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Abstract—**We study the concept of locally controlled invariant submanifolds for nonlinear differential-algebraic/descriptor systems. In contrast to classical approaches, we define controlled invariance as the property of solution trajectories to evolve in a given submanifold whenever they start in it. It is then proved that this concept is equivalent to the existence of a feedback which renders the closed-loop vector field invariant in the descriptor sense. This result is exploited to show that the outcome of the differential-algebraic version of the zero dynamics algorithm yields a maximal output zeroing submanifold. The latter is then used to characterize the zero dynamics of the system. In order to guarantee that the zero dynamics are locally autonomous (i.e., locally resemble the behavior of an autonomous dynamical system), sufficient conditions involving the locally maximal output zeroing submanifold are derived.**

Index Terms—**Differential-algebraic equations, nonlinear systems, descriptor systems, controlled invariance, output zeroing submanifold, zero dynamics.**

Nomenclature:

$\mathbb{N}$, $\mathbb{N}_0$	set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{R}^{n\times m}$	the set of real $n\times m$ matrices
$\operatorname{rk} A$, $\operatorname{im} A$	rank and image of $A \in \mathbb{R}^{n\times m}$
$\mathbf{Gl}_n(\mathbb{R})$	the group of invertible matrices in $\mathbb{R}^{n\times n}$
$A^\dagger$	$= (A^\top A)^{-1}A^\top$ for $A \in \mathbb{R}^{n\times m}$ with $\operatorname{rk} A = m$
$\mathscr{C}^k(X;Y)$	set of k-times continuously differentiable functions $f : X \to Y$, $k \in \mathbb{N}_0 \cup \{\infty\}$; if $k = \infty$ the function f is called *smooth*
$\operatorname{dom} f$	the domain of the function f
$f\|_I$	restriction of the function f to the set I

I. Introduction

We consider nonlinear descriptor systems governed by differential-algebraic equations (DAEs) of the form

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} E(x(t)) &= f(x(t)) + g(x(t))\,u(t), \\ y(t) &= h(x(t)), \end{aligned} \tag{1}$$

where $X \subseteq \mathbb{R}^n$ is open, $0 \in X$, $f \in \mathscr{C}(X;\mathbb{R}^l), h \in \mathscr{C}(X;\mathbb{R}^p)$, $E \in \mathscr{C}^1(X;\mathbb{R}^l)$ are vector fields such that $f(0) = 0$, $h(0) = 0$, and $g \in \mathscr{C}(X;\mathbb{R}^{l\times m})$ is a matrix-valued function. The set of these systems is denoted by $\Sigma^X_{l,n,m,p}$; and we write $[E, f, g, h] \in \Sigma^X_{l,n,m,p}$. Note that the class $\Sigma^X_{l,n,m,p}$ encompasses any linear singular descriptor system and various important classes of nonlinear singular descriptor systems (e.g. chemical process systems [1], mechanical systems [2], [3] and modified nodal analysis models of electrical circuits [4]). Nonlinear descriptor systems seem to have been first considered by Luenberger [5]; see also the recent textbooks [6], [7].

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The functions $u : I \to \mathbb{R}^m$ and $y : I \to \mathbb{R}^p$ are called *input* and *output* of the system, resp. Since solutions not necessarily exist globally (e.g. finite escape times may arise) we consider maximal solutions of (1).

Definition I.1 (Solutions)**.** For $[E, f, g, h] \in \Sigma^X_{l,n,m,p}$ a trajectory $(x, u, y) \in \mathscr{C}(I;X \times \mathbb{R}^m \times \mathbb{R}^p)$ is called a *solution* of (1), if $I = \operatorname{dom} x \subseteq \mathbb{R}$ is an open interval, $E \circ x \in \mathscr{C}^1(I;\mathbb{R}^l)$ and (x, u, y) solves (1) for all $t \in I$. A solution (x, u, y) of (1) is called *maximal*, if any other solution $(\tilde{x}, \tilde{u}, \tilde{y})$ of (1) satisfies

$$J := \operatorname{dom} \tilde{x} \cap \operatorname{dom} x \neq \emptyset \ \wedge\ \tilde{x}|_J = x|_J \quad \Longrightarrow \quad \operatorname{dom} \tilde{x} \subseteq \operatorname{dom} x.$$

The *behavior* of (1) is defined as the set of maximal solution trajectories

$$\begin{aligned} \mathfrak{B}_{(1)} := \{\, (x,u,y) \in \mathscr{C}(I;X \times \mathbb{R}^m \times \mathbb{R}^p) \ \mid\ & I \subseteq \mathbb{R} \text{ is an open} \\ & \text{interval and } (x,u,y) \text{ is a maximal solution of (1)} \}. \end{aligned}$$

In the present paper, we consider questions related to controlled invariance and the zero dynamics of (1). The concept of (locally) controlled invariant submanifolds has been introduced by Isidori and Moog [8] (see also the textbooks [9], [10]) as a counterpart to controlled invariant distributions [11], [12] and it is an extension of the well-studied concept of controlled invariant subspaces for linear systems. Loosely speaking, a locally controlled invariant submanifold M is a connected submanifold which is invariant under the flow of the closed-loop vector field $f(x) + g(x)u(x)$ for some feedback $u(x)$; in the case of DAEs this invariance has to be formulated with respect to $E(\cdot)$. In the present paper, we show that the above "classical" definition in terms of feedback is equivalent to the "natural" definition, that for any initial value in M there exists an input such that the corresponding state trajectory remains in M for all times.

Locally controlled invariant submanifolds which are output zeroing (i.e., $M \subseteq h^{-1}(0)$) are related to the zero dynamics of the system (1). The zero dynamics are, loosely speaking, those dynamics that are not visible at the output and they are defined as the set of trajectories

$$\mathscr{Z}\mathscr{D}_{(1)} := \{\ (x,u,y) \in \mathfrak{B}_{(1)} \ \mid y = 0\ \}.$$

If the system (1) is governed by an ordinary differential equation (ODE), i.e., $n = l$ and $E(x) = x$, then the concept of zero dynamics has been introduced by Byrnes and Isidori [13] and studied extensively since then, see e.g. the textbooks [9], [10]. For linear DAEs, the zero dynamics are a real vector space (which is not true for nonlinear systems) and have been investigated in detail recently [14], [15]; for nonlinear DAEs some results for a class of semi-explicit systems [16] and for semi-explicit index-1 systems [17] are available.

The present paper is organized as follows: We consider the concept of controlled invariant subspaces for linear descriptor systems in Section II and prove characterizations of it. In Section III we give a brief summary of the differential geometric concepts used in the remainder of the paper. Motivated by the results for linear systems, local controlled invariance of submanifolds for nonlinear DAE systems is defined and characterized in Section IV; crucial preliminary results on constant rank matrix functions and the existence and extension of solutions to an important class of DAEs are provided. In Section V we consider locally controlled invariant submanifolds that are output zeroing and derive an extension of the zero dynamics algorithm (see e.g. [9], [10]) to DAE systems in order to compute a locally maximal output zeroing submanifold. This submanifold is exploited for a characterization of the zero dynamics of the system. The concept of locally autonomous zero dynamics, which has been successively used for the analysis of linear time-varying ODEs in [18] and of linear time-invariant DAEs in [15], is introduced in Section VI. We prove a sufficient condition for locally autonomous zero dynamics in terms of the locally maximal output zeroing submanifold. A conclusion is given in Section VII.

II. The linear case

In the present section we study controlled invariance for linear differential-algebraic systems of the form

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t), \tag{2}$$

where $E, A \in \mathbb{R}^{l\times n}$ and $B \in \mathbb{R}^{l\times m}$. The set of these systems is denoted by $\Sigma_{l,n,m}$ and we write $[E,A,B] \in \Sigma_{l,n,m}$. Since $[E,A,B]$ is linear we consider only global solutions and hence define the *behavior* of (2) as the set

$$\begin{aligned}\mathfrak{B}_{(2)} := \{\, (x,u) \in \mathscr{C}(\mathbb{R};\mathbb{R}^n\times\mathbb{R}^m) \;\mid\; & Ex \in \mathscr{C}^1(\mathbb{R};\mathbb{R}^l) \\ & \text{and } (x,u) \text{ satisfies (2) for all } t\in\mathbb{R} \,\}.\end{aligned}$$

If (2) is an ODE, i.e., $l = n$ and $E = I$, then a subspace $\mathscr{V} \subseteq \mathbb{R}^n$ is called controlled invariant (see e.g. [19]) if, loosely speaking, for all initial values in $\mathscr{V}$ there exists an input such that the corresponding state trajectory remains in $\mathscr{V}$ for all times. It is well-known that this is the case if, and only if, there exists a *friend* $F \in \mathbb{R}^{m\times n}$ such that $(A+BF)\mathscr{V} \subseteq \mathscr{V}$, or, equivalently, $A\mathscr{V} \subseteq \mathscr{V} + \operatorname{im} B$.

We introduce controlled invariance for linear DAEs (2) and generalize the above characterizations. To the best of the author's knowledge these characterizations are new.

Definition II.1 (Controlled invariant subspaces)**.** Let $[E,A,B] \in \Sigma_{l,n,m}$ and $\mathscr{V} \subseteq \mathbb{R}^n$ be a subspace. Then $\mathscr{V}$ is called *controlled invariant*, if

$$\begin{aligned}&\forall x^0 \in \mathscr{V}\ \exists (x,u) \in \mathfrak{B}_{(2)}\ \forall t \ge 0: \\ &\qquad x \in \mathscr{C}^1(\mathbb{R};\mathbb{R}^n) \ \wedge\ x(0) = x^0 \ \wedge\ x(t) \in \mathscr{V}.\end{aligned}$$

In order to prove that controlled invariance is equivalent to the existence of a friend we need the following crucial lemma which guarantees existence of solutions to a certain class of linear DAEs.

