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A NOTE ON LOCAL CENTER MANIFOLDS FOR DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY

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ABSTRACT. In this note we consider local invariant manifolds of functional differential equations $x'(t) = f(x_t)$ representing differential equations with state-dependent delay. Starting with a local center-stable and a local center-unstable manifold of the functional differential equation at a stationary point, we construct, by a straightforward application of the Implicit Mapping Theorem, a local center manifold.

1. INTRODUCTION

Although the first studies of differential equations with state-dependent delay go back at least to the very beginning of 19th century, a significant research interest in that type of equations began not till the second half of the last century. In the light of this, it is unsurprising that only in the last descent there were developed a general framework for studying differential equations with state-dependent delay in the context of the dynamical systems theory. The starting point here was the work [8] of Walther proving, under some mild smoothness assumptions, the existence of a continuous semiflow with C^1-smooth time-t-maps for differential equations with state-dependent delay. In subsequent years this semiflow and its properties were studied in different papers under slightly modified smoothness conditions, and the best general reference here is the survey article [2] of Hartung et al. and the references therein.

Now it is well known that the mentioned semiflow particularly has different types of local invariant manifolds at a stationary point. For instance, the survey article [2] of Hartung et al. contains the construction of so-called *local center manifolds* by employing the Lyapunov-Perron method (with appropriate changes). The C^1-smoothness of these finite-dimensional invariant manifolds is shown in Krisztin [4]. Some years later the approaches in Hartung et al. [2] and in Krisztin [4] were adopted in [6], in order to prove the existence and C^1-smoothness of another type of finite-dimensional local invariant manifolds, namely, of so-called *local center-unstable manifolds*. Contrary to that, a proof of C^1-smooth *local center-stable manifolds*, which are infinite-dimensional, could not be obtained by employing a variation of the Lyapunov-Perron technique. However, as shown in Qesmi and Walther [5], they arise from local center-stable manifolds of time-t-maps of some global semiflows that are modifications of the original semiflow.

In the situation of a flow generated by an ordinary differential equation in a finite-dimensional space, of course, all three mentioned types of locally invariant manifolds at a stationary point are well understood. In this case, each of them is finite-dimensional and can be constructed, for instance, by applying the Lyapunov-Perron method as can be found in Vanderbauwhede [7], or by using the graph-transformation method as discussed in Kelley [3]. Additionally, it is well known that a local center manifold at a stationary point may also be obtained by intersecting a local center-unstable and a local center-stable manifold at the same stationary point.

Returning to differential equations with state-dependent delay, it suggests itself to ask whether the last point does also hold in the situation of the discussed semiflow; that is, does an intersection between a local center-unstable manifold from [6] and a local center-stable manifold from Qesmi and Walther [5] at the same stationary point form or contain a local center manifold as constructed in Hartung et al. [2]? The present paper answers this question in the affirmative and that was to be expected. By using local representations of the involved manifolds as graphs of maps and applying the Implicit Mapping Theorem, we show in a purely analytical way – in particular, without discussing properties such as transversality etc. from the geometric theory of differentiable manifolds – that the intersection of a local center-unstable and a local center-stable manifold indeed contains a local center manifold. A simple consequence of this is the fact that given a local center-stable and/or a local center-unstable manifold of the discussed semiflow at a stationary point we always find a local center manifold contained in the considered local center-stable and/or local center-unstable manifold.

The general approach applied here, that is, the construction of a local manifold via intersecting two non-disjoint manifolds by means of the Implicit Mapping Theorem, is certainly not new and it works of course also in the case of a flow induced by an ordinary differential equation. But observe that for a differential equation with state-dependent the situation is somewhat more subtil than for an ordinary differential equation in finite-dimensional space, since the state-space itself is a submanifold with finite codimension of an infinite dimensional Banach space.

The remaining part of this note is organized as follows. The next section contains a brief summary of the general setting. After introducing the differential equation together with the smoothness assumptions under consideration, we discuss here the mentioned semiflow and some of its properties. In the end, we state the local center manifold theorem – compare Theorem 1 – obtained in [2, 4].

Section 3 forms the main part. Starting with a local center-unstable and a local center-stable manifold, we construct a local center manifold and give so an alternative proof of the local center manifold theorem for differential equations with state-dependent delay.

2. OUTLINE OF A SEMIFLOW FRAMEWORK FOR DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY

In the sequel we give a short summary of a general concept for studying differential equations with state-dependent delay in the context of the dynamical systems

theory. For the left proofs as well as for a thorough discussion of the topic we refer the reader to the survey work [2] of Hartung at al. and the references therein.

Let $n \in \mathbb{N}$ and $h > 0$ be fixed. Further, let $\| \cdot \|_{\mathbb{R}^n}$ stand for any fixed norm on the n-dimensional Euclidean space $\mathbb{R}^n$, and C for the Banach space of all continuous functions $\varphi : [-h, 0] \to \mathbb{R}^n$ equipped with the usual norm $\|\varphi\|_C := \sup_{-h \le s \le 0} \|\varphi(s)\|_{\mathbb{R}^n}$. Similarly, we write C^1 for the Banach space of all continuously differentiable functions $\varphi : [-h, 0] \to \mathbb{R}^n$ with the norm given by $\|\varphi\|_{C^1} := \|\varphi\|_C + \|\varphi'\|_C$.

