of* (A,B,C) if $s \in D(G)$ and $G(s) = 0$.

Remark 3.2 (Transfer function)

a) Existence and uniqueness of the analytic extension of G to $D(G)$ is guaranteed by Riemann's theorem [18, Thm. 10.21].
b) The definition of the space $V \subset X$ yields $\operatorname{im}(sI-A)^{-1}B \subset V$ for all $s \in \rho(A)$. In particular, the assumption $V \subset D(C)$ implies that $C(sI-A)^{-1}B$ is well-defined for all $s \in \rho(A)$.

In the subsequent results we collect some properties of the transfer function of the operators (A,B,C) defined in (2.1).

Lemma 3.3 *Let $X := L^2(\Omega)$, define A, B and C by* (2.1)*, and*

$$V := \{x \in X \mid Ax \in X + \operatorname{im} B\}.$$

(i) For all $x \in V$ there exists exactly one $u \in \mathbb{C}$ such that $Ax + Bu \in X$. This u is given by $u = -\frac{\int_\Omega \Delta x(\xi)\mathrm{d}\xi}{|\partial\Omega|}$ and there holds $Ax + Bu = \Delta x \in L^2(\Omega)$.
(ii) $V = \left\{x \in H^2(\Omega) \;\middle|\; \exists u \in \mathbb{C} : \partial_\nu x|_{\partial\Omega} \equiv u.\right\}$, and $\|\cdot\|_V = \|\cdot\|_{L^2(\Omega)} + \|\Delta\cdot\|_{L^2(\Omega)}$.
(iii) A, B, C fulfills the prerequisites (a)–(d) in Definition 3.1.

Proof (i) If x is in V, then there exist $z \in L^2(\Omega)$ and $u \in \mathbb{C}$ with $Ax = z - Bu$. Explicitly, this means

$$\int_\Omega x\overline{\Delta\varphi}\mathrm{d}\xi = \int_\Omega z\overline{\varphi}\mathrm{d}\xi - u\int_{\partial\Omega}\overline{\varphi}\mathrm{d}\sigma_\xi \quad \forall\varphi \in D(A). \tag{3.1}$$

For $\varphi \in C_c^\infty(\Omega)$ this reduces to

$$\int_\Omega x\overline{\Delta\varphi}\mathrm{d}\xi = \int_\Omega z\overline{\varphi}\mathrm{d}\xi,$$

which shows that $\Delta x = z \in L^2(\Omega)$. For $\varphi \equiv 1 \in D(A)$, equation (3.1) reads

$$\int_\Omega \Delta x 1\mathrm{d}\xi + u\int_{\partial\Omega} 1\mathrm{d}\sigma_\xi = \int_\Omega x\Delta 1\mathrm{d}\xi = 0,$$

whence $u = -\frac{\int_\Omega \Delta x(\xi)\mathrm{d}\xi}{|\partial\Omega|}$. In particular, u is uniquely determined for every $x \in V$.

(ii) "$\supset$": Let $x \in H^2(\Omega)$ and $\partial_\nu x|_{\partial\Omega} = u \in \mathbb{C}$. Then for all $\varphi \in D(A)$ holds

$$\langle Ax, \varphi\rangle = \int_\Omega x\overline{\Delta\varphi}\mathrm{d}\xi = \int_\Omega (\Delta x)\overline{\varphi}\mathrm{d}\xi - \int_{\partial\Omega}(\partial_\nu x)\overline{\varphi}\mathrm{d}\sigma_\xi.$$

Hence Ax is represented by the sum of $\Delta x \in L^2(\Omega)$ and $-B(\partial_\nu x)|_{\partial\Omega} \in \operatorname{im} B$.

"$\subset$": Let $x \in V$. Then we have $Ax + Bu = \Delta x$ with u as in (i). Since B maps into $X_{-\frac{1}{2}}$ for some $\lambda > 0$ this leads to

$$(\lambda I - A)x = \lambda x - \Delta x + Bu \in X_{-\frac{1}{2}}.$$

Since $(sI - A)^{-1}$ is an isomorphism from $X_{-\frac{1}{2}}$ onto $X_{\frac{1}{2}}$ by Lemma 2.3, we conclude

$$x = (\lambda I - A)^{-1}(\lambda x - \Delta x + Bu) \in X_{\frac{1}{2}} = H^1(\Omega).$$

Now pick $h \in H^2(\Omega)$ such that $\partial_\nu h|_{\partial\Omega} \equiv u$. Then for all $\varphi \in D(A)$ holds

$$\begin{aligned}\int_\Omega \nabla(x-h)\overline{\nabla\varphi}\,\mathrm{d}\xi &= \int_\Omega (x-h)\overline{\Delta\varphi}\,\mathrm{d}\xi \\ &= \int_\Omega \Delta x\overline{\varphi}\,\mathrm{d}\xi - u\int_{\partial\Omega}\overline{\varphi}\,\mathrm{d}\sigma_\xi - \int_\Omega (\Delta h)\overline{\varphi}\,\mathrm{d}\xi + \int_{\partial\Omega}\underbrace{(\partial_\nu h)}_{=u}\overline{\varphi}\,\mathrm{d}\sigma_\xi \\ &= \int_\Omega (\Delta x + \Delta h)\overline{\varphi}\,\mathrm{d}\xi.\end{aligned}$$

Noting $\Delta x + \Delta h \in L^2(\Omega)$, we conclude from Lemma 2.1 that $x \in H^2(\Omega)$. Hence $\partial_\nu x$ is well-defined almost everywhere on $\partial\Omega$ and in $L^2(\partial\Omega)$. We claim that it equals u. Equation (3.1) implies

$$\int_{\partial\Omega} (u - \partial_\nu x)\,\varphi\,\mathrm{d}\sigma_\xi = 0 \qquad \forall\varphi \in D(A).$$

At least for all ψ in the dense subset $\mathscr{C}^\infty(\partial\Omega)$ of $L^2(\partial\Omega)$ we can construct functions $\varphi \in D(A)$ with $\varphi|_{\partial\Omega} = \psi$. It follows that $\partial_\nu x = u$ almost everywhere on $\partial\Omega$ and the inclusion is shown.
The considerations above show that the infimum in the definition of $\|x\|_V$ is obsolete and we get the asserted representation of the norm.

(iii) Observe that by Lemma 2.3 (i) the norm in (ii) is equivalent to the $H^2(\Omega)$ norm. Therefore, C is continuous from V into $\mathbb{C}$. The other prerequisites of Definition 3.1 are satisfied by assumption. $\square$

Lemma 3.4 *Let A and B be defined as in* (2.1)*. For all $s \in \rho(A)$ there holds*

$$(sI - A)^{-1}B = \sum_{k=0}^{\infty} \frac{\int_{\partial\Omega}\overline{v_k(\xi)}\,\mathrm{d}\sigma_\xi}{s + \lambda_k} \cdot v_k \quad \in \quad H^1(\Omega). \tag{3.2}$$

This series converges in $H^1(\Omega)$ and

$$\sum_{k=0}^{\infty}\left(\int_{\partial\Omega} v_k(\xi)\,\mathrm{d}\sigma_\xi\right)^2 \frac{\lambda_k}{(s+\lambda_k)^2} < \infty. \tag{3.3}$$

Proof Since $\operatorname{im} B \subset X_{-\frac{1}{2}}$ and $(sI-A)^{-1}$ is an isomorphism from $X_{-\frac{1}{2}}$ onto $X_{\frac{1}{2}}$ by Lemma 2.3, we have $(sI-A)^{-1}B \in H^1(\Omega)$. The spectral decomposition (2.5) implies that for all $s \in \rho(A)$, $x \in L^2(\Omega)$ the representation

$$(sI-A)^{-1}x = \sum_{k=0}^{\infty} \frac{\langle x, v_k \rangle}{s+\lambda_k} \cdot v_k \tag{3.4}$$

holds and it extends continuously to all $z \in X_{-1}$ via

$$(sI-A)^{-1}z = \sum_{k=0}^{\infty} \frac{\langle z, v_k \rangle}{s+\lambda_k} \cdot v_k.$$

For $z = B \in X_{-1}$ this yields the equality in (3.2). This equation and the representation of $H^1(\Omega)$ found in (2.8) imply that the series in (3.3) is finite and consequently, that (3.2) converges in $H^1(\Omega)$. □

Remark 3.5 The expression $x = (sI-A)^{-1}Bu$ is the solution of the Helmholtz equation

$$\begin{aligned} s \cdot x(\xi) &= \Delta x(\xi) \quad && \xi \in \Omega, \\ u &= \partial_\nu x(\xi) \quad && \xi \in \partial\Omega, \end{aligned}$$

see [7].