Lemma II.2 (Existence lemma)**.** *Let $E, A \in \mathbb{R}^{l\times n}$ be such that $\operatorname{im} A \subseteq \operatorname{im} E$. Then, for all $x^0 \in \mathbb{R}^n$, there exists $x \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^n)$ such that $x(0) = x^0$ and $E\dot x(t) = Ax(t)$ for all $t \in \mathbb{R}$.*

Proof. Since $\operatorname{im} A \subseteq \operatorname{im} E$ there exists $R \in \mathbb{R}^{n\times n}$ such that $A = ER$. Let $S, T \in \mathbf{Gl}_n(\mathbb{R})$ be such that $SET = \left[\begin{smallmatrix} I_r & 0 \\ 0 & 0\end{smallmatrix}\right]$, where $r = \operatorname{rk} E$. Then

$$SAT = SETT^{-1}RT = \begin{bmatrix} I_r & 0 \\ 0 & 0\end{bmatrix} T^{-1}RT =: \begin{bmatrix} T_1 & T_2 \\ 0 & 0\end{bmatrix}.$$

Now let $x^0 \in \mathbb{R}^n$ and let $x_1 \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^r)$, $x_2 \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^{n-r})$ be such that, for all $t \in \mathbb{R}$,

$$\dot x_1(t) = T_1x_1(t) + T_2x_2(t), \quad \text{and} \quad \begin{pmatrix} x_1(0) \\ x_2(0)\end{pmatrix} = T^{-1}x^0.$$

Define $x(\cdot) := T\left(\begin{smallmatrix} x_1(\cdot) \\ x_2(\cdot)\end{smallmatrix}\right) \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^n)$ and observe that $x(0) = x^0$ and, for all $t \in \mathbb{R}$,

$$\begin{aligned} E\dot x(t) &= S^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0\end{bmatrix} T^{-1}T\begin{pmatrix}\dot x_1(t) \\ \dot x_2(t)\end{pmatrix} \\ &= S^{-1}\begin{pmatrix}\dot x_1(t) \\ 0\end{pmatrix} = S^{-1}\begin{pmatrix} T_1x_1(t) + T_2x_2(t) \\ 0\end{pmatrix} \\ &= S^{-1}\begin{bmatrix} T_1 & T_2 \\ 0 & 0\end{bmatrix}\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix} = Ax(t). \end{aligned}$$

□

Note that the solution x in Lemma II.2 is not unique. The non-uniqueness amounts to the freedom in choosing x_2 in the proof of the lemma.

We are now in the position to state and prove the main result of the present section. This is the differential-algebraic analog of [19, Thm. 4.2]; note that its proof is also new in the ODE case as it uses Lemma II.2.

Theorem II.3 (Controlled invariance)**.** *Let $[E,A,B] \in \Sigma_{l,n,m}$ and $\mathscr{V} \subseteq \mathbb{R}^n$ be a subspace. Then the following statements are equivalent:*

(i) *$\mathscr{V}$ is controlled invariant.*
(ii) *$A\mathscr{V} \subseteq E\mathscr{V} + \operatorname{im} B$.*
(iii) *There exists $F \in \mathbb{R}^{m\times n}$ such that $(A+BF)\mathscr{V} \subseteq E\mathscr{V}$.*

Proof. (i)⇒(ii): Let $x^0 \in \mathscr{V}$. Then there exists $(x,u) \in \mathfrak{B}_{(2)}$ with $x \in \mathscr{C}^1(\mathbb{R};\mathbb{R}^n)$, $x(0) = x^0$ and $x(t) \in \mathscr{V}$ for all $t \ge 0$. The latter implies that $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) \in E\mathscr{V}$ for all $t > 0$ and continuity gives $\frac{\mathrm{d}}{\mathrm{d}t}Ex(0) \in E\mathscr{V}$, thus $Ax^0 = \frac{\mathrm{d}}{\mathrm{d}t}Ex(0) + Bu(0) \in E\mathscr{V} + \operatorname{im} B$. This implies $A\mathscr{V} \subseteq E\mathscr{V} + \operatorname{im} B$.

(ii)⇒(iii): Let $v_1,\ldots,v_k \in \mathbb{R}^n$ be basis vectors of $\mathscr{V}$, where $k = \dim\mathscr{V}$. By assumption there exist $w_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$ such that $Av_i = Ew_i + Bu_i$ for $i = 1,\ldots,k$. Now let $F \in \mathbb{R}^{m\times n}$ be the matrix representation of a linear map which is uniquely defined on $\mathscr{V}$ by $Fv_i = -u_i$, $i = 1,\dots,k$. Therefore, $(A+BF)v_i = Av_i - Bu_i = Ew_i \in E\mathscr{V}$ for $i = 1,\dots,k$ and thus $(A+BF)\mathscr{V} \subseteq E\mathscr{V}$.

(iii)⇒(i): Let $x^0 \in \mathscr{V}$ and $V \in \mathbb{R}^{n\times k}$ be such that $\operatorname{rk} V = k$ and $\operatorname{im} V = \mathscr{V}$. Then there exists $w^0 \in \mathbb{R}^k$ such that $x^0 = Vw^0$. Since $(A+BF)\mathscr{V} \subseteq E\mathscr{V}$ we have that $\operatorname{im}(A+BF)V \subseteq \operatorname{im} EV$ and by Lemma II.2 there exists $w \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^k)$ such that $w(0) = w^0$ and $EV\dot{w}(t) = (A+BF)Vw(t)$ for all $t \in \mathbb{R}$. Then $x(\cdot) := Vw(\cdot)$ and $u(\cdot) := FVw(\cdot)$ satisfy $(x,u) \in \mathfrak{B}_{(2)}$, $x(0) = x^0$ and $x(t) = Vw(t) \in \mathscr{V}$ for all $t \geq 0$. □

Note that a subspace $\mathscr{V}$ satisfying property (ii) in Theorem II.3 is usually called a (A,E,B)-invariant subspace, see [20] and also the survey [21].

III. Differential geometric preliminaries

We use the terminology of smooth manifolds and other differential geometric concepts as in [22]. Apart from that, by a *submanifold* we will always mean an embedded smooth k-submanifold of $\mathbb{R}^n$ for some $k \leq n$. Furthermore, we define the *tangent space* to a submanifold M of $\mathbb{R}^n$ at $x \in M$ as the linear subspace

$$T_xM := \left\{ v \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists I \subseteq \mathbb{R} \text{ open interval } \exists \gamma \in \mathscr{C}^\infty(I;M): \\ \gamma(0) = x \;\wedge\; \dot{\gamma}(0) = v \end{array} \right\}.$$

The above definition is different from the standard concept of the tangent space, usually introduced as the set of all derivations at x. However, by [22, Lem. 3.11] the derivations can be identified with tangent vectors to smooth curves, which in turn can be embedded into $\mathbb{R}^n$; cf. also [23, Thm. 2.2].

Let $X \subseteq \mathbb{R}^n$ be an open set (which is a manifold) and $M \subseteq X$ be submanifold. For any $x^0 \in M$ there exist $U \subseteq X$ open with $x^0 \in U$, $W \subseteq \mathbb{R}^k$ open for $k = \dim M \leq n$ and a diffeomorphism $\varphi : M \cap U \to W$. Without loss of generality, W and φ can be chosen such that $0 \in W$ and $\varphi(x^0) = 0$. (U,φ) is a *coordinate chart* for M at x^0 and φ is a *coordinate map*. Since φ is a diffeomorphism between submanifolds (in the sense of [22]) and $M \subseteq \mathbb{R}^n$, φ is a diffeomorphism in the sense of classical calculus, i.e., $\varphi \in \mathscr{C}^\infty(M\cap U;W)$ and $\varphi^{-1} \in \mathscr{C}^\infty(W;M\cap U)$. We call $\psi := \varphi^{-1}$ a *parametrization* for M at x^0 and record the following result which is important in due course.

Lemma III.1 (Parametrization and tangent space)**.** *Let M be a submanifold of an open set $X \subseteq \mathbb{R}^n$ and let $\psi : W \to M \cap U$ be a parametrization of M at $x^0 \in M$. Then*

$$\forall x \in M\cap U: \; T_xM = \operatorname{im} \psi'(\psi^{-1}(x)).$$

IV. Local controlled invariance

In Section II controlled invariant subspaces have been characterized for linear descriptor systems (2). In the present section we extend this approach to nonlinear systems (1) by considering a local version of controlled invariance for submanifolds of X (instead of subspaces). In classical textbooks [9], [10] on nonlinear ODE systems a locally controlled invariant submanifold M is, loosely speaking, defined by the existence of a feedback $u(x)$ such that the vector field $f(x)+g(x)u(x)$ is locally tangent to the M. In the linear case $f(x) = Ax$, $g(x) = B$, as considered in Section II, this is equivalent to the existence of a friend F such that $(A+BF)M \subseteq M$, i.e., property (iii) in Theorem (II.3). However, the characterization in terms of solution trajectories as in Definition II.1 is usually not considered for nonlinear systems.

In the following we extend Definition II.1 to nonlinear DAE systems (1) by considering controlled invariance locally on a conected submanifold of X. Then we derive, as a characterization, the existence of a feedback which (in some sense) renders the closed-loop vector field invariant, see Theorem IV.5. The idea for the proof comes from the consideration of the linear case discussed in Section II: First we generalize Lemma II.2 to nonlinear DAE systems, where additionally some care must be taken of the extendability of solutions, which results in Lemma IV.4. Then we prove the characterizations of locally controlled invariant submanifolds in Theorem IV.5.

