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Abstract. We propose a computational approach for the solution of an optimal control problem governed by the wave equation. We aim at obtaining approximate feedback laws by means of the application of the dynamic programming principle. Since this methodology is only applicable for low-dimensional dynamical systems, we first introduce a reduced-order model for the wave equation by means of Proper Orthogonal Decomposition. The coupling between the reduced-order model and the related dynamic programming equation allows to obtain the desired approximation of the feedback law. We discuss numerical aspects of the feedback synthesis and provide numerical tests illustrating this approach.

Keywords. Optimal control, Feedback control, Dynamic programming, Hamilton-Jacobi-Bellman equation, Proper Orthogonal Decomposition, Wave equation.

1. Introduction and description of the problem

In this paper, we shall be concerned with the approximation of feedback laws for abstract infinite-dimensional optimal control problems of the type:

$$\min_{u\in L^2(0,\infty;\mathcal{U})} J(u) := \int_0^\infty L(y(t),u(t))e^{-\lambda s}\,dt\,, \quad \lambda > 0, \tag{1.1}$$

subject to

$$\dot{y} = Ay(t) + Bu(t), \qquad t \in (0,+\infty)\,, \tag{1.2}$$

$$y(0) = y_0\,. \tag{1.3}$$

Although the methodology that we will introduce can in principle be applied to general evolutive problems governed by partial differential equations, we will focus on the setting corresponding to a wave equation over a space domain $\Omega \subset \mathbb{R}$:

$$\begin{cases} w_{tt} - w_{xx} - \beta w_{xxt} = u(x,t) & \text{in } \Omega\times(0,\infty), \\ w(\cdot,0) = w_0,\, w_t(\cdot,0) = w_1 & \text{in } \Omega, \\ w(\cdot,t) = 0 & \text{in } \partial\Omega\times(0,\infty), \end{cases}, \tag{1.4}$$

which can be cast in the canonical state-space representation by defining

$$y(t) := \begin{pmatrix} w(x,t) \\ w_t(x,t) \end{pmatrix}, \quad A := \begin{pmatrix} 0 & I \\ \Delta & \beta\Delta \end{pmatrix}, \quad Bu := \begin{pmatrix} 0 \\ \chi_\omega(x)u(t) \end{pmatrix}. \tag{1.5}$$

Here $\beta > 0$ denotes a damping parameter, and χ_ω corresponds to the indicator function of a subset $\omega \subset \Omega$, i.e. the control force is a constant load in space. The running cost is given by

$$L(y,u) := \|w(\cdot,t;u)\|^2_{H^1(\Omega)} + \|w_t(\cdot,t;u)\|^2_{L^2(\Omega)} + \alpha|u(t)|^2 . \tag{1.6}$$

Among the available solution techniques for the aforementioned problem, we are interested in the design of optimal feedback control laws, where the optimal control $u^*(t)$ can by expressed through the action of a mapping $\mathcal{K}(y)$ acting on the current state of the system,

$$u^*(t) = -\mathcal{K}(y(t)) . \tag{1.7}$$

Despite the vast literature concerning the analysis and numerical approximation of optimal control problems for the wave equation, the amount of works devoted to the synthesis of feedback controllers is rather reduced. In this direction, the application of the dynamic programming principle (DPP) is a powerful technique. It establishes a global link between the feedback mapping and the value function of the optimal control problem, which in turn is characterized as the viscosity solution of a Hamilton-Jacobi-Bellman (HJB) equation which is solved over the state space of the system dynamics. This latter fact is a major bottleneck for the application of DPP-based techniques in the optimal control of PDE's, as the natural approach for this class of control problems is to consider a semi-discretization (in space) via finite elements or finite differences of the abstract governing equations, leading to an inherently high-dimensional state space. This *curse of the dimensionality* has been a strong limitation for the computation of optimal feedback controllers in infinite-dimensional systems. However, in the last years several steps have been made to obtain reduced-order models for rather complicated dynamics and by the application of these techniques it is now possible to have a reasonable approximation of large-scale dynamics using a rather small number of basis functions. This can open the way to the DPP approach in high-dimensional systems. The aim of this paper is to study the interplay between reduced-order dynamics, the associated dynamic programming equation, the resulting feedback controller and its performance over the high-dimensional system. To set this paper into perspective we recall that a huge literature has been devoted to various aspects of control problems for the wave equation. Although a detailed picture of these contributions goes beyond the goals of this paper we want to mention some recent and relevant contributions on controllability by Lasiecka and Triggiani [16], Privat, Trélat and Zuazua [17] and Zhang, Zheng and Zuazua [20]. A thorough analysis of the approximation of the linear-quadratic regulator problem for the wave equation can be found in [10]. The interested reader can also find a comprehensive presentation of control problems for waves and their approximation in [6] (see also the long list of references therein).

