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Abstract

We study disturbance decoupling for linear differential-algebraic systems which are not necessarily regular. Compared to previous approaches, where state feedback is used, we use the concept of behavioral feedback which allows to study a larger class of systems. We derive geometric characterizations for solvability of the disturbance decoupling problem following the classical approach. Exploiting the freedom in the choice of the behavioral feedback we show that whenever disturbance decoupling can be achieved by behavioral feedback we may additionally achieve autonomous zero dynamics. Finally we solve Lebret's twenty year old open problem concerning disturbance decoupling with output uniqueness using behavioral feedback.

Key words: differential-algebraic systems; descriptor systems; disturbance decoupling; behavioral feedback; Wong sequences; zero dynamics.

1 Introduction

We study linear time-invariant systems given by differential-algebraic equations (DAEs) of the form

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= Ax(t) + Bu(t)\,,\\ y(t) &= Cx(t)\,, \end{aligned} \tag{1}$$

where $E, A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{p\times n}$. Systems of that type are also called *descriptor systems*. The set of systems (1) is denoted by $\Sigma_{l,n,m,p}$ and we write $[E, A, B, C] \in \Sigma_{l,n,m,p}$. DAE systems of the form (1) occur for example when modeling dynamical systems subject to algebraic constraints; for a further motivation we refer to [5, 18, 26, 27, 31] and the references therein. In the present paper we put special emphasis on the non-regular case, i.e., we do not assume that $sE - A$ is *regular*, which would mean that $l = n$ and $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$.
The functions $u : \mathbb{R} \to \mathbb{R}^m$ and $y : \mathbb{R} \to \mathbb{R}^p$ are called *input* and *output* of the system (1), resp. The tuple $(x, u, y) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ is said to be a *solution* of (1), if it belongs to the *behavior* of (1):

$$\mathfrak{B}_{[E,A,B,C]} := \left\{ \begin{array}{l} (x,u,y) \in \\ \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p) \end{array} \middle|\, \begin{array}{l} Ex \in \mathcal{AC}(\mathbb{R}\to\mathbb{R}^l) \text{ and} \\ (x,u,y) \text{ satisfies (1)} \\ \text{for almost all } t\in\mathbb{R} \end{array} \right\}.$$

Based on the above behavior, DAE control systems have been studied in detail e.g. in [5]. We assume that the states, inputs and outputs of the systems in $\Sigma_{l,n,m,p}$ are fixed a priori by the designer. This is different from other approaches based on the behavioral setting, see [13, 20]. Our aim in the present paper is to characterize the influence of disturbances on the system (1), i.e., for a given disturbance matrix $Q \in \mathbb{R}^{l\times q}$ we consider the disturbed system

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= Ax(t) + Bu(t) + Qw(t)\,,\\ y(t) &= Cx(t)\,, \end{aligned} \tag{2}$$

where $w \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^q)$ represents a smooth disturbance, which may be due to noise, modeling or measuring errors, or by higher terms in linearization.
In the case of ODE systems, the *disturbance decoupling problem (DDP)* is the problem of finding, for a given system $[I, A, B, C]$ and disturbance matrix Q, a proportional state feedback $u = Fx$ with $F \in \mathbb{R}^{m\times n}$ for (2)

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such that the transfer function of the closed-loop system satisfies $C(sI-(A+BF))^{-1}Q=0$. This problem has first been treated and solved by Wonham and Morse [42], see also the classical textbooks [3, 34, 41]; several other versions have been considered by Willems [36, 37]. In order to pursue a similar approach for DAEs, it must be required that the closed-loop pencil $sE-(A+BF)$ is regular. Then the transfer function $C(sE-(A+BF))^{-1}Q$ exists and the DDP can be investigated; this has been done in [22] where additionally derivative feedback is allowed and it is required that the closed-loop system has index at most one. Extensions of this problem have been considered e.g. in [21] and [24].