Definition IV.1 (Controlled invariant submanifolds)**.** Let $[E,f,g,h] \in \Sigma^X_{l,n,m,p}$ and M be a connected submanifold of X such that $0 \in M$. Then M is called *locally controlled invariant*, if there exists an open neighborhood $U \subseteq X$ of the origin in $\mathbb{R}^n$ such that

$$\begin{aligned} &\forall x^0 \in M\cap U \;\exists (x,u,y) \in \mathfrak{B}_{(1)} \;\exists t_0 \in \operatorname{dom} x \;\forall t \in \operatorname{dom} x, t \geq t_0: \\ &\qquad x \in \mathscr{C}^1(\operatorname{dom} x;\mathbb{R}^n) \;\wedge\; x(t_0) = x^0 \;\wedge\; x(t) \in M\cap U. \end{aligned}$$

In Definition IV.1 only the existence of a maximal solution (x,u,y) with x starting at x^0 and staying in $M\cap U$ is required. For DAE systems, this solution is not unique, not even if we fix x^0 and u. Therefore, in contrast to ODE systems, it is possible to find at the same time solutions staying in $M\cap U$ and solutions leaving $M\cap U$ generated by the same input. Furthermore, possible state constraints restrict the set of locally controlled invariant submanifolds in a natural way (for ODEs, the whole set X is always locally controlled invariant, but not necessarily for DAEs) and possible input constraints make it harder to find a suitable control which establishes evolution in the submanifold. The following example illustrates the situation.

Example IV.2. Consider the DAE system (1) with $X = \mathbb{R}^4$ and

$$E(x) = \begin{pmatrix} x_1 \\ 0 \\ x_4 \\ 0 \end{pmatrix}, \quad f(x) = \begin{pmatrix} x_1+x_2 \\ x_3 \\ 0 \\ x_4 \end{pmatrix}, \quad g(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

for $x = (x_1,x_2,x_3,x_4)^\top$; we omit the output equation $y(t) = h(x(t))$ here. The state constraint $x_3(t) = 0$ implies that any locally controlled invariant submanifold must be a subset of $\mathbb{R}^2 \times \{0\} \times \mathbb{R}$. We show that the subspace $\mathscr{V} := \operatorname{im}[1,0,0,1]^\top$ is (locally) controlled invariant. To this end, we observe that the equations $\dot{x}_4(t) = u_2(t)$ and $0 = x_4(t) + u_2(t)$ impose an input constraint of the form $\dot{u}_2(t) = -u_2(t)$ and this restricts the choice of input functions. Let $(x^0,0,0,x^0)^\top \in \mathscr{V}$. We set $x_2(\cdot) = 0$ and $x_3(\cdot) = 0$ and observe that the remaining equations read

$$\begin{aligned} \dot{x}_1(t) = x_1(t) + u_1(t), \quad x_4(t) = -u_2(t), \\ \dot{x}_4(t) = u_2(t), \quad x_1(0) = x_4(0) = x^0. \end{aligned}$$

Since the solution must evolve in $\mathcal{V}$ we get the additional condition $x_1(t) = x_3(t)$ and we may hence simplify the resulting system to

$$\dot{u}_2(t) = -u_2(t),\quad u_2(0) = -x^0,\quad u_1(t) = 2u_2(t),$$
$$x_1(t) = -u_2(t),\quad x_3(t) = -u_2(t),$$

thus the choice of inputs $u_1(t) = -2e^{-t}x^0$, $u_2(t) = -e^{-t}x^0$ yields that $x_1(t) = -e^{-t}x^0 = x_3(t)$ and establishes that $\mathcal{V}$ is indeed (locally) controlled invariant. Note that the state $(x_1(t), x_2(t), x_3(t), x_4(t))^\top$ corresponding to the initial value $(x^0, 0, 0, x^0)^\top$ and to the input $(u_1(t), u_2(t))^\top$ is not unique: Any choice of $x_2 \in \mathcal{C}(\mathbb{R};\mathbb{R})$ with $x_2(0) = 0$ does also yields a solution to this initial value problem. This also shows that it is in general not possible to ensure that any solution evolves in $\mathcal{V}$.

We record a result on smooth constant rank matrix functions as an important lemma. This result is a consequence of the implicit function theorem. It is also mentioned in [10, Exercise 2.4], however there the constant transformation S from the left is missing.

Lemma IV.3 (Constant rank matrix functions)**.** *Let $U \subseteq \mathbb{R}^n$ be open, $x^0 \in U$ and $A \in \mathcal{C}^k(U;\mathbb{R}^{p\times q})$, $k \in \mathbb{N}\cup\{\infty\}$, be such that* $\operatorname{rk} A(x) = r$ *for all $x \in U$. Then there exists an open neighborhood $V \subseteq U$ of x^0 and $S \in \mathbf{Gl}_p(\mathbb{R})$, $T \in \mathcal{C}^k(V;\mathbf{Gl}_q(\mathbb{R}))$, $L \in \mathcal{C}^k(V;\mathbb{R}^{(p-r)\times r})$ such that*

$$\forall x \in V:\ SA(x)T(x) = \begin{bmatrix} I_r & 0 \\ L(x) & 0 \end{bmatrix}.$$

If moreover $\operatorname{rk} A(x) = p$ *for all $x \in U$, then there exists $T \in \mathcal{C}^k(V;\mathbf{Gl}_q(\mathbb{R}))$ such that $A(x)T(x) = [I_r, 0]$ for all $x \in V$.*

The proof of the equivalence between controlled invariance and the existence of a desired feedback as explained above relies on the following lemma which guarantees existence and extendability of solutions to a certain class of DAE systems; this is the nonlinear version of Lemma II.2.

Lemma IV.4 (Existence and extension lemma)**.** *Let $U \subseteq \mathbb{R}^n$ be open and $E \in \mathcal{C}^{k+1}(U;\mathbb{R}^l)$, $f \in \mathcal{C}^k(U;\mathbb{R}^l)$, $k \in \mathbb{N}\cup\{\infty\}$, be such that, for all $x \in U$, $f(x) \in E'(x)T_xU$ and* $\operatorname{rk} E'(x) = r$*. Then the following statements are true:*

a) For all $(t_0, x^0) \in \mathbb{R}\times U$, there exists an open interval $I \subseteq \mathbb{R}$, $t_0 \in I$, and $x \in \mathcal{C}^{k+1}(I;U)$ such that $\frac{\mathrm{d}}{\mathrm{d}t}E(x(t)) = f(x(t))$ for all $t \in I$ and $x(t_0) = x^0$.

b) If $x \in \mathcal{C}^{k+1}((a,b);U)$ is such that $\frac{\mathrm{d}}{\mathrm{d}t}E(x(t)) = f(x(t))$ for all $t \in (a,b)$ and $x^0 := \lim_{t\to b} x(t) \in U$ exists, then there exists $\varepsilon > 0$ and $\tilde{x} \in \mathcal{C}^{k+1}((a,b+\varepsilon);U)$ with $\frac{\mathrm{d}}{\mathrm{d}t}E(\tilde{x}(t)) = f(\tilde{x}(t))$ for all $t \in (a,b+\varepsilon)$ and $\tilde{x}|_{(a,b)} = x$.

Proof. a): Lemma IV.3 applied to the transpose of $E'(\cdot)$ yields existence of an open neighborhood $V \subseteq U$ of x^0 and $S \in \mathbf{Gl}_n(\mathbb{R})$, $T \in \mathcal{C}^k(V;\mathbf{Gl}_l(\mathbb{R}))$, $L \in \mathcal{C}^k(V;\mathbb{R}^{r\times(n-r)})$ such that

$$\forall x \in V:\ T(x)E'(x)S = \begin{bmatrix} I_r & L(x) \\ 0 & 0 \end{bmatrix}.$$

Since $f(x) \in E'(x)T_xU \subseteq \operatorname{im} E'(x)$ it follows that $\left[\begin{smallmatrix} I_r & L(x) \\ 0 & 0 \end{smallmatrix}\right] T(x)f(x) = T(x)f(x)$ for all $x \in V$, and hence with

$$w(\cdot) := S\begin{bmatrix} I_r & L(\cdot) \\ 0 & 0 \end{bmatrix} T(\cdot)f(\cdot) \in \mathcal{C}^k(V;\mathbb{R}^n)$$

we have that there exists $f_1 \in \mathcal{C}^k(V;\mathbb{R}^r)$ such that for all $x \in V$

$$T(x)f(x) = T(x)E'(x)w(x) = \begin{bmatrix} I_r & L(x) \\ 0 & 0 \end{bmatrix} S^{-1}w(x) = \begin{pmatrix} f_1(x) \\ 0 \end{pmatrix}.$$

Partition $S^{-1}x^0 = \left(\begin{smallmatrix} z_1^0 \\ z_2^0 \end{smallmatrix}\right) \in \mathbb{R}^r \times \mathbb{R}^{n-r}$ and let $z_1 \in \mathcal{C}^{k+1}(I;\mathbb{R}^r)$, $I \subseteq \mathbb{R}$ an open interval with $t_0 \in I$, be a local solution of the ODE

$$\dot{z}_1(t) = f_1\left(S\left(\begin{smallmatrix} z_1(t) \\ z_2^0 \end{smallmatrix}\right)\right),\quad z_1(t_0) = z_1^0,$$

which exists since $x^0 \in \operatorname{dom} f_1$ and f_1 is continuously differentiable. Then $x(\cdot) := S\left(\begin{smallmatrix} z_1(\cdot) \\ z_2^0 \end{smallmatrix}\right) \in \mathcal{C}^{k+1}(I;\mathbb{R}^n)$ satisfies $x(t) \in V \subseteq U$ for all $t \in I$ by construction and, furthermore,

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}E(x(t)) &= E'(x(t))\dot{x}(t) = T(x(t))^{-1}\begin{bmatrix} I_r & L(x(t)) \\ 0 & 0 \end{bmatrix}\begin{pmatrix} \dot{z}_1(t) \\ 0 \end{pmatrix} \\ &= T(x(t))^{-1}\begin{pmatrix} f_1(x(t)) \\ 0 \end{pmatrix} = f(x(t))\end{aligned}$$

for all $t \in I$ as well as $x(t_0) = x^0$.

b): Using the notation from *a)* and choosing $V \subseteq U$ as a neighborhood of x^0, we find that there exists $h > 0$ such that $\left(\begin{smallmatrix} z_1(\cdot) \\ z_2(\cdot) \end{smallmatrix}\right) := S^{-1}x(\cdot) \in \mathcal{C}^{k+1}((b-h,b);S^{-1}V)$ and it can be extended continuously to $t = b$. Let $z_1^0 := \lim_{t\to b} z_1(t)$. Similar to *a)* we obtain that, for all $t \in (b-h,b)$,

$$\dot{z}_1(t) = f_1\left(S\left(\begin{smallmatrix} z_1(t) \\ z_2(t) \end{smallmatrix}\right)\right) - L\left(S\left(\begin{smallmatrix} z_1(t) \\ z_2(t) \end{smallmatrix}\right)\right)\dot{z}_2(t). \tag{3}$$

