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Abstract

For a smooth flow on $\mathbb{R}^n$ we investigate non-asymptotic stability properties of compact invariant sets, in particular of heteroclinic cycles and networks. As our main result we show that predominant asymptotic stability of a cycle or network is equivalent to all local stability indices along its connections being positive. We exploit this to give a complete picture of possible stability configurations for simple cycles in $\mathbb{R}^4$ that undergo transverse bifurcations. Furthermore, we deduce that cycles of types B and C are generically predominantly asymptotically stable after a transverse eigenvalue becomes positive and that no type B cycle can be predominantly unstable (unlike type C cycles).

Keywords: equivariant dynamics, heteroclinic cycle, (non-asymptotic) stability, stability index

Mathematics Subject Classification: 34C37, 37C29, 37C80, 37C75

1 Introduction

The discovery that heteroclinic cycles can be structurally stable in systems with symmetry sparked interest in conditions for their existence and attraction properties. As a result, asymptotic stability of a large class of cycles has been understood to a satisfactory extent, see the work of Krupa and

Melbourne [1, 2, 3]. Melbourne [4] was the first to discover that heteroclinic cycles often exhibit more diverse stability properties than the classic dichotomy between asymptotic stability and complete instability, which is typical, for instance, of hyperbolic equilibria. He introduced the term *essential asymptotic stability (e.a.s.)* to describe attraction of a large measure set that is not a full neighbourhood. More recently, Podvigina and Ashwin [5] contributed to this topic by introducing stability indices σ and σ_{loc} as tools to quantify stability and attraction along a trajectory. They coined the term *predominant asymptotic stability (p.a.s.)* – for the same attraction property that Melbourne [4] called e.a.s. Since there was a slight inaccuracy in Melbourne's definition, we stick with the terminology of Podvigina and Ashwin to avoid any possibility for confusion. Note that both have been used throughout the literature, usually referring to the same property.

In this paper we show how predominant asymptotic stability and its unstable counterpart *predominant instability* are related to the index σ_{loc}, our main result being that for a heteroclinic cycle or network $X \subset \mathbb{R}^n$ the following holds (theorem 3.1):

(i) X is p.a.s. $\quad\Leftrightarrow\quad$ $\sigma_{\text{loc}}(x) > 0$ along all connecting trajectories.

(ii) X is p.u. $\quad\Leftrightarrow\quad$ $\sigma_{\text{loc}}(x) < 0$ along all connecting trajectories.

This work is structured as follows. In section 2 we recall the well-known setting in which heteroclinic cycles occur as robust phenomena and provide relevant definitions of stability properties and indices. Section 3 contains the main result as indicated above, its proof is deferred to appendix A. In section 4 we apply our results to heteroclinic cycles in $\mathbb{R}^4$, obtaining a complete picture of stability configurations during transverse bifurcations for the cycles classified as *simple* in [3]. This yields general results about predominant (in)stability of simple cycles of types B and C in $\mathbb{R}^4$. For ease of reference, appendix B contains relevant statements from [5], which we need to prove our results in section 4. These proofs can be found in appendix C.

2 Preliminaries

Consider a vector field on $\mathbb{R}^n$ given through a smooth differential equation $\dot{y} = f(y)$, where f is Γ-equivariant under the action of a finite group $\Gamma \subset O(n)$, that is,

$$f(\gamma.y) = \gamma.f(y), \quad \forall\, \gamma \in \Gamma \quad \forall\, y \in \mathbb{R}^n.$$

A *heteroclinic cycle* is a collection of finitely many equilibria ξ_i, $i = 1, \ldots, m$, together with trajectories connecting them:

$$[\xi_i \to \xi_{i+1}] \subset W^u(\xi_i) \cap W^s(\xi_{i+1}) \neq \emptyset.$$

We set $\xi_{m+1} = \xi_1$ and write X to represent the heteroclinic cycle, i.e. the union of equilibria and connections. It is well-known that if the connections $[\xi_i \to \xi_{i+1}]$ are of saddle-sink type in a fixed-point subspace, then the cycle persists under perturbations respecting the Γ-equivariance and is called *robust*.

In the simplest case these fixed-point spaces are two-dimensional. We slightly adapt the definition of [3, p. 1181]: let $\Sigma_j \subset \Gamma$ be an isotropy subgroup and $P_j = \mathrm{Fix}(\Sigma_j)$. Assume that for all $j = 1, \ldots, m$ the connection $[\xi_j \to \xi_{j+1}]$ is a saddle-sink connection in P_j. Write $L_j = P_{j-1} \cap P_j$. A robust heteroclinic cycle $X \subset \mathbb{R}^4 \backslash \{0\}$ is called *simple* if

(i) $\dim(P_j) = 2$ for each j,

(ii) X intersects each connected component of $L_j \backslash \{0\}$ in at most one point,

(iii) the linearization $df(\xi_j)$ has no double eigenvalues.

It is these cycles that we focus our attention on in section 4. Note that condition (iii) was not part of the definition in [3], but seems to have been silently assumed in most of the literature. This was noticed by Podvigina and Chossat [6] who subsequently introduced the term *pseudo-simple* for cycles fulfilling only (i) and (ii).

Chossat *et al.* [7] classify simple cycles in $\mathbb{R}^4$ into types A, B and C and study bifurcations for each type. The same partitioning is also used in the context of asymptotic stability by Krupa and Melbourne [3] as well as Podvigina and Ashwin [5]. We reproduce this classification here from [3].

Definition 2.1 ([3], definition 3.2)**.** Let $X \subset \mathbb{R}^4$ be a simple robust heteroclinic cycle.

(i) X is of *type* A if $\Sigma_j \cong \mathbb{Z}_2$ for all j.

(ii) X is of *type* B if there is a three-dimensional fixed-point subspace Q with $X \subset Q$.

(iii) X is of *type* C if it is not of type A or B.

All cycles of types B and C in $\mathbb{R}^4$ are enumerated in [3]. We recall this result in the next lemma, employing the usual notation $B_m^{\pm}$ and $C_m^{\pm}$, where m indicates the number of equilibria in the cycle and the superscript $\pm$ denotes whether $-\mathbb{1} \in \Gamma\,(-)$ or $-\mathbb{1} \notin \Gamma\,(+)$. For example, a B_3^--cycle has three equilibria and $-\mathbb{1} \in \Gamma$, while a B_2^+-cycle consists of two equilibria and $-\mathbb{1} \notin \Gamma$.

Lemma 2.2 ([3])**.** *There are seven distinct simple heteroclinic cycles of type B and C in $\mathbb{R}^4$ and the only finite groups $\Gamma \subset O(n)$ that allow them are the ones denoted in parentheses:*

- $B_1^+(\mathbb{Z}_2 \ltimes \mathbb{Z}_2^3)$, $B_2^+(\mathbb{Z}_2^3)$, $B_1^-(\mathbb{Z}_3 \ltimes \mathbb{Z}_2^4)$, $B_3^-(\mathbb{Z}_2^4)$
- $C_1^-(\mathbb{Z}_4 \ltimes \mathbb{Z}_2^4)$, $C_2^-(\mathbb{Z}_2 \ltimes \mathbb{Z}_2^4)$, $C_4^-(\mathbb{Z}_2^4)$

Proof. See [3], section 3 (b). □

Krupa and Melbourne [1, 3] derive criteria for asymptotic stability of cycles in $\mathbb{R}^n$ (with a suitable generalisation of types A, B and C) depending on the eigenvalues of the vector field at each equilibrium. In a *heteroclinic network* (a connected union of more than one cycle), none of the individual cycles is asymptotically stable due to the presence of a connection with at least one other cycle. To deal with this, intermediate notions of stability have been introduced by Melbourne [4] and Brannath [8] – note that they use *essential asymptotic stability* differently, Brannath's definition being a corrected version of Melbourne's. The literal definition by Melbourne [4] is taken up by Podvigina and Ashwin [5], they rename that of Brannath [8] *predominant asymptotic stability (p.a.s.).* We use Podvigina and Ashwin [5] as a reference for our results and therefore use *p.a.s.*

In the rest of this work, let $B_\varepsilon(X)$ be an ε-neighbourhood of a set $X \subset \mathbb{R}^n$. We write $\mathcal{B}(X)$ for the basin of attraction of X, i.e. the set of points $x \in \mathbb{R}^n$ with $\omega(x) \subset X$. For $\delta > 0$ the δ-local basin of attraction is

$$\mathcal{B}_\delta(X) := \{x \in \mathcal{B}(X) \mid \phi_t(x) \in B_\delta(X)\, \forall t > 0\},$$

where $\phi_t(.)$ is the flow generated by the system of equations. By $\ell(.)$ we denote Lebesgue measure, using a subscript to indicate the respective dimension where necessary.

The following is the strongest intermediate notion of stability.

Definition 2.3 ([5], definition 4)**.** A compact invariant set X is called *predominantly asymptotically stable (p.a.s.)* if it is asymptotically stable relative to a set $N \subset \mathbb{R}^n$ with the property that

$$\lim_{\varepsilon \to 0} \frac{\ell(B_\varepsilon(X) \cap N)}{\ell(B_\varepsilon(X))} = 1. \tag{1}$$

Note that in light of the abbreviation *p.a.s.* we sometimes write *a.s.* for *asymptotically stable.* We now define a similar term for instability, adapting a definition from Krupa and Melbourne [2].

Definition 2.4 (adapted from [2], definition 1.2)**.** A compact invariant set X is called *completely unstable* if there is a neighbourhood of X such that all points in it leave U in finite positive time.

We say that X is *predominantly unstable*[1] *(p.u.)* if it is completely unstable relative to a set $N \subset \mathbb{R}^n$ with property (1).

Finally, we call a set that is neither p.a.s. nor p.u. *properly fragmentarily asymptotically stable (p.f.a.s.).*

The last term is in reference to Podvigina's *fragmentary asymptotic stability* in [9] which is used for sets that attract anything of positive measure. In the next section, this terminology allows us to translate statements about stability of an entire cycle into statements about the stability indices along its connections and vice versa.

Podvigina and Ashwin also introduced the following stability index to quantify the attractiveness of a compact, invariant set X, section 2.3 in [5].