Given a function $x : I \to \mathbb{R}^n$ defined on some interval $I \subset \mathbb{R}$ and $t \in \mathbb{R}$ with $[t - h, t] \subset I$, we will write x_t for the function $[-h, 0] \ni s \mapsto x(t + s) \in \mathbb{R}^n$, which is also known as the *segment* of x at t.

From now on, we consider the functional differential equation

$$(1) \qquad\qquad x'(t) = f(x_t)$$

defined by a map $f : C^1 \supset U \to \mathbb{R}^n$ on some open neighborhood U of $0 \in C^1$ with $f(0) = 0$. Under a *solution* of Eq. (1) we understand either a continuously differentiable function $x : [t_0 - h, t_+) \to \mathbb{R}^n$, $t_0 < t_+ \le \infty$, such that $x_t \in U$ for all $t_0 \le t < t_+$ and x satisfies Eq. (1) as $t_0 < t < t_+$, or a continuously differentiable function $x : \mathbb{R} \to \mathbb{R}^n$ satisfying $x_t \in U$ and Eq. (1) for all of $t \in \mathbb{R}$. In this sense, the function $x : \mathbb{R} \ni t \mapsto 0 \in \mathbb{R}^n$ is clearly a solution of Eq. (1) due to the assumption $f(0) = 0$.

Before discussing the assumptions on f which will ensure the existence of further solutions of Eq. (1), we should mention briefly the connection between differential equations with state-dependent delay and equations of the form (1). For this purpose, consider the differential equation

$$(2) \qquad\qquad x'(t) = g(x(t - r(x(t))))$$

defined by some function $g : \mathbb{R} \to \mathbb{R}^n$ with $g(0) = 0$ and involving a state-dependent delay given by another function $r : \mathbb{R}^n \to [-h, 0]$. Introducing

$$\hat{f} : C^1 \ni \varphi \mapsto g(\varphi(-r(\varphi(0)))) \in \mathbb{R}^n,$$

we see that

$$(3) \qquad x'(t) = g(x(t - r(x(t)))) = g(x(t - r(x_t(0)))) = g(x_t(-r(x_t(0))) = \hat{f}(x_t),$$

that is, the differential equation (2) with state-dependent delay takes the more abstract form of Eq. (1). Consequently, instead of studying the original equation (2), we may just as well study Eq. (3).

The presented transformation also works in many other cases of differential equations with state-dependent delay than the simple one discussed above. However, having in mind that Eq. (1) represents a differential equation with a state-dependent delay, we should particularly impose only those smoothness assumptions on f, which are typically fulfilled by differential equations with state-dependent delay. With this regard, we follow the further development of the ideas contained in Walther [8], and suppose that f satisfies the following conditions:

(S1) f is continuously differentiable, and

(S2) for each $\varphi \in U$ the derivative $Df(\varphi) : C^1 \to \mathbb{R}^n$ extends to a linear map $D_e f(\varphi) : C \to \mathbb{R}^n$ such that the map $U \times C \ni (\varphi, \psi) \mapsto D_e f(\varphi)\psi \in \mathbb{R}^n$ is continuous.

Then the closed and, in view of the zero function, nonempty subset

$$X_f := \{\varphi \in U \mid \varphi'(0) = f(\varphi)\}$$

of $U \subset C^1$ forms a continuously differentiable submanifold of U with codimension n. For each $\varphi \in X_f$, Eq. (1) has a unique (in the forward time-direction) noncontinuable solution $x^\varphi : [-h, t_+(\varphi)) \to \mathbb{R}^n$ satisfying $x_0^\varphi = \varphi$. Furthermore, for all $\varphi \in X_f$ and all $0 \le t < t_+(\varphi)$ the segments x_t^φ belong to X_f, and the relations

$$F(t, \varphi) := x_t^\varphi$$

as $\varphi \in X_f$ and $0 \le t < t_+(\varphi)$ define a domain $\Omega \subset [0, \infty) \times X_f$ and a continuous semiflow $F : \Omega \to X_f$ with C^1-smooth time-t-maps $F_t := F(t, \cdot)$.