Now, let $\tilde{h} > 0$ and $\tilde{z}_2 \in \mathcal{C}^{k+1}((b-h,b+\tilde{h});\mathbb{R}^{n-r})$ be such that $\tilde{z}_2|_{(b-h,b]} = z_2$ and for all $t \in (b-h,b+\tilde{h})$ there exists $v \in \mathbb{R}^r$ such that $\left(\begin{smallmatrix} v \\ z_2(t) \end{smallmatrix}\right) \in S^{-1}V$, which is clearly possible. Then there exists a local solution $\tilde{z}_1 \in \mathcal{C}^{k+1}((b-\varepsilon,b+\varepsilon);\mathbb{R}^r)$ of the initial value problem (3), $\tilde{z}_1(b) = z_1^0$. Define the continuous function

$$\hat{z}_1 : (b-h,b+\varepsilon) \to \mathbb{R}^r,\ t \mapsto \begin{cases} z_1(t), & t \in (b-h,b] \\ \tilde{z}_1(t), & t \in (b,b+\varepsilon). \end{cases}$$

Since $\hat{z}_1$ satisfies (3) on $(b-h,b)$ and on $(b,b+\varepsilon)$ it follows from continuity of $\hat{z}_1, \tilde{z}_2, \frac{\mathrm{d}}{\mathrm{d}t}\tilde{z}_2, f_1$ and L that $\frac{\mathrm{d}}{\mathrm{d}t}\hat{z}_1$ is continuous and hence, because $\hat{z}_1$ is a continuously differentiable solution of (3), $\hat{z}_1 \in \mathcal{C}^{k+1}((b-h,b+\varepsilon);\mathbb{R}^r)$. Similar to *a)*, we may now calculate that $\tilde{x}(\cdot) := S\left(\begin{smallmatrix} \hat{z}_1(\cdot) \\ \tilde{z}_2(\cdot) \end{smallmatrix}\right) \in \mathcal{C}^{k+1}((b-h,b-\varepsilon);V)$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t}E(\tilde{x}(t)) = f(\tilde{x}(t))$ for all $t \in \operatorname{dom}\tilde{x}$ and $\tilde{x}|_{(b-h,b)} = x|_{(b-h,b)}$. Gluing together x and $\tilde{x}$ yields an extension of x on $(a,b+\varepsilon)$ and finishes the proof of the lemma. □

We are now in the position to prove the main result of this section. This is the local, nonlinear analog of Theorem IV.5.

Theorem IV.5 (Local controlled invariance)**.** *Let $[E,f,g,h] \in \Sigma^X_{l,n,m,p}$ be such that $E \in \mathcal{C}^2(X;\mathbb{R}^l)$, $f \in \mathcal{C}^1(X;\mathbb{R}^l)$ and $g \in \mathcal{C}^1(X;\mathbb{R}^{l\times m})$ and let M be a smooth connected submanifold of X such that $0 \in M$. Suppose that there exists an open neighborhood V of $0 \in X$ such that both* $\dim E'(x)T_xM$ *and* $\dim\left(E'(x)T_xM + \operatorname{im} g(x)\right)$ *are constant for $x \in M\cap V$. Then the following statements are equivalent:*

(i) *M is locally controlled invariant.*

(ii) *There exists an open neighborhood U of $0 \in X$ such that $f(x) \in E'(x)T_xM + \operatorname{im} g(x)$ for all $x \in M \cap U$.*

(iii) *There exists an open neighborhood U of $0 \in X$ and $u \in \mathscr{C}^1(M \cap U; \mathbb{R}^m)$ such that $f(x) + g(x)u(x) \in E'(x)T_xM$ for all $x \in M \cap U$.*

Proof. (i)⇒(ii): Let U be as in Definition IV.1 and $x^0 \in M \cap U$. Then there exists $(x,u) \in \mathfrak{B}_{(1)}$ with $x \in \mathscr{C}^1(\operatorname{dom} x; \mathbb{R}^n)$ and some $t_0 \in \operatorname{dom} x$ such that $x(t_0) = x^0$ and $x(t) \in M \cap U$ for all $t \in \operatorname{dom} x \cap [t_0, \infty) =: I$. Therefore, $\frac{\mathrm{d}}{\mathrm{d}t}E(x(t)) = E'(x(t))\dot{x}(t) \in E'(x(t))T_{x(t)}M$ for all $t \in I$ and hence

$$f(x^0) = \tfrac{\mathrm{d}}{\mathrm{d}t}E(x(t))\big|_{t=0} - g(x^0)u(0) \in E'(x^0)T_{x^0}M + \operatorname{im} g(x^0).$$

(ii)⇒(iii): Let $\psi : G \to M \cap W$ be a parametrization of M at $0 \in M$ and let $U_1 := U \cap V \cap W$, $G_1 := \psi^{-1}(M \cap U_1)$. Then, by Lemma III.1 and the assumption we have

$$\forall x \in U_1 \cap M: \quad f(x) \in \operatorname{im} E'(x)\psi'(\psi^{-1}(x)) + \operatorname{im} g(x) = \operatorname{im} K(\psi^{-1}(x)),$$

where $K(\cdot) := [E'(\psi(\cdot))\psi'(\cdot), g(\psi(\cdot))] \in \mathscr{C}^1(G_1; \mathbb{R}^{l \times (q+m)})$ and $q = \dim M$. Since $\dim\big(E'(x)T_xM + \operatorname{im} g(x)\big)$ is constant for $x \in M \cap V$, we have that, for some $r \le q+m$, $\operatorname{rk} K(z) = r$ for all $z \in G_1$. From Lemma IV.3 it then follows that there exists an open neighborhood $G_2 \subseteq G_1$ of $0 \in \mathbb{R}^q$ and $S \in \mathbf{Gl}_l(\mathbb{R})$, $T \in \mathscr{C}^1(V_3; \mathbf{Gl}_{q+m}(\mathbb{R}))$, $L \in \mathscr{C}^1(V_3; \mathbb{R}^{(l-r)\times r})$ such that

$$\forall z \in G_2: \; SK(z)T(z) = \begin{bmatrix} I_r & 0 \\ L(z) & 0 \end{bmatrix}.$$

Let the open set U_2 be such that $M \cap U_2 = \psi(G_2)$ and observe that $0 \in U_2$. Now, we find that

$$\forall x \in M \cap U_2: \quad Sf(x) \in \operatorname{im} SK(\psi^{-1}(x)) = \operatorname{im} \begin{bmatrix} I_r & 0 \\ L(\psi^{-1}(x)) & 0 \end{bmatrix} T(\psi^{-1}(x))^{-1},$$

by which $\left[\begin{smallmatrix} I_r & 0 \\ L(x) & 0 \end{smallmatrix}\right] Sf(x) = Sf(x)$ for all $x \in M \cap U_2$. Therefore, with

$$v(\cdot) := T(\psi^{-1}(\cdot)) \begin{bmatrix} I_r & 0 \\ L(\psi^{-1}(\cdot)) & 0 \end{bmatrix} Sf(\cdot) \in \mathscr{C}^1(M \cap U_2; \mathbb{R}^{q+m}),$$

we obtain that $K(x)v(x) = f(x)$ for all $x \in M \cap U_2$. Partitioning $v(x) = (v_1(x)^\top, u(x)^\top)^\top$ with $v_1(x) \in \mathbb{R}^q$ and $u(x) \in \mathbb{R}^m$ for all $x \in M \cap U_2$ yields that

$$\forall x \in M \cap U_2: \; f(x) + g(x)u(x) \in E'(x)T_xM$$

with $u \in \mathscr{C}^1(M \cap U_2; \mathbb{R}^m)$.

(iii)⇒(i): Let ψ, U_1 and G_1 be as above.

Step 1: We show that for all $x^0 \in M \cap U_1$ there exists a local solution $(x,u,y) \in \mathscr{C}^1(I; X \times \mathbb{R}^m \times \mathbb{R}^p)$ of (1) with $x(t) \in M \cap U_1$ for all $t \in I$. Define

$$\tilde{E} : G_1 \to \mathbb{R}^l, \; x \mapsto E(\psi(x)),$$
$$\tilde{f} : G_1 \to \mathbb{R}^l, \; x \mapsto f(\psi(x)) + g(\psi(x))u(\psi(x)).$$

Since $\psi : G_1 \to M \cap U_1$ is a diffeomorphism, it is immediate that

$$\forall x \in G_1: \; \psi'(x)T_xG_1 = T_{\psi(x)}(M \cap U_1) = T_{\psi(x)}M.$$

By assumption we obtain

$$\forall x \in G_1: \quad \tilde{f}(x) \in E'(\psi(x))T_{\psi(x)}M = E'(\psi(x))\psi'(x)T_xG_1 = \tilde{E}'(x)T_xG_1.$$

Furthermore,

$$\begin{aligned} \operatorname{rk} \tilde{E}'(x) &= \operatorname{rk} E'(\psi(x))\psi'(x) \\ &= \dim \operatorname{im} E'(\psi(x))\psi'(x) \overset{\text{Lem. III.1}}{=} \dim E'(\psi(x))T_{\psi(x)}M \end{aligned}$$

for all $x \in G_1$ and since $\psi(x) \in V$ it follows that $\tilde{E}'$ has constant rank. We may now conclude from Lemma IV.4 a) that for arbitrary $x^0 \in M \cap U_1$ there exists an open interval $I \subseteq \mathbb{R}$, $0 \in I$, and $z \in \mathscr{C}^2(I; G_1)$ such that $z(0) = \psi^{-1}(x^0)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{E}(z(t)) = \tilde{f}(z(t))$. Then $x(\cdot) := \psi(z(\cdot)) \in \mathscr{C}^2(I; U_1)$ satisfies, for all $t \in I$,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}E(x(t)) = f(x(t)) + g(x(t))u(x(t)), \quad x(0) = x^0,$$

and $x(t) \in \operatorname{im} \psi|_{G_1} = M \cap U_1$ for all $t \in I$ with $t \ge 0$.

