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Unstable Attractors: Existence and Stability Indices

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Abstract

We show that unstable attractors do not exist for smooth invertible dynamics. In systems lacking these properties we draw simple conclusions about their stability indices and look at examples highlighting extreme cases of stability and attractiveness – characterised in terms of stability indices. In particular, we investigate the possibilities for great discrepancies between the local and non-local indices $\sigma_{\mathrm{loc}}(x)$ and $\sigma(x)$, also depending on properties of the system. We show that while $\sigma_{\mathrm{loc}}(x) = -\infty$ holds for all unstable attractors, it is not straightforward to uniquely identify them using stability indices.

Keywords: unstable attractors, (non-asymptotic) stability, stability index

AMS classification: 34D20, 34D45, 37C70, 37C75

1 Introduction

The stability index was introduced by Podvigina and Ashwin [1] as a tool for quantifying stability properties of attractors in dynamical systems. It has subsequently been used by several authors in different contexts: Mohd Roslan [2] considers skew product systems and uses the stability index to gain insight in the nature of riddled basins of attraction, while Castro and Lohse [3] look at heteroclinic cycles and networks, where the stability index enhances understanding of competition between cycles in a network.

Unstable attractors have been rigorously defined by Ashwin and Timme [4] as sets with seemingly contradictory stability properties: attracting a set of positive measure, while repelling almost everything in its immediate neighbourhood. Numerical simulations indicate that these occur naturally in networks of pulse-coupled oscillators, see Timme et. al. [5] and [6], for instance. In [4] the authors give analytical examples of unstable attractors, but only for semiflows that are not invertible, i.e. cannot be extended to flows. They conjecture that unstable attractors do not exist in smooth invertible systems.

The main result of this work is to confirm their conjecture through the proof of theorem 3.1. Moreover, we investigate unstable attractors in terms of their stability indices. Both is done in section 3, after recalling standard definitions and terminology in section 2. We also provide several examples illustrating limitations of the stability index when it comes to characterising extreme cases of (unstable) attractiveness. This means we focus on situations where local and non-local stability indices differ, $\sigma(x) \neq \sigma_{\mathrm{loc}}(x)$. In fact, we look for the greatest possible discrepancies between the two. This is in contrast to most of the other work on stability indices, especially [3] and [7], where $\sigma(x) = \sigma_{\mathrm{loc}}(x)$ is often an appropriate assumption.

2 Preliminaries

We are concerned with finite-dimensional dynamics given by a (semi)flow

$$\phi_t : M \to M \tag{1}$$

on a (compact) manifold M with Lebesgue/Riemann measure $\ell(\cdot)$. Time is either continuous ($t \in \mathbb{R}_{(\geq 0)}$) or discrete ($t \in \mathbb{Z}_{(\geq 0)}$). We are interested in the stability properties of compact, invariant sets $X \subset M$, which are commonly called *attractors* if their basin of attraction $\mathcal{B}(X) = \{x \in M \mid \omega(x) \subset X\}$ has positive measure $\ell(\mathcal{B}(X)) > 0$. By $B_\varepsilon(X)$ we denote an ε-neighbourhood of X, and by $\mathcal{B}_\varepsilon(X)$ its ε-local basin of attraction – the set of points $x \in \mathcal{B}(X)$ that for positive times do not leave $B_\varepsilon(X)$.

There are attractors in the above sense that at the same time exhibit strong repelling properties. The following precise definition was made by Ashwin and Timme [4].

Definition 2.1 ([4], definitions 1&2)**.** An attractor $X \subset M$ is called *unstable attractor* if there is a neighbourhood U of X such that

$$\ell(\mathcal{A}(U)) = 0,$$

where $\mathcal{A}(U) := \{x \in U \mid \phi_t(x) \in U \ \forall t \geq 0\}$ is the *lingering subset* of $U \subset M$.

Note that any neighbourhood U of X contains $B_\varepsilon(X)$ for $\varepsilon > 0$ small enough. Then the ε-local basin of attraction is contained in the lingering subset of U, $\mathcal{B}_\varepsilon(X) \subset \mathcal{A}(U)$.

Ashwin and Timme further distinguish unstable attractors with *positive* and *zero measure local basin*, depending on whether or not there is a neighbourhood U of X with $\ell(U \cap \mathcal{B}(X)) = 0$, see definitions 3 and 4 in [4]. They give examples for both types and show that unstable attractors with zero measure local basin do not occur in certain cases (proposition 1 in [4]). In theorem 3.1 we generalise their result and prove that unstable attractors – regardless of the size of their local basin – do not exist for smooth invertible flows.