Definition 2.5 ([5], definition 5)**.** For $x \in X$ and $\varepsilon, \delta > 0$ define

$$\Sigma_\varepsilon(x) := \frac{\ell(B_\varepsilon(x) \cap \mathcal{B}(X))}{\ell(B_\varepsilon(x))}, \qquad \Sigma_{\varepsilon,\delta}(x) := \frac{\ell(B_\varepsilon(x) \cap \mathcal{B}_\delta(X))}{\ell(B_\varepsilon(x))}.$$

Then the *stability index* at x (with respect to X) is set to be

$$\sigma(x) := \sigma_+(x) - \sigma_-(x),$$

where

$$\sigma_-(x) := \lim_{\varepsilon \to 0} \left[\frac{\ln(\Sigma_\varepsilon(x))}{\ln(\varepsilon)} \right], \qquad \sigma_+(x) := \lim_{\varepsilon \to 0} \left[\frac{\ln(1 - \Sigma_\varepsilon(x))}{\ln(\varepsilon)} \right].$$

The convention that $\sigma_-(x) = \infty$ if $\Sigma_\varepsilon(x) = 0$ for some $\varepsilon > 0$ and $\sigma_+(x) = \infty$ if $\Sigma_\varepsilon(x) = 1$ is introduced. Therefore, $\sigma(x) \in [-\infty, \infty]$. In the same way the *local stability index* at $x \in X$ is defined to be

$$\sigma_{\text{loc}}(x) := \sigma_{\text{loc},+}(x) - \sigma_{\text{loc},-}(x),$$

with

$$\sigma_{\text{loc},-}(x) := \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \left[\frac{\ln(\Sigma_{\varepsilon,\delta}(x))}{\ln(\varepsilon)} \right], \sigma_{\text{loc},+}(x) := \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \left[\frac{\ln(1 - \Sigma_{\varepsilon,\delta}(x))}{\ln(\varepsilon)} \right].$$

For $X \subset \mathbb{R}^n$ and $x \in X$ the index $\sigma(x)$ quantifies attraction to X near x, while the local index $\sigma_{\text{loc}}(x)$ does the same for stability: if $\sigma_{(\text{loc})}(x) > 0$, then

[1] corresponding to *almost completely unstable* in [2], with the same drawback as e.a.s.

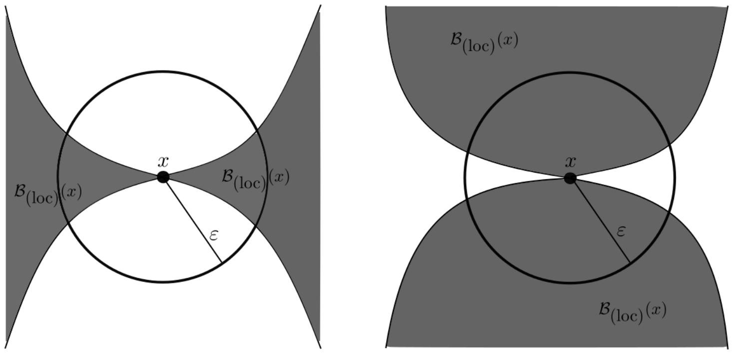

Figure 1: $\sigma_{(\mathrm{loc})}(x) < 0$ left and $\sigma_{(\mathrm{loc})}(x) > 0$ right.

in a small neighbourhood of x an increasingly large portion of points is in the (local) basin of attraction $\mathcal{B}_{(\delta)}(X)$ (and therefore attracted to X), see the schematic illustration in figure 1 (right). If on the other hand $\sigma_{(\mathrm{loc})}(x) < 0$, then the portion of such points goes to zero as the neighbourhood shrinks, also shown in figure 1 (left).

The fact that both $\sigma(x)$ and $\sigma_{\mathrm{loc}}(x)$ are constant along trajectories (theorem 2.2 in [5]) allows us to characterise attraction properties of heteroclinic cycles and networks in terms of the stability index by calculating only a finite number of indices. Moreover, Podvigina and Ashwin also show that the calculation of the indices can be simplified by restricting to a transverse section (theorem 2.4 in [5]). We use this in theorem 3.1, the proof of which is found in appendix A.

In the generic cases considered in [5] local and non-local indices are equal, which is why we drop the subscript *loc*, when it does not make a difference.

3 Stability index and attraction properties

Stability indices for heteroclinic cycles can be calculated by iterating return maps around the cycle, composed of local and global maps in the standard way, see e.g. [3]. Examples of such calculations can be found in [10]. Computations by Podvigina and Ashwin [5] show that for simple cycles in $\mathbb{R}^4$, generically, the basin of attraction is bounded by an exponential curve, which means that $\sigma(x) \neq 0$ along connecting trajectories. In degenerate cases, it is possible for a cycle to be p.a.s. even though there is a connection where $\sigma(x) = 0$. However, this requires a highly unusual geometry of the basin of attraction, see example 1.41 and its subsequent remarks in [11].

We now state our main result which relates predominant (in)stability to the sign of local stability indices.

Theorem 3.1. *Let $X \subset \mathbb{R}^n$ be a heteroclinic cycle or network with finitely many equilibria and connecting trajectories. Suppose that $\ell_1(X) < \infty$ and that the local stability index $\sigma_{\text{loc}}(x)$ exists and is not equal to zero for all $x \in X$. Then we have*

$$(i)\ X \text{ is p.a.s.} \quad \Leftrightarrow \quad \sigma_{\text{loc}}(x) > 0 \text{ along all connecting trajectories.}$$

If, in addition, X is an isolated invariant set, then we also have

$$(ii)\ X \text{ is p.u.} \quad \Leftrightarrow \quad \sigma_{\text{loc}}(x) < 0 \text{ along all connecting trajectories.}$$

Proof. See appendix A. □

Without X being isolated, negative local stability indices do not imply predominant instability: imagine a cycle where along all connections the local basin of attraction is the thin side of a cusp-shaped region, yielding a negative local index. Let all other points be periodic orbits in the full system, i.e. stationary points of the return map. Then X is not p.u. since there is no neighbourhood that all periodic orbits exit.

The need for this extra condition can be understood intuitively: the stability properties p.a.s. and p.u. both make statements about "many" trajectories in a neighbourhood of X – p.a.s. meaning most of them converge to X and p.u. meaning most of them leave the neighbourhood. The same is true for a positive stability index – it means that most for initial values the trajectories are eventually attracted to X. For negative stability indices, though, all we know is that "very few" trajectories converge to X. We do not know what happens to the rest, so without further assumptions we cannot expect all (or most) of them to leave a neighbourhood of X.

Multiple heteroclinic cycles combined in a network – as e.g. studied by Kirk and Silber in [12] – are not isolated, so we cannot apply (ii). However, this is not much of a problem for two reasons: first, when it comes to networks it is of greater interest to identify stable (p.a.s.) subcycles than unstable (p.u.) ones, since these are "visible" (in the sense that they attract almost everything close to them). So the p.a.s.-equivalence is the more important one. Second, as long as for each cycle the only other invariant set in its neighbourhood is another cycle in the same network, this restricts the behaviour of the remaining trajectories, so that the p.u.-equivalence holds, as well. In this case, stability indices with respect to single cycles or the entire network can be used to study relative stability in a network and competition between connected cycles, as done in [10] and [11].

Since $\sigma(x) \geq \sigma_{\text{loc}}(x)$ by construction, positive non-local stability indices also imply predominant asymptotic stability of the cycle. The reversed implication fails, though, as can be seen from standard examples of invariant sets that are stable but not attractive.

4 Stability of simple cycles in $\mathbb{R}^4$

In this section we investigate the stability configurations of simple cycles in $\mathbb{R}^4$. Concerning the explicit calculation of stability indices we rely on several results from Podvigina and Ashwin [5] – they are listed in appendix B for reference. We also follow the notation in [5] by using $r_j, c_j, e_j > 0$ and $t_j \gtrless 0$ for the (radial, contracting, expanding and transverse) eigenvalues of the linearization $df(\xi_j)$ and by introducing the quantities $a_j = c_j/e_j$ and $b_j = -t_j/e_j$ to define

$$\rho_j := \min(a_j,\ 1 + b_j) \quad \text{and} \quad \rho := \rho_1 \cdot \ldots \cdot \rho_m.$$

For simple cycles in $\mathbb{R}^4$ there is one eigenvalue of each type.

We denote the index along the trajectory leading to ξ_j by σ_j. Trajectories leaving a neighbourhood of the cycle are assumed to stay away from it for all positive times, so that $\sigma(x) = \sigma_{\text{loc}}(x)$. This allows us to use theorem 3.1 to deduce attraction properties of the cycles, i.e. necessary and sufficient criteria for their predominant (in)stability, from the sign of the indices.

Lemma 4.1. *Assume that for a simple heteroclinic cycle X in $\mathbb{R}^4$ all stability indices exist. Then generically the following equivalences hold:*

- $\sigma_i = +\infty$ *for all* $i \Leftrightarrow X$ *is asymptotically stable.*
- $\sigma_i = -\infty$ *for all* $i \Leftrightarrow X$ *is completely unstable.*
- $\sigma_i > 0$ *for all* $i \Leftrightarrow X$ *is predominantly asymptotically stable.*
- $\sigma_i < 0$ *for all* $i \Leftrightarrow X$ *is predominantly unstable.*

Proof. For the first two statements the implications from right to left are trivial. The other directions follow from results for the different types of cycles in subsections 4.2.1 and 4.2.2 of [5], most of which are listed in appendix B. The third and fourth statement are just theorem 3.1, keeping in mind that for simple cycles in $\mathbb{R}^4$ the basin of attraction generically is an algebraic cusp shaped by ratios of the respective eigenvalues. □

In the following subsections we investigate the different types of cycles one by one.

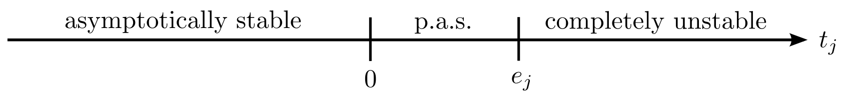

Figure 2: Stability change for type A cycles when t_j turns positive.

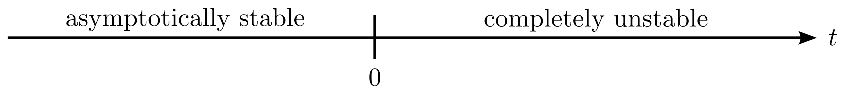

Figure 3: Stability change for homoclinic A- and B-cycles.

4.1 Stability of type A cycles

For type A cycles in $\mathbb{R}^4$ the situation is rather simple. The following is a consequence of our theorem 3.1 and the results from Podvigina and Ashwin [5]. It is for the most part only a reformulation of theorem 2.4 by Krupa and Melbourne [2]. In fact, their statement even holds for type A cycles in $\mathbb{R}^n$. Nonetheless, we state it for the sake of completeness and because in the special case of $\mathbb{R}^4$ we can make the stronger statement that a type A cycle cannot be predominantly unstable, unless it is completely unstable.

Theorem 4.2 ([2], theorem 2.4)**.** *In $\mathbb{R}^4$ a simple heteroclinic cycle of type A is generically*

(a) asymptotically stable if and only if $\rho > 1$ and $t_j < 0$ for all j,

(b) p.a.s. (but not a.s.) if and only if $\rho > 1$ and there is at least one positive transverse eigenvalue, but $t_j < e_j$ holds for all j,

(c) completely unstable if $\rho < 1$ or there is j with $t_j > e_j$.