The main stream for the optimal control of the wave equation is still related to open-loop controls based on the Pontryagin Maximum Principle (this approach has been extensively discussed in the monograph [19]). More recently, the Proper Orthogonal

Decomposition (POD) has been proposed for PDE control problems to open the way to the application of the dynamic programming approach. The dimensional reduction of the dynamics is necessary to bring the number of state variable to a manageable scale. The starting point has been the optimal control of the heat equation which exploits the regularity of the solutions for that equation [5] and then attacking more difficult problems as the advection-diffusion equation [1, 2, 11], Burgers's equation [14, 15] and Navier-Stokes [4]. We also mention that the semi-linear wave equation has been recently addressed in [13] with the goal of obtaining an approximate feedback law, by means of an spectral finite element discretization is used for the wave dynamics. Instead, we construct a POD approximation based on a finite element scheme.

2. Dynamic programming equations and its approximation

We illustrate the dynamic programming approach for optimal control problems of the form

$$\begin{cases} \min\limits_{u\in\mathcal{U}} J_x(u(t)) := \int\limits_0^\infty L(y(s),u(s))\, e^{-\lambda s}\, ds \\ \text{constrained by the dynamics} \\ \dot{y}(t) = f(y(t),u(t)) \\ y(0) = x \end{cases} \tag{2.8}$$

with system dynamics in $\mathbb{R}^n$ and a control signal $u(t) \in \mathcal{U} \equiv \{u(\cdot)\,\text{measurable}\,, u : [0,T] \to U\}$, where U is a compact subset of $\mathbb{R}^m$; we assume $\lambda > 0$, while $L(\cdot,\cdot)$ and $f(\cdot,\cdot)$ are Lipschitz-continuous, bounded functions. Note that the optimal control problem (1.1)-(1.3) fits into the more general setting (2.8) provided that the system dynamics are finite-dimensional. In this setting, a standard solution tool is the application of the dynamic programming principle, which leads to a characterization of the value function

$$v(x) := \inf_{u\in\mathcal{U}} J_x(u)$$

as unique viscosity solution of the HJB equation:

$$\lambda v(x) + \sup_{u\in\mathcal{U}}\{-Dv\cdot f(x,u) - L(x,u)\} = 0\,. \tag{2.9}$$

Equation (2.9) may be approximated in several ways, we consider a fully-discrete semi-Lagrangian scheme which is based on the discretization of the system dynamics with time step h, and a mesh parameter k, leading to a fully discrete approximation $V_{h,k}(x)$ satisfying

$$V_{h,k}(x_i) = \min_{u\in\mathcal{U}}\{(1-\lambda h)I_1[V_{h,k}](x_i + hf(x_i,u)) + L(x_i,u)\}\,, \tag{2.10}$$

for every element x_i of the discretized state space. Note that in general, the arrival point $x_i + hf(x_i,u)$ is not a node of the state space grid, and therefore the value is computed by means of an linear interpolation operator, denoted by $I_1[V_{h,k}]$.

The bottleneck of this approach is related to the so-called *curse of the dimensionality*, namely, the computational cost increases dramatically as soon as the dimension does. One way to overcome the dimensionality issue is the construction of efficient iterative solvers for (2.10). Note that it will not be enough for the purpose of the control of a

parial differential equation as explained in the next section.
The simplest iterative solver is based on a direct fixed point iteration of the value function (VI)

$$V_{h,k}^{j+1} = T(V_{h,k}^{j}) \tag{2.11}$$

$$[T(V)]_i := \min_{u \in \mathcal{U}} \{(1-\lambda h) I_1[V](x_i + hf(x_i,u)) + L(x_i,u)\}, \quad i = 1,\dots,N_p \tag{2.12}$$

where N_p denotes the number of nodes of the grid. Convergence is guaranteed for any initial condition $V_{h,k}^0$ since the operator on the right hand side is a contraction mapping. Although simple and reliable, this algorithm is computationally demanding and slow when fine meshes are considered.
A more efficient formulation is the so-called *policy iteration algorithm* (PI), which starting from an initial guess u_i^0 of the control at every node, performs the following iterative procedure:

$$[V_{h,k}^j]_i = (1-\lambda_h) I_1[V_{h,k}^j](x_i + hf(x_i,u_i^j)) + hL(x_i,u_i^j)$$

$$[u^{j+1}]_i = \operatorname*{argmin}_{u \in \mathcal{U}} \{(1-\lambda h) I_1[V_{h,k}^j](x_i + hf(x_i,u)) + hL(x_i,u)\}$$

where we first have to solve a linear system, since we freeze the control, in order to find the value function correspondent to the given control and then update the control. We iterate until we have convergence for the value function. It is well-known that the PI algorithm has a quadratic convergence provided a good initial guess. However, its convergence is only local (as for the Newton method), so there is a need for good initialization. This point is very delicate since this would require to know a reasonable approximation of the optimal feedback control. To solve this problem in [3] was proposed an acceleration mechanism based on a (VI) solution on a coarse grid, which is used to generate an initial guess for (PI) on the fine grid. This is based on the fact that (VI) generates a fast error decay when applied over coarse meshes, without depending on a good starting point. Therefore, the proposed approach is a way to enhance (PI) with both efficiency and robustness features. We adopt the aforementioned approach (shortly API) for the approximation of the HJB equation.

3. POD-Model Reduction for the controlled problem

In this section, we explain the POD method for the approximate solution of the optimal control problem. The approach is based on projecting the nonlinear dynamics onto a low dimensional manifold utilizing projectors which contain information of the dynamics. A common approach in this framework is based on the snapshot form of POD proposed by Sirovich in [18], which works as follows.
The snapshots are computed on the basis of a finite element discretization in space of the system dynamics (1.4) which leads to a semi-discrete system of ODEs of the form:

$$M^N \dot{y}^N = A^N y^N(t) + B^N u(t), \qquad t \in (0,+\infty), \tag{3.13}$$

$$y^N(0) = y_0^N. \tag{3.14}$$

where A^N, B^N, M^N correspond to the finite-dimensional projections of the abstract operators in (1.4). Let us assume to know the finite element solutions $y(t_i) \in \mathbb{R}^N$ for given time instances. We define the snapshot matrix $Y = [y(t_1), \ldots, y(t_n)]$ and determine its singular value decomposition $W^{1/2}Y = \Psi\Sigma V$. The POD basis functions $\{\psi_i\}_{i=1}^{\ell}$ of rank ℓ are the first ℓ columns of the matrix Ψ and we define the POD ansatz of order ℓ for the state y as

$$y^N(t) \approx \sum_{i=1}^{\ell} y_i^\ell(t)\psi_i \,. \tag{3.15}$$

The reduced optimal control problem is obtained through replacing (3.13) by a dynamical system obtained from a Galerkin approximation with basis functions $\{\psi_i\}_{i=1}^{\ell}$ and ansatz (3.15) for the state. This leads to a $\ell-$dimensional system for the unknown coefficients $\{y_i^\ell\}_{i=1}^{\ell}$, namely

$$M^\ell \dot{y}^\ell + A^\ell y^\ell = B^\ell u, \quad y^\ell(0) = y_0^\ell. \tag{3.16}$$

Here the entries of the mass M^ℓ and the stiffness A^ℓ are given by $\langle \psi_j, M^N \psi_i \rangle_W$ and $\langle \psi_j, A^N \psi_i \rangle_W$, respectively. The weighted inner product is defined as: $\langle q, p \rangle_W = q^\top W p$ where $p, q \in \mathbb{R}^N$ and $W \in \mathbb{R}^{N\times N}$ is induced from the approximation of the abstract state space, which is our case is considered to be $H^1(\Omega) \times L^2(\Omega)$. The error of the Galerkin projection is governed by the singular values associated to the truncated states of the SVD.
The POD-Galerkin approximation leads to the optimization problem

$$\inf_{u\in\mathcal{U}} J^\ell_{y_0^\ell}(u) := \int_0^\infty L(y^\ell(s), u(s))e^{-\lambda s}\, ds, \tag{3.17}$$

where $u \in \mathcal{U}$, y^ℓ solves the reduced dynamics (3.16) . The value function v^ℓ, defined for the initial state $y_0^\ell \in \mathbb{R}^\ell$ is given by

$$v^\ell(y_0^\ell) = \inf_{u\in\mathcal{U}} J^\ell_{y_0^\ell}(u)\,.$$

Note that the resulting HJB equations are defined in $\mathbb{R}^\ell$, but for computational purposes we need to restrict our numerical domain to a bounded subset of $\mathbb{R}^\ell$. We refer the interested reader to [1] for a details on this issue.