Theorem 3.6 *Let A, B and C be defined as in* (2.1) *and let* (λ_k), (v_k) *be as in Lemma 2.4. Define*

$$c_k := \left| \int_{\partial\Omega} v_k(\xi) \,\mathrm{d}\sigma_\xi \right|^2 \quad \forall k \in \mathbb{N}_0 \qquad \textit{and} \qquad J_c := \{k \in \mathbb{N}_0 \mid c_k \neq 0\}. \tag{3.5}$$

Then for all $s \in \rho(A)$ *holds*

$$G(s) = \sum_{k=0}^{\infty} \frac{c_k}{s+\lambda_k} = \sum_{k \in J_c} \frac{c_k}{s+\lambda_k}. \tag{3.6}$$

Furthermore, we have $0 \in J_c$, *and*

$$\left(\frac{c_k}{\lambda_k} \right) \in \ell_1(\mathbb{N}). \tag{3.7}$$

Proof We express $(sI-A)^{-1}B$ using the series in (3.2). Since this series converges in $H^1(\Omega)$, we may interchange the order of limit and application of C to obtain

$$\begin{aligned} C(sI-A)^{-1}B =& C \sum_{k=0}^{\infty} \frac{\int_{\partial\Omega} \overline{v_k(\xi)} \,\mathrm{d}\sigma_\xi}{s+\lambda_k} \cdot v_k = \sum_{k=0}^{\infty} \frac{\int_{\partial\Omega} \overline{v_k(\xi)} \,\mathrm{d}\sigma_\xi}{s+\lambda_k} \cdot C v_k \\ =& \sum_{k=0}^{\infty} \frac{\left| \int_{\partial\Omega} v_k(\xi) \,\mathrm{d}\sigma_\xi \right|^2}{s+\lambda_k} = \sum_{k \in J_c} \frac{c_k}{s+\lambda_k}. \end{aligned}$$

Therefore, (3.6) holds true on $\rho(A)$.
We have $0 \in J_c$ because the first eigenvector v_0 in Lemma 2.4 is the constant function; more precisely,

$$v_0(\cdot) \equiv \frac{1}{\sqrt{\int_\Omega 1\,\mathrm{d}\xi}}, \qquad \text{whence} \quad c_0 = \frac{(\int_{\partial\Omega} 1\,\mathrm{d}\sigma_\xi)^2}{\int_\Omega 1\,\mathrm{d}\xi} > 0.$$

Finally, (3.7) is a consequence of (3.3): Since $\lambda_0 = 0$ is an isolated eigenvalue, we may pick an $s \in (-\lambda_1, 0)$ that is in the resolvent of A and get

$$\sum_{k=1}^{\infty} \frac{c_k}{\lambda_k} \le \sum_{k=1}^{\infty} \frac{c_k}{s+\lambda_k} \le \sum_{k=1}^{\infty} \frac{c_k \lambda_k}{|s+\lambda_k|^2} \overset{(3.3)}{<} \infty.$$

□

Remark 3.7

(i) The self-adjointness of A and relation (2.10) imply that if $\lambda_k \in \sigma(A)$ is a removable singularity of $C(sI-A)^{-1}$, then there exists some $x \in D(A) \setminus \{0\}$ with $Ax = \lambda_k x$ and $Cx = 0$. In other words, λ_k is an unobservable mode. The *Hautus test* for infinite-dimensional systems [19, Corollary 9.6.2] then gives rise to the property that (A,B,C) is not approximately observable.

(ii) For all $u \in L^2_{\mathrm{loc}}(\mathbb{R}_{\ge 0})$ with $\mathrm{e}^{-\alpha\cdot}u(\cdot) \in L^2(\mathbb{R}_{\ge 0})$ for some $\alpha \in \mathbb{R}_{>0}$, the output of (1.2) with $x_0 = 0$ fulfills $\mathrm{e}^{-\alpha\cdot}y(\cdot) \in L^2(\mathbb{R}_{\ge 0})$, and the Laplace transforms $\widehat{u}$ and $\widehat{y}$ of u and y are related by

$$\widehat{y}(s) = G(s)\widehat{u}(s) = C(sI-A)^{-1}B\widehat{u}(s) \quad \text{for all } s \in \mathbb{C} \text{ with } \mathrm{Re}(s) > \alpha. \tag{3.8}$$

The remaining part of this section is devoted to a detailed characterization of the locations of the transmission zeros of (A,B,C) with A, B and C as in (2.1).

Theorem 3.8 *Let G be the transfer function of (A,B,C) in (2.1) and (c_k), (λ_k), J_c as in Theorem 3.6, i.e. $\sigma(A) = \{-\lambda_k : k \in \mathbb{N}_0\}$. Then the following holds:*

(i) $\lambda \in \mathbb{C}$ is a non-removable singularity of G if and only if there exists an index $k \in J_c$ with $\lambda = -\lambda_k$. In this case $\lambda = -\lambda_k$ is a pole of first order.

(ii) Let λ_k be a pole of G and let $\lambda_{\tilde{k}}$ be its consecutive pole determined by $\widetilde{k} := \min\{j \in J_c \mid j > k \wedge \lambda_j \neq \lambda_k\}$. Then there is exactly one transmission zero μ of (A,B,C) in $(-\lambda_{\tilde{k}}, -\lambda_k)$. This zero is simple, i.e. $G'(\mu) \neq 0$.

(iii) Every transmission zero is of the form described in (ii). That is, $G(\mu) = 0$ implies that $\mu \in (-\lambda_{\tilde{k}}, -\lambda_k)$ where the two consecutive poles $\lambda_k, \lambda_{\tilde{k}}$ are determined by $k := \max\{j \in J_c : \mu \le -\lambda_j\}$ and $\widetilde{k}$ as in (ii).

Remark 3.9 In other words, part (i) states that the set of poles of G has the characterization

$$\mathbb{C} \setminus D(G) = \{\lambda \in \sigma(A) \mid \exists k \in \mathbb{N}_0 : \lambda = -\lambda_k \ \wedge \ c_k \neq 0\}.$$

The property of the singularities of $C(sI-A)^{-1}B$ being either removable or a pole of first order implies that the transfer function is meromorphic in $\mathbb{C}$.

Proof (i) Equation (3.6) shows that $\rho(A) \subset D(G)$. Let λ be a non-removable singularity of G. Then it lies in $\sigma(A)$, which means that it equals $-\lambda_k$ for some $k \in \mathbb{N}_0$. In the decomposition

$$G(s) = \sum_{j=0}^{\infty} \frac{c_j}{s+\lambda_j} = \sum_{\{j \in \mathbb{N}_0 \mid \lambda_j = \lambda_k\}} \frac{c_j}{s+\lambda_k} + \sum_{\{j \in \mathbb{N}_0 \mid \lambda_j \neq \lambda_k\}} \frac{c_j}{s+\lambda_j} \tag{3.9}$$

the first series is a finite sum and the second series is continuous at the point λ_k, since it converges absolutely by (3.7). If all c_j in the first series were zero, then G could be extended analytically to λ_k, a contradiction. So there must be at least one j with $\lambda_j = \lambda_k = \lambda$ and $c_j \neq 0$, which means $j \in J_c$. Multiplying (3.9) by $(\lambda + \lambda_k)$ removes the singularity, hence the pole λ_k is of first order.

On the other hand, let λ_k be such that $c_k \neq 0$. Then (3.9) shows that the limit

$$\lim_{h \downarrow 0} G(-\lambda_k + h) = \lim_{h \downarrow 0} \sum_{\{j \in \mathbb{N}_0 \mid \lambda_j = \lambda_k\}} \frac{c_j}{h} + \sum_{\{j \in \mathbb{N}_0 \mid \lambda_j \neq \lambda_k\}} \frac{c_j}{\lambda_j - \lambda_k}, \qquad h > 0$$

does not exist because the second series is a finite number and all summands in the first sum are strictly positive, with at least one of some diverging to ∞. So λ_k is a non-removable singularity.