Proof. This follows directly from combining theorem B.1 with lemma 4.1. □

So for an A-cycle in $\mathbb{R}^4$ either all stability indices are positive or they are all equal to $-\infty$. This change of stability as one of the transverse eigenvalues becomes positive (while $\rho > 1$) is schematically depicted in figure 2. Note that for a homoclinic cycle there is no p.a.s. region, because $\rho < 1$ if all transverse eigenvalues are positive, see figure 3. So the condition for asymptotic stability reduces to $c > e$ and $t < 0$ in the case of homoclinic cycles. The same holds for homoclinic B-cycles. For homoclinic C-cycles the condition changes to $t < \min(0, c - e)$, and $c > e$ is not necessary anymore. All this was shown in theorems 2.3 and 4.3 in [3] and section 4.2 in [5], which is why from now on we look only at non-homoclinic cycles.

4.2 Stability of type B cycles

Recall from lemma 2.2 that there are two non-homolinic simple cycles of type B. We begin with the B_2^+-cycles, consisting of two equilibria and two connecting trajectories.

Theorem 4.3. *In $\mathbb{R}^4$ a simple heteroclinic cycle of type B_2^+ is generically*

(a) asymptotically stable if and only if the following two conditions hold.

+ $c_1c_2 > e_1e_2$
+ $t_1, t_2 < 0$

(b) p.a.s. (but not a.s.) if and only if the following three conditions hold.

+ $c_1c_2 > e_1e_2$
+ $t_2 < 0$
+ $0 < t_1 < \min\left(e_1, -\dfrac{e_1t_2}{c_2}\right)$

(c) p.f.a.s. if and only if the following three conditions hold.

+ $c_1c_2 > e_1e_2$
+ $t_2 < 0$
+ $0 < e_1 < t_1 < -\dfrac{e_1t_2}{c_2}$

(d) completely unstable if and only if one of the following conditions holds.

∘ $c_1c_2 < e_1e_2$
∘ $t_1, t_2 > 0$
∘ $t_2 < 0 < t_1$ *and* $t_1 > -\dfrac{e_1t_2}{c_2}$.

Proof. See appendix C. □

This gives us a description of the stability changes that a B_2^+-cycle undergoes when one of its transverse eigenvalues becomes positive, see figure 4. In the two diagrams $t_2 < 0$ is fixed and we assume that $c_1c_2 > e_1e_2$. Note that for such a cycle to be p.f.a.s it is necessary that $-c_2 > t_2$, i.e. the flow at ξ_2 has to be more strongly contracting in the transverse direction than in the direction of the cycle. In this case $\min(e_1, -e_1t_2/c_2) = e_1$ and the p.a.s./p.f.a.s. question is decided by $t_1 \lesseqgtr e_1$, depending on which direction (transverse or expanding along the cycle) is more unstable.

Melbourne [4] studied a B_2^+-cycle and gave sufficient conditions for its predominant asymptotic stability. In theorem 4.3 we have generalised his results, giving necessary and sufficient conditions for all stability properties.

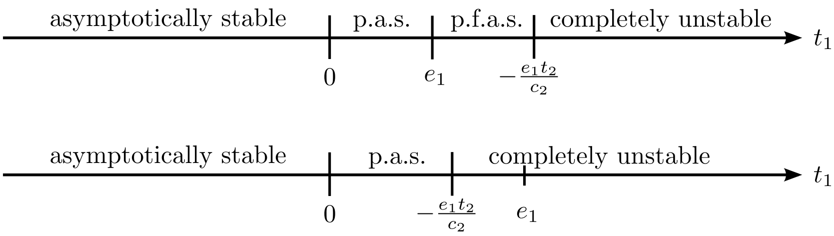

Figure 4: Stability change for B_2^+-cycles when t_1 turns positive ($t_2 < 0$).

We now do the same for B_3^--cycles. There are three transverse eigenvalues, resulting in a two-dimensional picture for stability changes, but the calculations are similar.

Theorem 4.4. *In $\mathbb{R}^4$ a simple heteroclinic cycle of type B_3^- is generically*

(a) asymptotically stable if and only if the following two conditions hold.

+ $c_1c_2c_3 > e_1e_2e_3$
+ $t_1, t_2, t_3 < 0$

(b) p.a.s. (but not a.s.) if and only if the following four conditions hold.

+ $c_1c_2c_3 > e_1e_2e_3$
+ $t_3 < 0$
+ $0 < t_1 < \min\left(e_1, \dfrac{e_1(e_3 - t_3)}{c_3}, \dfrac{e_1(-t_3c_2 - e_3t_2)}{c_2c_3}\right)$
+ $0 < t_2 < \min\left(e_2, \dfrac{e_2(e_1 - t_1)}{c_1}, \dfrac{e_2(-t_1c_3 - e_1t_3)}{c_1c_3}\right)$ *or* $t_2 < 0$

(c) completely unstable if and only if one of the following four conditions is satisfied.

∘ $c_1c_2c_3 < e_1e_2e_3$
∘ $t_1, t_2, t_3 > 0$
∘ $t_1 > 0 > t_3$ *and* $t_1 > -\dfrac{e_1(t_3c_2 + t_2e_3)}{c_2c_3}$
∘ $t_1, t_2 > 0 > t_3$ *and* $t_2 > -\dfrac{e_2(t_1c_3 + t_3e_1)}{c_1c_3}$.

(d) In all other cases the cycle is p.f.a.s. In particular, the cycle is never predominantly unstable.

Proof. See appendix C. □

In the same way as for the B_2^+-cycles we determine what happens when transverse eigenvalues become positive for type B_3^--cycles. Here we have not only t_1 but also t_2 as a varying parameter. We assume $c_1c_2c_3 > e_1e_2e_3$ and without loss of generality fix $t_3 < 0$. The diagram we obtain and the extent of the regions of different stability properties depend on the slopes of the boundary lines, and therefore on the eigenvalues at the equilibria. However, there are only two qualitatively different cases, depicted in figure 5.

Theorem 4.4 implies that the stability properties for a parameter combination (t_1, t_2) are determined by its position relative to the six lines given by:

$$t_1(t_2) = e_1 \tag{1}$$

$$t_1(t_2) = \frac{e_1(e_3 - t_3)}{c_3} \tag{2}$$

$$t_1(t_2) = \frac{e_1(-t_3c_2 - e_3t_2)}{c_2c_3} \tag{3}$$

$$t_2(t_1) = e_2 \tag{4}$$

$$t_2(t_1) = \frac{e_2(e_1 - t_1)}{c_1} \tag{5}$$

$$t_2(t_1) = \frac{e_2(-t_1c_3 - e_1t_3)}{c_1c_3} \tag{6}$$

The first three lines bound the region of predominant asymptotic stability where $t_1 > 0$, the other three where $t_2 > 0$. Lines (5) and (6) are parallel, they have the same slope $-e_2/c_1$. The intersections of all lines with the t_1- and t_2-axis are collected in table 1.

Table 1: Intersections with the t_i-axes.

	(1)	(2)	(3)	(4)	(5)	(6)
t_1-axis	e_1	$\frac{e_1(e_3-t_3)}{c_3}$	$-\frac{e_1t_3}{c_3}$	–	e_1	$-\frac{e_1t_3}{c_3}$
t_2-axis	–	–	$-\frac{c_2t_3}{e_3}$	e_2	$\frac{e_1e_2}{c_1}$	$-\frac{e_1e_2t_3}{c_1c_3}$

Since $e_i, c_i > 0$ and $c_1c_2c_3 > e_1e_2e_3$ we have

$$\frac{e_1(e_3 - t_3)}{c_3} > -\frac{e_1t_3}{c_3} \quad \text{and} \quad -\frac{e_1e_2t_3}{c_1c_3} < -\frac{c_2t_3}{e_3},$$

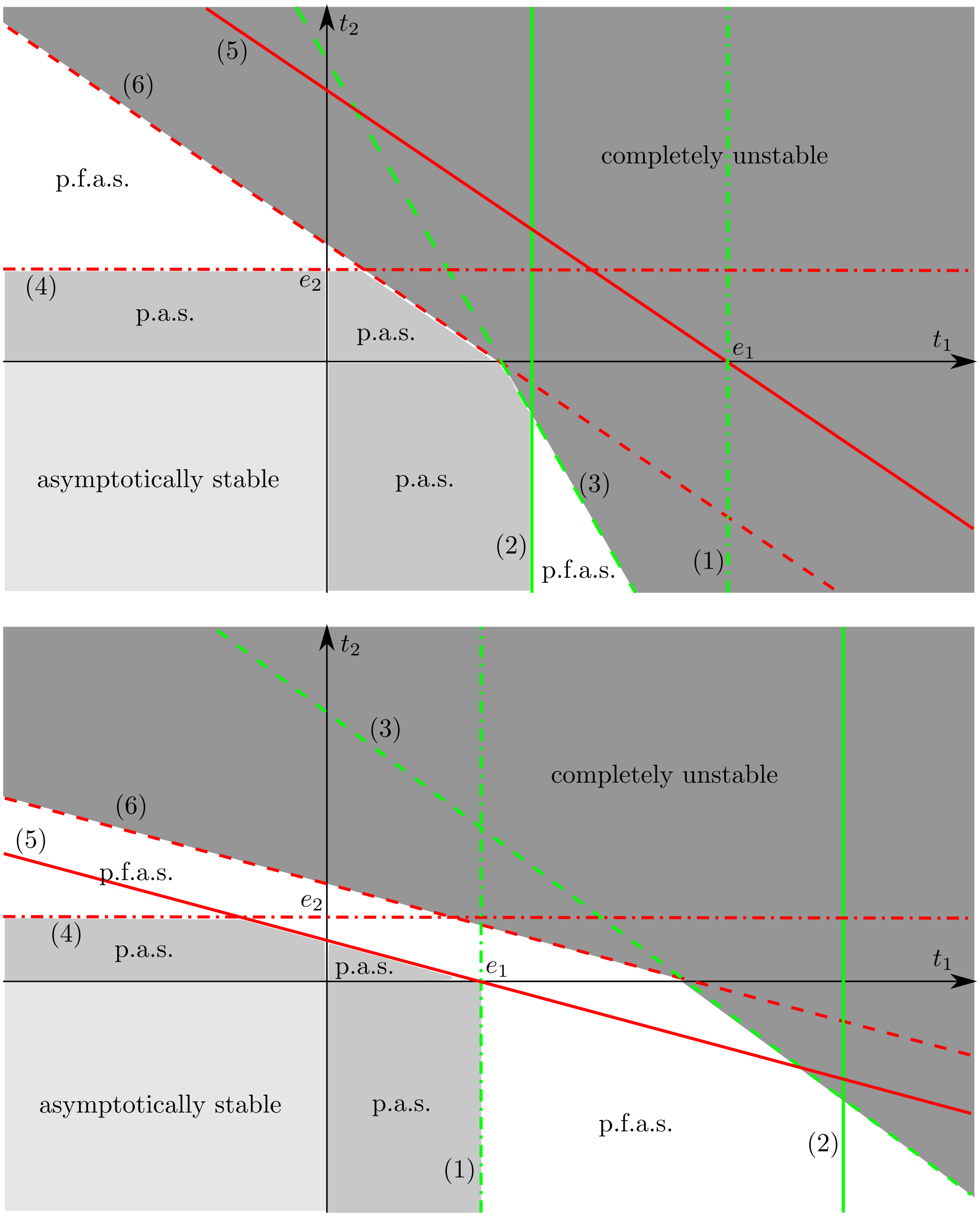

Figure 5: Stability of B_3^--cycles: $-c_3 < t_3$ (top), $-c_3 > t_3$ (bottom) .

restricting the number of possibilities for ordering the intersection points along the axes. Now consider two cases, depending on whether the contracting or transverse direction at ξ_3 dominates:

(a) $-c_3 < t_3$. Then $\frac{e_1 e_2}{c_1} > -\frac{e_1 e_2 t_3}{c_1 c_3}$ and $e_1 > -\frac{e_1 t_3}{c_3}$.