However, the class of regular DAE systems is not closed under the action of a feedback group [1], thus requiring $sE-(A+BF)$ to be regular is a restriction in the choice of F. The first version of the DDP for DAEs, which has been formulated by Fletcher and Aasaraai [23], does not impose any regularity assumptions. However, it was assumed that the output is independent of the disturbance in the sense that there is a set of admissible initial conditions such that the output vanishes. But admissibility of an initial condition depends on the disturbance, which is usually unknown, and hence it is not known a priori if a given initial condition is admissible. The appropriate version of the DDP with proportional state feedback for DAEs has been introduced and solved by Banaszuk et al. [2]; some alternative characterizations have also been given in [28]. However, the definition of the DDP in [2] already requires a zero output, which does not reflect the intuitive notion of a disturbance not influencing the output.

To define disturbance decoupling we follow an intuitive approach. In the case $B=0$ we may treat the disturbance w as the input of system (2) and define disturbance decoupling in terms of the set-valued input-output map of the system $[E,A,Q,C]\in\Sigma_{l,n,q,p}$.

Definition 1 *For a system $[E,A,B,C]\in\Sigma_{l,n,m,p}$, we call the set-valued map*

$$\Phi_{[E,A,B,C]}:\mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^m)\to\mathcal{P}\big(\mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^p)\big),$$
$$u\mapsto\left\{\, y\in\mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^p)\;\middle|\;\begin{array}{l}\exists\, x\in\mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^n):\\ (x,u,y)\in\mathfrak{B}_{[E,A,B,C]}\end{array}\right\},$$

the input-output map *of $[E,A,B,C]$. Here $\mathcal{P}(\mathcal{M})$ denotes the power set of a set $\mathcal{M}$.*

Definition 2 *Let $[E,A,0,C]\in\Sigma_{l,n,0,p}$ and $Q\in\mathbb{R}^{l\times q}$. Then we call $[E,A,Q,C]$* disturbance decoupled*, if*

$$\forall\, w_1,w_2\in\mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q):\quad \Phi_{[E,A,Q,C]}(w_1)=\Phi_{[E,A,Q,C]}(w_2).$$

Roughly speaking, $[E,A,Q,C]$ is disturbance decoupled, if any two disturbances cannot be distinguished using knowledge of the output. In Section 3 we show that this is equivalent to the concept introduced in [2].

Compared to the approaches in [2, 23, 28], we do not consider proportional state feedback of the form $u=Fx$ for the solution of the DDP. Instead, we consider a behavioral feedback of the form $K_1x+K_2u=0$, where $K=[K_1,K_2]\in\mathbb{R}^{k\times n}\times\mathbb{R}^{k\times m}$, which is not always solvable for u in general. The concept of feedback in the behavioral sense has its origin in the works by Willems, Polderman and Trentelman [4, 30, 35, 38, 39], where differential behaviors and their stabilization via *control by interconnection* is considered. The latter means a systematic addition of some further equations such that a desired behavior is achieved. The interconnection of system (2) with the behavioral feedback is depicted in Figure 1.

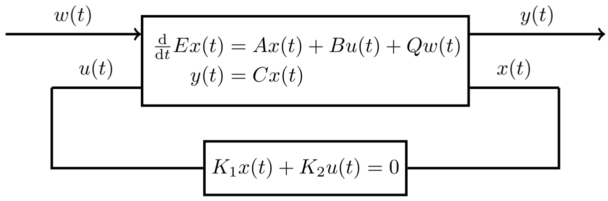

Fig. 1. Interconnection of system and behavioral feedback

The closed-loop system of (2) with the behavioral feedback $K_1x+K_2u=0$ is given by

$$[E^K,A^K,Q^K,C^K]=\left[\begin{bmatrix}E&0\\0&0\end{bmatrix},\begin{bmatrix}A&B\\K_1&K_2\end{bmatrix},\begin{bmatrix}Q\\0\end{bmatrix},[C,0]\right]\in\Sigma_{l+k,n+m,q,p}\qquad(3)$$

with state $\binom{x}{u}$, input w and output y. If $[K_1,K_2]=[F,-I_m]$, then $[E^K,A^K,Q^K,C^K]$ is equivalent to $[E,A+BF,Q,C]$ and we are in the case of proportional state feedback. In Theorem 9 we derive a geometric characterization of solvability of the DDP with behavioral feedback. This result is displayed in Figure 2 and compared to the classical ODE result, see [41], and to the DAE result from [2], which both consider proportional state feedback.