(ii) Let λ_k, and $\lambda_{\tilde{k}}$ be as in assertion (ii). Then according to (i), both are poles and $c_k, c_{\tilde{k}} \neq 0$. Thus, we see

$$\lim_{h \downarrow 0} G(-\lambda_{\tilde{k}} + h) = \lim_{h \downarrow 0} \sum_{\{j \in J_c \mid \lambda_j = \lambda_{\tilde{k}}\}} \frac{c_j}{h} + \sum_{\{j \in J_c \mid \lambda_j \neq \lambda_{\tilde{k}}\}} \frac{c_j}{\lambda_j - \lambda_{\tilde{k}}} = \infty$$

and

$$\lim_{h \downarrow 0} G(-\lambda_k - h) = \lim_{h \downarrow 0} \sum_{\{j \in J_c \mid \lambda_j = \lambda_k\}} \frac{c_j}{-h} + \sum_{\{j \in J_c \mid \lambda_j \neq \lambda_k\}} \frac{c_j}{\lambda_j - \lambda_k} = -\infty.$$

Since G maps real values to real values, the intermediate value theorem implies that there exists at least one $\mu \in (-\lambda_{\tilde{k}}, -\lambda_k)$ with $G(\mu) = 0$. Note that for all j with $\lambda_j \geq 1 + 2\lambda_{\tilde{k}}$, and all $s \in (-\lambda_{\tilde{k}}, -\lambda_k)$ holds

$$\frac{c_j}{(s+\lambda_j)^2} \leq \frac{c_j}{(\lambda_j - \lambda_{\tilde{k}})^2} \leq \frac{c_j}{\lambda_j}.$$

Together with (3.7) this implies that the derivative of G along the real axis is given by the absolutely convergent series

$$\frac{\mathrm{d}}{\mathrm{d}s} G(s) = -\sum_{k=0}^{\infty} \frac{c_k}{(s+\lambda_k)^2}.$$

Since this expression is greater than zero, we deduce that G strictly decreasing on the interval $(-\lambda_{\tilde{k}}, -\lambda_k)$. Thus, μ is the only zero of G in this interval, and it is simple.

(iii) Let $\mu \in \mathbb{C}$ be a transmission zero of (1.2). Then $\mu \in D(G)$, and

$$0 = \operatorname{Im} G(\mu) = \sum_{k=0}^{\infty} c_k \operatorname{Im}\left(\frac{1}{\mu + \lambda_k}\right) = \sum_{k=0}^{\infty} c_k \frac{-\operatorname{Im}\mu}{|\mu + \lambda_k|^2}$$

implies that $\operatorname{Im}\mu = 0$, whence $\mu \in \mathbb{R}$. For $\mu > 0$, the positivity of $c_0 > 0$ leads to the contradiction

$$0 < \sum_{k=0}^{\infty} c_k \frac{1}{\mu + \lambda_k} = \sum_{k=0}^{\infty} c_k \frac{\mu + \lambda_k}{|\mu + \lambda_k|^2} = G(\mu) = 0.$$

From (i) and $c_0 > 0$ we know that $0 \notin D(G)$, whence μ must be in $\mathbb{R}_{<0}$. Since $\lambda_0 = 0$, the integer $k := \max\{j \in \mathbb{N}_0 : \mu \le -\lambda_j \wedge c_j \neq 0\}$ exists and the claim (iii) follows from (ii). $\square$

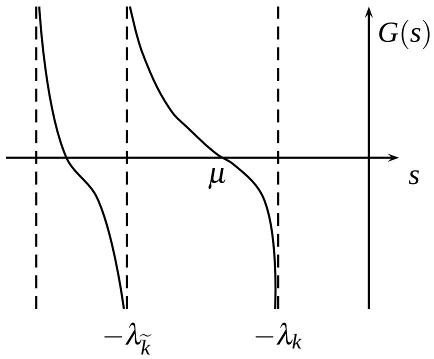

Fig. 3.1 The transfer function on the negative real axis

4 Invariant zeros

The following definition of invariant zeros is a direct generalization of the finite-dimensional case in [20, 25]. This concept has been introduced by *Rosenbrock* [17], where it is called *input-output-decoupling zeros*. For infinite-dimensional systems with bounded input and output operators, invariant zeros have been considered in [16]. Invariant zeros for boundary control systems are treated in [8].

Definition 4.1 (Invariant zero) Let X and (A,B,C) be as in Definition 3.1. Then $\lambda \in \mathbb{C}$ is called an *invariant zero of* (A,B,C) if there exist $u \in \mathbb{C}$ and $x \in V$ such that

$$x \neq 0 \ \wedge \ (\lambda I - A)x = Bu \ \wedge \ Cx = 0. \tag{4.1}$$

The set of all invariant zeros of (A,B,C) is denoted by $\mathrm{inv}(A,B,C)$ and the *multiplicity* of an invariant zero λ is

$$\mathrm{mult}(\lambda) := \dim\{x \in V \mid (\lambda I - A)x \in \mathrm{im}\, B \wedge Cx = 0\}.$$

It is well-known for finite-dimensional systems that transmission zeros are contained in the set of invariant zeros. The difference between these two sets consists of eigenvalues of A. The following result shows that this holds true for systems that are subject in Definition 3.1. We will see that argumentation for the finite-dimensional SISO case can be used in a straightforward manner.

Proposition 4.2 *Let A, B and C be operators satisfying the prerequisites of Definition 3.1 and let $\lambda \in \mathbb{C}$ be an invariant zero of (A,B,C). Then at least one of the following assertions hold true:*

(i) *λ is a spectral value of A;*
(ii) $G(\lambda) = 0$.

Proof Assume that $\lambda \in \rho(A)$ and $(x,u) \in V \times \mathbb{C}$ fulfill (4.1). Then $\lambda \in \rho(A)$ implies $u \neq 0$. Multiplication of $(\lambda I - A)x + Bu = 0$ from the left with $(\lambda I - A)^{-1}$ further implies $x = -(\lambda I - A)Bu$, and thus

$$G(\lambda)u = C(\lambda I - A)^{-1}Bu = -Cx = 0.$$

□

In other words, Proposition 4.2 states that for $\lambda \in \mathbb{C}$ with $\mathrm{mult}(\lambda) > 0$, there holds $\lambda \in \sigma(A) \cup \mathrm{inv}(A,B,C)$. The following stronger result gives a detailed expression for the multiplicity of an invariant zero.

Theorem 4.3 *Let A, B, C be as in* (2.1)*. Then*

$$\mathrm{inv}(A,B,C) = \{\mu \in \mathbb{C} \mid G(\mu) = 0\} \cup \{-\lambda_k \mid k \in \mathbb{N}_0 \setminus J_c\}, \tag{4.2}$$

Moreover, the multiplicity of each invariant zero $\lambda \in \mathrm{inv}(A,B,C)$ satisfies

(i) $\mathrm{mult}(\lambda) = \dim(\ker(\lambda I - A) \cap \ker C)$, *if $G(\lambda) \neq 0$,*
(ii) $\mathrm{mult}(\lambda) = \dim(\ker(\lambda I - A) \cap \ker C) + 1$, *if $G(\lambda) = 0$.*

Proof
(i) Let $\lambda \in \mathrm{inv}(A,B,C)$ and assume $G(\lambda) \neq 0$. By the definition of multiplicity, it suffices to prove that

$$\{x \in V \mid (\lambda I - A)x \in \mathrm{im}\, B \;\wedge\; Cx = 0\} = \ker(\lambda I - A) \cap \ker C.$$

The subset relation "⊃" is trivial. To prove the reverse inclusion, we have to show that for all $x \in V$ and $u \in \mathbb{C}$ with (4.1) holds $u = 0$. By Proposition 4.2 λ must be an eigenvalue of A. If there is a $k \in J_c$ with $\lambda = -\lambda_k$, then the corresponding eigenvector $v_k \in D(A)$ satisfies

$$\langle u, Cv_k\rangle = \langle Bu, v_k\rangle = \langle x, (\lambda I - A)^* v_k\rangle = \langle x, (-\lambda_k I - A)v_k\rangle = 0.$$

Since $Cv_k \neq 0$ for $k \in J_c$, we obtain $u = 0$.
Now we consider the case where $\lambda \neq -\lambda_k$ for all $k \in J_c$. Note that we have

$$0 = \langle x, (\lambda I - A)^* \varphi \rangle - \langle Bu, \varphi \rangle \qquad \forall \varphi \in D(A).$$