Now, recall that $x(t) = 0$, $t \in \mathbb{R}$, is a solution of Eq. (1). For this reason, $\varphi_0 = 0 \in X_f$ is a stationary point of the semiflow F, that is, $F(t, \varphi_0) = 0$ for all $t \ge 0$. The linearization of F at φ_0 is the strongly continuous semigroup $T = \{T(t)\}_{t \ge 0}$ of bounded linear operators $T(t) := DF_t(0) : T_0 X_f \to T_0 X_f$ on the tangent space

$$T_0 X_f := \{\chi \in C^1 \mid \chi'(0) = Df(0)\chi\}$$

of F at φ_0, which forms a Banach space with respect to the norm $\| \cdot \|_{C^1}$ of C^1. Given $t \ge 0$ and $\chi \in T_0 X_f$, the operator $T(t)$ maps χ to the segment v_t^χ of the uniquely determined solution $v^\chi : [-h, \infty) \to \mathbb{R}^n$ of the linear variational equation

$$v'(t) = Df(0)v_t$$

with initial value $v_0^\chi = \chi$. The infinitesimal generator of T is the linear operator $G : \mathcal{D}(G) \ni \chi \mapsto \chi' \in T_0 X_f$ defined on the closed subset

$$\mathcal{D}(G) := \{\chi \in C^2 \mid \chi'(0) = Df(0)\chi, \chi''(0) = Df(0)\chi'\}$$

of the space C^2 of all twice continuously differentiable functions from $[-h, 0]$ into $\mathbb{R}^n$.

In order to describe the spectrum of the linearization T, or more precisely, the spectrum $\sigma(G) \subset \mathbb{C}$ of its generator G, remember that due to condition (S2) the derivative $Df(0)$ is extendible to a bounded linear operator $D_e f(0) : C \to \mathbb{R}^n$. In particular, $D_e f(0)$ defines the linear retarded functional differential equation

$$v'(t) = D_e f(0)v_t$$

on C. As shown, for instance, in Diekmann et al. [1], the solutions of the associated Cauchy problems induce a strongly continuous semigroup $T_e := \{T_e(t)\}_{t \ge 0}$ of linear bounded operators $T_e(t) : C \to C$, and $G_e : \mathcal{D}(G_e) \ni \varphi \mapsto \varphi' \in C$ with

$$\mathcal{D}(G_e) := \{\chi \in C^1 \mid \chi'(0) = Df_e(0)\chi\}$$

forms the generator of T_e. Obviously, $T_0 X_f = \mathcal{D}(G_e)$. But even more is true: For all $t \ge 0$ and all $\varphi \in \mathcal{D}(G_e)$ we have $T(t)\varphi = T_e(t)\varphi$, and the spectra $\sigma(G)$ and $\sigma(G_e)$ coincide. The spectrum $\sigma(G_e) \subset \mathbb{C}$ of G_e is given by the zeros of a familiar characteristic function. In particular, it is discrete, contains only eigenvalues with finite dimensional generalized eigenspaces, and for each $\beta \in \mathbb{R}$ the intersection $\sigma(G_e) \cap \{\lambda \in \mathbb{C} \mid \Re(\lambda) \ge \beta\}$ is either finite or empty. For this reason, each of the spectral parts $\sigma_u(G) := \{\lambda \in \sigma(G_e) \mid \Re(\lambda) > 0\}$ and $\sigma_c(G) := \{\lambda \in \sigma(G_0) \mid \Re(\lambda) = 0\}$ is either finite or empty as well. Moreover, it follows that the associated realified generalized eigenspaces C_u and C_c, which are called the *unstable* and the

center space of G_e, respectively, are finite dimensional subspaces of $T_0X_f \subset C^1$. In contrast to those, the *stable space* of G_e, that is, the realified generalized eigenspace given by the eigenvalues with negative real part, is infinite dimensional and belongs not to T_0X_f. In all, these subspaces provide the decomposition

$$C = C_u \oplus C_c \oplus C_s$$

of C. Since $C_s^1 := C^1 \cap C_s$ is closed in C^1 we also have the decomposition

(4) $$C^1 = C_u \oplus C_c \oplus C_s^1$$

of the smaller Banach space C^1. The unstable and center space of G coincide with C_u and C_c, respectively, whereas the stable space of G is given by the intersection $C_s \cap T_0X_f = C_s^1 \cap T_0X_f$. In particular, we get

$$T_0X_f = C_u \oplus C_c \oplus (C_s^1 \cap T_0X_f).$$

Next, we repeat some facts about local invariant manifolds of F at $\varphi_0 = 0$. For doing so, recall that a *trajectory* of F is a map $\gamma : I \to X_f$, $I \subset \mathbb{R}$ an interval, with $\gamma(t) = F(t - s, \gamma(s))$ whenever $s, t \in I$ and $t \geq s$. We begin with a statement contained in Qesmi and Walther [5, Theorem 1.1]: There exist an open neighborhood N_{cs} of 0 in U and a continuously differentiable submanifold $W_{cs} \subset X_f$ with

$$T_0W_{cs} = C_c \oplus (C_s^1 \cap T_0X_f)$$

which has the following properties.

(CS 1) W_{cs} is positively invariant with respect to F relative to N_{cs}; that is, for all $\varphi \in W_{cs}$ and all $0 \leq t < t_\varphi$ with $F([0,t] \times \{\varphi\}) \subset N_{cs}$ we have $F(s, \varphi) \in W_{cs}$ as $0 \leq s \leq t$.

(CS 2) W_{cs} contains all initial values $\varphi \in X_f$ with $F(t, \varphi) \in N_{cs}$ for all $0 \leq t < \infty$.