Note that the behavioral feedback allows to avoid the "cumbersome" dimensionality condition (D2) derived in [2] for the characterization of solvability of the DDP. In our framework, solvability of the DDP can be characterized by a single geometric condition on the generalized Wong sequences (introduced in Section 2).

The vast freedom in choosing the control matrix $K=[K_1,K_2]$ for the behavioral feedback can be exploited to obtain several additional properties such as autonomous zero dynamics of the (undisturbed) closed-loop system $[E^K,A^K,0,C^K]$. Roughly speaking, the zero dynamics of a system are those dynamics which are not visible at the output; they are called autonomous, if every trajectory is is uniquely determined by its values on any open interval, see Section 4 for more details. Furthermore, the

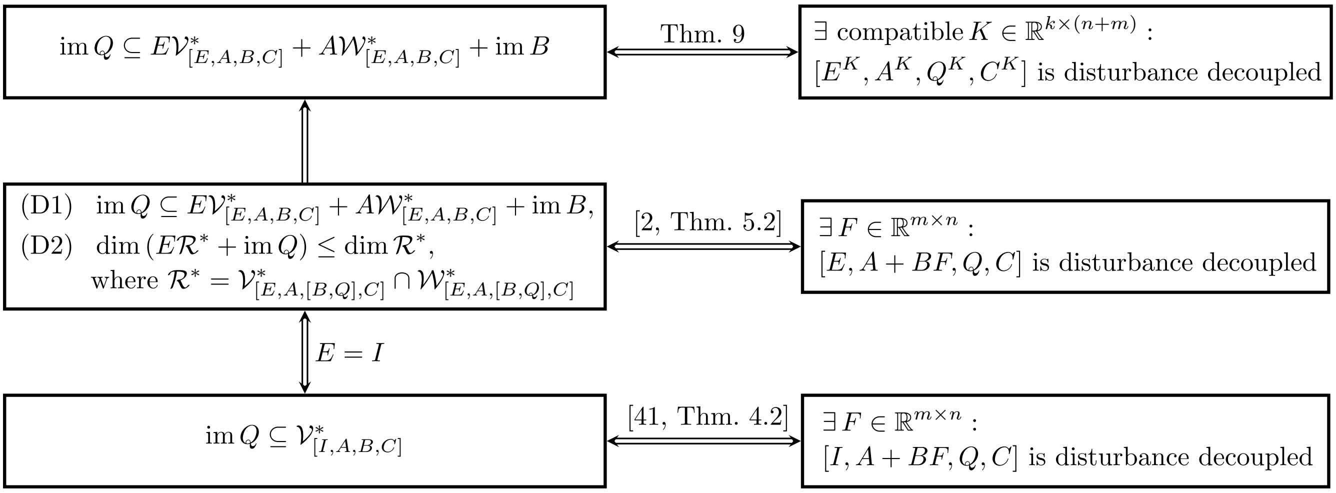

Fig. 2. Theorem 9 compared to earlier results on disturbance decoupling. The spaces $\mathcal{V}^*_{[E,A,B,C]}$ and $\mathcal{W}^*_{[E,A,B,C]}$ are introduced in Section 2. For compatibility of a control matrix K see Definition 8.

behavioral feedback approach allows to solve Lebret's twenty year old open problem [28], that is disturbance decoupling with output uniqueness. In the present paper we present two possible solutions: One is to derive additional conditions on $[E, A, B, C]$ and Q which guarantee output uniqueness, see Theorem 19. For the second solution we relax the compatibility assumption on the control matrix K – a trade-off between requirements on the data and properties of the control – see Theorem 22. These results and their relations are depicted in Figure 3.

We like to stress that the behavioral feedback approach is new even for ODE systems and its power may be illustrated by the following simple example. Consider the ODE system $\dot{x} = x+u+w$, $y = x$, where $w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R})$ is a disturbance. If we seek a state feedback $u = Fx$ such that the closed-loop system $\dot{x} = (1+F)x + w$, $y = x$, is disturbance decoupled, then this problem is not solvable. However, if we consider the larger class of behavioral feedback laws $K_1x + K_2u = 0$, then the choice $[K_1, K_2] = 0 \in \mathbb{R}^{0\times 2}$ guarantees that disturbances are not distinguishable at the output, i.e.,

$$\forall\, w_1, w_2 \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}) :\ \Phi_{[[1,0],[1,1],[1],[1,0]]}(w_1) = \Phi_{[[1,0],[1,1],[1],[1,0]]}(w_2) = \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}).$$

In this example it is important that we have the possibility of not restricting the input u, which then serves to compensate the disturbance in the system so that the outputs corresponding to two different disturbances are not distinguishable anymore.