Choose an arbitrary $k \in J_c$. Then the eigenvector $\frac{1}{\overline{\lambda}+\lambda_k} v_k$ is in $D(A)$, and

$$0 = \left\langle x, (\lambda I - A)^* \frac{1}{\overline{\lambda} + \lambda_k} v_k \right\rangle - \left\langle Bu, \frac{1}{\overline{\lambda} + \lambda_k} v_k \right\rangle = \langle x, v_k \rangle - \left\langle u, \frac{Cv_k}{\overline{\lambda} + \lambda_k} \right\rangle.$$

Hence, for all $k \in J_c$ holds $\langle x, v_k \rangle = \frac{\overline{Cv_k}}{\lambda+\lambda_k}$. With this we obtain

$$\begin{aligned} 0 = Cx &= C \sum_{k \in J_c} \langle x, v_k \rangle v_k + C \underbrace{\sum_{k \in \mathbb{N} \setminus J_c} \langle x, v_k \rangle v_k}_{\in \ker C} \\ &= C \sum_{k \in J_c} \langle x, v_k \rangle C v_k = \sum_{k \in J_c} \frac{|Cv_k|^2}{\lambda + \lambda_k} u = G(\lambda) u. \end{aligned}$$

Then $G(\lambda) \neq 0$ implies $u = 0$.

(ii) Assume that $\lambda \in \mathbb{C}$ with $G(\lambda) = 0$.
Step 1: We show that $\operatorname{mult}(\lambda) \geq \dim \ker(\lambda I - A) + 1$.
Observe that the assumption $G(\lambda) = 0$ includes that λ is not a pole of G. Define the set $J_\lambda = \{ \, k \in \mathbb{N}_0 \mid v_k \in \ker(\lambda I - A) \, \}$. Theorem 3.8 (i) implies

$$\ker(\lambda I - A) = \operatorname{span}\{ v_k \mid k \in J_\lambda \} = \ker(\lambda I - A) \cap \ker(C).$$

This gives rise to $\ker(\lambda I - A) \subset V \cap \ker C$. The inequality $\operatorname{mult}(\lambda) \geq \ker(\lambda I - A) + 1$ therefore holds true, if we find some vector $x \in \ker(\lambda I - A)^\perp \cap V \setminus \{0\}$ with $x \in \ker C$. Define x via the series

$$x_p := \sum_{k \in \mathbb{N} \setminus J_\lambda} \frac{\int_{\partial \Omega} \overline{v_k(\xi)} \, \mathrm{d}\sigma_\xi}{\lambda + \lambda_k} \cdot v_k \quad \in \ker(\lambda I - A)^\perp, \tag{4.3}$$

which converges in $H^1(\Omega)$ according to Lemma 3.4. Further, $x \neq 0$ since the density of $\operatorname{span}\{ \, v_k \mid k \in \mathbb{N} \, \}$ in $D(A)$ and the fact that boundary integration is not the zero operator on $D(A)$ implies that there exists some $k_0 \in \mathbb{N} \setminus J_\lambda$ with $\int_{\partial\Omega} v_{k_0}(\xi) \, \mathrm{d}\sigma_\xi \neq 0$. Further, the spectral decomposition of A in Lemma 2.4 and the fact that $v_k \in \ker C$ for $k \in J_\lambda$ yields for all $\varphi \in D(A)$ holds

$$\begin{aligned} \langle (\lambda I - A) x_p, \varphi \rangle &= \langle x_p, (\lambda I - A) \varphi \rangle \\ &= \left\langle \sum_{k \in \mathbb{N} \setminus J_\lambda} \frac{\int_{\partial\Omega} \overline{v_k(\xi)} \, \mathrm{d}\sigma_\xi}{\lambda + \lambda_k} v_k \, , \, \sum_{l \in \mathbb{N}} (\lambda + \lambda_l) \langle \varphi, v_l \rangle v_l \right\rangle \\ &= \sum_{k \in \mathbb{N} \setminus J_\lambda} \int_{\partial\Omega} \overline{v_k(\xi)} \, \mathrm{d}\sigma_\xi \, \langle \varphi, v_k \rangle \\ &= \overline{\int_{\partial\Omega} \sum_{k \in \mathbb{N}} \langle v_k, \varphi \rangle v_k(\xi) \, \mathrm{d}\sigma_\xi} \\ &= \langle B \cdot 1, \varphi \rangle \, . \end{aligned}$$

This shows that $x_p \in V$. We further obtain

$$Cx_p = \sum_{k\in\mathbb{N}\setminus J_\lambda} \frac{\overline{Cv_k}}{\lambda+\lambda_k} Cv_k = \sum_{k\in J_c} \frac{c_k}{\lambda+\lambda_k} = G(\lambda) = 0.$$

Altogether, we have $x_p \in V \cap \ker C \cap \ker(\lambda I - A)^\perp \setminus \{0\}$. This leads to $\operatorname{mult}(\lambda) \geq \dim\ker(\lambda I - A) + 1$.

Step 2: Now we prove the reverse inequality $\operatorname{mult}(\lambda) \leq \dim\ker(\lambda I - A) + 1$: Assume that $(x,u) \in V \times \mathbb{C}$, such that $Cx = 0$ and $(\lambda I - A)x = Bu$. By Lemma 2.4 and Theorem 3.8, we have $\lambda_k, \lambda \in \mathbb{R}$. Then, by using (2.10) and the self-adjointness of A, we obtain for all $l \in \mathbb{N} \setminus J_\lambda$ that

$$\begin{aligned}(\lambda+\lambda_l)\cdot\langle x, v_l\rangle &= \langle x, (\lambda I - A)v_l\rangle = \langle x, (\lambda I - A)^* v_l\rangle = \langle(\lambda I - A)x, v_l\rangle \\ =&\langle Bu, v_l\rangle = \langle u, Cv_l\rangle = u\cdot\int_{\partial\Omega} \overline{v_k(\xi)}\,\mathrm{d}\sigma_\xi = u\cdot\overline{Cv_k}\end{aligned}$$

holds. This yields

$$\langle x, v_k\rangle = \frac{\overline{Cv_k}}{\lambda+\lambda_k} u \qquad \forall k \in \mathbb{N}\setminus J_\lambda.$$

Hence, for x_p as in (4.3), we have

$$x \in \ker(\lambda I - A) + \operatorname{span}\{x_p\},$$

and therefore $\operatorname{mult}(\lambda) \leq \dim\ker(\lambda I - A) + 1$. □

5 Zero dynamics

Here we study zero dynamics which consist of the trajectories of (1.2) resulting in a trivial output $y(\cdot) \equiv 0$. Zero dynamics play a central role in (adaptive) output regulation [15, 11].

Definition 5.1 (Zero dynamics) Let A, B and C be operators with properties as in Definition 3.1. Let $T(\cdot)$ be the semigroup generated by A. The *zero dynamics of* (A,B,C) are the pairs $(x,u) \in \mathscr{C}(\mathbb{R}_{\geq 0}; X) \times L^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$ with

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)Bu(\tau)\mathrm{d}\tau, \qquad t \in \mathbb{R}_{\geq 0}, \quad \text{and} \tag{5.1}$$

$$Cx(t) = 0 \qquad \text{for almost all } t \in \mathbb{R}_{\geq 0}. \tag{5.2}$$

In this part we prove that the zero dynamics of the heat equation system (1.1) are completely described by an exponentially stable, contractive and analytical semigroup on $L^2(\Omega)$. First we consider an operator which turns out to be the generator of this semigroup. In particular, we show that this operator admits an eigenvalue decomposition; the spectrum of this operator is the set of invariant zeros of (A,B,C).