A useful way to quantify stability of any attractor is the stability index, we recall its definition from Podvigina and Ashwin [1].

Definition 2.2 ([1], definition 5)**.** For $x \in X$ and $\varepsilon, \delta > 0$ define

$$\Sigma_\varepsilon(x) := \frac{\ell(B_\varepsilon(x) \cap \mathcal{B}(X))}{\ell(B_\varepsilon(x))}, \qquad \Sigma_{\varepsilon,\delta}(x) := \frac{\ell(B_\varepsilon(x) \cap \mathcal{B}_\delta(X))}{\ell(B_\varepsilon(x))}.$$

Then the *stability index* at x (with respect to X) is set to be

$$\sigma(x) := \sigma_+(x) - \sigma_-(x),$$

where

$$\sigma_-(x) := \lim_{\varepsilon \to 0} \left[\frac{\ln(\Sigma_\varepsilon(x))}{\ln(\varepsilon)} \right], \qquad \sigma_+(x) := \lim_{\varepsilon \to 0} \left[\frac{\ln(1 - \Sigma_\varepsilon(x))}{\ln(\varepsilon)} \right].$$

The convention that $\sigma_-(x) = \infty$ if $\Sigma_\varepsilon(x) = 0$ for some $\varepsilon > 0$ and $\sigma_+(x) = \infty$ if $\Sigma_\varepsilon(x) = 1$, leads to $\sigma(x) \in [-\infty, \infty]$. In the same way the *local stability index* is defined to be

$$\sigma_{\text{loc}}(x) := \sigma_{\text{loc},+}(x) - \sigma_{\text{loc},-}(x),$$

with

$$\sigma_{\text{loc},-}(x) := \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \left[\frac{\ln(\Sigma_{\varepsilon,\delta}(x))}{\ln(\varepsilon)} \right], \; \sigma_{\text{loc},+}(x) := \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \left[\frac{\ln(1 - \Sigma_{\varepsilon,\delta}(x))}{\ln(\varepsilon)} \right].$$

Stability indices quantify attraction ($\sigma_{\text{loc}}(x)$) and stability ($\sigma(x)$) properties of a set X locally near a point $x \in X$. Positive indices represent strong attraction/stability, while negative indices indicate that most trajectories near x are repelled by X, see figure 1.

The next lemma relates $\sigma(x)$ to the behaviour of $\Sigma_\varepsilon(x)$ for small $\varepsilon > 0$. Most of it was already established in lemma 2.2 of [1], a proof is given in [7].

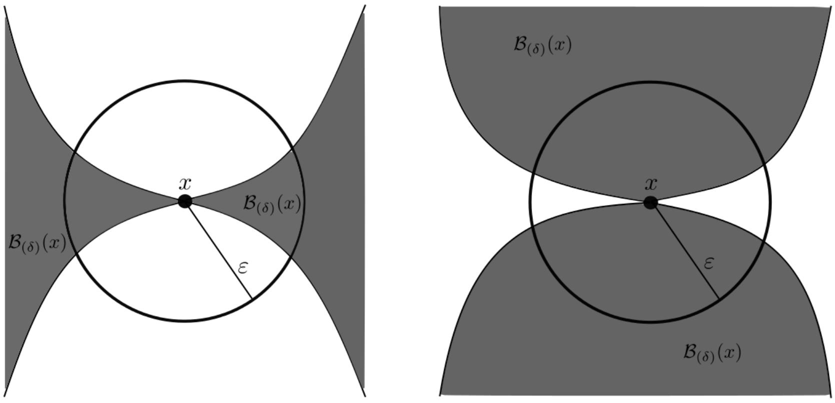

Figure 1: $\sigma_{(\mathrm{loc})}(x) < 0$ left and $\sigma_{(\mathrm{loc})}(x) > 0$ right.