4. Remarks on the computation of the feedback control

The main advantage of the dynamic programming approach is the possibility to have a synthesis of feedback controls. Once the discretized value function $V_{h,k}$ has been obtained, the approximated optimal control $u^*_{h,k}(x)$ for a point x of the state space is given by:

$$u^*_{h,k}(x) = \arg\min_{u\in\mathcal{U}}\{(1-\lambda h)I_1[V_{h,k}](x + hf(x,u)) + l(x,u)\} \tag{4.18}$$

This choice is quasi-optimal provided some additional condition on the dynamics are satisfied, a typical example is a linear dependence on control variable as it has been shown in [8, p. 231] . However, some difficulties arise in this approximation, we examine them and provide some possible solutions.

Discretization of the control space. A first step for the computation of the argmin appearing in (4.18) is to replace the continuous control space $\mathcal{U}$ by a discrete number of steps, i.e. considering the discrete control space $\widehat{\mathcal{U}} = \{u_1, \ldots, u_m\}$. This, of course, introduces an additional discretization error which could be avoided working in a continuous space as in [12] or by a descent method without derivatives (like Brent's algorithm) . The discretization error can be reduced by an iterative bisection method described in [7].

Non uniqueness of the argmin. When the minimum is obtained via a discretization of the control space one should also take into account the error in the discretization of the value function. So all the controls which give an approximate value of the right-hand side of (4.18) close to the minimum by a tolerance $\varepsilon(k)$ should also be taken into account, we will denote this set by $\widehat{\mathcal{U}}_\varepsilon$. Then, we have to choose among them following another rule, e.g. by taking the control which is optimal with respect to another convex criterium like the squared norm of the control.

Chattering along the optimal trajectories. Even with the adoption of an additional selection rule it could be that the feedback control jumps introducing a chattering behavior along the optimal trajectories. This is typical around the curves where the value function is not regular (remember that the value function is only Lipschitz continuous). To avoid the chattering and stabilize the trajectory we can introduce an inertia penalization in the numerical realization of the feedback reconstruction (4.18), which we illustrate in Algorithm 4.1.

Algorithm 4.1 Feedback regularization algorithm

```
Require: V_{h,k}, tolerance ϵ, current state x, discretized control set U = {u_1, ..., u_M},
    admissible optimal control set Û_ε = ∅, previous optimal control u_old.
for i=1,...,M do
    Val(i) = {(1 − λh) I_1[V](x + h f(x, u_i) − L(x, u_i)}.
end for
Val* = min{Val}
for i = 1, ..., M do
    if |Val* − Val(i)| < ϵ then
        Û_ε = Û_ε ⋃ u_i
    end if
end for
u*_{h,k}(x) = argmin_{u ∈ Û_ε} |u − u_old|
```

5. Numerical Tests

This section presents numerical tests aiming at assessing the performance of the proposed method. We will present two test cases. The first experiment shows for the damped case $\beta > 0$, the performance of the coupling between HJB and POD methods and the study of the chattering of the optimal control. In the second test we focus on the undamped case $\beta = 0$. Both examples are compared with a linear-quadratic regulator (LQR) controller implemented as in [10]. This latter approach is based on the solution of a large-scale Riccati equation and does not consider a dimensional reduction step. For the relation between this approach and the HJB-based control, we refer to [8, Chapter 8].

Test 1: damped wave equation. In (1.4) we set: $\Omega = (0,1), \omega = (0.4, 0.6), w_0 = \sin(\pi x), w_1 \equiv 0, \beta = 0.05$. In (2.10) we take $k = \{0.1, 0.05\}, h = 0.1k, \lambda = 1, U = [-1.2, 0, 6], \alpha = 1$. The discrete control set contains 19 equidistant controls. The reduced domain is $[-2.25, 2.55] \times [-2.15, 1.95] \times [-0.65, 1.05] \times [-0.35, 0.25] \times [-0.25, 0.25]$. The cost functional is expressed in (1.6). The snapshots are computed from a $\mathbb{P}^1$ finite element discretization in space with an implicit time integrator for given constant control inputs $u \equiv \{-1.2, 0, 0.6\}$. In order to compare our proposed algorithm, we also compute the solution with a discounted LQR control, and fix control set U from the active set of the LQR result.
Figure 5.1, presents a comparison between the uncontrolled evolution, the controlled states by means of an LQR control for the full-order dynamics, and the HJB-POD approach with 5 reduced states. This is consistent with the fact that for inactive control constraints and similar settings, both controllers are equivalent. The effect of the dissipative damping term guarantees the fast decay of the singular values related to the POD approximation, and therefore a good approximation of the system dynamics can be obtained with a reduced number of states. Table 5.1 shows the error