Step 2: We show that $(x, u \circ x, h \circ x)$ can be extended to a differentiable maximal solution of (1) of the same structure which also evolves in $M \cap U_1$. We prove this by using a standard technique which invokes Zorn's Lemma (cf. for instance [24, Thm. 4.8]): Denote $I = (a,b)$ and define

$$\mathscr{E} := \left\{ (\omega, z) \;\middle|\; \begin{array}{l} \omega \ge b, \; J = (a,\omega), \; z \in \mathscr{C}^1(J; X), \; z(t) \in \\ M \cap U_1 \text{ for all } 0 \le t < \omega, \; (z, u \circ z, h \circ z) \\ \text{is a solution of (1)}, \; z|_I = x \end{array} \right\}.$$

Since $(b,x) \in \mathscr{E}$, the set is nonempty. We endow $\mathscr{E}$ with a partial order $\preceq$ defined by

$$\begin{aligned} &(\omega_1, z_1) \preceq (\omega_2, z_2) \\ &\quad :\Longleftrightarrow \quad \omega_1 \le \omega_2 \;\wedge\; \forall t \in (a, \omega_1): \; z_1(t) = z_2(t). \end{aligned}$$

Now let $\mathscr{O}$ be a totally ordered subset of $\mathscr{E}$. Define $\omega^* := \sup\{\, \omega \mid (\omega, z) \in \mathscr{O} \,\}$ and $z^* \in \mathscr{C}^1((a,\omega^*); X)$ by $z^*|_{(a,\omega)} = z$ for all $(\omega, z) \in \mathscr{O}$; (ω^*, z^*) is well-defined since $\mathscr{O}$ is totally ordered. It is clear that $(\omega^*, z^*) \in \mathscr{E}$ is an upper bound for $\mathscr{O}$. Zorn's Lemma now implies existence of at least one maximal element of $\mathscr{E}$. Let $(\omega, \tilde{x})$ denote such an element and let $(\tilde{x}, u \circ \tilde{x}, h \circ \tilde{x})$ be a corresponding solution of (1). Assume that there exists a maximal solution $(\hat{x}, \hat{u}, \hat{y}) \in \mathscr{C}((\tilde{a}, \tilde{\omega}); X \times \mathbb{R}^m \times \mathbb{R}^p)$ of (1) with $\tilde{a} \le a$, $\omega < \tilde{\omega}$ and $\hat{x}|_{(a,\omega)} = \tilde{x}$. Since $\tilde{x}(t) \in M \cap U_1$ for all $t \in (a, \omega)$ and $\tilde{x}$ is extended by $\hat{x}$, continuity implies that $\alpha := \lim_{t \to \omega} \tilde{x}(t) \in M \cap U_1$ exists. Then $\tilde{z}(\cdot) := \psi^{-1}(\tilde{x}(\cdot)) \in \mathscr{C}^1((a,\omega); G_1)$ and similar to Step 1 we find that $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{E}(\tilde{z}(t)) = \tilde{f}(\tilde{z}(t))$ for all $t \in (a, \omega)$. Since $z^0 = \psi^{-1}(\alpha) = \lim_{t \to \omega} \tilde{z}(t) \in G_1$ exists, it follows from Lemma IV.4 b) that there exists $\varepsilon > 0$ and $\hat{z} \in \mathscr{C}^1((a, \omega + \varepsilon); G_1)$ with $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{E}(\hat{z}(t)) = \tilde{f}(\hat{z}(t))$ for all $t \in (a, \omega + \varepsilon)$ and $\hat{z}|_{(a,\omega)} = \tilde{z}$. But then $\check{x}(\cdot) := \psi(\hat{z}(\cdot)) \in \mathscr{C}^1((a, \omega+\varepsilon); M \cap U_1)$ satisfies $(\omega + \varepsilon, \check{x}) \in \mathscr{E}$ as can be easily checked and this contradicts the fact that $(\omega, \tilde{x})$ is a maximal element of $\mathscr{E}$.

We have thus proved that there is no maximal solution $(\hat{x}, \hat{u}, \hat{y})$ of (1) such that $\hat{x}$ extends $\tilde{x}$ to the right. With a similar procedure as above we can extend $\tilde{x}$ to the left preserving continuous differentiability; here we do not have to ensure that the solution stays within a certain submanifold. In the end, we obtain a maximal solution of (1) with continuously differentiable state that extends $(x, u \circ x, h \circ x)$ and evolves in $M \cap U_1$ as desired. This finishes the proof of the theorem. $\square$

Remark IV.6. If, under the assumptions of Theorem IV.5, additionally E is $(k+1)$- and f, g, h are k-times continuously differentiable, then the feedback $u(x)$ for the locally controlled invariant submanifold M in (iii) can be chosen to be k-times continuously differentiable, $k \in \mathbb{N}_0 \cup \{\infty\}$. Furthermore, the implication (ii)⇒(iii) is true without the assumption that $\dim E'(x)T_xM$ is constant in a certain region, (iii)⇒(i) holds true without the assumption that $\dim\big(E'(x)T_xM + \operatorname{im} g(x)\big)$ is constant in a certain region, and the implication (i)⇒(ii) does not need any of these assumptions.

V. Output zeroing submanifolds

In this section we investigate the concept of output zeroing submanifolds for nonlinear DAEs (1). This is important for the characterization of the zero dynamics of the system.

Definition V.1 (Output zeroing submanifold)**.** Let $[E, f, g, h] \in \Sigma^X_{l,n,m,p}$ and M be a connected submanifold of X such that $0 \in M$. Then M is called *output zeroing*, if M is locally controlled invariant and $h(x) = 0$ for all $x \in M$.

To illustrate the above definition we consider the following example.

Example V.2. Consider the DAE system (1) with $X = \mathbb{R}^2$ and

$$E(x) = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \quad f(x) = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$h(x) = x_1 - x_2^2 \quad \text{for} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

It is clear that the submanifold $M := \left\{ (x_1, x_2)^\top \in \mathbb{R}^2 \;\middle|\; x_1 = x_2^2 \right\}$ is a subset of $h^{-1}(0)$. For any $x^0 = (x_1^0, x_2^0)^\top \in M$, the choice $x_1(\cdot) \equiv x_1^0$, $x_2(\cdot) \equiv x_2^0$ and $u(\cdot) \equiv -x_1^0$ yields a solution $((x_1, x_2)^\top, u, 0) \in \mathfrak{B}_{(1)}$ which is globally defined, smooth and evolves in M for all times, starting at x^0. Therefore, M is an output zeroing submanifold. Note that it was necessary to make an appropriate choice of $u(\cdot)$ so that the algebraic constraint of the DAE system is satisfied.

In the following we seek an output zeroing submanifold M that is *locally maximal*, i.e., there exists an open neighborhood U of $0 \in X$ such that any output zeroing submanifold $\tilde{M}$ satisfies $\tilde{M} \cap U \subseteq M \cap U$. To this end, we extend the zero dynamics algorithm developed in [8], [25] to nonlinear DAE systems (1), where we stay close to the representation in [9, Sec. 6.1] and [10, Sec. 11.1].

Theorem V.3 (Zero dynamics algorithm)**.** *Let $[E, f, g, h] \in \Sigma^X_{l,n,m,p}$ be such that E, f, g and h are smooth. Define $M_0 := h^{-1}(0)$ and for any $k \in \mathbb{N}$ the set M_k recursively as follows: Suppose that for some open neighborhood U_{k-1} of $0 \in X$, $M_{k-1} \cap U_{k-1}$ is a submanifold, define*

$$\tilde{M}_{k-1} := \bigcup \left\{ M_{k-1} \cap U \;\middle|\; \begin{array}{l} U \subseteq X \text{ open, } M_{k-1} \cap U \\ \text{is a submanifold} \end{array} \right\},$$

let M_{k-1}^c be the connected component of $\tilde{M}_{k-1}$ which contains $0 \in X$ and define

$$M_k := \left\{ x \in M_{k-1}^c \;\middle|\; f(x) \in E'(x) T_x M_{k-1}^c + \operatorname{im} g(x) \right\}. \tag{4}$$

Then we have the following:

(i) *The sequence (M_k) is nested, terminates and satisfies*
$$\exists k^* \in \mathbb{N}_0 \ \forall j \in \mathbb{N}: \ M_0 \supsetneq M_1 \supsetneq \ldots \supsetneq M_{k^*} \supseteq M_{k^*}^c = M_{k^*+j} = M_{k^*+j}^c.$$

(ii) *If $Z^* := M_{k^*}^c$ satisfies, for some open neighborhood U of $0 \in \mathbb{R}$, that $\dim E'(x) T_x Z^*$ and $\dim\big(E'(x) T_x Z^* + \operatorname{im} g(x)\big)$ are both constant for $x \in Z^* \cap U$, then Z^* is a locally maximal output zeroing submanifold.*

(iii) *There exists an open neighborhood U of $0 \in X$ such that for all open $O \subseteq U$ and all $(x, u, y) \in \mathfrak{B}_{(1)}$ with $x \in \mathscr{C}^1(\operatorname{dom} x; X)$ and $x(t) \in O$ for all $t \in \operatorname{dom} x$ we have*
$$(x, u, y) \in \mathscr{ZD}_{(1)} \iff x(t) \in Z^* \cap O \text{ for all } t \in \operatorname{dom} x.$$

Proof. Step 1: We show (i). It is clear that for all $k \in \mathbb{N}_0$, $M_k \supseteq M_k^c \supseteq M_{k+1} \supseteq M_{k+1}^c$.