(b) $-c_3 > t_3$. Then $\frac{e_1 e_2}{c_1} < -\frac{e_1 e_2 t_3}{c_1 c_3}$ and $e_1 < -\frac{e_1 t_3}{c_3}$.

For both cases the (t_1, t_2)-plane is shown in figure 5. From (a) it is clear that the inequalities

$$e_1 \gtrless \frac{e_1(e_3 - t_3)}{c_3} \quad \text{and} \quad \frac{e_1 e_2}{c_1} \gtrless -\frac{c_2 t_3}{e_3}$$

as well as the relative position of the line $t_2(t_1) = e_2 > 0$ do not qualitatively affect the dynamics. In (b) all relative positions are fixed except for that of $t_2(t_1) = e_2 > 0$, which is qualitatively irrelevant in this case, too.

From figure 5 we deduce the following result.

Corollary 4.5. *Let X be a simple heteroclinic cycle of type B_3^- in $\mathbb{R}^4$. Suppose $c_1 c_2 c_3 > e_1 e_2 e_3$ and $t_3 < 0$. In the (t_1, t_2)-plane consider paths leading from the region of asymptotic stability to that of complete instability.*

- *If $-c_3 < t_3$, then no such path that is sufficiently close to the origin leads through an open region where X is p.f.a.s.*
- *If $-c_3 > t_3$, then every such path leads through an open region where X is p.f.a.s.*

In other words, in the second case the p.f.a.s.-region in the (t_1, t_2)-plane is connected while in the first it is not. This corresponds to what we learned about cycles of type B_2^+ above: such a cycle can only be p.f.a.s if $-c_2 > t_2$. For B_3^--cycles, along a path that is sufficiently close to the origin, proper fragmentary asymptotic stability only occurs if $-c_3 > t_3$.

In terms of stability indices corollary 4.5 means that in the first case, along a path close to the origin in the (t_1, t_2)-plane, all indices along the cycle have the same sign. In particular, the cycle goes from all indices equal to $+\infty$, to all indices positive (but some finite) directly to all indices equal to $-\infty$. This behaviour is very similar to type A cycles. In the second case, there is a region where there are indices with opposite signs. From these considerations we deduce the following two statements about all type B cycles in $\mathbb{R}^4$.

Corollary 4.6. *A simple heteroclinic cycle of type B in $\mathbb{R}^4$ is generically p.a.s. after a transverse bifurcation.*

Corollary 4.7. *In $\mathbb{R}^4$ a simple heteroclinic cycle of type B is never predominantly unstable.*

4.3 Stability of type C cycles

For C_2^--cycles, the corresponding transition matrix product – a useful way of writing the return maps, see subsection 4.2.2 in [5] for an explanation – is given by

$$M = \begin{pmatrix} b_1b_2 + a_2 & b_1 \\ a_1b_2 & a_1 \end{pmatrix}.$$

We denote its eigenvalues by λ_1, λ_2 such that $\lambda_1 \geq \lambda_2$ if they are real.

Theorem 4.8. *In $\mathbb{R}^4$ a simple heteroclinic cycle of type C_2^- is generically*

(a) asymptotically stable if and only if the following two conditions hold.

+ $t_1, t_2 < 0$
+ $\max(\mathrm{tr}M,\ 2(\mathrm{tr}M - \det M)) > 2$

(b) completely unstable if and only if one of the following holds.

- $t_1, t_2 > 0$
- $t_1, t_2 < 0$ *and* $\max(\mathrm{tr}M,\ 2(\mathrm{tr}M - \det M)) < 2$
- $t_2 < 0 < t_1$ *and one of the following three:*
  - $(\mathrm{tr}M)^2 - 4\det M < 0$
  - $\max(\mathrm{tr}M,\ 2(\mathrm{tr}M - \det M)) < 2$
  - $b_1b_2 - a_1 + a_2 < 0$

(c) p.a.s. (but not a.s.) if and only if the following three conditions hold.

+ $t_2 < 0 < t_1$
+ *none of the conditions in (b) hold*
+ $\lambda_2 \in (b_1b_2 + a_1,\ b_1b_2 + \min(a_1 + b_2,\ a_2 + b_1))$

(d) p.u. if and only if the following three conditions hold.

+ $t_2 < 0 < t_1$
+ *none of the conditions in (b) hold*
+ $\lambda_2 \in (b_1b_2 + \max(a_1 + b_2,\ a_2 + b_1),\ b_1b_2 + a_2)$

(e) In all other cases the cycle is p.f.a.s.

Proof. See appendix C. □

Case (a) was already covered by Krupa and Melbourne [3]. It follows from their theorem 4.3 on asymptotic stability for type C cycles in $\mathbb{R}^4$. Our theorem 4.8 extends their result to the other stability properties.

The conditions for λ_2 in (c) and (d) in theorem 4.8 cannot be visualised as easily as for the cycles of type B, since $\lambda_2 = \lambda_2(t_1)$ and the boundaries of the intervals also depend on t_1. However, we give a different graphic interpretation in figure 6: just like for the B_2^+-cycle, we vary only one transverse eigenvalue, t_1, yet the diagram is two-dimensional, as in the case of the B_3^--cycle. The second axis is necessary since the stability changes depend on the position of $\lambda_2(t_1)$ relative to four linear functions of t_1. These are the boundaries of the intervals for λ_2 and they are given by:

$$\begin{aligned}
L_1(t_1) &= b_1 b_2 + a_1 \\
L_2(t_1) &= b_1 b_2 + a_1 + b_2 \\
L_3(t_1) &= b_1 b_2 + a_2 + b_1 \\
L_4(t_1) &= b_1 b_2 + a_2
\end{aligned}$$

We have $L_1(0) = a_1 < a_1 + b_2 = L_2(0)$ and $L_3(0) = L_4(0) = a_2$. Since $b_1 b_2 - a_1 + a_2 > 0$ implies $a_2 > a_1 - b_1 b_2 > a_1$, we are left with two cases for the relative positions of the L_i:

(i) $a_1 < a_2 < a_1 + b_2$
(ii) $a_1 < a_1 + b_2 < a_2$

Note that L_1, L_2, L_4 all have the same slope $t_2/e_1 e_2 < 0$, while that of L_3 is smaller since $b_1 < 0$. In terms of the L_i, the λ_2-conditions for predominant (in)stability can now be reformulated as

$$\begin{aligned}
\text{p.a.s.} \quad &\Leftrightarrow \quad \lambda_2 \in (L_1(t_1),\ \min(L_2(t_1), L_3(t_1))) \\
\text{p.u.} \quad &\Leftrightarrow \quad \lambda_2 \in (\max(L_2(t_1), L_3(t_1)),\ L_4(t_1)).
\end{aligned}$$

We have

$$\lambda_2(t_1) = \frac{b_1 b_2 + a_1 + a_2}{2} - \sqrt{\left(\frac{b_1 b_2 + a_1 + a_2}{2}\right)^2 - a_1 a_2},$$

which gives $\lambda_2(0) = a_1 > 0$. Note that if $\lambda_2(t_1) > L_4(t_1)$, then $b_1 b_2 - a_1 + a_2 < 0$, which according to case (b) implies complete instability. We have $\lambda_2(t_1) \in \mathbb{R}$ as long as $(\mathrm{tr} M)^2 - 4 \det M > 0$. This expression depends quadratically on t_1, its zeros are given through

$$t_1 = \frac{e_1}{b_2} \left(\sqrt{a_2} - \sqrt{a_1}\right)^2 > 0 \quad \text{and} \quad t_1 = \frac{e_1}{b_2} \left(\sqrt{a_2} + \sqrt{a_1}\right)^2 > 0.$$

For small $t_1 > 0$ the eigenvalue is therefore real. Moreover, $\lambda_2(t_1)$ increases monotonically as long as $t_1 \in \left(0, e_1/b_2 \left(\sqrt{a_2} - \sqrt{a_1}\right)^2\right)$. This completes the

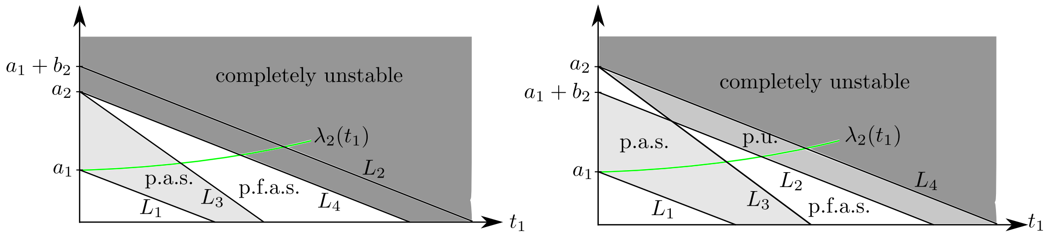

Figure 6: Stability of C_2^--cycles: $a_2 < a_1 + b_2$ (left), $a_2 > a_1 + b_2$ (right).

derivation of figure 6. Note that we have neglected the other conditions in (b) that lead to complete instability. Each of them constitutes an upper bound on t_1, above which the cycle is completely unstable. We assume all of these bounds to be sufficiently large so that they do not influence the picture. In case they are smaller, the dynamics are simplified in the sense that figure 6 is "cut off" at the respective value and the cycle is completely unstable for larger t_1.