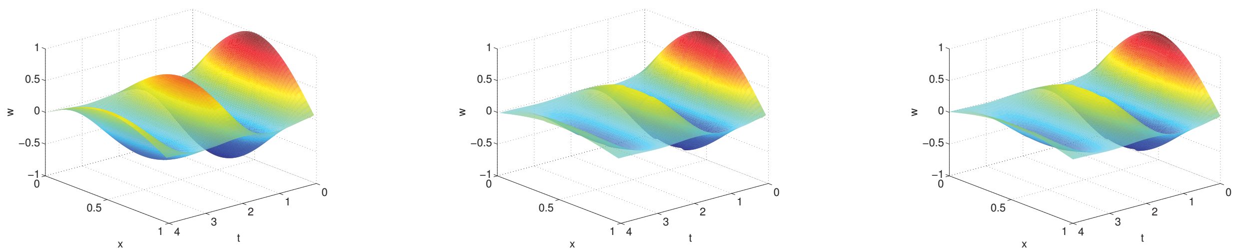

FIGURE 5.1. Test 1: Uncontrolled solution (left), controlled state with full-order LQR control (middle), controlled state with HJB-POD design with 5 reduced states (right).

between LQR, considered as *truth solution*, and the HJB-POD approximation. As expected, the error decays as soon as we increase the number of basis functions ℓ and decrease the step size k. It is worth to comment on the CPU time of the API scheme for the approximation of the HJB equation. The value function with 5 basis functions is computed in $722[s]$ for the API scheme, compared to $6.63 \times 10^3[s]$ if computed with a standard Value Iteration scheme.
Finally, we would like to put our attention on the optimal control output. A common problem dealing with numerical feedback control, and in particular with numerical approximations of HJB equations is the chattering of the control output. Our goal is

	$k = 0.1$	$k = 0.05$
$\ell = 3$	0.1224	0.0953
$\ell = 4$	0.1009	0.0886
$\ell = 5$	0.0648	0.0468

TABLE 5.1. H_0^1- error for w between LQR control and HJB-POD approximations at time $t = 4$. The number of basis functions is ℓ and k is the step size for the approximation of the value function.

to minimize the jumps of the control signal in order to obtain a smoother approximation. Figure 5.2 shows the difference between the numerical HJB feedback control, the LQR control signal and the regularized HJB control following the algorithm presented in Section 4. In particular, our regularization algorithm considerably reduced the chattering of the control signal, mimicking in a better way the LQR control. Note that the LQR feedback is obtained over a continuous control set $\mathcal{U}$, whereas the HJB approach uses a discrete approximation. Finally, we note that the error between the HJB-POD solution correspondent to $\ell = 5$ and $k = 0.05$ with a regularized control and the LQR controlled state is 0.0304 instead of 0.0468, yielding a more accurate approximation of the control trajectory.

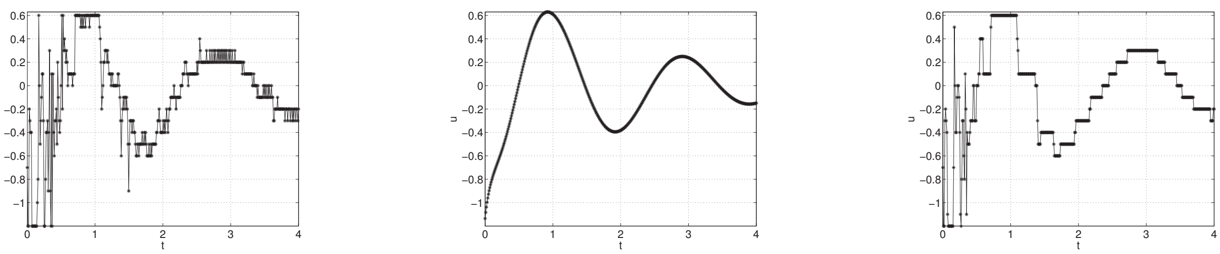

FIGURE 5.2. Test 1: HJB-POD feedback control with chattering (left) LQR control (middle), HJB-POD feedback control with chattering reduction (right).