Theorem 5.2 *Consider the operator $A_0 : D(A_0) \subset L^2(\Omega) \to L^2(\Omega)$ with*

$$A_0 x = \Delta x \qquad \forall x \in D(A_0) = \left\{ x \in H^2(\Omega) \;\middle|\; \begin{array}{l} \partial_\nu x|_{\partial\Omega} \equiv \dfrac{\int_\Omega \Delta x(\xi) \mathrm{d}\xi}{|\partial\Omega|} \\ \text{and} \displaystyle\int_{\partial\Omega} x(\xi)\, \mathrm{d}\sigma_\xi = 0. \end{array} \right\}. \tag{5.3}$$

Then the following holds true:

(i) *A_0 is self-adjoint and has compact resolvent;*
(ii) *A_0 generates an analytical, contractive, and exponentially stable semigroup on $L^2(\Omega)$;*
(iii) *For the operators A, B and C as in* (2.1) *holds*

$$\sigma(A_0) = \{\, \lambda \in \mathbb{C} \mid \lambda \text{ is an invariant zero of } (A,B,C) \,\}.$$

Proof *Step 1:* We construct an associated sesquilinear form for A_0 (cf. Theorem 1.1): Define the space

$$H = \left\{ x \in H^1(\Omega) \;\middle|\; \int_{\partial\Omega} x(\xi)\, \mathrm{d}\sigma_\xi = 0 \right\}. \tag{5.4}$$

Then H is dense in $L^2(\Omega)$. We obtain from the trace theorem [1, Theorem 5.5.36] that H is a closed subspace of $H^1(\Omega)$. In other words, H is a Hilbert space inheriting the inner product of $H^1(\Omega)$. We define the sesquilinear form

$$a_0 : H \times H \to \mathbb{C}, \qquad (x,z) \mapsto \int_\Omega \nabla x(\xi) \cdot \overline{\nabla z(\xi)} \mathrm{d}\xi, \tag{5.5}$$

which is continuous and symmetric. We prove that there is an $\alpha > 0$ with

$$\operatorname{Re} a_0(x,x) \geq \alpha \langle x,x \rangle_H \qquad \forall x \in H. \tag{5.6}$$

Assume that this is false. Then there exists a bounded sequence (x_n) in H with

$$\|x_n\|_{H^1(\Omega)} = 1 \qquad \forall n \in \mathbb{N}, \tag{5.7}$$

and

$$\lim_{n\to\infty} a_0(x_n, x_n) = 0. \tag{5.8}$$

The Rellich-Kondrachov Theorem [10, Theorem 4.17 (i)] implies that there exists some $z \in L^2(\Omega)$ and a subsequence (x_{n_k}) with

$$\lim_{k\to\infty} \|z - x_{n_k}\|_{L^2(\Omega)} = 0.$$

Together with (5.8) and (5.5) this implies that (x_{n_k}) is a Cauchy sequence in $H^1(\Omega)$, whence $z \in H^1(\Omega)$ and

$$\lim_{k\to\infty} \|z - x_{n_k}\|_{H^1(\Omega)} = 0.$$

Since differentiation as well as boundary evaluation are continuous with respect to the $H^1(\Omega)$ norm, it follows that

$$\nabla z = 0 \qquad \text{and} \qquad \int_{\partial\Omega} z(\xi) d\xi = 0.$$

Hence, z is a constant function whose boundary integral vanishes. This implies $z=0$, wich is a contradiction to (5.7).

Step 2: With the definition of H and a_0 as in Step 1 and A_0 as in (5.3), we show that

$$D(A_0)=\left\{\, x\in H \;\middle|\; \exists z\in L^2(\Omega): a_0(x,\varphi)=\langle z,\varphi\rangle_{L^2(\Omega)} \quad \forall \varphi\in H \,\right\}, \tag{5.9}$$

and

$$\langle A_0x,\varphi\rangle=-a_0(x,\varphi) \quad \forall\varphi\in D(A_0). \tag{5.10}$$

For $x\in D(A_0)$ the equation (5.10) follows by Green's formula, since for all $\varphi\in H$ holds

$$\begin{aligned} a_0(x,\varphi) &= -\int_\Omega \nabla x(\xi)\cdot\overline{\nabla\varphi(\xi)}\mathrm{d}\xi \\ &= -\int_\Omega \Delta x(\xi)\cdot\overline{\varphi(\xi)}\mathrm{d}\xi+\int_{\partial\Omega}\partial_\nu x(\xi)\cdot\overline{\varphi(\xi)}\mathrm{d}\sigma_\xi \\ &= -\int_\Omega \Delta x(\xi)\cdot\overline{\varphi(\xi)}\mathrm{d}\xi+u\cdot\underbrace{\overline{\int_{\partial\Omega}\varphi(\xi)\mathrm{d}\sigma_\xi}}_{=0} \\ &= -\langle\Delta x,\varphi\rangle_{L^2(\Omega)}. \end{aligned}$$

This computation also gives rise to the inclusion "$\subset$" in (5.9). To prove the converse inclusion, assume that $x\in H$ and there exists some $z\in L^2(\Omega)$ with

$$\int_\Omega \nabla x(\xi)\overline{\nabla\varphi(\xi)}\mathrm{d}\xi=a_0(x,\varphi)=\langle z,\varphi\rangle_{L^2(\Omega)} \qquad \forall\varphi\in H. \tag{5.11}$$

Then (5.11) holds true for all $\varphi\in\mathscr{C}_c^\infty(\Omega)$ in particular. Consequently, $z=-\Delta x$. We choose an $H^2(\Omega)$-function h with

$$\partial_\nu h|_{\partial\Omega}\equiv\frac{\int_\Omega\Delta x(\xi)\mathrm{d}\xi}{|\partial\Omega|}$$

and claim that $x-h$ fulfills

$$\int_\Omega\nabla(x-h)(\xi)\overline{\nabla\psi(\xi)}\mathrm{d}\xi=-\int_\Omega\Delta(x-h)(\xi)\overline{\psi(\xi)}\mathrm{d}\xi \qquad \forall\psi\in H^1(\Omega).$$

Let $\psi \in H^1(\Omega)$. Then $\varphi := \psi - \frac{\int_{\partial\Omega}\psi(\xi)\mathrm{d}\xi}{|\partial\Omega|}$ is in H and $\nabla\psi = \nabla\varphi$. Thus we have

$$\begin{aligned}
&\int_\Omega \nabla(x-h)(\xi)\overline{\nabla\psi(\xi)}\mathrm{d}\xi\\
&= \int_\Omega \nabla x(\xi)\overline{\nabla\varphi(\xi)}\mathrm{d}\xi - \int_\Omega \nabla h(\xi)\overline{\nabla\psi(\xi)}\mathrm{d}\xi\\
&= \int_\Omega z(\xi)\overline{\varphi(\xi)}\mathrm{d}\xi - \int_\Omega \nabla h(\xi)\overline{\nabla\psi(\xi)}\mathrm{d}\xi\\
&= \int_\Omega z(\xi)\overline{\varphi(\xi)}\mathrm{d}\xi + \int_\Omega \Delta h(\xi)\overline{\psi(\xi)}\mathrm{d}\xi - \int_{\partial\Omega} \partial_\nu h(\xi)\overline{\psi(\xi)}\mathrm{d}\xi\\
&= -\int_\Omega \Delta x(\xi)\overline{\varphi(\xi)}\mathrm{d}\xi + \int_\Omega \Delta h(\xi)\overline{\psi(\xi)}\mathrm{d}\xi - \int_{\partial\Omega} \frac{\int_\Omega \Delta x(\zeta)\mathrm{d}\zeta}{|\partial\Omega|}\overline{\psi(\xi)}\mathrm{d}\xi\\
&= -\int_\Omega \Delta x(\xi)\overline{\left(\varphi(\xi) + \frac{\int_{\partial\Omega}\psi(\xi)}{|\partial\Omega|}\right)}\mathrm{d}\xi + \int_\Omega \Delta h(\xi)\overline{\psi(\xi)}\mathrm{d}\xi\\
&= \int_\Omega \Delta(h(\xi) - x(\xi))\overline{\psi(\xi)}\mathrm{d}\xi.
\end{aligned}$$

Now Lemma 2.1 implies that $x-h \in H^2(\Omega)$ and $\partial_\nu(x-h)|_{\partial\Omega} = 0$. Hence, $x \in H^2(\Omega)$ and $\partial_\nu x|_{\partial\Omega} \equiv \frac{\int_\Omega \Delta x \mathrm{d}\xi}{|\partial\Omega|}$.

Step 3: We conclude statement (i) and (ii): The relations (5.6) and (5.10), together with the symmetry of $a_0(\cdot,\cdot)$, imply that A_0 is self-adjoint and negative definite. In particular, $0 \in \rho(A_0)$, and

$$A_0^{-1}L^2(\Omega) \subset D(A_0).$$

Since $H^2(\Omega)$ is compactly embedded in $L^2(\Omega)$ by the Rellich-Kondrachov Theorem, we infer that A_0 has compact resolvent. Therefore, its spectrum consists of isolated eigenvalues [13, Theorem 6.29, p.187], which must be strictly negative because of (5.6). This shows that A_0 is a sectorial operator, and by [19, Theorem 3.10.5], A_0 generates an analytical semigroup $T_0(\cdot)$. The property that its largest eigenvalue $-\omega_0$ is negative further implies that

$$\|T_0(t)\|_{\mathscr{B}(L^2(\Omega))} \leq e^{-\omega_0 t} \qquad \forall t \in \mathbb{R}_{\geq 0},$$

see [24, Proposition 2.6.5]. Hence, the semigroup is contractive and exponentially stable.