(CS 3) If $\gamma : [t, 0] \to X_f$, $t < 0$, is a trajectory of F with $\gamma(0) \in W_{cs}$ and with $\gamma([t, 0]) \subset N_{cs}$, then $\gamma(s) \in W_{cs}$ as $t \leq s \leq 0$.

The submanifold W_{cs} of X_f is called a *local center-stable manifold* of F at φ_0. In terms of solutions of Eq. (1), the assertion (CS 1) means that all the segments of a solution of Eq. (1) with initial value in W_{cs} remain in W_{cs} as long as the solution does not leave the neighborhood N_{cs} of φ_0, whereas assertion (CS 2) says that W_{cs} contains the segments of all sufficiently small solutions $x : [-h, \infty) \to \mathbb{R}^n$ of Eq. (1).

The counterpart of a local center-stable manifold W_{cs} is formed by a so-called *local center-unstable manifold* W_{cu} which does exist under the additional assumption that $\{\lambda \in \sigma(G_e) \mid \Re(\lambda) \geq 0\} \neq \emptyset$ and so $C_{cu} := C_u \oplus C_c \neq \{0\}$ holds as discussed in [6, Theorems 1 & 2]: There exist an open neighborhood N_{cu} of 0 in U and a continuously differentiable submanifold $W_{cu} \subset X_f$ of dimension $\dim C_{cu} \geq 1$ with

$$T_0W_{cu} = C_{cu} = C_u \oplus C_c$$

which has the following properties:

(CU 1) W_{cu} is positively invariant with respect to F relative to N_{cu}; that is, for all $\varphi \in W_{cu}$ and all $0 \leq t < t_\varphi$ with $F([0,t] \times \{\varphi\}) \subset N_{cu}$ we have $F(s, \varphi) \in W_{cu}$ as $0 \leq s \leq t$.

(CU 2) If $\gamma : (-\infty, 0] \to X_f$ is a trajectory of F with $\gamma((-\infty, 0]) \subset N_{cu}$ then $\gamma(t) \in W_{cu}$ for all $t \leq 0$.

Concerning solutions of Eq. (1), the property (CU 1) of W_{cu} is completely similar to the one of the local center-stable manifold W_{cs}, whereas assertion (CU 2) means that W_{cu} contains the segments of all sufficiently small solutions $x : (-\infty, 0] \to \mathbb{R}^n$ of Eq. (1). But let us now state the Local Center Manifold Theorem which is proven in [2, 4] and whose alternative construction is the main ingredient of this note.

Theorem 1 (see Theorem 4.1.1 in [2] and Theorem 2.1 in [4]). *Suppose that, in addition to the stated assumptions on f, $\sigma(G_e) \cap i\mathbb{R} \neq \emptyset$, that is, $\dim C_c \geq 1$ holds. Then there exist open neighborhoods $C_{c,0}$ of 0 in C_c and $C^1_{su,0}$ in $C^1_s \oplus C_u$ with $N_c = C_{c,0} + C^1_{su,0} \subset U$ and a continuously differentiable map $w_c : C_{c,0} \to C^1_{su,0}$ such that $w_c(0) = 0$, $Dw_c(0) = 0$, and such that for the graph*

$$W_c = \{\varphi + w_c(\varphi) \mid \varphi \in C_{c,0}\}$$

of w_c, which is called a local center manifold of F at $\varphi_0 = 0$, the following holds.

(C 1) *$W_c \subset X_f$, and W_c is a $\dim C_c$-dimensional continuously differentiable submanifold of X_f.*

(C 2) *W_c is positively invariant with respect to F relative to N_c; that is, for all $\varphi \in W_c$ and all $0 \leq t < t_\varphi$ with $F([0, t] \times \{\varphi\}) \subset N_c$ we have $F(s, \varphi) \in W_c$ as $0 \leq s \leq t$.*

(C 3) *If $\gamma : \mathbb{R} \to X_f$ is a trajectory of F with $\gamma(\mathbb{R}) \subset N_c$, then $\gamma(t) \in W_c$ for all $t \in \mathbb{R}$.*

In contrast to the discussion of W_{cs} and W_{cu}, the above result is formulated in terms of a local representation. Observe that we have

$$T_0 W_c = C_c$$

and that by (C 3) the local center manifold W_c contains the segments of all globally defined and small enough solutions of Eq. (1). As already mentioned in the introduction, the original proof of Theorem 1 is based on the Lyapunov-Perron method. In the next section, we deduce the statement of Theorem 1 from the existence of a local center-stable W_{cs} and a local center-unstable manifold W_{cu} by means of the Implicit Mapping Theorem.

3. Alternative proof of Theorem 1

From now on, let the assumptions of Theorem 1 be satisfied. Then our discussion in the last section implies that we find both a local center-stable W_{cs} as well as a local center-unstable manifold W_{cu} of F at $\varphi_0 = 0$. Let N_{cs} and N_{cu} denote the corresponding open neighborhoods of 0 in U where W_{cs} and W_{cu} are positively invariant, respectively. The remaining proof will be divided into five parts as follows: In the first one, we prepare the application of the Implicit Mapping Theorem by introducing a (local) manifold chart for X_f and representing both W_{cs} and W_{cu} locally at $\varphi_0 = 0$ as graphs of appropriate maps. The second part then contains the application of the Implicit Mapping Theorem, whereas in the third one we define a local center manifold W_c. In the final two steps we give a representation of W_c as a graph of a continuously differentiable map and show that W_c has properties (C 1) – (C 3) of Theorem 1.