Test 2: undamped wave equation. The second test differs from the first in the choice of the parameter $\beta = 0$ and the control set $U = [-1.5, 1.5]$. In this case the reduced domain is $[-2.7, 2.7] \times [-3.1, 3.1] \times [-1.3, 1.3] \times [-0.7, 0.7] \times [-0.8, 0.8]$. In the undamped case, the numerical approximation of the LQR problem is a difficult task due to the number of purely complex eigenvalues of the matrix A^N. The snapshots are computed with a finite element scheme for given constant control inputs $u \equiv \{-1.5, 0, 1.5\}$ in (1.4), and the discrete control set contains 31 equidistant elements. Figure 5.3 presents the uncontrolled and controlled states for the different settings. The LQR control exhibits a suboptimal stabilizing behaviour, due to innacuracies in the solution of the large-scale algebraic Riccati equation for the undamped system. Instead, the HJB-POD does not suffer from this problem and the system is stabilized at a faster rate, although the control signal generates some spurious oscillations on the state. Due to the non-dissipative dynamics, it is known that a larger number of POD basis functions would be required for a more accurate control synthesis. The different control signals are shown in Figure 5.4. We can observe the chattering of LQR control which explains the slow stabilization rate. The HJB-POD feedback control also exhibits chattering, but it is regularized by means of the same approach applied in the previous test.

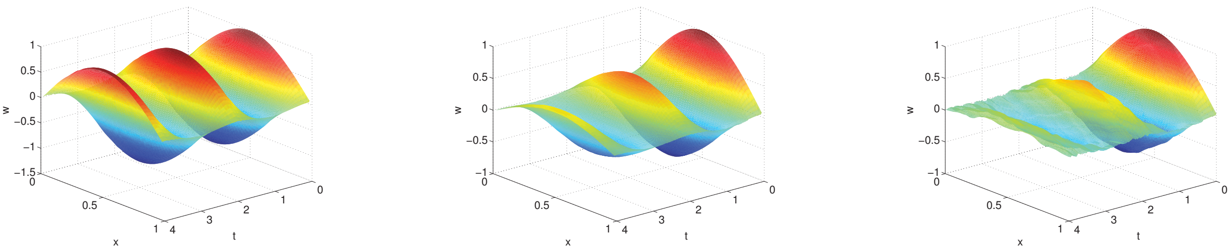

FIGURE 5.3. Test 2: Uncontrolled solution (left), controlled state with full-order LQR control (middle), controlled state with HJB-POD design with 5 reduced states (right).

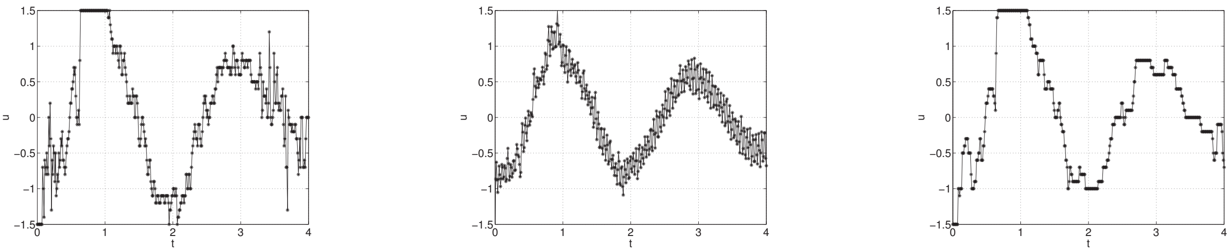

FIGURE 5.4. Test 2: HJB-POD feedback control with chattering (left) LQR control (middle), HJB-POD feedback control with chattering reduction (right).

Concluding remarks. We have presented a computational approach for the approximation of feedback controllers in the wave equation. The approach is based on a projection of the abstract system onto a low-order model. A dynamic programming-based controller is then synthesized for the reduced system. In general the approach has proven satisfactory in the sense that is able to generate stabilizing feedback controllers based on low-dimensional dynamics, which perform in a consistent way when applied to the full system. Future directions of research consider the inclusion on nonlinearities in the dynamics and multi-dimensional extensions of the problem.