The present paper is organized as follows: In Section 2 we introduce the generalized Wong sequences as the crucial geometric tool for the characterization of solvability of the DDP. Disturbance decoupled systems are characterized in Section 3 and we investigate when disturbance decoupling can be achieved by behavioral feedback. The considerations in Section 3 reveal that there is a lot of freedom in the choice of the behavioral feedback. In Section 4 we exploit this freedom. We recall the concept of zero dynamics and show that whenever disturbance decoupling can be achieved by behavioral feedback we may additionally achieve autonomous zero dynamics. In Section 5 we investigate Lebret's open problem [28] and solve it using the concept of behavioral feedback.

We close the introduction with the nomenclature used in this paper:

$\mathbb{N}$, $\mathbb{N}_0$	set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{R}[s], \mathbb{R}(s)$	the ring of polynomials with coefficients in $\mathbb{R}$ and its quotient field, resp.
$R^{n\times m}$	the set of $n \times m$ matrices with entries in a ring R
$\mathbf{Gl}_n(R)$	the group of invertible matrices in $R^{n\times n}$
$M\mathcal{S}$	$= \{\, Mx \in \mathbb{R}^l \mid x \in \mathcal{S} \,\}$, the image of $\mathcal{S} \subseteq \mathbb{R}^n$ under $M \in \mathbb{R}^{l\times n}$
$M^{-1}\mathcal{S}$	$= \{\, x \in \mathbb{R}^n \mid Mx \in \mathcal{S} \,\}$, the preimage of $\mathcal{S} \subseteq \mathbb{R}^l$ under $M \in \mathbb{R}^{l\times n}$
$\mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^n)$	the set of infinitely times differentiable functions $f : \mathbb{R}\to\mathbb{R}^n$; f is also called *smooth*
$\mathcal{AC}(\mathbb{R}\to\mathbb{R}^n)$	the set of absolutely continuous functions $f : \mathbb{R}\to\mathbb{R}^n$
$\mathcal{L}^1_{\text{loc}}(\mathbb{R}\to\mathbb{R}^n)$	the set of locally (Lebesgue) integrable functions $f : \mathbb{R}\to\mathbb{R}^n$
$f \overset{\text{a.e.}}{=} g$	means that $f, g \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}\to\mathbb{R}^n)$ are equal "almost everywhere", i.e., $f(t) = g(t)$ for almost all $t \in \mathbb{R}$
$f\vert_I$	restriction of the function $f : \mathbb{R}\to\mathbb{R}^n$ to $I \subseteq \mathbb{R}$