3.1. **Local representations of X_f, W_{cs} and W_{cu}.** Set $Y := T_0 X_f$ and fix some subspace $Z \subset C^1$ with $\dim Z = n$ such that

$$C^1 = Y \oplus Z$$

holds. Let $P : C^1 \to C^1$ denote the continuous projection of C^1 along Z onto Y. Then recall the decomposition (4) of C^1 and that we have $C_u, C_c \subset Y$. Hence, introducing the closed subspace $Y_s := C_s^1 \cap Y = C_s \cap Y$ of Y, we obtain

$$C_s^1 = Y_s \oplus Z$$

and so the additional decomposition

$$C^1 = C_u \oplus C_c \oplus Y_s \oplus Z$$

of C^1. In particular, there are continuous projections $P_u, P_c, P_{Y_s} : C^1 \to C^1$ of C^1 onto C_u, C_c, and Y_s, respectively.

Now, observe that we find an open neighborhood $U_0 \subset U$ of $\varphi_0 = 0$ in C^1 and an open neighborhood Y_0 of 0 in Y such that the equation

$$K(\varphi) = P(\varphi - \varphi_0) = P\varphi$$

defines a manifold chart for X_f with $K(\varphi_0) = 0 \in Y$ and $K(U_0 \cap X_f) = Y_0$. The inverse of K is given by a C^1-smooth map $R : Y_0 \to U_0 \subset C^1$, and both derivatives $DK(\varphi_0)$ and $DR(0)$ are equal to the identity operator on Y.

Next, we represent both W_{cs} as well as W_{cu} locally at $\varphi_0 = 0$ as a graph of a map defined on some open neighborhood of 0 in the corresponding tangent space. Consider the submanifold W_{cs} first: There are open neighborhoods $C_{cs,0}^{cs}$ of 0 in $C_c \oplus Y_s = T_0 W_{cs}$ and $C_{uz,0}^{cs}$ of 0 in $C_u \oplus Z$ with

$$\tilde{N}_{cs} := C_{cs,0}^{cs} + C_{uz,0}^{cs} \subset N_{cs} \cap U_0$$

and a continuously differentiable map $w_{cs} : C_{cs,0}^{cs} \to C_{uz,0}^{cs}$ with $w_{cs}(0) = 0$ and $Dw_{cs}(0) = 0$ such that we have

$$W_{cs} \cap \tilde{N}_{cs} = \{\varphi + w_{cs}(\varphi) \mid \varphi \in C_{cs,0}^{cs}\}.$$

Similarly, we find open neighborhoods $C_{cu,0}^{cu}$ of 0 in $C_c \oplus C_u = T_0 W_{cu}$ and $C_{sz,0}^{cu}$ of 0 in $Y_s \oplus Z$ with

$$\tilde{N}_{cu} := C_{cu,0}^{cu} + C_{sz,0}^{cu} \subset N_{cu} \cap U_0$$

and a continuously differentiable map $w_{cu} : C_{cu,0}^{cu} \to C_{sz,0}^{cu}$ with $w_{cu}(0) = 0$ and $Dw_{cu}(0) = 0$ such that

$$W_{cu} \cap \tilde{N}_{cu} = \{\varphi + w_{cu}(\varphi) \mid \varphi \in C_{cu,0}^{cu}\}$$

holds.

3.2. **Application of the Implicit Mapping Theorem.** Choose open neighborhoods U_c, U_u, and U_{Y_s} of the origin in C_c, C_u, and Y_s, respectively, such that all subset relations

$$U_c + U_{Y_s} \subset C_{cs,0}^{cs}, \quad U_c + U_u \subset C_{cu,0}^{cu}, \quad \text{and} \quad U_c + U_u + U_{Y_s} \subset Y_0$$

are satisfied. Then define the map

$$G : U_c \times U_u \times U_{Y_s} \to C_u \times Y_s$$

by

$$G(\varphi^c, \varphi^u, \varphi^{Y_s}) := \begin{pmatrix} \varphi^u - P_u \, w_{cs}(\varphi^c + \varphi^{Y_s}) \\ \varphi^{Y_s} - P_{Y_s} w_{cu}(\varphi^c + \varphi^u) \end{pmatrix}.$$

We clearly have $G(0,0,0) = (0,0)^T$, and as a composition of C^1-smooth maps the map G is C^1-smooth as well. Now, a straightforward calculation shows that the derivative of G with respect to the last two components can be represented by the matrix

$$D_{(2,3)}G(\varphi^c, \varphi^u, \varphi^{Y_s}) = \begin{pmatrix} \mathrm{id}_{C_u} & -P_u Dw_{cs}(\varphi^c + \varphi^{Y_s}) \\ -P_{Y_s} Dw_{cu}(\varphi^c + \varphi^u) & \mathrm{id}_{Y_s} \end{pmatrix}.$$