Lemma 2.3 ([7], lemma 1.32)**.** *Suppose that the stability index $\sigma(x)$ exists for some $x \in X$ and let $c > 0$. Then the following is true.*

(a) *If $\sigma_{\pm}(x) > 0$, then $\sigma_{\mp}(x) = 0$.*

(b) $\Sigma_\varepsilon(x) = O(\varepsilon^c) \quad \Leftrightarrow \quad \sigma(x) \leq -c$

(c) $1 - \Sigma_\varepsilon(x) = O(\varepsilon^c) \quad \Leftrightarrow \quad \sigma(x) \geq c$

(d) $\Sigma_\varepsilon(x)$ *is bounded away from* 0 *and* $1 \quad \Rightarrow \quad \sigma(x) = 0$

3 Results

We begin this section by excluding the existence of unstable attractors in the smooth invertible case.

Theorem 3.1 ([7], corollary 1.27)**.** *For a smooth invertible flow (1) there exist no unstable attractors.*

Proof. If $X \subset M$ is an attractor, then by definition $\ell(\mathcal{B}(X)) > 0$. We argue that this implies $\ell(\mathcal{B}_\varepsilon(X)) > 0$ for any $\varepsilon > 0$ and therefore $\ell(\mathcal{A}(U)) > 0$ for any neighbourhood U of X, so X is not unstable.

Let $\varepsilon > 0$. Since ϕ_t is smooth and $\mathcal{B}(X)$ is flow-invariant, it contains a set $A \subset B_\varepsilon(X)$ with $\ell(A) > 0$. For $T \in \mathbb{N}$ we set

$$A_T := \{x \in A \mid \exists t > T : \phi_t(x) \notin B_\varepsilon(X)\}.$$

Then A_T contains all points in A that still leave $B_\varepsilon(X)$ after time T. Now there are two cases:

1. $\exists T \in \mathbb{N} : \ell(A_T) < \ell(A)$
2. $\forall T \in \mathbb{N} : \ell(A_T) = \ell(A)$

In the first case we have $\ell(A \setminus A_T) > 0$ and $\phi_t(x) \in B_\varepsilon(X)$ for all $t > T$ and $x \in A \setminus A_T$, so the set $A_\varepsilon := \phi_T(A \setminus A_T)$ has the required properties, i.e. $A_\varepsilon \subset \mathcal{B}_\varepsilon(X)$ and $\ell(A_\varepsilon) > 0$.

In the second case we have $\ell(A \setminus A_T) = 0$ for all $T \in \mathbb{N}$. However, for all $x \in A$ there exists $T \in \mathbb{N}$ such that $x \notin A_T$, so $A = \bigcup_{T\in\mathbb{N}} A \setminus A_T$. To see this, suppose there is $x \in A \subset \mathcal{B}(X)$ such that the trajectory through x leaves (and enters) $B_\varepsilon(X)$ infinitely many times. Then, since the boundary $\partial B_\varepsilon(X)$ is compact, the set $\{\phi_t(x) \mid t \in \mathbb{R}\} \cap \partial B_\varepsilon(X)$ has an accumulation point x_0. But then $x_0 \in \omega(x)$, in contradiction with $\omega(x) \subset X$ for $x \in \mathcal{B}(X)$. It follows that

$$0 < \ell(A) = \ell\left(\bigcup_{T\in\mathbb{N}} A \setminus A_T\right) \leq \sum_{T\in\mathbb{N}} \ell(A \setminus A_T) = 0.$$

This is a contradiction, so the second case cannot occur and the proof is complete. $\square$

We have thus confirmed the conjecture in [4] and know that unstable attractors can only occur for (semi)flows failing to be smooth and invertible. Clearly, then the local stability indices must be equal to $-\infty$.

Lemma 3.2. *Let X be an unstable attractor. Then $\sigma_{\mathrm{loc}}(x) = -\infty$ for all $x \in X$.*

Proof. There is a neighbourhood U of X with $\ell(\mathcal{A}(U)) = 0$. For $\delta > 0$ small enough we have $B_\delta(X) \subset U$, so $\mathcal{B}_\delta(X) \subset \mathcal{A}(U)$ and thus $\ell(\mathcal{B}_\delta(X)) = 0$. Then by definition $\Sigma_{\varepsilon,\delta}(x) = 0$ for any $x \in X$ and $\varepsilon, \delta > 0$ small enough, so $\sigma_{\mathrm{loc}}(x) = -\infty$. $\square$

The same can not be said about the non-local indices $\sigma(x)$, because $\ell(\mathcal{B}(X)) > 0$ holds for any (unstable) attractor. However, it is not impossible to have $\sigma(x) = -\infty$, i.e. a positive measure basin of attraction does not prevent X from appearing completely unstable in terms of stability indices. This is illustrated by example 3.3.