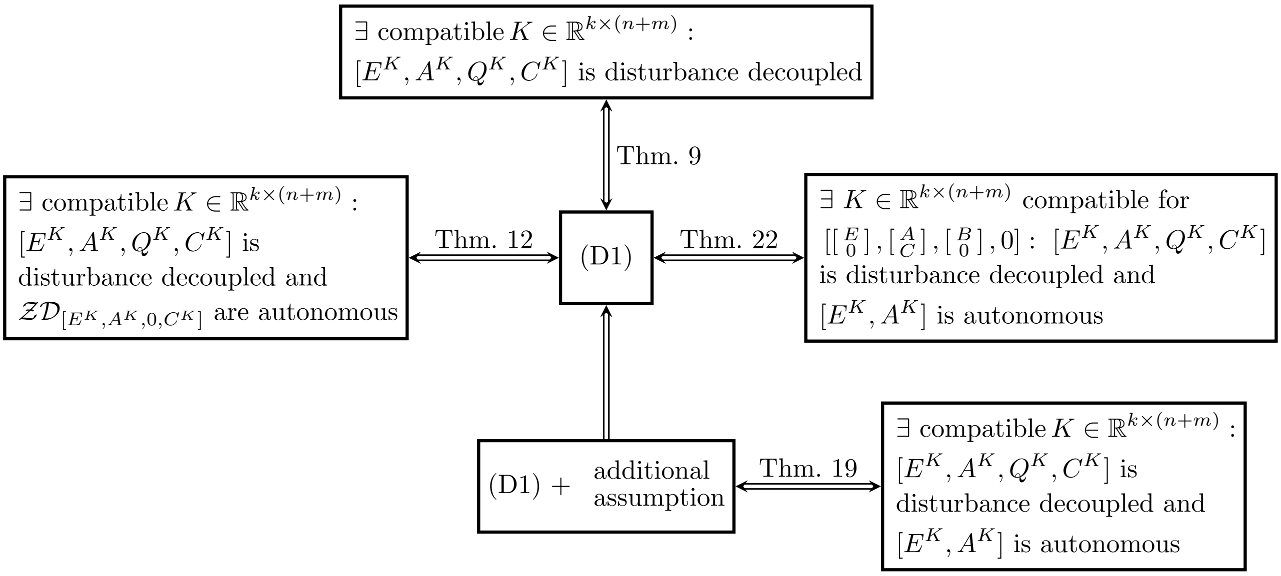

Fig. 3. Disturbance decoupling results and their relations. (Autonomous) zero dynamics $\mathcal{ZD}_{[E^K,A^K,0,C^K]}$ are defined in Section 4. Autonomy of DAEs $[E^K, A^K]$ is defined in Section 5.

2 Generalized Wong sequences

In this section we introduce the crucial geometric tools for the characterization of disturbance decoupling. Since the seminal work by Wonham and Morse [42] the existence of a feedback which achieves disturbance decoupling is usually characterized by a geometric condition involving the limits of certain subspace sequences. For DAE systems $[E, A, B, C] \in \Sigma_{l,n,m,p}$ we define the sequences

$$\begin{aligned}
\mathcal{V}^0_{[E,A,B,C]} &= \ker C,\\
\mathcal{V}^{i+1}_{[E,A,B,C]} &= A^{-1}(E\mathcal{V}^i_{[E,A,B,C]} + \operatorname{im} B) \cap \ker C,\\
\mathcal{W}^0_{[E,A,B,C]} &= \{0\},\\
\mathcal{W}^{i+1}_{[E,A,B,C]} &= E^{-1}(A\mathcal{W}^i_{[E,A,B,C]} + \operatorname{im} B) \cap \ker C.
\end{aligned}$$

The sequence $(\mathcal{V}^i_{[E,A,B,C]})_{i\in\mathbb{N}_0}$ is non-increasing and $(\mathcal{W}^i_{[E,A,B,C]})_{i\in\mathbb{N}_0}$ is non-decreasing and both sequences terminate after finitely many steps, thus we may set

$$\mathcal{V}^*_{[E,A,B,C]} = \bigcap_{i\in\mathbb{N}_0} \mathcal{V}^i_{[E,A,B,C]},\qquad \mathcal{W}^*_{[E,A,B,C]} = \bigcup_{i\in\mathbb{N}_0} \mathcal{W}^i_{[E,A,B,C]}.$$