In particular,

$$D_{(2,3)}G(0,0,0) = \begin{pmatrix} \mathrm{id}_{C_u} & 0 \\ 0 & \mathrm{id}_{Y_s} \end{pmatrix},$$

which is obviously a linear automorphism of $C_u \times Y_s$. Applying the Implicit Mapping Theorem, we obtain open neighborhoods $C_{c,0}^c$, $C_{u,0}^c$, and $C_{Y_s,0}^c$ of 0 in U_c, U_u, and U_{Y_s}, respectively, and a continuously differentiable map

$$g : C_{c,0}^c \to C_{u,0}^c \times C_{Y_s,0}^c$$

such that $g(0) = (0,0)^T$ and

$$G(\varphi^c, \varphi^u, \varphi^{Y_s}) = (0,0)^T \qquad \Longleftrightarrow \qquad g(\varphi^c) = (\varphi^u, \varphi^{Y_s})^T$$

for all $(\varphi^c, \varphi^u, \varphi^{Y_s}) \in C_{c,0}^c \times C_{u,0}^c \times C_{Y_s,0}^c$.

3.3. **Definition of W_c.** Given $\varphi^c \in C_{c,0}^c$, set $(\varphi^u, \varphi^{Y_s})^T := g(\varphi^c)$ with the map g obtained in the last part. Then we have

$$G(\varphi^c, \varphi^u, \varphi^{Y_s}) = (0,0)^T,$$

that is,

$$\begin{cases} 0 = \varphi^u - P_u \, w_{cs}(\varphi^c + \varphi^{Y_s}), \\ 0 = \varphi^{Y_s} - P_{Y_s} w_{cu}(\varphi^c + \varphi^u). \end{cases}$$

By using the map g and denoting by π_i for $i \in \{1,2\}$ the canonical projection mapping an element $(u,s)^T \in C_u \times Y_s$ to its i-th component, the last system of equations may equivalently be written as

$$\begin{cases} (\pi_1 \circ g)(\varphi^c) = P_u w_{cs}(\varphi^c + (\pi_2 \circ g)(\varphi^c)), \\ (\pi_2 \circ g)(\varphi^c) = P_{Y_s} w_{cu}(\varphi^c + (\pi_1 \circ g)(\varphi^c)). \end{cases}$$

But the more important point is that on the one hand

$$\begin{aligned} R\left(\varphi^c + \varphi^u + \varphi^{Y_s}\right) &= R\left(\varphi^c + \varphi^{Y_s} + P_u w_{cs}\left(\varphi^c + \varphi^{Y_s}\right)\right) \\ &= (R \circ P)\left(\varphi^c + \varphi^{Y_s} + w_{cs}\left(\varphi^c + \varphi^{Y_s}\right)\right) \\ &= (R \circ K)\left(\varphi^c + \varphi^{Y_s} + w_{cs}\left(\varphi^c + \varphi^{Y_s}\right)\right) \\ &= \varphi^c + \varphi^{Y_s} + w_{cs}\left(\varphi^c + \varphi^{Y_s}\right) \in W_{cs} \cap \tilde{N}_{cs} \end{aligned}$$

and on the other hand

$$
\begin{aligned}
R\left(\varphi^c + \varphi^u + \varphi^{Y_s}\right) &= R\left(\varphi^c + \varphi^u + P_{Y_s} w_{cu}\left(\varphi^c + \varphi^u\right)\right) \\
&= (R \circ P)\left(\varphi^c + \varphi^u + w_{cu}\left(\varphi^c + \varphi^u\right)\right) \\
&= (R \circ K)\left(\varphi^c + \varphi^u + w_{cu}\left(\varphi^c + \varphi^u\right)\right) \\
&= \varphi^c + \varphi^u + w_{cu}\left(\varphi^c + \varphi^u\right) \in W_{cu} \cap \tilde{N}_{cu}.
\end{aligned}
$$

Thus, starting with any point $\varphi \in C_{c,0}^c$ we get a point, namely,

$$
(5) \qquad \psi = R\left(\varphi + (\pi_1 \circ g)(\varphi) + (\pi_2 \circ g)(\varphi)\right),
$$

belonging to the intersection $W_{cu} \cap W_{cs}$. From now on, let W_c denote the set of all points $\psi \in W_{cu} \cap W_{cs}$ obtained in this way.