Note that here and in the following we often consider equilibria in $\mathbb{R}^2$ as the invariant set X. This may seem like a rather specialised setting, but it is very common for heteroclinic cycles in $\mathbb{R}^4$, where the stability indices may be computed with respect to a plane that is transverse to the flow, see theorem 2.4 in [1]. The intersection with the cycle is then an equilibrium of the return map. This is frequently exploited and explained in more detail in [3].

Example 3.3 ([7], section 1.2.3)**.** Consider a (not necessarily unstable) attractor $X \subset M$ for which locally near $x \in X$ the basin of attraction $\mathcal{B}(X)$ is a superalgebraic cusp, i.e. $\Sigma_\varepsilon(x) = O(\varepsilon^c)$ for all $c > 0$. For instance, let $\mathcal{B}(X)$ be shaped in such a way that $\Sigma_\varepsilon(x) = \exp(-1/\varepsilon)$. Then for any $c > 0$ we get

$$\varepsilon^c \exp\left(-1/\varepsilon\right) = \sum_{k=0}^{\infty} \frac{\varepsilon^{c-k}}{k!} \xrightarrow{\varepsilon \to 0} \infty,$$

so $\Sigma_\varepsilon(x) = O(\varepsilon^c)$. Thus, lemma 2.3 yields $\sigma(x) \leq -c$, so $\sigma(x) = -\infty$, even though $\mathcal{B}(X)$ is of positive measure in any neighbourhood of x.

We look at this in more detail in $\mathbb{R}^2$: let X consist of the origin only. Then such a cusp is bounded by the graph of $f(x) = (2x+1)\exp\left(-1/x\right)$ and the x-axis. If the basin of attraction is shaped by f, then for small $\varepsilon > 0$ we have (up to a constant factor)

$$\Sigma_\varepsilon(0) = \frac{\ell(B_\varepsilon(0) \cap \mathcal{B}(0))}{\ell(B_\varepsilon(0))} \approx \varepsilon^{-2} \int_0^\varepsilon (2x+1)\exp\left(-1/x\right)\,\mathrm{d}x = \exp\left(-1/\varepsilon\right).$$

Therefore, X being unstable in the sense that $\sigma(x) = -\infty$ does not imply $\ell(\mathcal{B}(X)) = 0$, so it does not prevent X from being an attractor. Moreover, this shows that $\sigma_{\mathrm{loc}}(x) = -\infty$ is merely a necessary condition for X being an unstable attractor – it is not sufficient, not even when $\ell(\mathcal{B}(X)) > 0$.

In the other extreme it is similar: adapting the flow such that everything except for the cusp is contained in the basin of attraction, we get $\sigma(x) = +\infty$ even though X is not asymptotically stable in the classical sense.

We take this one step further by modifying a dynamical system that serves as an example for an unstable but attractive invariant set, so that $\sigma(x) = +\infty$ and $\sigma_{\mathrm{loc}}(x) = -\infty$ simultaneously.

Example 3.4. Consider the following ordinary differential equation on the upper half plane $\mathbb{H}^+ \subset \mathbb{R}^2$ that Hahn discusses in §40 of [8].

$$\dot{\xi}_1 = \frac{\xi_1^2(\xi_2 - \xi_1) + \xi_2^5}{(\xi_1^2 + \xi_2^2)(1 + (\xi_1^2 + \xi_2^2)^2)}, \qquad \dot{\xi}_2 = \frac{\xi_2^2(\xi_2 - 2\xi_1)}{(\xi_1^2 + \xi_2^2)(1 + (\xi_1^2 + \xi_2^2)^2)}.$$

This creates a smooth flow with a phase portrait as in figure 2. The invariant set $X = \{0\}$ again consists of a single point, the origin. It is unstable but attractive, its basin $\mathcal{B}(X)$ is the entire space and thus has full measure in $B_\varepsilon(0)$ for any $\varepsilon > 0$, so $\sigma(0) = +\infty$. In the first quadrant there are infinitely many homoclinic orbits. For small $\delta > 0$ the local basin of attraction $\mathcal{B}_\delta(0)$ is confined between the longest homoclinic orbit and the ξ_1-axis. In fact,