References

[1] A. Alla, and M. Falcone. *An adaptive POD approximation method for the control of advection-diffusion equations,* in K. Kunisch, K. Bredies, C. Clason, G. von Winckel, (eds) *Control and Optimization with PDE Constraints,* International Series of Numerical Mathematics, **164**, Birkhäuser, Basel, 2013, 1-17.

[2] A. Alla and M. Falcone. *A Time-Adaptive POD Method for Optimal Control Problems,* in the Proceedings of the 1st IFAC Workshop on Control of Systems Modeled by Partial Differential Equations, **1**, 2013, 245-250.

[3] A. Alla, M. Falcone and D. Kalise. *An efficient policy iteration algorithm for dynamic programming equations*, SIAM J. Sci. Comput., **37**(1), 2015, 181-200.

[4] A. Alla and M. Hinze. *HJB-POD feedback control for Navier-Stokes equations* preprint, available at: `http://preprint.math.uni-hamburg.de/public/papers/hbam/hbam2014-09.pdf`

[5] J.A. Atwell and B.B. King. *Proper Orthogonal Decomposition for Reduced Basis Feedback Controllers for Parabolic Equations*, Matematical and Computer Modelling. **33**, 2001, 1-19.

[6] S. Ervedoza andE. Zuazua, *The wave equation: control and numerics*, Control of partial differential equations, Lecture Notes in Math., **2048**, Springer, Heidelberg, 2012, 245–339.

[7] M. Falcone. *Some remarks on the synthesis of feedback controls via numerical methods*, in J.L. Menaldi, E. Rofman, A. Sulem (eds), Optimal Control and Partial Differential Equations, IOS Press, 2001, 456-465.

[8] M. Falcone and R. Ferretti. *Semi-Lagrangian Approximation Schemes for Linear and Hamilton-Jacobi Equations,* SIAM, 2014.

[9] M. Gubisch and S. Volkwein. Proper Orthogonal Decomposition for Linear-Quadratic Optimal Control, preprint, available at: `http://kops.ub.uni-konstanz.de/handle/urn:nbn:de:bsz:352-250378`.

[10] E. Hernández, D. Kalise and E. Otárola. Numerical approximation of the LQR problem in a strongly damped wave equation, Computational Optimization and Applications, **47**(1), 2010, 161-178.

[11] D. Kalise and A. Kröner. Reduced-order minimum time control of advection-reaction-diffusion systems via dynamic programming Proceedings of the 21st International Symposium on Mathematical Theory of Networks and Systems, 2014, 1196-1202.

[12] D. Kalise, A. Kröner and K. Kunisch. Local minimization algorithms for dynamic programming equations preprint, available at: `http://www.ricam.oeaw.ac.at/publications/reports/15/rep15-04.pdf`.

[13] A. Kröner, K. Kunisch and H. Zidani. *Optimal feedback control of undamped wave equations by solving a HJB equation,* preprint, available at: `http://www.cmap.polytechnique.fr/~kroener/KroenerKunischZidani2013.pdf`

[14] K. Kunisch, S. Volkwein and L. Xie. *HJB-POD Based Feedback Design for the Optimal Control of Evolution Problems,* SIAM J. on Applied Dynamical Systems, **4**, 2004, 701-722.

[15] K. Kunisch and L. Xie. *POD-Based Feedback Control of Burgers Equation by Solving the Evolutionary HJB Equation,* Computers and Mathematics with Applications. **49**, 2005, 1113-1126.

[16] I. Lasiecka and R. Triggiani. *Global exact controllability of semilinear wave equations by a double compactness/uniqueness argument*, Discrete Contin. Dyn. Syst., 2005, 556–565.

[17] Y. Privat, E. Trélat and E. Zuazua. *Optimal location of controllers for the one–dimensional wave equation*, Ann. Inst. H. Poincar'é Anal. Non Linéaire **30**, 2013, 1097–1126.

[18] L. Sirovich. *Turbulence and the dynamics of coherent structures. Parts I-II,* Quarterly of Applied Mathematics, **XVL**, 1987, 561-590.

[19] F. Tröltzsch. *Optimal Control of Partial Differential Equations: Theory, Methods and Application*, American Mathematical Society, 2010.

[20] X. Zhang, C. Zheng and E. Zuazua. *Exact controllability of the time discrete wave equation: a multiplier approach*, Comput. Methods Appl. Sci. **15**, 2010, 229–245.

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