Step 1a: We show that if, for some $k \in \mathbb{N}$, $\dim M_k^c = \dim M_{k-1}^c$, then $M_k^c = M_{k+j} = M_{k+j}^c$ for all $j \in \mathbb{N}$. Let (U_i, φ_i), $i \in I$, be an atlas for M_k^c. Since $\dim M_k^c = \dim M_{k-1}^c$, for all $i \in I$, (U_i, φ_i) is also a coordinate chart for M_{k-1}^c, and hence U_i is open in M_{k-1}^c. Since the U_i cover M_k^c, it follows that each point in M_k^c has an open neighborhood in M_{k-1}^c, thus M_k^c is open in M_{k-1}^c. This implies that there exists an open subset U of X such that $M_{k-1}^c \cap U = M_k^c$. Then we find that

$$\begin{aligned}
& x \in M_k \cap U \\
\Leftrightarrow\ & x \in M_{k-1}^c \ \wedge \ f(x) \in E'(x) T_x M_{k-1}^c + \operatorname{im} g(x) \ \wedge x \in U \\
\Leftrightarrow\ & x \in M_{k-1}^c \cap U \ \wedge \ f(x) \in E'(x) T_x (M_{k-1}^c \cap U) + \operatorname{im} g(x) \\
\Leftrightarrow\ & x \in M_k^c \ \wedge \ f(x) \in E'(x) T_x M_k^c + \operatorname{im} g(x) \\
\Leftrightarrow\ & x \in M_{k+1},
\end{aligned}$$

by which $M_k \cap U = M_{k+1}$. Therefore, $M_k^c = M_k^c \cap U \subseteq M_k \cap U$ and $M_k^c = M_{k-1}^c \cap U \supseteq M_k \cap U$, thus $M_k^c = M_k \cap U = M_{k+1}$. Hence, M_{k+1} is a connected submanifold containing zero, by which $M_{k+1}^c = \tilde{M}_{k+1} = M_{k+1}$. By the formula (4) it then follows that $M_{k+1}^c = M_{k+2}$ and continuing these arguments finally gives the assertion.

Step 1b: Since each of the M_k^c is a finite dimensional submanifold and they are nested, there exists some $k^* \in \mathbb{N}$ such that $\dim M_{k^*}^c = \dim M_{k^*-1}^c$. Then, by Step 1a the sequence (M_k) terminates and the proof of (i) is complete.

Step 2: We show (ii). Since $Z^* = M_{k^*+1}$ it follows from (4) that

$$\forall x \in Z^* \cap U: \ f(x) \in E'(x) T_x Z^* + \operatorname{im} g(x).$$

Then Theorem IV.5 implies that Z^* is locally controlled invariant. As $Z^* \subseteq M_0 = h^{-1}(0)$ it follows that Z^* is an output zeroing submanifold. It remains to show that Z^* is locally maximal. To this end, let Z' be any other output zeroing submanifold. By local controlled invariance of Z', there exists an open neighborhood O of $0 \in X$ with the property as in Definition IV.1. We show that $Z' \cap O \subseteq M_k$ by induction over $k \in \mathbb{N}_0$. Since Z' is output zeroing, it follows that $Z' \subseteq h^{-1}(0) = M_0$. Assume that $Z' \cap O \subseteq M_k$ for some $k \in \mathbb{N}_0$. Then $Z' \cap O \subseteq M_k^c$ since $Z' \cap O$ is a submanifold with $0 \in Z' \cap O$. Now, for $x^0 \in Z' \cap O$ there exists $(x,u,y) \in \mathfrak{B}_{(1)}$ with $x \in \mathscr{C}^1(I;X)$, $I \subseteq \mathbb{R}$ an open interval with $t_0 \in I$, such that $x(t_0) = x^0$ and $x(t) \in Z' \cap O$ for all $t \in I$ with $t \geq t_0$. Therefore, $\dot{x}(0) \in T_{x^0} Z'$ and hence

$$\begin{aligned} f(x^0) &= \tfrac{\mathrm{d}}{\mathrm{d}t} E(x(t))\big|_{t=0} - g(x^0)u(0) \\ &\in E'(x^0)T_{x^0}Z' + \operatorname{im} g(x^0) \subseteq E'(x^0)T_{x^0}M_k^c + \operatorname{im} g(x^0), \end{aligned}$$

thus $x^0 \in M_{k+1}$. We may now deduce that in particular $Z' \cap O \subseteq M_{k^*+1} \cap O = Z^* \cap O$, thus Z^* is locally maximal.

Step 3: We show (iii). Choose the open set U small enough so that $M_k \cap U = M_k^c \cap U$ for all $k = 0, \ldots, k^*$. Now, it is easy to see the implication "$\Leftarrow$". For "$\Rightarrow$", observe that since $O \subseteq U$ we have

$$M_k \cap O = (M_k \cap U) \cap O = (M_k^c \cap U) \cap O = M_k^c \cap O$$

for all $k = 0, \ldots, k^*$. Furthermore, we have from $(x,u,y) \in \mathscr{ZD}_{(1)}$ and differentiability of x that

$$f(x(t)) = E'(x(t))\dot{x}(t) + g(x(t))u(t) \quad \text{and} \quad h(x(t)) = 0$$

for all $t \in \operatorname{dom} x$. Therefore, $x(t) \in M_0 \cap O = M_0^c \cap O$ and $\dot{x}(t) \in T_{x(t)} M_0^c$ for all $t \in \operatorname{dom} x$, whence $x(t) \in M_1 \cap O = M_1^c \cap O$. Inductively, we obtain that $x(t) \in M_{k^*}^c \cap O = Z^* \cap O$ for all $t \in \operatorname{dom} x$. □

Note that, if the system (1) is linear, then the sequence (M_k) becomes a modification of the first Wong sequence [26], [27], see [14, Lem. 4.1.2]. Furthermore, Theorem V.3 (iii) has been proved in [14, Prop. 4.1.4] in the case of a linear DAE system.

Remark V.4 (Zero dynamics algorithm)**.** We consider the algorithm for the construction of the sequence (M_k) in Theorem V.3. Note that in the corresponding algorithm for ODE systems as in Isidori's book [9, p. 294] and several other papers on that topic, usually the statement "suppose that, for some neighborhood U_{k-1} of 0, $M_{k-1} \cap U_{k-1}$ is a smooth submanifold, let M_{k-1}^c denote the connected component of $M_{k-1} \cap U_{k-1}$ which contains the point 0 [...]" can be found, which defines M_{k-1}^c in a different way than in Theorem V.3, but still the claim is that (i) is true. However, this is not quite correct, since the "dimensionality argument" used by Isidori in his proof does not apply to submanifolds; in general his construction of (M_k) does not lead to a terminating sequence, if the open sets U_k are not chosen maximal so that $M_k \cap U_k = \tilde{M}_k$. For instance, for the system $\dot{x}(t) = x(t) + u(t)$, $y(t) = 0$, we have $M_0 = \mathbb{R}$ and the choice $U_k = (-\frac{1}{k+1}, \frac{1}{k+1})$, $k \geq 0$, leads to $M_{k+1} = U_k$ and therefore to a nested sequence of submanifolds of the same dimension which does not terminate. Hence, the intermediate step of defining $\tilde{M}_k$ as in Theorem V.3 is indispensable.

That the assumption of constant dimension of $E'(x)T_x Z^*$ and $E'(x)T_x Z^* + \operatorname{im} g(x)$ in Theorem V.3 (ii) cannot be omitted in general has been shown in [10, p. 325]. However, it is not necessary for Z^* to be locally maximal output zeroing as the following example illustrates.

Example V.5 (Example V.2 revisited)**.** Consider the system $[E,f,g,h] \in \Sigma_{2,2,1,1}^{\mathbb{R}^2}$ from Example V.2. The output zeroing submanifold $M = \{\ (x_1,x_2)^\top \in \mathbb{R}^2 \ |\ x_1 = x_2^2\ \}$ is locally maximal, since $M = M_0 \supseteq Z^*$, which implies $M \cap U = Z^* \cap U$ for some open $U \subseteq X$ with $0 \in U$ by Theorem V.3. A simple calculation actually yields that $M = M_0 = M_1 = Z^*$. However,

$$\begin{aligned} &\dim E'(x)T_x Z^* = \dim \operatorname{im} \begin{bmatrix} 2x_2 \\ 0 \end{bmatrix} \\ \text{and} \quad &\dim \big(E'(x)T_x Z^* + \operatorname{im} g(x)\big) = \dim \operatorname{im} \begin{bmatrix} 2x_2 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

are not constant on $Z^* \cap U$ for any open neighborhood U of $0 \in \mathbb{R}^2$, since there is a drop of dimension in $x = 0$.

VI. Locally autonomous zero dynamics

Statement (iii) of Theorem V.3 shows that the submanifold Z^* allows to characterize the zero dynamics of (1) locally. However, this does not imply that the zero dynamics are (locally) autonomous, i.e., are the (local) behavior of a dynamical system governed ODEs. This problem is treated in the present section.

Here we use the behavioral approach [28] to dynamical systems and treat them as a set of trajectories; the solution behavior $\mathfrak{B}_{(1)}$ and the zero dynamics $\mathscr{ZD}_{(1)}$ have already been defined as behaviors. In the following we introduce the notion of local autonomy for behaviors by generalizing the concept of autonomy introduced for linear behaviors in [29, Sec. 3.2]. Note that in the present paper, autonomy always refers to the autonomy of the underlying behavior and not to time-invariance of the considered system.