Step 4: We prove (iii): First assume that $\lambda \in \mathbb{C}$ is an invariant zero of (A,B,C). Then there exists some nontrivial pair $(x,u) \in V \times \mathbb{C}$ with $\lambda x - Ax + Bu = 0$ and $Cx = 0$. From Lemma 3.3 we obtain that $x \in H^2(\Omega)$ with

$$u = \partial_\nu x(\xi) \quad \forall \xi \in \partial\Omega, \qquad \text{and} \qquad \Delta x = Ax + Bu.$$

In particular, there holds

$$(\lambda I - \Delta)x = (\lambda I - A)x - Bu = 0.$$

The equation $Cx = 0$ furthermore gives rise to the fact that the boundary integral of x vanishes. Altogether, we obtain that $x \in D(A_0)$ and $A_0 x = \lambda x$, whence $\lambda \in \sigma(A_0)$.

Conversely, assume $\lambda \in \sigma(A_0)$, i.e. there exists some $x \in D(A_0) \setminus \{0\}$ with $A_0 x = \lambda x$. Lemma 3.3 shows that $x \in V$ and $\Delta x = A_0 x = Ax + Bu$. Thus,

$$(\lambda I - A)x - Bu = (\lambda I - A_0)x = 0,$$

and λ is an invariant zero of (A,B,C). □

The following result shows that the semigroup generated by A_0 indeed gives a full characterization of the zero dynamics of the heat equation system with boundary control.

Theorem 5.3 *Let A, B, C be given by* (2.1) *and A_0 as in Theorem 5.2. Let $T_0(\cdot)$ be the semigroup generated by A_0. Then the following holds true:*

(i) If $(x,u) \in \mathscr{C}(\mathbb{R}_{\geq 0};X) \times L^1_{\text{loc}}(\mathbb{R}_{\geq 0})$ is in the zero dynamics of (A,B,C), then $T_0(t)x(0) = x(t)\ \forall t \in \mathbb{R}_{\geq 0}$.
(ii) Let $x_0 \in L^2(\Omega)$ and define the function $x(\cdot) := T_0(\cdot)x_0$. Then there exists some $u \in L^1_{\text{loc}}(\mathbb{R}_{\geq 0};\mathbb{C})$ such that (x,u) is in the zero dynamics of the system (A,B,C). The function $u : \mathbb{R}_{>0} \to \mathbb{C}$ is analytical.

Proof (i) Assume that $(x,u) \in \mathscr{C}(\mathbb{R}_{\geq 0};X) \times L^1_{\text{loc}}(\mathbb{R}_{\geq 0})$ is in the zero dynamics of (A,B,C). Since B maps into the space $X_{-\frac{1}{2}}$ and T is an analytic semigroup, the solution x of (5.1) satisfies $x(t) \in X_{\frac{1}{2}}$ for all $t > 0$ [14, Proposition 2.2.2]. Therefore, for all $\varphi \in D(A)$ holds [24, Remark 4.1.2]

$$\langle x(t) - x(0), \varphi\rangle = \int_0^t \langle x(t), A^*\varphi\rangle_X + \langle u, B^*\varphi\rangle \,\mathrm{d}\tau = \int_0^t -a(x(t),\varphi) + \langle u, B^*\varphi\rangle \,\mathrm{d}\tau.$$

Since the right hand side depends continuously on φ with respect to the $X_{\frac{1}{2}}$ norm, this equation extends to all $\varphi \in X_{\frac{1}{2}}$. The assumption $Cx(t) = 0$ implies that for $t > 0$, $x(t)$ is even in the domain of a_0 defined in (5.4). Hence, for $\varphi \in D(A_0) \subset X_{\frac{1}{2}}$ the equation above becomes

$$\langle x(t) - x(0), \varphi\rangle = -\int_0^t a(x(t),\varphi)\mathrm{d}\tau = -\int_0^t a_0(x(t),\varphi)\mathrm{d}\tau = \int_0^t \langle x(t), A_0^*\varphi\rangle \mathrm{d}\tau.$$

This implies $x(t) = T_0(t)x(0)$.

(ii) Let $x_0 \in L^2(\Omega)$ and define $x(t) := T_0(t)x_0$. Since the semigroup T_0 is analytic we have $x(t) \in D(A_0)$ for all $t \in \mathbb{R}_{>0}$,

$$\|T_0(t)x_0\|_X \leq e^{-\omega t}, \qquad \text{and} \qquad \|T_0(t)x_0\|_{D(A_0)} \leq \frac{1}{t}e^{-\omega t}.$$

For the interpolation space $(X, D(A))_{\theta,\infty}$ [14, Proposition 2.2.2] there exists some $c > 0$ with

$$\|T_0(t)x_0\|_{(X,D(A))_{\theta,\infty}} \leq c\left(\frac{1}{t}e^{-\omega t}\right)^{\theta}\left(e^{-\omega t}\right)^{1-\theta} = \frac{c}{t^{\theta}}e^{-\omega t}.$$

For $\theta \in \left(\frac{3}{4},1\right)$ it follows that the expression on the left is integrable over finite intervals. Due to the facts that $(X,D(A))_{\theta,\infty} \hookrightarrow H^{2\theta}(\Omega)$ [21, Theorem 4.3.3, see also Theorem 4.3.1.1 and Remark 2.4.2.2] and the mapping

$$\mathrm{tr}_\nu : H^{2\theta}(\Omega) \to H^{2\theta-\frac{3}{2}}(\partial\Omega), \quad x \mapsto (\xi \mapsto \partial_\nu x(\xi)), \qquad 2\theta \geq 3/2,$$

is bounded [10, Theorem 4.24 (ii)], we get that $u(t) := \mathrm{tr}_\nu\, x(t)$ defines a $L^1_{\mathrm{loc}}(\mathbb{R}_{\geq}0)$-function.

Furthermore, $x(t) \in D(A_0)$ implies that $x(t)$ is an element of the space V defined in Lemma 3.3(ii), and by Lemma 3.3 (i),

$$A_0 x(t) = \Delta x(t) = Ax(t) + Bu(t).$$

Thus, we have

$$x(t) - x_0 = \int_0^t A_0 x(\tau)\mathrm{d}\tau = \int_0^t Ax(\tau) + Bu(\tau)\mathrm{d}\tau,$$

and $Cx(t) = 0$ for all $t > 0$. So (x,u) is in the zero dynamics of (A,B,C). □

Remark 5.4 (Zero dynamics for the heat equation) Theorem 5.3 gives rise to an interesting effect for zero dynamics of the heat equation system (1.1): For each x_0 in the state space, there exists some unique trajectory $(x,u) \in \mathscr{C}(\mathbb{R}_{\geq 0};X) \times L^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$ in the zero dynamics with $x(0) = x_0$. For finite-dimensional SISO systems, this is not true in general: The zero dynamics evolve in some proper subspace of the state-space. The dimension of the zero dynamics is determined by the relative degree of the transfer function. The zero dynamics of finite-dimensional systems can be fully characterized by the *Byrnes-Isidori form* [5].

6 Proportional output feedback and root locus

We consider the heat equation system with output feedback $u(t) = v(t) - ky(t)$. Here, the *proportional output gain k* is given, and $v \in L^2(\mathbb{R}_{\geq 0})$ corresponds to the input of the closed-loop system. We show that for the system with operators as in (2.1), proportional output feedback results in a well-posed linear system. The evolution operator A_k of the closed-loop system will turn out to be self-adjoint and to have a compact resolvent. We moreover prove the *root-locus property*: As k tends to ∞, the eigenvalues of A_k move towards the invariant zeros of (A,B,C) in a certain sense.
Well-posedness of infinite-dimensional output feedback systems is well understood by the results of WEISS in [22]. In this context the notion of *regularity* plays an important role. We recall the following definition from [23].