We will call the sequences $(\mathcal{V}^i_{[E,A,B,C]})_{i\in\mathbb{N}_0}$ and $(\mathcal{W}^i_{[E,A,B,C]})_{i\in\mathbb{N}_0}$ *generalized Wong sequences*. In [10,15,16] the Wong sequences for matrix pencils (i.e., $B = 0$ and $C = 0$) are investigated, the name chosen this way since Wong [40] was the first who used both sequences for the analysis of matrix pencils. In [11,13,17] the case $C = 0$ is considered and the sequences $(\mathcal{V}^i_{[E,A,B,0]})_{i\in\mathbb{N}_0}$ and $(\mathcal{W}^i_{[E,A,B,0]})_{i\in\mathbb{N}_0}$ are called *augmented Wong sequences*. Similarly, in [14] the sequences $(\mathcal{V}^i_{[E,A,0,C]})_{i\in\mathbb{N}_0}$ and $(\mathcal{W}^i_{[E,A,0,C]})_{i\in\mathbb{N}_0}$ (i.e., $B = 0$) are called *restricted Wong sequences*. Using the invariance concepts introduced in [29], $\mathcal{V}^*_{[E,A,B,C]}$ is the supremal (A, E, B)-invariant subspace of $\ker C$, i.e., the largest subspace $\mathcal{V} \subseteq \ker C$ with the property $A\mathcal{V} \subseteq E\mathcal{V} + \operatorname{im} B$. Furthermore, $\mathcal{W}^*_{[E,A,B,C]}$ is the infimal restricted (E, A, B)-invariant subspace of $\ker C$, i.e., the smallest subspace $\mathcal{W} \subseteq \ker C$ with the property $\mathcal{W} = E^{-1}(A\mathcal{W} + \operatorname{im} B) \cap \ker C$. For more details on the Wong sequences see the surveys [11,14] and the references therein.
Note that in geometric control theory for ODE systems (i.e., $E = I$), see e.g. [41], the sequence $(\mathcal{V}^i_{[I,A,B,C]})_{i\in\mathbb{N}_0}$ is called *invariant subspace algorithm*, and the sequence $(\mathcal{W}^i_{[I,A,B,C]})_{i\in\mathbb{N}_0}$ is called *controllability subspace algorithm*.
In the remainder of this section we derive some important relations for the generalized Wong sequences which are the basis for the results on disturbance decoupling.

Lemma 3 *Let* $[E, A, B, C] \in \Sigma_{l,n,m,p}$ *and choose* $T \in \mathbb{R}^{n\times k}$ *with* $\operatorname{rk} T = k$ *such that* $\operatorname{im} T = \ker C$. *Then*

$$(ET)\mathcal{V}^*_{[ET,AT,B,0]} = E\mathcal{V}^*_{[E,A,B,C]}$$
$$\text{and}\quad (AT)\mathcal{W}^*_{[ET,AT,B,0]} = A\mathcal{W}^*_{[E,A,B,C]}.$$

PROOF. First we prove that

$$\forall\, i \in \mathbb{N}_0 :\ ET\mathcal{V}^i_{[ET,AT,B,0]} = E\mathcal{V}^i_{[E,A,B,C]}.$$

For $i=0$ we have $E\mathcal{V}^0_{[E,A,B,C]} = E(\ker C) = \operatorname{im} ET = ET\mathcal{V}^0_{[ET,AT,B,0]}$. Suppose that the assertion is true for some $i \in \mathbb{N}_0$. Then

$$\begin{aligned}
&ET\mathcal{V}^{i+1}_{[ET,AT,B,0]}\\
&= ET(AT)^{-1}(ET\mathcal{V}^i_{[ET,AT,B,0]} + \operatorname{im} B)\\
&= ET(AT)^{-1}(E\mathcal{V}^i_{[E,A,B,C]} + \operatorname{im} B)\\
&= ET\left\{\, x \in \mathbb{R}^k \;\middle|\; ATx \in E\mathcal{V}^i_{[E,A,B,C]} + \operatorname{im} B \,\right\}\\
&= E\left\{\, y \in \operatorname{im} T \;\middle|\; Ay \in E\mathcal{V}^i_{[E,A,B,C]} + \operatorname{im} B \,\right\}\\
&= E(A^{-1}(E\mathcal{V}^i_{[E,A,B,C]} + \operatorname{im} B) \cap \ker C) = E\mathcal{V}^{i+1}_{[E,A,B,C]}.
\end{aligned}$$

The proof for $AT\mathcal{W}^i_{[ET,AT,B,0]} = A\mathcal{W}^i_{[E,A,B,C]}$ for all $i \in \mathbb{N}_0$ is analogous and omitted. □

Lemma 4 *Let $[E,A,B,C] \in \Sigma_{l,n,m,p}$. Then*

$$\begin{aligned}
&[E,0]\mathcal{V}^*_{\left[[E,0],[A,B],0,0\right]} = E\mathcal{V}^*_{[E,A,B,0]}\\
\text{and}\quad &[A,B]\mathcal{W}^*_{\left[[E,0],[A,B],0,0\right]} = A\mathcal{W}^*_{[E,A,B,0]} + \operatorname{im} B.
\end{aligned}$$