Definition VI.1 (Locally autonomous behavior)**.** A behavior

$$\mathfrak{B} \subseteq \{\ f \in \mathscr{C}(I;\mathbb{R}^q) \ |\ I \subseteq \mathbb{R} \text{ an open interval }\ \}$$

is called *locally autonomous* with respect to an open neighborhood U of $0 \in \mathbb{R}^q$, if for all $f_1, f_2 \in \mathfrak{B}$, $J := \operatorname{dom} f_1 \cap \operatorname{dom} f_2 \neq \emptyset$, and for all open intervals $I \subseteq J$ we have:

$$\big(\forall t \in J:\ f_1(t), f_2(t) \in U \ \wedge\ f_1|_I = f_2|_I\big) \implies f_1|_J = f_2|_J.$$

Remark VI.2 (Locally autonomous zero dynamics and vector fields)**.**

(i) Let $[E,f,g,h] \in \Sigma_{l,n,m,p}^X$. We call the zero dynamics $\mathscr{ZD}_{(1)}$ *locally autonomous*, if the behavior $\mathscr{ZD}_{(1)}$ is locally autonomous with respect to $U \times \mathbb{R}^m \times \mathbb{R}^p$ for some open neighborhood U of $0 \in X$. The latter is the case if, and only if, for all $(x_1,u_1,0), (x_2,u_2,0) \in \mathscr{ZD}_{(1)}$, $J := \operatorname{dom} x_1 \cap \operatorname{dom} x_2 \neq \emptyset$, and for all open intervals $I \subseteq J$ we have:

$$\begin{aligned} &\left(\forall t \in J:\ x_1(t), x_2(t) \in U \ \wedge\ \begin{pmatrix} x_1 \\ u_1 \end{pmatrix}\bigg|_I = \begin{pmatrix} x_2 \\ u_2 \end{pmatrix}\bigg|_I\right) \\ &\qquad \implies \operatorname{dom} x_1 = \operatorname{dom} x_2 \ \wedge\ \begin{pmatrix} x_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ u_2 \end{pmatrix}. \end{aligned}$$

The equality of the domains in the implication follows from the fact that $(x_1,u_1,0),(x_2,u_2,0)$ are both maximal solutions of (1) and

$$(\tilde{x},\tilde{u},0):\operatorname{dom}x_1\cup\operatorname{dom}x_2\to X\times\mathbb{R}^m\times\mathbb{R}^p,\qquad t\mapsto\begin{cases}(x_1(t),u_1(t),0), & t\in\operatorname{dom}x_1,\\(x_2(t),u_2(t),0), & t\in\operatorname{dom}x_2,\end{cases}$$

is well-defined (since the solutions intersect on I) and a solution of (1) with $\operatorname{dom}x_i\cap\operatorname{dom}\tilde{x}=\operatorname{dom}x_i$ and $\tilde{x}|_{\operatorname{dom}x_i}=x_i$, $i=1,2$, thus $\operatorname{dom}x_1\cup\operatorname{dom}x_2=\operatorname{dom}x_1=\operatorname{dom}x_2=J$.

(ii) Consider a vector field $F\in\mathscr{C}(X;\mathbb{R}^n)$, $X\subseteq\mathbb{R}^n$ open, $0\in X$, with $F(0)=0$. Then F is called *locally autonomous*, if the behavior

$$\mathfrak{B}=\left\{\,x\in\mathscr{C}^1(I;X)\;\middle|\;\begin{array}{l}I\subseteq\mathbb{R}\text{ open interval,}\\\forall t\in I:\ \dot{x}_i(t)=F(x_i(t))\end{array}\right\}$$

is locally autonomous with respect to some open neighborhood U of $0\in X$.

Next we show that locally autonomous zero dynamics carry in a certain sense the structure of a dynamical system.

Remark VI.3 (Locally autonomous zero dynamics are a dynamical system)**.** Let $[E,f,g,h]\in\Sigma^X_{l,n,m,p}$ be such that the zero dynamics $\mathscr{ZD}_{(1)}$ are locally autonomous and let U be a corresponding open subset of X as in Remark VI.2. We show that, in some sense, the zero dynamics carry the structure of a dynamical system as defined in [30, Def. 2.1.1]. For $t_0\in\mathbb{R}$, $x^0\in X$, $u^0\in\mathbb{R}^m$ and $(x,u,0)\in\mathscr{ZD}_{(1)}$ we will use the following property in due course:

$$t_0\in\operatorname{dom}x\ \wedge\ \begin{pmatrix}x(t_0)\\u(t_0)\end{pmatrix}=\begin{pmatrix}x^0\\u^0\end{pmatrix}\ \wedge\ \forall t\in\operatorname{dom}x: x(t)\in U. \tag{5}$$

Define

$$Z:=\left\{\begin{pmatrix}x^0\\u^0\end{pmatrix}\in Z\times\mathbb{R}^m\;\middle|\;\begin{array}{l}\exists t_0\in\mathbb{R}\ \exists(x,u,0)\in\mathscr{ZD}_{(1)}:\\\text{(5) holds}\end{array}\right\}$$

and

$$\mathscr{D}_\varphi:=\left\{\left(t,t_0,\begin{pmatrix}x^0\\u^0\end{pmatrix},0\right)\;\middle|\;\begin{array}{l}\exists(x,u,0)\in\mathscr{ZD}_{(1)}:\\t\in\operatorname{dom}x\text{ and (5) holds}\end{array}\right\}\subseteq\mathbb{R}\times\mathbb{R}\times Z\times\{0\},$$

which is the domain of the state transition map

$$\varphi:\mathscr{D}_\varphi\to Z,\ \left(t,t_0,\begin{pmatrix}x^0\\u^0\end{pmatrix},0\right)\mapsto\begin{pmatrix}x(t)\\u(t)\end{pmatrix}\quad\text{s.t. }(x,u,0)\in\mathscr{ZD}_{(1)}\text{ satisfies (5).}$$

We show that φ is well-defined. Let $\left(t,t_0,\begin{pmatrix}x^0\\u^0\end{pmatrix},0\right)\in\mathscr{D}_\varphi$ and $(x_1,u_1,0),(x_2,u_2,0)\in\mathscr{ZD}_{(1)}$ be such that they satisfy (5). Define

$$(\tilde{x},\tilde{u}):(\inf\operatorname{dom}x_1,\sup\operatorname{dom}x_2)\to X,\qquad t\mapsto\begin{cases}(x_1(t),u_1(t)), & t<t_0\\(x_2(t),u_2(t)), & t\ge t_0.\end{cases}$$

Then $\tilde{x}\in\mathscr{C}(\operatorname{dom}\tilde{x};X)$ and $\tilde{u}\in\mathscr{C}(\operatorname{dom}\tilde{x};\mathbb{R}^m)$. Furthermore, $E(\tilde{x}(\cdot))$ is continuously differentiable everywhere in $\operatorname{dom}\tilde{x}$, except for possibly t_0. But then

$$\begin{aligned}\lim_{t\nearrow t_0}\tfrac{\mathrm{d}}{\mathrm{d}t}E(\tilde{x}(t))&=\lim_{t\nearrow t_0}f(x_1(t))+g(x_1(t))u_1(t)\\&=f(x^0)+g(x^0)u^0=\lim_{t\searrow t_0}f(x_2(t))+g(x_2(t))u_2(t)\\&=\lim_{t\searrow t_0}\tfrac{\mathrm{d}}{\mathrm{d}t}E(\tilde{x}(t)),\end{aligned}$$

and hence $E(\tilde{x}(\cdot))$ is also continuously differentiable in t_0, thus $(\tilde{x},\tilde{u},0)$ is a solution of (1). Since $(x_1,u_1,0)$ and $(x_2,u_2,0)$ are both maximal solutions, it follows that $(\tilde{x},\tilde{u},0)$ is maximal and hence $(\tilde{x},\tilde{u},0)\in\mathscr{ZD}_{(1)}$ and satisfies (5). But $(\tilde{x},\tilde{u})$ coincides with (x_i,u_i), $i=1,2$ on some open interval contained in the intersection of their domains, thus local autonomy of the zero dynamics implies that $\operatorname{dom}x_1=\operatorname{dom}\tilde{x}=\operatorname{dom}x_2$ and $(x_1,u_1)=(\tilde{x},\tilde{u})=(x_2,u_2)$. We have hence shown that φ is well-defined, since it is independent of the choice of $(x,u,0)\in\mathscr{ZD}_{(1)}$. Now, define the output map by

$$\eta:\mathbb{R}\times Z\times\{0\}\to\mathbb{R}^p,\ \left(t,\begin{pmatrix}x^0\\u^0\end{pmatrix},0\right)\mapsto 0.$$

It is then readily verified that the tuple $(\mathbb{R},\mathbb{R}^m,\{0\},Z,X,\mathbb{R}^p,\varphi,\eta)$ satisfies the axioms of a dynamical system as given in [30, Def. 2.1.1]. Now let $t_0\in\mathbb{R}$ and define

$$\mathscr{ZD}^{U,t_0}_{(1)}:=\left\{\,(x,u,0)\in\mathscr{ZD}_{(1)}\;\middle|\;\begin{array}{l}t_0\in\operatorname{dom}x,\ \forall t\in\operatorname{dom}x:\\x(t)\in U\end{array}\right\}.$$

Then observe that

$$\mathscr{ZD}^{U,t_0}_{(1)}=\left\{\left(\varphi\left(\,\cdot\,,t_0,\begin{pmatrix}x^0\\u^0\end{pmatrix},0\right),0\right)\;\middle|\;\begin{array}{l}\exists\begin{pmatrix}x^0\\u^0\end{pmatrix}\in X\times\mathbb{R}^m:\\\left(t_0,t_0,\begin{pmatrix}x^0\\u^0\end{pmatrix},0\right)\in\mathscr{D}_\varphi\end{array}\right\},$$

and hence we find that the local zero dynamics $\mathscr{ZD}^{U,t_0}_{(1)}$ evolving through t_0 are the set of trajectories of a dynamical system, and can thus be viewed as a dynamical system themselves. This justifies to say that the zero dynamics locally carry the structure of a dynamical system.