Definition 6.1 An analytical function $G : D(G) \subset \mathbb{C} \to \mathbb{C}$ is called *regular*, if there exists some $D \in \mathbb{C}$ such that transfer function fulfills

$$\lim_{s>0, s\to\infty} G(s) = D.$$

Proposition 6.2 [6] *The transfer function of the operators* (A,B,C) *given by* (2.1) *is regular with* $D=0$.

We are now able to formulate and prove the result that the heat equation (1.1) with output feedback $u(t)=v(t)-ky(t)$ defines a well-posed linear system.

Theorem 6.3 *Let* (A,B,C) *be defined by* (2.1), *and define* $T,\mathfrak{B}_t,\mathfrak{C},\mathfrak{D}$ *as in* (2.13), (2.14) *respectively. Let* $u\in L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$, $x_0\in L^2(\Omega)$, $k\in\mathbb{R}$ *and set* $v(t):=u(t)+ky(t)$, *with* y *defined in* (2.15b). *Then the state* x *defined in* (2.15a) *satisfies*

$$x(t)\ =T_k(t)x_0+\mathfrak{B}_{k,t}v, \tag{6.1}$$

where T_k *is a strongly continuous semigroup on* $L^2(\Omega)$ *generated by*

$$A_kx=(A-kBC)x,\quad D(A_k)=\{\ x\in D(C)\ \mid (A-kBC)x\in X\ \}, \tag{6.2}$$

and

$$\mathfrak{B}_{k,t}v=\int_0^t T_k(t-\tau)|_{D(A_k^*)'}Bv(\tau)\mathrm{d}\tau \qquad \text{in } D(A_k^*)'.$$

Here $T_k(t)|_{D(A_k^*)'}$ *is the extension of* $T_k(t)$ *to* $D(A_k^*)'$. *In particular, the range of* B *is contained in this space.*

Proof By Proposition 6.2, the transfer function of (A,B,C) is regular with $D=0$. In particular, $1-k\cdot D\neq 0\ \ \forall k\in\mathbb{R}$. The overall statement now follows from [22, Theorem 6.1 & Theorem 7.2]. □

Theorem 6.4 *Let* (A,B,C) *be defined by* (2.1) *and let* G *be the corresponding transfer function. The following holds true for the operator* A_k *in* (6.2) *with* $k>0$*:*

(i) $A_kx=\Delta x$, *and*

$$D(A_k)=\left\{x\in H^2(\Omega)\ \middle|\ \partial_\nu x(\xi)=-k\int_{\partial\Omega}x(\zeta)\mathrm{d}\sigma_\zeta\quad \forall\xi\in\partial\Omega\right\}. \tag{6.3}$$

(ii) The operator A_k *is self-adjoint,* $\sigma(A_k)\subset(-\infty,0)$, *and* A_k *has a compact resolvent. For all* $\{s\in\rho(A)\,|\,kG(s)\neq -1\}$ *we have* $s\in\rho(A_k)$ *and*

$$(sI-A_k)^{-1}=(sI-A)^{-1}-(sI-A)^{-1}B\left(\frac{1}{k}+G(s)\right)^{-1}C(sI-A)^{-1}. \tag{6.4}$$

(iii) A_k *generates an exponentially stable semigroup* $T_k(\cdot)$ *on* $L^2(\Omega)$.

Proof (i) We show that the set defined in (6.3) is a subset of the domain given in (6.2). Let x be in the former set. Then x is in $D(C)$ because the trace operator is well-defined on $H^2(\Omega)$. Moreover, we have the following equation for all $\varphi \in D(A^*) = D(A)$:

$$\begin{aligned}
\langle Ax, \varphi\rangle - \langle BkCx, \varphi\rangle &= \int_\Omega x(\xi)\cdot\overline{\Delta\varphi(\xi)}\mathrm{d}\xi - k\int_{\partial\Omega} x(\zeta)\mathrm{d}\sigma_\zeta \int_{\partial\Omega}\overline{\varphi(\xi)}\mathrm{d}\sigma_\xi \\
&= \int_\Omega \Delta x(\xi)\cdot\overline{\varphi(\xi)}\mathrm{d}\xi - \int_{\partial\Omega}(\partial_\nu x(\xi))\cdot\overline{\varphi(\xi)}\mathrm{d}\sigma_\xi \\
&\quad + \int_{\partial\Omega} x(\xi)\cdot(\partial_\nu\overline{\varphi(\xi)})\mathrm{d}\sigma_\xi - \int_{\partial\Omega} k\int_{\partial\Omega} x(\zeta)\mathrm{d}\sigma_\zeta\ \overline{\varphi(\xi)}\mathrm{d}\sigma_\xi \\
&= \int_\Omega \Delta x(\xi)\cdot\overline{\varphi(\xi)}\mathrm{d}\xi \\
&\quad - \int_{\partial\Omega}\left(\partial_\nu x(\xi)\mathrm{d}\xi + k\int_{\partial\Omega} x(\zeta)\mathrm{d}\sigma_\zeta\right)\ \overline{\varphi(\xi)}\mathrm{d}\sigma_\xi \\
&= \int_\Omega \Delta x(\xi)\cdot\overline{\varphi(\xi)}\mathrm{d}\xi.
\end{aligned}$$

This shows that $Ax - BkCx \in X$ because it can be represented by the function $\Delta x \in L^2(\Omega)$. For the converse inclusion, take any $x \in D(C)$ with $Ax - BkCx \in X$. Then x is by definition an element of the space V in Lemma 3.3. Therefore, part (i) and (ii) of Lemma 3.3 imply $x \in H^2(\Omega)$ and $\partial_\nu x \equiv -kCx = -k\int_{\partial\Omega} x(\xi)\,\mathrm{d}\xi$.

(ii) Let $s \in \rho(A)$ and $kG(s) \neq -1$. Then

$$\begin{aligned}
&(sI - A_k)\left((sI-A)^{-1} - (sI-A)^{-1}B\left(\frac{1}{k}+G(s)\right)^{-1}C(sI-A)^{-1}\right) \\
&= (sI - A + kBC)\left((sI-A)^{-1} - (sI-A)^{-1}B\left(\frac{1}{k}+G(s)\right)^{-1}C(sI-A)^{-1}\right) \\
&= I + kBC(sI-A)^{-1} - (I + kBC(sI-A)^{-1})B\left(\frac{1}{k}+G(s)\right)^{-1}C(sI-A)^{-1} \\
&= I + \left(kB - (I + kBC(sI-A)^{-1})B\left(\frac{1}{k}+G(s)\right)^{-1}\right)C(sI-A)^{-1} \\
&= I + \left(kB - (B + kBG(s))\left(\frac{1}{k}+G(s)\right)^{-1}\right)C(sI-A)^{-1} \\
&= I + \left(kB - kB\left(\frac{1}{k}+G(s)\right)\left(\frac{1}{k}+G(s)\right)^{-1}\right)C(sI-A)^{-1} = I
\end{aligned}$$

and

$$\begin{aligned}
&\left((sI-A)^{-1} - (sI-A)^{-1}B\left(\frac{1}{k}+G(s)\right)^{-1}C(sI-A)^{-1}\right)(sI - A_k) \\
&= \left((sI-A)^{-1} - (sI-A)^{-1}B\left(\frac{1}{k}+G(s)\right)^{-1}C(sI-A)^{-1}\right)(sI - A + kBC)
\end{aligned}$$

$$= I + (sI-A)^{-1}kBC - (sI-A)^{-1}B\left(\frac{1}{k}+G(s)\right)^{-1}C(I+(sI-A)^{-1}kBC)$$
$$= I + (sI-A)^{-1}kBC - (sI-A)^{-1}B\left(\frac{1}{k}+G(s)\right)^{-1}\left(\frac{1}{k}+G(s)\right)kC = I$$

prove that $s \in \rho(A_k)$ and (6.4) holds true. Since the resolvent of A is compact, this formula shows furthermore that the resolvent of A_k is compact as well. Therefore, the spectrum of A_k is a countable set of isolated eigenvalues [13, Theorem 6.29, p.187]. With Gauß's Theorem we get for all $x,z \in D(A_k)$

$$\langle A_k x, z\rangle = \int_\Omega \Delta x(\xi)\cdot \overline{z(\xi)}\,\mathrm{d}\xi = -\int_\Omega \nabla x(\xi)\cdot \nabla\overline{z(\xi)}\,\mathrm{d}\xi + \int_{\partial\Omega} \partial_\nu x(\xi)\cdot \overline{z(\xi)}\,\mathrm{d}\sigma_\xi$$
$$= -\int_\Omega \nabla x(\xi)\cdot \overline{\nabla z(\xi)}\,\mathrm{d}\xi - k\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi \int_{\partial\Omega}\overline{z(\xi)}\,\mathrm{d}\sigma_\xi.$$