PROOF. First we prove that

$$\forall\, i \in \mathbb{N}_0 :\ [E,0]\mathcal{V}^i_{\left[[E,0],[A,B],0,0\right]} = E\mathcal{V}^i_{[E,A,B,0]}.$$

For $i = 0$ we have $E\mathcal{V}^0_{[E,A,B,0]} = \operatorname{im} E = [E,0]\mathcal{V}^0_{\left[[E,0],[A,B],0,0\right]}$. Suppose that the assertion is true for some $i \in \mathbb{N}_0$. Then

$$\begin{aligned}
&[E,0]\mathcal{V}^{i+1}_{\left[[E,0],[A,B],0,0\right]}\\
&= [E,0]([A,B])^{-1}\left([E,0]\mathcal{V}^i_{\left[[E,0],[A,B],0,0\right]}\right)\\
&= [E,0]([A,B])^{-1}(E\mathcal{V}^i_{[E,A,B,0]})\\
&= [E,0]\left\{\begin{pmatrix}x_1\\x_2\end{pmatrix} \in \mathbb{R}^{n+m} \;\middle|\; Ax_1 + Bx_2 \in E\mathcal{V}^i_{[E,A,B,0]}\right\}\\
&= E\left\{x_1 \in \mathbb{R}^n \;\middle|\; \exists\, x_2 \in \mathbb{R}^m :\ Ax_1 + Bx_2 \in E\mathcal{V}^i_{[E,A,B,0]}\right\}\\
&= EA^{-1}(E\mathcal{V}^i_{[E,A,B,0]} + \operatorname{im} B) = E\mathcal{V}^{i+1}_{[E,A,B,0]}.
\end{aligned}$$

Now we prove that

$$\begin{aligned}
\forall\, i \in \mathbb{N}_0 :\ [A,B]\mathcal{W}^i_{\left[[E,0],[A,B],0,0\right]} &\subseteq A\mathcal{W}^i_{[E,A,B,0]} + \operatorname{im} B\\
&\subseteq [A,B]\mathcal{W}^{i+1}_{\left[[E,0],[A,B],0,0\right]}.
\end{aligned}$$

For $i = 0$ we have

$$\begin{aligned}
[A,B]\mathcal{W}^0_{\left[[E,0],[A,B],0,0\right]} &= \{0\} \subseteq A\mathcal{W}^0_{[E,A,B,0]} + \operatorname{im} B\\
&\subseteq A\ker E + \operatorname{im} B = [A,B]\mathcal{W}^1_{\left[[E,0],[A,B],0,0\right]}.
\end{aligned}$$

Suppose that the assertion is true for some $i \in \mathbb{N}_0$. Then

$$\begin{aligned}
&[A,B]\mathcal{W}^{i+1}_{\left[[E,0],[A,B],0,0\right]}\\
&= [A,B]([E,0])^{-1}\left([A,B]\mathcal{W}^i_{\left[[E,0],[A,B],0,0\right]}\right)\\
&\subseteq [A,B]([E,0])^{-1}\left(A\mathcal{W}^i_{[E,A,B,0]} + \operatorname{im} B\right)\\
&= [A,B]\left\{\begin{pmatrix}x_1\\x_2\end{pmatrix} \in \mathbb{R}^{n+m} \;\middle|\; Ex_1 \in A\mathcal{W}^i_{[E,A,B,0]} + \operatorname{im} B\right\}\\
&= \left\{ Ax_1 + Bx_2 \;\middle|\; \begin{array}{l} x_1 \in \mathbb{R}^n,\ x_2 \in \mathbb{R}^m,\\ x_1 \in E^{-1}(A\mathcal{W}^i_{[E,A,B,0]} + \operatorname{im} B)\\ \qquad = \mathcal{W}^{i+1}_{[E,A,B,0]} \end{array}\right\}\\
&= A\mathcal{W}^{i+1}_{[E,A,B,0]} + \operatorname{im} B
\end{aligned}$$

and analogously, just with the opposite inclusion sign, we obtain $[A,B]\mathcal{W}^{i+1}_{\left[[E,0],[A,B],0,0\right]} \supseteq A\mathcal{W}^i_{[E,A,B,0]} + \operatorname{im} B$. □

3 Disturbance decoupling

In this section we derive a characterization of disturbance decoupled systems. Furthermore, we recall the concepts of behavioral feedback and compatible control and characterize solvability of the DDP by behavioral feedback.
Recall Definition 2 of disturbance decoupled systems. We derive the following characterization of disturbance decoupled systems which matches the definition for the DDP used in [2], but differs from the concepts used in [21–24] where regularity is required.