From these considerations we conclude the following result.

Corollary 4.9. *A simple heteroclinic cycle of type C_2^- in $\mathbb{R}^4$ is generically predominantly asymptotically stable after a transverse eigenvalue becomes positive. As the eigenvalue becomes larger, generically there exists an open interval where the cycle is p.f.a.s.*

Proof. The first statement is clear from figure 6. Concerning the second one: the only possibility for $\lambda_2(t_1)$ not to enter the region of proper fragmentary asymptotic stability is when it passes through the intersection point of $L_2(t_1)$ and $L_3(t_1)$. But that is a degenerate configuration. $\square$

In contrast to type B cycles predominant instability is also possible for certain configurations of eigenvalues. In case $a_2 > a_1 + b_2$ it is a generic state along each path from asymptotic stability to complete instability.

For the remaining simple heteroclinic cycle, type C_4^-, it is in principle possible to do the same and reformulate the classification in [5] as necessary and sufficient conditions for the different stability properties. However, for a C_4^--cycle there are four transverse eigenvalues, meaning we have to consider three of them becoming positive, making it impossible to illustrate the results graphically in the same way as before. This graphical interpretation was our main achievement for the other cycles – without it, simply stating the algebraic conditions would add little insight to the results already given in [5]. There is, however, a useful conclusion that can be reached for the case where one transverse eigenvalue becomes positive.

Theorem 4.10. *Let X be a simple heteroclinic cycle of type C_4^- in $\mathbb{R}^4$. Suppose that $b_1 < 0 < b_j$ for $j \neq 1$, and assume that none of the conditions (a), (b) and (c) in lemma B.5 are satisfied. Then there is $\varepsilon_0 > 0$ such that for $0 < t_1 < \varepsilon_0$ the cycle is p.a.s.*

Proof. See appendix C. □

Together with the detailed study of the C_2^--cycle above, this allows us to close with the following conclusion about (non-homoclinic) type C cycles.

Corollary 4.11. *A simple heteroclinic cycle of type C in $\mathbb{R}^4$ is generically p.a.s. after a transverse bifurcation.*

In this paper, we have given a detailed picture of what happens when simple heteroclinic cycles in $\mathbb{R}^4$ lose their stability through a transverse eigenvalue that becomes positive. Our results can be (in fact, have already been, see [10]) used to investigate competition of heteroclinic cyles within a network, where no individual cycle can be asymptotically stable and there is a delicate interplay of non-asymptotic attraction properties of single cycles and the entire network. We have not addressed the "real bifurcation question", i.e. the emergence of new branches of solutions after the cycle has lost (some of) its stability. While partial results have already been obtained in this area by others, e.g. for homoclinic cycles by Chossat et al. [7], this shall be the subject of future work.

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A Proof of theorem 3.1

We prove theorem 3.1, through a combination of two lemmas.

Lemma A.1. *Let $X \subset \mathbb{R}^n$ be a heteroclinic cycle or network consisting of finitely many equilibria $\xi_1, \dots, \xi_m$ and connecting trajectories and suppose that $\ell_1(X) < \infty$. Assume that the local stability index $\sigma_{\text{loc}}(x)$ exists and is unequal to zero for all $x \in X$. Then the following holds.*

(a) X is p.a.s. $\Rightarrow \ell_1(\{x \in X \mid \sigma_{\text{loc}}(x) < 0\}) = 0$

(b) X is p.u. $\Rightarrow \ell_1(\{x \in X \mid \sigma_{\text{loc}}(x) > 0\}) = 0$

Proof. We begin with the first statement. Let X be p.a.s., in particular asymptotically stable relative to some $N \subset \mathbb{R}^n$. Assume that the implication is not true, so that $\ell_1(\hat{X}_a) > 0$, where

$$\hat{X}_a := \left\{ x \in X \setminus \bigcup_{j=1}^{m} \{\xi_j\} \;\middle|\; \sigma_{\mathrm{loc}}(x) < 0 \right\}.$$

Note that $\hat{X}_a$ is flow-invariant since the index is constant on trajectories. For $x \in \hat{X}_a$ we have $\sigma_{\mathrm{loc},-}(x) > 0$ and therefore $\Sigma_{\varepsilon,\delta}(x) \to 0$ when $\varepsilon, \delta \to 0$. By theorem 2.4 in [5] stability indices can be calculated relative to a codimension one surface S_x, transverse to the flow at x, so the same is true for

$$\Sigma_{\varepsilon,\delta,S_x}(x) = \frac{\ell_{n-1}(B_\varepsilon(x) \cap \mathcal{B}_\delta(X) \cap S_x)}{\ell_{n-1}(B_\varepsilon(x) \cap S_x)}.$$

Thus, for all $x \in \hat{X}_a$ there exists $\gamma(x) > 0$ such that for $\delta, \varepsilon < \gamma(x)$ we have $\Sigma_{\varepsilon,\delta,S_x}(x) < \frac{1}{2}$. Since

$$\begin{aligned} 0 < \ell_1(\hat{X}_a) &= \ell_1\left(\bigcup_{n \in \mathbb{N}} \left\{ x \in \hat{X}_a \;\middle|\; \gamma(x) \geq \frac{1}{n} \right\} \right) \\ &\leq \sum_{n \in \mathbb{N}} \ell_1\left(\left\{ x \in \hat{X}_a \;\middle|\; \gamma(x) \geq \frac{1}{n} \right\} \right), \end{aligned}$$

the measure of the sets in the sum cannot be zero for all $n \in \mathbb{N}$. So there is a set $Y_a \subset \hat{X}_a$ with $\ell_1(Y_a) > 0$, where the bound is uniform, i.e. there is $\gamma > 0$ such that

$$\forall y \in Y_a \quad \forall \varepsilon, \delta < \gamma \quad \Sigma_{\varepsilon,\delta,S_y}(y) < \frac{1}{2}.$$

Without loss of generality we assume that the transverse sections S_y are disjoint and of uniform size for all $y \in Y_a$, by excluding small neighbourhoods of the equilibria if necessary, without losing the property $\ell_1(Y_a) > 0$. We write $\ell_1(Y_a) = \alpha \ell_1(X)$ with $\alpha \in (0, 1]$ and look at

$$W_\varepsilon(Y_a) := \bigcup_{y \in Y_a} (B_\varepsilon(y) \cap S_y) \subset B_\varepsilon(Y_a)$$

to find

$$\frac{\ell(W_\varepsilon(Y_a))}{\ell(B_\varepsilon(X))} = \frac{\ell_{n-1}(B_\varepsilon)\ell_1(Y_a)}{\ell_{n-1}(B_\varepsilon)\ell_1(X) + O(\varepsilon^n)} \xrightarrow{\varepsilon \to 0} \frac{\ell_1(Y_a)}{\ell_1(X)} = \alpha,$$

because the volume of an $(n-1)$-dimensional ε-ball, $\ell_{n-1}(B_\varepsilon)$, is of order ε^{n-1}. Now for $\varepsilon, \delta < \gamma$ and small enough, Fubini's theorem gives

$$\begin{aligned}
\ell(W_\varepsilon(Y_a) \cap \mathcal{B}_\delta(X)) &= \int\limits_{W_\varepsilon(Y_a)} \chi_{\mathcal{B}_\delta(X)} \, \mathrm{d}\ell_n \\
&= \int\limits_{Y_a} \ell_{n-1}(B_\varepsilon(y) \cap S_y \cap \mathcal{B}_\delta(X)) \, \mathrm{d}\ell_1 \\
&< \frac{1}{2} \int\limits_{Y_a} \ell_{n-1}(B_\varepsilon(y) \cap S_y) \, \mathrm{d}\ell_1 \\
&= \frac{1}{2} \ell(W_\varepsilon(Y_a)).
\end{aligned}$$

Since X is asymptotically stable relative to N, for $\delta < \gamma$ we find $\varepsilon < \gamma$ such that $\ell(B_\varepsilon(X) \cap N) \leq \ell(B_\varepsilon(X) \cap \mathcal{B}_\delta(X))$. Then by the above

$$\begin{aligned}
\frac{\ell(B_\varepsilon(X) \cap \mathcal{B}_\delta(X))}{\ell(B_\varepsilon(X))} &= \frac{\ell(W_\varepsilon(Y_a) \cap \mathcal{B}_\delta(X))}{\ell(B_\varepsilon(X))} + \frac{\ell(B_\varepsilon(X) \setminus W_\varepsilon(Y_a) \cap \mathcal{B}_\delta(X))}{\ell(B_\varepsilon(X))} \\
&< \frac{1}{2} \frac{\ell(W_\varepsilon(Y_a))}{\ell(B_\varepsilon(X))} + \frac{\ell(B_\varepsilon(X) \setminus W_\varepsilon(Y_a))}{\ell(B_\varepsilon(X))} \\
&= 1 - \frac{1}{2} \frac{\ell(W_\varepsilon(Y_a))}{\ell(B_\varepsilon(X))}.
\end{aligned}$$

Since X is p.a.s., taking the limit $\varepsilon \to 0$ now gives

$$1 = \lim_{\varepsilon \to 0} \frac{\ell(B_\varepsilon(X) \cap N)}{\ell(B_\varepsilon(X))} \leq \lim_{\varepsilon \to 0} \frac{\ell(B_\varepsilon(X) \cap \mathcal{B}_\delta(X))}{\ell(B_\varepsilon(X))} \leq 1 - \frac{\alpha}{2}.$$

This is a contradiction, so $\ell_1(\hat{X}_a) = 0$ as claimed.

Now let X be p.u., in particular let it be completely unstable relative to $M \subset \mathbb{R}^n$. Denote by U a neighbourhood such that all points in $U \cap M \setminus X$ leave U in finite positive time. In the same way as above, assume that $\ell_1(\hat{X}_b) > 0$, where

$$\hat{X}_b := \left\{ x \in X \setminus \bigcup_{j=1}^{m} \{\xi_j\} \mid \sigma_{\mathrm{loc}}(x) > 0 \right\}.$$

We obtain a contradiction to this assumption in a similar way as before, so we just point out the steps where the reasoning is different. As before we find a set $Y_b \subset \hat{X}_b$ with $\ell_1(Y_b) = \beta \ell_1(X) > 0$ and $\gamma > 0$ such that

$$\forall y \in Y_b \quad \forall \varepsilon, \delta < \gamma \qquad \Sigma_{\varepsilon,\delta,S_y}(y) > \frac{1}{2}.$$

Again with Fubini's theorem we obtain $\ell(W_\varepsilon(Y_b) \cap \mathcal{B}_\delta(X)) > \frac{1}{2}\ell(W_\varepsilon(Y_b))$, for $\varepsilon, \delta > 0$ small enough and $W_\varepsilon(Y_b)$ as above. Therefore

$$\ell(W_\varepsilon(Y_b) \cap (\mathcal{B}_\delta(X))^c) < \frac{1}{2}\ell(W_\varepsilon(Y_b)). \tag{2}$$

Since all points in $U \cap M \setminus X$ leave U, for ε, δ small enough we have

$$B_\varepsilon(X) \cap M \setminus X \subset B_\varepsilon(X) \cap (\mathcal{B}_\delta(X))^c.$$

In the same way as above this leads to a contradiction

$$1 = \lim_{\varepsilon \to 0} \frac{\ell(B_\varepsilon(X) \cap M)}{\ell(B_\varepsilon(X))} \leq \lim_{\varepsilon \to 0} \frac{\ell(B_\varepsilon(x) \cap (\mathcal{B}_\delta(X))^c)}{\ell(B_\varepsilon(X))} \leq 1 - \frac{\beta}{2},$$

completing the proof also for (b). $\square$

We now take a look at the converse of this result.