In the following, we derive sufficient conditions for locally autonomous zero dynamics. To this end, we use the submanifold Z^* from Theorem V.3, which is a locally maximal output zeroing submanifold if $\dim E'(x)T_xZ^*$ and $\dim\big(E'(x)T_xZ^*+\operatorname{im}g(x)\big)$ are constant for $x\in Z^*\cap U$. In order to obtain uniqueness of the feedback $u(x)$ in the characterization of local controlled invariance in Theorem IV.5 (iii), we need to strengthen the latter condition to $\dim\big(E'(x)T_xZ^*+\operatorname{im}g(x)\big)=q+m$ for all $x\in Z^*\cap U$, where $q=\dim Z^*$; in fact, for this it is sufficient to assume $\dim\big(E'(0)T_0Z^*+\operatorname{im}g(0)\big)=q+m$. Furthermore, we require that, loosely speaking, those components of $f|_{Z^*\cap U}$ corresponding to the zero dynamics, form a locally autonomous vector field. It is also necessary to consider only those trajectories in the zero dynamics which have a continuously differentiable state trajectory, i.e., under the above assumptions (specified in the following theorem) we prove that

$$\mathscr{ZD}^{\mathscr{C}^1}_{(1)}:=\left\{\ (x,u,0)\in\mathscr{ZD}_{(1)}\;\middle|\;x\in\mathscr{C}^1(\operatorname{dom}x;X)\ \right\}$$

is locally autonomous, using the same definition as for local autonomy of $\mathscr{ZD}_{(1)}$. This requirement seems unsatisfactory (since it is not necessary in the linear case) and it is an open problem whether it can be omitted.

Theorem VI.4 (Sufficient condition for locally autonomous zero dynamics)**.** *Let $[E,f,g,h]\in\Sigma^X_{l,n,m,p}$ be such that E,f,g and h are smooth and assume, for the sets M_k as in (4), that for some open neighborhood U_k of $0\in X$, $M_k\cap U_k$ is a submanifold, for all $k\in\mathbb{N}_0$. Use the notation from Theorem V.3, let $\psi: V\to Z^*\cap U$ be a parametrization of Z^* at $0\in Z^*$, and assume furthermore that*

(1) $\dim\big(E'(0)T_0Z^*+\operatorname{im} g(0)\big)=q+m$, where $q=\dim Z^*$, and

(2) $F:=\big(E'(\psi(\cdot))\psi'(\cdot)\big)^\dagger f(\psi(\cdot))\in\mathscr{C}^\infty(V;\mathbb{R}^q)$ is locally autonomous.

Then the zero dynamics $\mathscr{ZD}^{\mathscr{C}^1}_{(1)}$ are locally autonomous.

Proof. By assumption (1) and the fact that by Lemma III.1 $T_xZ^*=\operatorname{im}\psi'(\psi^{-1}(x))$ for all $x\in Z^*\cap U$, it follows that $[E'(0)\psi'(\psi^{-1}(0)),g(0)]$ has full column rank $q+m$. From continuity we may infer existence of an open neighborhood $U_1\subseteq U$ of $0\in X$ such that $\operatorname{rk}[E'(x)\psi'(\psi^{-1}(x)),g(x)]=q+m$ for all $x\in Z^*\cap U_1$. Let $V_1:=\psi^{-1}(Z^*\cap U_1)$ and observe that by full column rank of $[E'(\psi(z))\psi'(z),g(\psi(z))]$ for all $z\in V_1$, Lemma IV.3 applied to its transpose gives existence of an open neighborhood $V_2\subseteq V_1$ of $0\in\mathbb{R}^q$ and $S\in\mathscr{C}^\infty(V_2;\mathbf{Gl}_l(\mathbb{R}))$ such that

$$\forall z\in V_2:\ S(z)[E'(\psi(z))\psi'(z),g(\psi(z))]=\begin{bmatrix}I_q&0\\0&I_m\\0&0\end{bmatrix}.$$

Note that $S(\cdot)$ can be chosen such that $[I_q,0]S(\cdot)=\big(E'(\psi(\cdot))\psi'(\cdot)\big)^\dagger$. Let the open neighborhood $U_2\subseteq U_1$, $0\in U_2$, be such that $Z^*\cap U_2=\psi(V_2)$. Furthermore, let V_3 be an open neighborhood of $0\in\mathbb{R}^q$ corresponding to the locally autonomous vector field F as in Remark VI.2 (ii) and let the open neighborhood U_3, $0\in U_3$, be such that $Z^*\cap U_3=\psi(V\cap V_3)$. Finally, let U_4 be an open neighborhood of $0\in X$ as in Theorem V.3 (iii).

Now, define $\tilde U:=U_2\cap U_3\cap U_4$, let $(x_1,u_1,0),(x_2,u_2,0)\in\mathscr{ZD}^{\mathscr{C}^1}_{(1)}$ be such that $J:=\operatorname{dom}x_1\cap\operatorname{dom}x_2\neq\emptyset$ and $x_1(t),x_2(t)\in\tilde U$ for all $t\in J$, and let $I\subseteq J$ be an open interval such that $\binom{x_1}{u_1}|_I=\binom{x_2}{u_2}|_I$. Let $i\in\{1,2\}$. Then Theorem V.3 (iii) implies that $x_i(t)\in Z^*\cap\tilde U$ for all $t\in J$. Therefore, $x_i(t)\in\psi(V_2)$ and thus there exists $z_i(t)\in V_2$ such that $x_i(t)=\psi(z_i(t))$, $t\in J$. Since x_i is continuously differentiable and ψ has a smooth inverse it follows that $z_i\in\mathscr{C}^1(J;V_2)$ and

$$\dot x_i(t)=\psi'(z_i(t))\dot z_i(t),\quad t\in J.$$

Furthermore, by $(x_i,u_i,0)\in\mathscr{ZD}_{(1)}$ we find that, for all $t\in J$,

$$E'(\psi(z_i(t)))\psi'(z_i(t))\dot z_i(t)-g(\psi(z_i(t)))u_i(t)=f(\psi(z_i(t))),$$

and a multiplication from the left by $S(z_i(t))$ yields

$$\begin{pmatrix}\dot z_i(t)\\-u_i(t)\\0_{l-q-m}\end{pmatrix}=S(z_i(t))f(\psi(z_i(t)))=:\begin{pmatrix}f_1(z_i(t))\\f_2(z_i(t))\\f_3(z_i(t))\end{pmatrix},$$

where $f_1\in\mathscr{C}^\infty(V_2;\mathbb{R}^q)$, $f_2\in\mathscr{C}^\infty(V_2;\mathbb{R}^m)$, $f_3\in\mathscr{C}^\infty(V_2;\mathbb{R}^{l-q-m})$. The vector field

$$f_1(\cdot)=[I_q,0]S(\cdot)f(\psi(\cdot))=\big(E'(\psi(\cdot))\psi'(\cdot)\big)^\dagger f(\psi(\cdot))=F(\cdot)$$

is locally autonomous by assumption with corresponding open neighborhood V_3. Since

$$z_i(t)=\psi^{-1}(x_i(t))\in\psi^{-1}(Z^*\cap U_2\cap U_3\cap U_4)\subseteq V\cap V_3,\quad t\in J,$$

and

$$z_1(t)=\psi^{-1}(x_1(t))=\psi^{-1}(x_2(t))=z_2(t),\quad t\in I,$$

it follows that $z_1|_J=z_2|_J$ and hence $x_1|_J=x_2|_J$ and $u_1|_J=u_2|_J$. This concludes the proof of the theorem. □

Note that it can be shown that the vector field $f_3\in\mathscr{C}^\infty(V_2;\mathbb{R}^{l-q-m})$ in the proof of Theorem VI.4 vanishes on some open subset $V_4\subseteq V_3$ by using local controlled invariance of Z^*: For all $z\in V_4$ there exist $\tilde z\in\mathbb{R}^q$, $\tilde u\in\mathbb{R}^m$ such that

$$\begin{pmatrix}f_1(z)\\f_2(z)\\f_3(z)\end{pmatrix}=S(z)f(\psi(z))=S(z)\Big(E(\psi(z))\psi'(z)\tilde z+g(\psi(z))\tilde u\Big)=\begin{pmatrix}\tilde z\\\tilde u\\0\end{pmatrix}.$$

In the case of a linear DAE system, (locally) autonomous zero dynamics are equivalent to the assumption (1) in Theorem VI.4, which is equivalent to the assumptions (A1)–(A3) in [15]; assumption (2) is always satisfied for linear systems. Although it is possible to show that locally autonomous zero dynamics always imply that $\operatorname{rk} g(0)=m$, the converse of the statement of Theorem VI.4 is not true in general for nonlinear DAE systems, not even in the case where $E'(\cdot)$ is constant. This is illustrated by the following example.

Example VI.5 (Examples V.2, V.5 revisited)**.** Consider the system $[E,f,g,h]\in\Sigma^{\mathbb{R}^2}_{2,2,1,1}$ from Examples V.2 and V.5. As already calculated, the locally maximal output zeroing submanifold Z^* satisfies

$$\dim\big(E'(0)T_0Z^*+\operatorname{im} g(0)\big)=1\neq 2=q+m,$$

and thus assumption (1) in Theorem VI.4 is violated. However, the zero dynamics are locally autonomous, since the system equations (1) read

$$\dot x_1(t)=0,\quad x_1(t)=-u(t),\quad x_1(t)=x_2(t)^2,$$

by which we may infer that any solution satisfies $x_2\equiv c$ for some $c\in\mathbb{R}$, $x_1\equiv c^2$ and $u\equiv -c^2$. Note that the system does not have a solution if the initial value for x_1 is negative.

VII. Conclusion

In the present paper we have introduced the concept of local controlled invariance for connected submanifolds as the property of local solution trajectories to evolve in a given submanifold whenever they start in it. Motivated by the observations in the linear case, we have shown that local controlled invariance is equivalent to the existence of a feedback which renders the closed-loop vector field invariant. Furthermore, the zero dynamics algorithm has been extended to DAE systems and the resulting locally maximal output zeroing submanifold has been exploited for a characterization of the zero dynamics. Under some appropriate conditions on the latter submanifold, the zero dynamics are proved to be locally autonomous.

The concept of (locally) autonomous zero dynamics can be used to derive conditions for the application of adaptive controllers to nonlinear DAE systems. For instance, in [15] it is shown for linear descriptor systems, that autonomous zero dynamics and right invertibility of the system are required for the application of funnel control. Further studies have the aim to derive a local zero dynamics form for nonlinear DAE systems (1) under the assumption of locally autonomous zero dynamics; this normal form would provide the basis for the application of adaptive control techniques. In particular, it is our aim to use the results of [31] and show feasibility of funnel control for nonlinear descriptor systems which encompass nonlinear electrical circuits, extending the results for the linear case [32].

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