3.4. **Representation of W_c as a graph of a C^1-smooth map.** Consider the set W_c constructed in the last step. Obviously, W_c may alternatively be defined as the image of the map

$$
C_{c,0}^c \ni \varphi \mapsto \varphi + (\pi_2 \circ g)(\varphi) + w_{cs}(\varphi + (\pi_2 \circ g)(\varphi)) \in \tilde{N}_{cu} \cap \tilde{N}_{cs}
$$

or of the map

$$
C_{c,0}^c \ni \varphi \mapsto \varphi + (\pi_1 \circ g)(\varphi) + w_{cu}(\varphi + (\pi_1 \circ g)(\varphi)) \in \tilde{N}_{cu} \cap \tilde{N}_{cs}.
$$

Therefore, W_c particularly coincides with the graph

$$
\{\varphi + w_c(\varphi) \mid \varphi \in C_{c,0}^c\}
$$

of the map

$$
w_c : C_{c,0}^c \ni \varphi \mapsto (\pi_1 \circ g)(\varphi) + w_{cu}(\varphi + (\pi_1 \circ g)(\varphi)) \in C_{su,0}^1
$$

from the open neighborhood $C_{c,0}^c$ of the origin in C_c into the open neighborhood $C_{su,0}^1 := C_{u,0}^c + C_{Y_s,0}^c + C_{Z,0}^c$ of the origin in $C_u \oplus C_s^1$ where $C_{Z,0}^c$ denotes the open subset

$$
\left\{z \in Z \mid C_{Y_s,0}^c + \{z\} \subset C_{sz,0}^{cu}, C_{u,0}^c + \{z\} \subset C_{uz,0}^{cs}\right\}
$$

of Z. In view of $g(0) = (0,0)^T$ and $w_{cu}(0) = 0$, we also have $w_c(0) = 0$ and so $0 \in W_c$. Moreover, as a sum and composition of continuously differentiable maps the map w_c is C^1-smooth as well. We claim that

$$
Dw_c(0) = 0.
$$

In order to see this, observe that for all $\varphi \in C_{c,0}^c$ and all $\psi \in C_c$ we have

$$
Dw_c(\varphi)\psi = \pi_1 Dg(\varphi)\psi + Dw_{cu}(\varphi + (\pi_1 \circ g)(\varphi))[\mathrm{id}_{C_c} + \pi_1 Dg(\varphi)]\psi.
$$

Especially, in case $\varphi = 0$

$$
Dw_c(0)\psi = \pi_1 Dg(0)\psi + Dw_{cu}(0)[\mathrm{id}_{C_c} + \pi_1 Dg(0)]\psi = \pi_1 Dg(0)\psi
$$

as $Dw_{cu}(0) = 0$. Thus, for the proof of $Dw_c(0) = 0$, it suffices to show $Dg(0) = 0$. But this point is easily seen as follows. The second part implies that for all $\varphi \in C_{c,0}^c$ and all $\psi \in C_c$ we have

$$
D[\tilde{\varphi} \mapsto G\left(\tilde{\varphi}, (\pi_1 \circ g)(\tilde{\varphi}), (\pi_2 \circ g)(\tilde{\varphi})\right)](\varphi)\psi = (0,0)^T \in C_u \times Y_s,
$$

that is, in matrix notation,

$$(0,0)^T$$
$$=D[\tilde{\varphi} \mapsto G(\tilde{\varphi}, (\pi_1 \circ g)(\tilde{\varphi}), (\pi_2 \circ g)(\tilde{\varphi}))](\varphi)\,\psi$$
$$=D_1 G(\varphi, (\pi_1 \circ g)(\varphi), (\pi_2 \circ g)(\varphi))\,\psi$$
$$\quad + D_{(2,3)}[\tilde{\varphi} \mapsto G(\tilde{\varphi}, (\pi_1 \circ g)(\tilde{\varphi}), (\pi_2 \circ g)(\tilde{\varphi}))](\varphi)\,\psi$$
$$=\begin{pmatrix} -P_u\,Dw_{cs}(\varphi + (\pi_2 \circ g)(\varphi))\,\psi \\ -P_{Y_s}\,Dw_{cu}(\varphi + (\pi_1 \circ g)(\varphi))\,\psi \end{pmatrix}$$
$$\quad + \begin{pmatrix} \mathrm{id}_{C_u} & -P_u\,Dw_{cs}(\varphi + (\pi_2 \circ g)(\varphi)) \\ -P_{Y_s}Dw_{cu}(\varphi + (\pi_1 \circ g)(\varphi)) & \mathrm{id}_{Y_s} \end{pmatrix} Dg(\varphi)\psi.$$

Hence, after taking into account $g(0) = (0,0)^T$, $Dw_{cu}(0) = 0$, and $Dw_{cs}(0) = 0$, for $\varphi = 0$ we get

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{C_u} & 0 \\ 0 & \mathrm{id}_{Y_s} \end{pmatrix} Dg(0)\psi \;=\; \begin{pmatrix} \pi_1\,(Dg(0)\psi) \\ \pi_2\,(Dg(0)\psi) \end{pmatrix}$$

for all $\psi \in C_c$. It follows that $Dg(0) = 0 \in \mathfrak{L}(C_c, C_u \times Y_s)$ and that finally proves $Dw_c(0) = 0$ as claimed.