Lemma A.2. *Under the same assumptions as in the previous lemma the following holds.*

(a) $[\sigma_{\mathrm{loc}}(x) > 0$ *along all connections*$] \Rightarrow X$ *is p.a.s.*

(b) If, in addition, X is an isolated invariant set, then we also have $[\sigma_{\mathrm{loc}}(x) < 0$ *along all connections*$] \Rightarrow X$ *is p.u.*

Proof. We start with (a). For all $x \in \hat{X} := X \setminus \bigcup_{j \in \mathbb{N}} \{\xi_j\}$ we have $\sigma_{\mathrm{loc}}(x) > 0$, so it follows that $\lim\limits_{\delta \to 0} \lim\limits_{\varepsilon \to 0} \Sigma_{\varepsilon,\delta}(x) = 1$. Again the same is then true for $\Sigma_{\varepsilon,\delta,S_x}(x)$ with S_x as above. So for all $\rho_1 > 0$ and all $x \in \hat{X}$ there is $\varepsilon(x) > 0$ such that $\Sigma_{\varepsilon,\delta,S_x}(x) > 1 - \rho_1$ for $\varepsilon, \delta < \varepsilon(x)$. So for all $x \in \hat{X}$ we have

$$\forall \varepsilon < \varepsilon(x) \quad \ell_{n-1}(\mathcal{B}_\varepsilon(X) \cap B_\varepsilon(x) \cap S_x) > (1 - \rho_1)\ell_{n-1}(B_\varepsilon(x) \cap S_x).$$

We find a uniform lower bound for $\varepsilon(x)$ in the same way as above. Since

$$\hat{X} = \bigcup_{n \in \mathbb{N}} \left\{ x \in \hat{X} \;\middle|\; \varepsilon(x) \geq \frac{1}{n} \right\},$$

for any given $\rho_2 > 0$ we find $n \in \mathbb{N}$ and $Y \subset \hat{X}$ with

$$\ell_1(Y) > (1 - \rho_2)\ell_1(X) \quad \text{and} \quad \forall y \in Y \quad \varepsilon(y) \geq \frac{1}{n}.$$

Thus, we have $\ell(W_\varepsilon(Y)) > (1-\rho_2)\ell(B_\varepsilon(X))$ for $W_\varepsilon(Y)$ as in the previous lemma and ε small enough. Again by Fubini's theorem we obtain

$$\begin{aligned}
\ell(\mathcal{B}_\varepsilon(X)) &\geq \int\limits_{W_\varepsilon(Y)} \chi_{\mathcal{B}_\varepsilon(X)}\,\mathrm{d}\ell_n \\
&= \int\limits_{Y} \ell_{n-1}(B_\varepsilon(y) \cap S_y \cap \mathcal{B}_\varepsilon(X))\,\mathrm{d}\ell_1 \\
&> (1-\rho_1) \int\limits_{Y} \ell_{n-1}(B_\varepsilon(y) \cap S_y)\,\mathrm{d}\ell_1 \\
&= (1-\rho_1)\ell(W_\varepsilon) \\
&> (1-\rho_1)(1-\rho_2)\ell(B_\varepsilon(X)) \\
&> (1-\rho)\ell(B_\varepsilon(X))
\end{aligned}$$

for suitable choices of $\rho_1, \rho_2 > 0$ and a given $\rho > 0$. As in lemma A.1 we exclude small neighbourhoods of the equilibria from Y to ensure uniform size of the transverse sections and also take ε small enough that the neighbourhoods do not overlap. So we have shown

$$\lim_{\varepsilon \to 0} \frac{\ell(\mathcal{B}_\varepsilon(X))}{\ell(B_\varepsilon(X))} = 1. \tag{3}$$

This is not yet sufficient for predominant asymptotic stability of X, we still have to construct a set N such that X is asymptotically stable relative to N and (1) holds. This can be done as follows: we construct two monotonically decreasing sequences $\alpha_j, \delta_j > 0$ with $\lim\limits_{j\to\infty} \alpha_j = \lim\limits_{j\to\infty} \delta_j = 0$ and set

$$N := \bigcup_{j \geq 2} N_j, \quad \text{where} \quad N_j := \mathcal{B}_{\delta_j}(X) \cap B_{\alpha_{j-1}}(X) \setminus B_{\alpha_j}(X).$$

For $j \in \mathbb{N}$ choose $\delta_j > 0$ such that for all $\delta \leq \delta_j$ we have

$$\ell(\mathcal{B}_\delta(X)) > \frac{j}{j+1}\ell(B_\delta(X)).$$

Then for $j \in \mathbb{N}$ pick $\alpha_j > 0$ in such a way that $\alpha_{j-1} \leq \delta_j$ and

$$\ell(B_{\alpha_j}(X)) < \frac{1}{j(j+1)}\ell(B_{\alpha_{j-1}}(X)).$$

This gives

$$\begin{aligned}
\ell(\mathcal{B}_{\alpha_{j-1}}(X) \setminus B_{\alpha_j}(X)) &\geq \ell(\mathcal{B}_{\alpha_{j-1}}(X)) - \ell(B_{\alpha_j}(X)) \\
&> \frac{j}{j+1}\ell(B_{\alpha_{j-1}}(X)) - \frac{1}{j(j+1)}\ell(B_{\alpha_{j-1}}(X)) \\
&= \frac{j-1}{j}\ell(B_{\alpha_{j-1}}(X)).
\end{aligned}$$

With this we calculate for $\varepsilon > 0$ and $\alpha_k < \varepsilon \leq \alpha_{k-1}$

$$\begin{aligned}
\ell(B_\varepsilon(X) \cap N) &= \ell\left(B_\varepsilon(X) \cap \bigcup_{j\geq 2} N_j\right) \\
&> \ell\left(B_\varepsilon(X) \cap \bigcup_{j\geq 2} \mathcal{B}_{\alpha_{j-1}}(X) \setminus B_{\alpha_j}(X)\right) \\
&= \sum_{j\geq 2} \ell\left(B_\varepsilon(X) \cap \mathcal{B}_{\alpha_{j-1}}(X) \setminus B_{\alpha_j}(X)\right) \\
&= \ell\left(B_\varepsilon(X) \cap \mathcal{B}_{\alpha_{k-1}}(X) \setminus B_{\alpha_k}(X)\right) + \sum_{j>k} \ell\left(\mathcal{B}_{\alpha_{j-1}}(X) \setminus B_{\alpha_j}(X)\right) \\
&\geq \ell(\mathcal{B}_\varepsilon(X)) - \ell(B_{\alpha_k}(X)) + \sum_{j>k} \ell\left(\mathcal{B}_{\alpha_{j-1}}(X) \setminus B_{\alpha_j}(X)\right) \\
&> \ell(\mathcal{B}_\varepsilon(X)) - \ell(B_{\alpha_k}(X)) + \ell\left(\mathcal{B}_{\alpha_k}(X) \setminus B_{\alpha_{k+1}}(X)\right) \\
&> \ell(\mathcal{B}_\varepsilon(X)) - \ell(B_{\alpha_k}(X)) + \frac{k}{k+1}\ell(B_{\alpha_k}(X)) \\
&= \ell(\mathcal{B}_\varepsilon(X)) - \frac{1}{k+1}\ell(B_{\alpha_k}(X)).
\end{aligned}$$

Now since $\alpha_k < \varepsilon$, we have $\ell(B_{\alpha_k}(X)) < \ell(B_\varepsilon(X))$, so

$$\frac{\ell(B_\varepsilon(X) \cap N)}{\ell(B_\varepsilon(X))} > \frac{\ell(\mathcal{B}_\varepsilon(X))}{\ell(B_\varepsilon(X))} - \frac{1}{k+1} \xrightarrow{\varepsilon\to 0} 1,$$

since $k = k(\varepsilon) \xrightarrow{\varepsilon\to 0} \infty$, and the first term converges to 1 by (3). This shows that (1) is satisfied, so X is p.a.s.

Now we prove (b), so assume that X is isolated and that $\sigma_{\mathrm{loc}}(x)$ is negative along all connections. This means that $\Sigma_{\varepsilon,\delta,S_x}(x) \to 0$ for $\delta, \varepsilon \to 0$, so the points in S_x converging directly to X form a thin cusp-shaped region at most. Since X is isolated, there is a neighbourhood U of X that contains no other invariant set. So all points in U, that are not in the thin part of the cusp-shaped set, leave U in finite positive time. If this was not the case, their ω-limit set would be contained in U, leading to a contradiction. Such points do not belong to $\mathcal{B}_\delta(X)$ for $\delta > 0$ so small that $B_\delta(X) \subset U$. With the same techniques as before, it follows that for fixed $\delta > 0$ small enough, the complement $\mathcal{B}_\delta(X)^c$ of the local basin of attraction satisfies

$$\lim_{\varepsilon \to 0} \frac{\ell(B_\delta(X)^c \cap B_\varepsilon(x))}{\ell(B_\varepsilon(X))} = 1,$$

proving predominant instability of X. □

B Results from Podvigina and Ashwin [5]

In this appendix we state the results from of Podvigina and Ashwin [5] that we used in section 4, starting with the one for type A cycles.

Theorem B.1 ([5], theorem 4.1)**.** *Generically, for a simple robust heteroclinic cycle of type A in $\mathbb{R}^4$ the stability indices are as follows.*
(a) If $\rho > 1$ and $b_j > 0$ for all j, then $\sigma_j = +\infty$ for all j.
(b) If $\rho > 1$, $b_j > -1$ for all j and $b_j < 0$ for some j, then $\sigma_j > 0$ for all j.
(c) If $\rho < 1$ or there exists j such that $b_j < -1$, then $\sigma_j = -\infty$ for all j.