By further reversing the roles of x and z in the above formula, we can conclude that

$$\langle A_k x, z\rangle = \langle x, A_k z\rangle \qquad \forall x,z \in D(A_k).$$

Since the spectrum of A_k consists of isolated eigenvalues, we have $\mathbb{R}\cap\rho(A_k) \neq \emptyset$. In other words, there exists some $\lambda \in \mathbb{R}$ such that $\lambda I - A$ is onto. Then we can conclude from [24, Proposition 3.2.4] that A_k is self-adjoint. Furthermore, A_k is non positive because for all $x \in D(A_k)$ holds

$$\langle A_k x, x\rangle = -\int_\Omega \nabla x(\xi)\cdot \overline{\nabla x(\xi)}\,\mathrm{d}\xi - k\left(\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi\right)^2 \le 0 \tag{6.5}$$

We show that zero is not an eigenvalue of A_k. Assume that $A_k x = 0$ for some function $x \in D(A_k)$, $x \not\equiv 0$. Then (6.5) implies $\nabla x = 0$ everywhere and $\int_{\partial\Omega} x(\xi)\,\mathrm{d}\sigma_\xi = 0$. Hence, x must be the constant zero function, which leads to a contradiction. Consequently zero is not an eigenvalue of A_k.

(iii) With the spectrum containing only isolated eigenvalues, statement (ii) implies $\sup_{\lambda\in\sigma(A_k)} \mathrm{Re}(\lambda) < 0$ and the claim follows with [24, Proposition 3.8.5]. □

Theorem 6.5 *Let* (A,B,C) *be defined by* (2.1) *with corresponding transfer function* G*. Let* λ_n*,* v_n *and* J_c *be as in Theorem 3.6 and define* A_k *by* (6.2) *with* $k>0$*. The eigenvalues of* A_k *are given by*

$$\sigma(A_k) = \left\{-\lambda_n \,\middle|\, n\in\mathbb{N}_0\setminus J_c\right\} \,\cup\, \left\{\lambda\in\mathbb{R}_{<0} \,\middle|\, G(\lambda) = -1/k\right\}.$$

Proof "$\supset$": Let $n \in \mathbb{N}_0\setminus J_c$, then $-\lambda_n$ is an eigenvalue of A and there exists at least one corresponding eigenvector $v_n \in \ker C$. Hence, v_n is in $D(A_k)$ by the definition (6.2) and the equation

$$A_k v_n = A v_n + kBC v_n = A v_n = -\lambda_n v_n$$

shows that λ_n is an eigenvalue of A_k.

Now let λ be such that $G(\lambda) = -\frac{1}{k}$. We distinguish the cases $\lambda \in \sigma(A)$ and $\lambda \in \rho(A)$: First assume that $\lambda\in\sigma(A)$. Then $\lambda = -\lambda_n$ for at least one $n\in\mathbb{N}$. If n

was in J_c, then Theorem 3.8 (i) would imply that λ is a pole of G, contradicting the assumption $G(\lambda) = -1/k$. So we have $n \in \mathbb{N}_0 \setminus J_c$ and the first part shows that $\lambda \in \sigma(A_k)$. Let us consider the case where $\lambda \in \rho(A)$: We have for all $x \in D(C)$

$$(\lambda I - A)^{-1}(\lambda I - A + kBC)x = x + (\lambda I - A)^{-1}kBCx.$$

Exploiting that $(\lambda I - A)^{-1}B$ maps into $D(C)$ by Remark 3.2 and Lemma 3.3, we get

$$C(\lambda I - A)^{-1}(\lambda I - A + kBC)x = Cx + kG(\lambda)Cx = Cx - Cx = 0.$$

If $\lambda I - A + kBC$ was bijective, this would lead to the contradiction $Cx = 0$ for all x in the dense subset $(\lambda I - A)^{-1}D(C)$ of $D(A)$. Thus, λ has to be in $\sigma(A_k)$.

"$\subset$": Let λ be an eigenvalue of A_k, i.e. $\lambda \in \mathbb{R}$ and some nontrivial $x \in D(A_k)$ satisfies $(\lambda I - A_k)x = (\lambda I - A + BkC)x = 0$. Assume further that $\lambda \neq -\lambda_n$ for all $n \in \mathbb{N} \setminus J_c$. Note that this implies $Cx \neq 0$ because otherwise, x would be an eigenvector of A contained in the kernel of C. We are going to show that $G(\lambda) = -1/k$. First, we determine the coefficients $\langle x, v_n \rangle$ for $n \in J_c$ by exploiting that

$$(\lambda + \lambda_n)\langle x, v_n \rangle = \langle x, (\overline{\lambda} - A)v_n \rangle = -\langle BkCx, v_n \rangle = -k\langle Cx, Cv_n \rangle.$$

Observe that $n \in J_c$ implies $Cv_n \neq 0$, and that we have $Cx \neq 0$. It follows that $\lambda \neq -\lambda_n$ and

$$\langle x, v_n \rangle = -k\frac{\langle Cx, Cv_n \rangle}{\lambda + \lambda_n} \qquad \forall n \in J_c.$$

Thus,

$$Cx = C\sum_{n\in\mathbb{N}_0}\langle x, v_n \rangle v_n = \sum_{n\in J_c}\langle x, v_n \rangle Cv_n = -k\sum_{n\in J_c}\frac{\langle Cx, Cv_n \rangle}{\lambda + \lambda_n}Cv_n = -kG(\lambda)Cx.$$

A division of this equation by $-k \cdot Cx \neq 0$ gives $G(\lambda) = -1/k$. $\square$

Corollary 6.6 *For $k \to \infty$ the eigenvalues of A_k converge from the right to the eigenvalues of A_0 in the following sense:*

$$\forall \mu \in \sigma(A_0)\ \forall \varepsilon > 0\ \exists K > 0\ \forall k \geq K\ \exists \lambda \in \sigma(A_k) \cap [\mu, \mu + \varepsilon).$$

Proof Let $\mu \in \sigma(A_0)$ and $J_c \subset \mathbb{N}$ be defined as in (3.5). By Theorem 5.2 (iii) and Theorem 4.3, at least one of the following scenarios is valid:

(i) $\mu = -\lambda_n$ for some $n \in \mathbb{N} \setminus J_c$, or
(ii) $G(\mu) = 0$.

In the first case, Theorem 6.5 implies that μ is an eigenvalue of A_k for all $k > 0$. In the second case, there exist two consecutive poles $-\lambda_{\tilde{n}}$, $-\lambda_n$ of G with $\mu \in (-\lambda_{\tilde{n}}, -\lambda_n)$ due to Theorem 3.8 (iii). The partial fraction decomposition (3.6), together with $c_k > 0\ \forall k \in J_c$, implies that the function $G|_{(-\lambda_{\tilde{n}}, -\lambda_n)}$ is real valued and monotonically decreasing with

$$\lim_{h\downarrow 0} G(-\lambda_{\tilde{n}} + h) = \infty, \qquad \lim_{h\downarrow 0} G(-\lambda_k - h) = -\infty,$$

(cf. proof of Theorem 3.8 and Figure 3.1). Hence, $\widetilde{G} := G|_{(-\lambda_{\tilde{n}},-\lambda_n)}$ has some continuous inverse $\widetilde{G}^{-1} : \mathbb{R} \to (-\lambda_{\tilde{n}}, -\lambda_n)$. Theorem 6.5 implies that $\widetilde{G}^{-1}(-1/k)$ is an eigenvalue of A_k. Further, $\mu = \widetilde{G}^{-1}(0)$ is the only transmission zero in $(-\lambda_{\tilde{n}}, -\lambda_n)$, whence, by Proposition 4.2 and Theorem 5.2 (iii), μ is an eigenvalue of A_0. Continuity of $\widetilde{G}$ on $(-\lambda_{\tilde{n}}, -\lambda_n)$ furthermore implies

$$\lim_{k\to\infty} \widetilde{G}^{-1}(-1/k) = \mu,$$

which proves the result. □

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