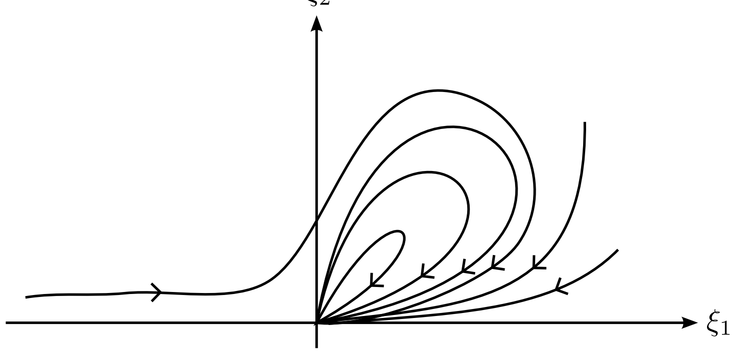

Figure 2: Unstable but attractive equilibrium

Hahn [8] shows that the sector between the ξ_1-axis and the ray through the point $({}^{25}\!/\!_{32}, {}^{5}\!/\!_{8})$ belongs to $\mathcal{B}_\delta(0)$. Denoting the opening angle of that sector by α this implies that for small $\varepsilon, \delta > 0$ we get

$$\frac{\alpha}{\pi} \leq \Sigma_{\varepsilon,\delta}(0) \leq \frac{1}{2},$$

and therefore $\sigma_{\mathrm{loc}}(0) = 0$ by lemma 2.3. We modify this example so that $\sigma_{\mathrm{loc}}(0) = -\infty$.

The idea is to transform the phase portrait into the one in figure 3 where all homoclinic orbits are located below the line $\xi_2 = (2\xi_1 + 1)\exp(-{}^{1}\!/\!_{\xi_1})$, the boundary of a superalgebraic cusp as described above. The local basin of the origin is then contained in this cusp, so $\sigma_{\mathrm{loc}}(0) = -\infty$, while the equilibrium is still globally attractive, hence $\sigma(0) = +\infty$.

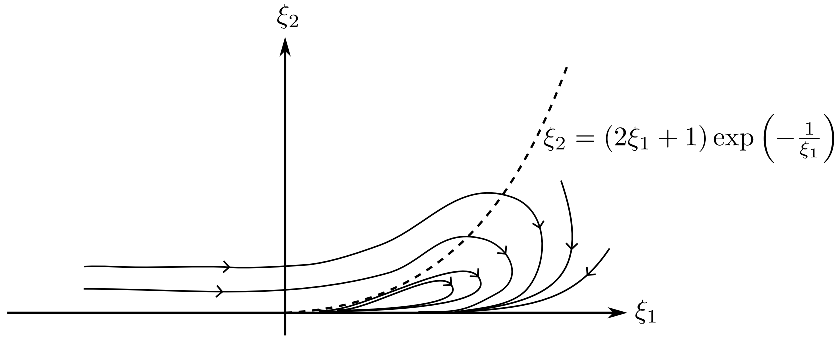

Figure 3: $\mathcal{B}_\delta(X)$ contained in a superalgebraic cusp

In order to see how this is possible we make the following general considerations. Let $\dot{x} = f(x)$ be an ordinary differential equation generating a flow on $\mathbb{H}^+$ with some phase portrait (A), and let $x = x(t, x_0)$ be the solution for an inital value $x_0 \in \mathbb{H}^+$. Then consider a continuously differentiable homeomorphism $\Phi : \mathbb{H}^+ \to \mathbb{H}^+$ transforming the phase portrait in such a way that $y = y(t, y_0) := \Phi(x(t, x_0))$, with $y_0 := \Phi(x_0)$, are the trajectories of a desired phase portrait (B). Then $y(t, y_0)$ is the solution to an equation $\dot{y} = g(y)$ generating the desired flow (corresponding to (B)), where g is obtained by transforming f with Φ in the following way:

$$\dot{y} = \frac{\mathrm{d}}{\mathrm{d}t}(\Phi(x(t))) = \mathrm{d}\Phi(x(t)) \cdot \dot{x}(t) = [\mathrm{d}\Phi \circ \Phi^{-1}](y(t)) \cdot [f \circ \Phi^{-1}](y(t)) =: g(y)$$

The right-hand side g is continuous because Φ is a homeomorphism and $\mathrm{d}\Phi$ is continuous. Now define Φ for our specific case:

$$\Phi : \mathbb{H}^+ \to \mathbb{H}^+, \quad (\xi_1, \xi_2) \mapsto (\xi_1 + \xi_2, (2\xi_2 + 1) \exp(-{}^1\!/\!_{\xi_2}))$$