This is essentially theorem 4.1 in [5]. In case (b) we do not give their full expression for σ_j, since we are only interested in the sign of the indices. In order to conclude that all indices are positive, one does not have to evaluate the expression for $\sigma_{j,+}$ by hand, since $\sigma_{j,+} \geq 0$ by construction and the case $\sigma_{j,+} = \sigma_{j,-} = 0$ is degenerate.

Finite stability indices for cycles of types B and C can be conveniently expressed through a function f^{index}, defined on page 905 in [5].

$$f^{\text{index}} : \mathbb{R}^2 \to [-\infty, \infty], \quad f^{\text{index}}(\alpha, \beta) := f^+(\alpha, \beta) - f^-(\alpha, \beta),$$

where $f^-(\alpha, \beta) := f^+(-\alpha, -\beta)$ and:

$$f^+(\alpha, \beta) := \begin{cases} +\infty, & \alpha, \beta \geq 0, \\ 0, & \alpha, \beta \leq 0, \\ -\frac{\beta}{\alpha} - 1, & \alpha < 0 < \beta, \ \frac{\beta}{\alpha} < -1 \\ 0, & \alpha < 0 < \beta, \ \frac{\beta}{\alpha} > -1 \\ -\frac{\alpha}{\beta} - 1, & \alpha > 0 > \beta, \ \frac{\alpha}{\beta} < -1 \\ 0, & \alpha > 0 > \beta, \ \frac{\alpha}{\beta} > -1 \end{cases}$$

We now list the results on stability indices for the four non-homoclinic B- and C-cycles, quoting the relevant passages from subsection 4.2.1 of [5].

Lemma B.2 ([5], p. 906)**.** *Generically, for a cycle of type B_2^+ in $\mathbb{R}^4$, the stability indices along connecting trajectories are as follows:*

(i) If $b_1 < 0$ and $b_2 < 0$, then the cycle is not an attractor and all stability indices are equal to $-\infty$.

(ii) Suppose $b_1 > 0$ and $b_2 > 0$.

(a) If $a_1 a_2 < 1$, then the cycle is not an attractor and all stability indices are equal to $-\infty$.

(b) If $a_1 a_2 > 1$, then the cycle is locally attracting and all stability indices are equal to $+\infty$.

(iii) Suppose $b_1 < 0$ and $b_2 > 0$.

(a) If $a_1 a_2 < 1$ or $b_1 a_2 + b_2 < 0$, then the cycle is not an attractor and all indices are equal to $-\infty$.

(b) If $a_1 a_2 > 1$ and $b_1 a_2 + b_2 > 0$, then the stability indices are

$$\sigma_1 = f^{\text{index}}(b_1, 1), \quad \sigma_2 = +\infty.$$

Lemma B.3 ([5], pp. 906–907)**.** *Generically, for a cycle of type B_3^- in $\mathbb{R}^4$, the stability indices along connecting trajectories are as follows:*

(i) If $b_1 < 0$, $b_2 < 0$ and $b_3 < 0$, then the cycle is not an attractor and all stability indices are equal to $-\infty$.

(ii) Suppose $b_1 > 0$, $b_2 > 0$ and $b_3 > 0$.

(a) If $a_1 a_2 a_3 < 1$, then the cycle is not an attractor and all stability indices are equal to $-\infty$.

(b) If $a_1 a_2 a_3 > 1$, then the cycle is locally attracting and all stability indices are equal to $+\infty$.

(iii) Suppose $b_1 < 0$, $b_2 > 0$ and $b_3 > 0$.

(a) If $a_1 a_2 a_3 < 1$ or $b_1 a_2 a_3 + b_3 a_2 + b_2 < 0$, then the cycle is not an attractor and all stability indices are equal to $-\infty$.

(b) If $a_1 a_2 a_3 > 1$ and $b_1 a_2 a_3 + b_3 a_2 + b_2 > 0$, then the stability indices are

$$\sigma_1 = f^{\text{index}}(b_1, 1), \quad \sigma_2 = +\infty, \quad \sigma_3 = f^{\text{index}}(b_3 + b_1 a_3, 1).$$

(iv) Suppose $b_1 < 0$, $b_2 < 0$ and $b_3 > 0$.

(a) If $a_1a_2a_3 < 1$ or $b_2a_1a_3 + b_1a_3 + b_3 < 0$ or $b_1a_2a_3 + b_3a_2 + b_2 < 0$, then the cycle is not an attractor and all stability indices are equal to $-\infty$.

(b) If $a_1a_2a_3 > 1$ and $b_2a_1a_3+b_1a_3+b_3 > 0$ and $b_1a_2a_3+b_3a_2+b_2 > 0$, then the stability indices are

$$\begin{aligned}&\sigma_1 = \min\left(f^{\text{index}}(b_1, 1),\ f^{\text{index}}(b_1 + b_2a_1, 1)\right),\\ &\sigma_2 = f^{\text{index}}(b_2, 1), \quad \sigma_3 = +\infty.\end{aligned}$$

Note that compared to the statement in [5], in lemma B.3 (iv) (b) we have replaced $\sigma_3 = f^{\text{index}}(b_3 + b_1a_3, 1)$ by $\sigma_3 = +\infty$. This is true since

$$b_2a_1a_3 + b_1a_3 + b_3 > 0 \quad \Rightarrow \quad b_1a_3 + b_3 > -b_2a_1a_3 > 0$$

and $f^{\text{index}}(\alpha, \beta) = +\infty$ for $\alpha, \beta > 0$.

Now we state the corresponding result for C_2^--cycles. Recall that λ_1, λ_2 are the eigenvalues of the transition matrix product

$$M := M_1M_2 = \begin{pmatrix} b_1b_2 + a_2 & b_1 \\ a_1b_2 & a_1 \end{pmatrix},$$

where $\lambda_1 \geq \lambda_2$ if both are real. Then $\mathrm{tr}M = b_1b_2 + a_1 + a_2$ and $\det M = a_1a_2$.

Lemma B.4 ([5], pp. 907–908)**.** *Generically, for a cycle of type C_2^- in $\mathbb{R}^4$, the stability indices along connecting trajectories are as follows:*

(i) If $b_1 < 0$ and $b_2 < 0$, then the cycle is not an attractor and all stability indices are equal to $-\infty$.

(ii) Suppose $b_1 > 0$ and $b_2 > 0$.

(a) If $\max(\mathrm{tr}M,\ 2(\mathrm{tr}M - \det M)) < 2$, then the cycle is not an attractor and all stability indices are equal to $-\infty$.

(b) Otherwise the cycle is locally attracting and all stability indices are equal to $+\infty$.

(iii) Suppose $b_1 < 0$ and $b_2 > 0$.

(a) If one of the following conditions is satisfied

- $(\mathrm{tr}M)^2 - 4\det M < 0$,
- $\max(\mathrm{tr}M,\ 2(\mathrm{tr}M - \det M)) < 2$,

- $b_1b_2 - a_1 + a_2 < 0$,

then the cycle is not an attractor and all stability indices are equal to $-\infty$.

(b) *If none of the conditions above are satisfied, then the stability indices are*

$$\sigma_1 = f^{\text{index}}\left(\frac{b_1b_2 + a_1 - \lambda_2}{b_2}, 1\right),$$
$$\sigma_2 = f^{\text{index}}\left(\frac{\lambda_2 - b_1b_2 - a_2}{b_1}, -1\right).$$

The last result of this kind that we need is for C_4^--cycles. We list the indices only for the case we are interested in and do not reproduce the entire classification from [5]. Again we need some more notation from [5]. The product of transition matrices in this case is given by:

$$\begin{aligned} M^{4,1} &= M_4M_3M_2M_1 \\ &= \begin{pmatrix} (b_1b_2 + a_1)(b_3b_4 + a_3) + b_1a_2b_4 & (b_3b_4 + a_3)b_2 + b_4a_2 \\ a_4b_3(b_1b_2 + a_1) + a_2a_4b_1 & a_4b_2b_3 + a_2a_4 \end{pmatrix} \end{aligned}$$

By $M^{j+3,j}$ we denote the matrix with cyclically permuted factors M_i. Here $j + 3$ is to be understood mod 4 as usual. Again we denote the eigenvalues by $\lambda_1 \geq \lambda_2$ if both are real. They are independent of j since all $M^{j+3,j}$ are similar matrices, thus they have equal eigenvalues, determinant and trace. The associated eigenvectors we write as

$$v_1^{j+3,j} = (v_{11}^{j+3,j}, v_{12}^{j+3,j}) \quad \text{and} \quad v_2^{j+3,j} = (v_{21}^{j+3,j}, v_{22}^{j+3,j}),$$

respectively. With this define

$$h^{j+3,j} := v_{11}^{j+3,j}v_{22}^{j+3,j} - v_{12}^{j+3,j}v_{21}^{j+3,j}.$$

To reduce the number of sub- and superscripts set $M := M^{4,1}$. We can now state the result.

Lemma B.5 ([5], pp. 908–909)**.** *For a cycle of type C_4^-, suppose that $b_1 < 0$ and $b_j > 0$ for $j \neq 1$.*

(i) *If one of the following holds, then the cycle is not an attractor and all stability indices are equal to $-\infty$.*

(a) $\max(\mathrm{tr}M,\ 2(\mathrm{tr}M - \det M)) < 2$

(b) $(\mathrm{tr} M)^2 - 4 \det M < 0$
(c) $v_{11}^{1,2} v_{12}^{1,2} < 0$

(ii) Otherwise the stability indices along connecting trajectories are as follows:

$$\sigma_1 = \min\left(f^{\text{index}}(v_{22}^{4,1}/h^{4,1}, -v_{21}^{4,1}/h^{4,1}),\ f^{\text{index}}(b_1, 1)\right)$$
$$\sigma_2 = \min\left(f^{\text{index}}(v_{22}^{1,2}/h^{1,2}, -v_{21}^{1,2}/h^{1,2}),\ f^{\text{index}}(b_1 b_4 + a_4, b_1)\right)$$
$$\sigma_3 = \min\left(f^{\text{index}}(v_{22}^{2,3}/h^{2,3}, -v_{21}^{2,3}/h^{2,3}),\ f^{\text{index}}(b_3(b_1 b_4 + a_4) + b_1 a_3, b_1 b_4 + a_4)\right)$$
$$\sigma_4 = f^{\text{index}}\left(v_{22}^{3,4}/h^{3,4}, -v_{21}^{3,4}/h^{3,4}\right)$$

Note that the expressions for the stability indices in [5] differ from the ones above. This has been noticed in private communication between this author and Olga Podvigina, who subsequently provided the corrected version above.

C Proofs of the theorems in section 4

In this appendix we prove theorems 4.3, 4.4, 4.8 and 4.10 from section 4. The proofs are straightforward calculations. Before we start, we facilitate usage of the function f^{index} by a lemma which follows directly from its definition.