3.5. **Proof of Properties (C 1) – (C 3).** Recall from the construction above that $0 \in W_c$ and $W_c \subset W_{cs} \cap W_{cu} \subset X_f$. Moreover, W_c is the graph of the continuously differentiable map $w_c : C_{c,0}^c \to C_{su,0}^1$ from the open neighborhood $C_{c,0}^c$ of 0 in C_c into the open neighborhood $C_{su,0}^1$ of 0 in $C_u \oplus C_s^1$, and it holds that $w_c(0) = 0$ and $Dw_c(0) = 0$. Hence, it is clear that W_c is not only a non-empty subset of the solution manifold X_f but forms a C^1-smooth submanifold of X_f with $\dim W_c = \dim C_c$.

Next, consider the open neighborhood $N_c := C_{c,0}^c + C_{su,0}^1$ of 0 in U_0 and note that

$$N_c \subset \tilde{N}_{cs} \cap \tilde{N}_{cu} \subset N_{cs} \cap N_{cu}.$$

Suppose now that $\gamma : \mathbb{R} \to X_f$ is a trajectory of F with $\gamma(\mathbb{R}) \subset N_c$. We claim that for each $t \in \mathbb{R}$ we have $\gamma(t) \in W_c$. In order to see this, fix an arbitrary $T \in \mathbb{R}$ and set $\psi := \gamma(T)$. Then, on the one hand, for each $t \geq 0$ we have

$$F(t, \psi) = F(t, \gamma(T)) = F(t + T - T, \gamma(T)) = \gamma(t + T) \in N_c \subset \tilde{N}_{cs} \subset N_{cs}$$

and therefore

$$\psi \in W_{cs} \cap \tilde{N}_{cs}$$

in view of property (CS 2) of the manifold W_{cs}. On the other hand, for the induced trajectory $\tilde{\gamma} : (-\infty, 0] \ni t \mapsto \gamma(t + T) \in X_f$ of F we have

$$\tilde{\gamma}(t) = \gamma(t + T) \in N_c \subset \tilde{N}_{cu} \subset N_{cu}$$

and for this reason

$$\psi = \gamma(T) = \tilde{\gamma}(0) \in W_{cu} \cap \tilde{N}_{cu}$$

due to property (CU 1) of W_{cu}. Consequently,

$$\psi \in (W_{cs} \cap \tilde{N}_{cs}) \cap (W_{cu} \cap \tilde{N}_{cu})$$

and from the local graph representations of W_{cs} and W_{cu} it follows that

$$\psi = P_c\psi + P_{Y_s}\psi + w_{cs}(P_c\psi + P_{Y_s}\psi) \quad \text{and} \quad \psi = P_c\psi + P_u\psi + w_{cu}(P_c\psi + P_u\psi).$$

Applying the projection operators P_u to the first and P_{Y_s} to the second representation of ψ above, we easily infer that

$$\begin{cases} P_u \psi = P_u \, w_{cs}(P_c \psi + P_{Y_s} \psi), \\ P_{Y_s} \psi = P_{Y_s} w_{cu}(P_c \psi + P_u \psi). \end{cases}$$

As $\psi \in N_c$ also implies that

$$(P_c \psi, P_u \psi, P_{Y_s} \psi) \in C^c_{c,0} \times C^c_{u,0} \times C^c_{Y_s,0}$$

we conclude that $G(P_c \psi, P_u \psi, P_{Y_s} \psi) = (0,0)^T$ and therefore

$$g(P_c \psi) = (P_u \psi, P_{Y_s} \psi)^T.$$

Hence,

$$\begin{aligned} \gamma(T) &= \psi \\ &= (R \circ K)(\psi) \\ &= R(P\psi) \\ &= R(P_c \psi + P_u \psi + P_{Y_s} \psi) \\ &= R(P_c \psi + (\pi_1 \circ g)(P_c \psi) + (\pi_2 \circ g)(P_c \psi)) \in W_c \end{aligned}$$

due to Eq. (5). As $T \in \mathbb{R}$ was arbitrary chosen, the above proves our claim that $\gamma(t) \in W_c$ for all $t \in \mathbb{R}$.

Finally, we assert that, for all $\varphi \in W_c$ and all $\alpha > 0$ such that $F(t, \varphi)$ is defined and contained in N_c as $0 \le t < \alpha$, we also have $F(t, \varphi) \in W_c$ for all $0 \le t < \alpha$. Indeed, under given assumptions we clearly have

$$F(t, \varphi) \in \tilde{N}_{cs} \cap \tilde{N}_{cu} \subset N_{cs} \cap N_{cu}$$

for all $0 \le t < \alpha$ and therefore, by property (CS 1) of W_{cs} and property (CU 1) of W_{cu}, $F(t, \varphi) \in W_{cs} \cap W_{cu}$ for each $0 \le t < \alpha$. Hence, it follows first that

$$F(t, \varphi) \in (W_{cs} \cap \tilde{N}_{cs}) \cap (W_{cu} \cap \tilde{N}_{cu})$$

and next, by using completely similar arguments as applied above to ψ, that $F(t, \varphi) \in W_c$ as $0 \le t < \alpha$. This finishes the proof of the assertion and completes the proof of Theorem 1. $\square$

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