It is simple to check that Φ is continuously differentiable. Moreover, it is bijective and its inverse Ψ is continuous, since $\xi_2 \mapsto (2\xi_2 + 1) \exp(-{}^1\!/\!_{\xi_2})$ is strictly monotonically increasing and unbounded, thus open. However, Ψ is not differentiable along $\xi_2 = 0$, so neither is g. Nevertheless, this does exactly what we wanted: for $\xi_1 > 0$ we have $\exp(-{}^1\!/\!_{\xi_2}) < \exp(-{}^1\!/\!_{(\xi_1+\xi_2)})$ and therefore

$$(2\xi_1 + 1) \exp(-{}^1\!/\!_{\xi_2}) < (2(\xi_1 + \xi_2) + 1) \exp(-{}^1\!/\!_{(\xi_1+\xi_2)}),$$

so the first quadrant (and thus the entire local basin of attraction) is mapped below the boundary of a superalgebraic cusp. So indeed we have $\sigma_{\mathrm{loc}}(0) = -\infty$. We emphasise again that this is achieved at the cost of differentiability.

We have seen that in systems with a continuous (but not differentiable) right-hand side it is possible to have $\sigma(x) = +\infty$ while $\sigma_{\mathrm{loc}}(x) = -\infty$, the two indices are independent in the same way that the classical definitions of stability and attractivity do not go hand in hand. This is not necessarily a feature of an unstable attractor (typically $\sigma(x) < +\infty$), but the stability properties of an attractor with indices like that are very similar to unstable attractors: locally strongly repelling while globally attracting. It is unclear, however, if such an example can be constructed with a smooth right-hand side. We consider this unlikely, since we were able to exclude the existence of unstable attractors in theorem 3.1. In any case, an example cannot be expected from a construction like ours, because by theorem 2.2 in [1] the stability index is invariant under topological equivalence. Therefore, we cannot use a diffeomorphism to alter the indices in the same way as above.

We conclude with an example for strong discrepancies between $\sigma(x)$ and $\sigma_{\mathrm{loc}}(x)$ in smooth systems. Consider a flow on the unit circle S^1, as depicted in figure 4. Again X consists of a single equilibrium $x \in S^1$ and the whole space belongs to $\mathcal{B}(X)$, so $\sigma(x) = +\infty$. But only one side of x (shaded grey in figure 4) belongs to the local basin $\mathcal{B}_\varepsilon(X)$ for $\varepsilon > 0$ small enough. Thus, $\Sigma_{\varepsilon,\delta}(X)$ is constant and $\sigma_{\mathrm{loc}}(x) = 0$ by lemma 2.3.

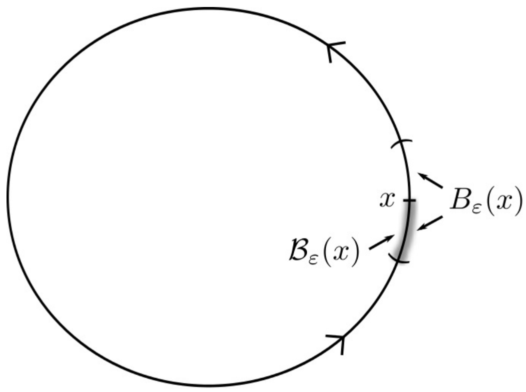

Figure 4: $\sigma(x) = +\infty$ and $\sigma_{\mathrm{loc}}(x) = 0$

We point out that in smooth systems it is also possible to have

$$-\infty < \sigma_{\mathrm{loc}}(x) < 0 < \sigma(x) < +\infty.$$

An example for this can be found in proposition 5.3 and lemma 5.5 in [3], where in a heteroclinic network the index along a trajectory that belongs to two cycles is positive with respect to the network ($\sigma(x)$), but negative with respect to both cycles ($\sigma_{\mathrm{loc}}(x)$). This is also commented on in remark 2.18 of [7].

In summary, we have proved that unstable attractors only occur in systems lacking smoothness or invertibility. While the stability index helps to characterise them by common features such as $\sigma_{\mathrm{loc}}(x) = -\infty$, our examples exhibit different intermediate cases with delicate configurations of $\sigma(x)$, $\sigma_{\mathrm{loc}}(x)$ and stability/attractiveness of X.

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