Lemma C.1. *For $\alpha, \beta \in \mathbb{R}$ we have*
(a) $f^{\text{index}}(\alpha, 1) \in (0, +\infty)$ *if and only if* $\alpha \in (-1, 0)$,
(b) $f^{\text{index}}(\alpha, 1) \in (-\infty, 0)$ *if and only if* $\alpha < -1$,
(c) $f^{\text{index}}(\beta, -1) \in (0, +\infty)$ *if and only if* $\beta > 1$,
(d) $f^{\text{index}}(\beta, -1) \in (-\infty, 0)$ *if and only if* $\beta \in (0, 1)$.

Now we proceed to the proofs of the four theorems.

Proof. [Theorem 4.3, B_2^+-cycles]. By lemma 4.1 it suffices to find out when the stability indices are
- all equal to $+\infty$ (asymptotically stable),
- positive, but not all equal to $+\infty$ (p.a.s.),
- negative, but not all equal to $-\infty$ (p.u.),
- all equal to $-\infty$ (completely unstable).

In the remaining cases the cycle is p.f.a.s. We calculate these conditions based on lemma B.2.
(a) Both $\sigma_1 = \sigma_2 = +\infty$ if and only if $a_1 a_2 > 1$ and $b_1, b_2 > 0$, i.e. if and only if $c_1 c_2 > e_1 e_2$ and t_1, $t_2 < 0$.

(b) We need $\sigma_1, \sigma_2 > 0$, but at least one of them not equal to infinity. This only happens in case (iii) of lemma B.2, where $b_1 < 0 < b_2$ and $a_1 a_2 > 1$, i.e. $t_2 < 0 < t_1$ and $c_1 c_2 > e_1 e_2$. Moreover, it is necessary that

$$b_1 a_2 + b_2 > 0 \quad \Leftrightarrow \quad -\frac{t_1 c_2}{e_1 e_2} - \frac{t_2}{e_2} > 0 \quad \Leftrightarrow \quad t_1 < -\frac{e_1 t_2}{c_2}.$$

Then we have $\sigma_2 = +\infty$ and $\sigma_1 = f^{\text{index}}(b_1, 1)$. The latter expression has to be in $(0, \infty)$, which by lemma C.1 is the case if and only if $b_1 \in (-1, 0)$, which is the same as $0 < t_1 < e_1$.

(d) Both $\sigma_1 = \sigma_2 = -\infty$ if and only if the following holds

$$a_1 a_2 < 1 \quad \vee \quad b_1, b_2 < 0 \quad \vee \quad (b_1 < 0 < b_2 \quad \wedge \quad b_1 a_2 + b_2 < 0).$$

This is equivalent to the conditions in case (d) of theorem 4.3.

(c) The cycle is never predominantly unstable since in lemma B.2 there is always either at least one index equal to $+\infty$ or all indices are equal to $-\infty$. Thus, in all remaining cases the cycle is p.f.a.s.

□

Proof. [Theorem 4.4, B_3^--cycles]. Just as before it suffices to check, with lemma B.3, when all stability indices are positive/negative or equal to $\pm\infty$.

(a) Clearly, $\sigma_1 = \sigma_2 = \sigma_3 = +\infty$ if and only if $a_1 a_2 a_3 > 1$ and $b_1, b_2, b_3 > 0$, i.e. if and only if $c_1 c_2 c_3 > e_1 e_2 e_3$ and $t_1, t_2, t_3 < 0$.

(b) For all stability indices to be positive but at least one of them not equal to $+\infty$, we need $a_1 a_2 a_3 > 1$, i.e. $c_1 c_2 c_3 > e_1 e_2 e_3$ and at least one positive and one negative transverse eigenvalue, say $t_1 > 0$ and $t_3 < 0$. Moreover, we must have

$$0 < b_1 a_2 a_3 + a_2 b_3 + b_2 \quad \Leftrightarrow \quad t_1 < -\frac{e_1(t_3 c_2 + e_3 t_2)}{c_2 c_3}.$$

If $t_2 < 0$, then we have $\sigma_2 = +\infty$ and for the other two indices to be positive we need

$$\sigma_1 = f^{\text{index}}(b_1, 1) > 0 \quad \Leftrightarrow \quad b_1 > -1 \quad \Leftrightarrow \quad t_1 < e_1,$$

and

$$\sigma_3 = f^{\text{index}}(b_3 + b_1 a_3, 1) > 0 \Leftrightarrow b_3 + b_1 a_3 > -1 \Leftrightarrow t_1 < \frac{e_1(e_3 - t_3)}{c_3}.$$

Since $t_1 > 0$ we have $\sigma_1 < +\infty$, so this is not a case of asymptotic stability.

If on the other hand $t_2 > 0$, we also require

$$b_2a_1a_3 + a_3b_1 + b_3 > 0 \quad \Leftrightarrow \quad t_2 < -\frac{e_2(t_1c_3 + e_1t_3)}{c_1c_3}.$$

For positive stability indices we then need

$$\begin{aligned}
\sigma_1 &= \min\left(f^{\text{index}}(b_1, 1),\ f^{\text{index}}(b_1 + b_2a_1, 1)\right) > 0, \\
\sigma_2 &= f^{\text{index}}(b_2, 1) > 0, \\
\sigma_3 &= f^{\text{index}}(b_3 + b_1a_3, 1) > 0,
\end{aligned}$$

which is equivalent to

$$t_1 < e_1 \ \wedge\ t_2 < \frac{e_2(e_1 - t_1)}{c_1}, \quad t_2 < e_2, \quad t_1 < \frac{e_1(e_3 - t_3)}{c_3}.$$

Again $\sigma_1 < +\infty$ and this is also not a case of asymptotic stability.

(c) We have $\sigma_1 = \sigma_2 = \sigma_3 = -\infty$ if and only if one of the following holds:

- $a_1a_2a_3 < 1$
- $b_1, b_2, b_3 < 0$
- $b_1 < 0 < b_3$ and $b_1a_2a_3 + b_3a_2 + b_2 < 0$
- $b_1, b_2 < 0 < b_3$ and $b_2a_1a_3 + b_1a_3 + b_3 < 0$

These conditions are equivalent to the ones in case (c) of theorem 4.4.

(d) Again it suffices to show that the cycle is never p.u. This follows from lemma B.3 for the same reason as in the previous proof.

□

Proof. [Theorem 4.8, C_2^--cycles]. The statements in (a) and (b) follow directly from lemma B.4. There is only one case in which the stability indices are not all equal to $\pm\infty$, namely when $b_1 < 0 < b_2$. The first conditions are the same in both (c) and (d), and just the ones keeping the cycle from being completely unstable. The stability indices then are

$$\begin{aligned}
\sigma_1 &= f^{\text{index}}(\alpha, 1) &\quad \text{with} \quad \alpha &= \frac{b_1b_2 + a_1 - \lambda_2}{b_2}, \\
\sigma_2 &= f^{\text{index}}(\beta, -1) &\quad \text{with} \quad \beta &= \frac{\lambda_2 - b_1b_2 - a_2}{b_1}.
\end{aligned}$$

As before we have to determine when $\sigma_1, \sigma_2 > 0$ for (c) and $\sigma_1, \sigma_2 < 0$ for (d), with at least one of them being finite in either case. By lemma C.1 we get

$$\sigma_1 > 0 \Leftrightarrow \alpha \in (-1, 0); \qquad \sigma_2 > 0 \Leftrightarrow \beta > 1$$
$$\sigma_1 < 0 \Leftrightarrow \alpha < -1; \qquad \sigma_2 < 0 \Leftrightarrow \beta \in (0, 1).$$

Solving this for λ_2 leads to

$$\sigma_1 > 0 \Leftrightarrow \lambda_2 \in (b_1 b_2 + a_1,\ b_1 b_2 + a_1 + b_2)$$
$$\sigma_1 < 0 \Leftrightarrow \lambda_2 > b_1 b_2 + a_1 + b_2$$

and

$$\sigma_2 > 0 \Leftrightarrow \lambda_2 < b_1 b_2 + a_2 + b_1$$
$$\sigma_2 < 0 \Leftrightarrow \lambda_2 \in (b_1 b_2 + a_2 + b_1,\ b_1 b_2 + a_2),$$

giving precisely the required conditions. □

Finally, we prove the statement about C_4^--cycles. This is qualitatively slightly different.

Proof. [Theorem 4.10, C_4^--cycles]. We use lemma B.5 and show that for $t_1 > 0$ sufficiently small all stability indices are positive, but not all of them equal to $+\infty$. To this end, we first convince ourselves that

$$f^{\text{index}}(v_{22}^{j+3,j}/h^{j+3,j}, -v_{21}^{j+3,j}/h^{j+3,j}) = +\infty \tag{4}$$

for all j: by construction, for $j \neq 2$ the eigenvalues of $M^{j+3,j}$ can be determined from those of $M^{1,2}$ by multiplying with the matrices M_j, $j \neq 1$, in the correct order. These have positive entries only. Thus, it follows from the converse of condition (c) in lemma B.5 that for all j the entries of $v_1^{j+3,j}$ have the same sign. For $t_1 > 0$ small enough all $M^{j+3,j}$ have only positive entries. Therefore, while eigenvectors corresponding to the greater one of their eigenvalues have same sign entries, those for the smaller one have opposite sign entries. So $v_{21}^{j+3,j} v_{22}^{j+3,j} < 0$, and thus the arguments of f^{index} above have the same sign. In fact, taking into account $h^{j+3,j}$, they are both positive. This proves (4), so $\sigma_4 = +\infty$ and the other stability indices are equal to the respective second expression in the minimum in lemma B.5.

Now we choose $t_1 > 0$ small enough such that

(i) $t_1 < e_1$, $\quad (ii)$ $b_1 b_4 + a_4 > -b_1$, $\quad (iii)$ $b_1 b_3 b_4 + b_3 a_4 + b_1 a_3, b_1 b_4 + a_4 > 0$.

This is possible because all quantities involved are positive except for b_1. Then the stability indices are

$$\sigma_1 = f^{\text{index}}(b_1, 1) = -\frac{1}{b_1} - 1 = \frac{e_1}{t_1} - 1 > 0,$$
$$\sigma_2 = f^{\text{index}}(b_1 b_4 + a_4, b_1) = -\frac{b_1 b_4 + a_4}{b_1} - 1 > 0,$$
$$\sigma_3 = f^{\text{index}}(b_1 b_3 b_4 + b_3 a_4 + b_1 a_3, b_1 b_4 + a_4) = +\infty.$$

So all indices are positive for $t_1 > 0$ small enough, so the C_4^--cycle is predominantly asymptotically stable as claimed. □

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