Proposition 5 *Let $[E,A,0,C] \in \Sigma_{l,n,0,p}$ and $Q \in \mathbb{R}^{l\times q}$. Then $[E,A,Q,C]$ is disturbance decoupled if, and only if,*

$$\begin{aligned}
\forall\, w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\ \exists\, x \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^n) :&\\
Cx = 0\ \wedge\ E\dot{x} = Ax + Qw\,.&\quad (4)
\end{aligned}$$

PROOF. ⇒: Let $w \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^q)$. Then $\Phi_{[E,A,Q,C]}(w) = \Phi_{[E,A,C,Q]}(0)$ and the assertion follows from $0 \in \Phi_{[E,A,Q,C]}(0)$.
⇐: It suffices to show that $\Phi_{[E,A,Q,C]}(w) = \Phi_{[E,A,Q,C]}(0)$ for all $w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)$. Let $w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)$ and observe that $\Phi_{[E,A,Q,C]}(w) \neq \emptyset$ by (4). We show $\Phi_{[E,A,Q,C]}(w) \subseteq \Phi_{[E,A,Q,C]}(0)$. Let $y \in \Phi_{[E,A,Q,C]}(w)$.

Then there exists $z_1 \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^n)$ such that $E\dot{z}_1 = Az_1 + Qw$ and $y = Cz_1$. By assumption, there exists $z_2 \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^n)$ such that $E\dot{z}_2 = Az_2 + Qw$ and $Cz_2 = 0$. Setting $x := z_1 - z_2 \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^n)$ yields

$$E\dot{x} = Az_1 + Qw - Az_2 - Qw = Ax$$

and $Cx = Cz_1 - Cz_2 = y$. Therefore, $y \in \Phi_{[E,A,Q,C]}(w)$. The opposite inclusion can be shown analogously and this finishes the proof. □

Remark 6 *Consider $[E, A, 0, C] \in \Sigma_{n,n,0,p}$ with regular $sE - A$ and let $Q \in \mathbb{R}^{l \times q}$. Using Laplace transform and Proposition 5 it is immediate that $[E, A, Q, C]$ is disturbance decoupled if, and only if, the transfer function from the disturbance to the output is zero, i.e., $C(sE - A)^{-1}Q = 0$. In particular, the concept introduced in Definition 2 generalizes the classical concept of disturbance decoupled ODE systems, see e.g. [41].*

We are now in the position to derive a geometric characterization for $[E, A, Q, C]$ being disturbance decoupled.

Theorem 7 *Let $[E, A, 0, C] \in \Sigma_{l,n,0,p}$ and $Q \in \mathbb{R}^{l \times q}$. Then*

$$\boxed{\begin{aligned} &[E, A, Q, C] \text{ is disturbance decoupled} \\ \iff\ &\operatorname{im} Q \subseteq E\mathcal{V}^*_{[E,A,0,C]} + A\mathcal{W}^*_{[E,A,0,C]}. \end{aligned}} \tag{5}$$

PROOF. *Step 1*: We reduce the problem to a problem of existence of solutions for a certain DAE. Let $T_1 \in \mathbb{R}^{n \times r}, T_2 \in \mathbb{R}^{n \times (n-r)}$ be such that $\operatorname{im} T_1 = \ker C$ and $T = [T_1, T_2]$ is invertible. Then $CT = [0, CT_2] = [0, C_2]$, where $C_2 \in \mathbb{R}^{p \times (n-r)}$ has full column rank: $C_2 x = 0 = CT_2 x$ for some $x \in \mathbb{R}^{n-r}$ implies $T_2 x \in \operatorname{im} T_2 \cap \ker C = \operatorname{im} T_2 \cap \operatorname{im} T_1 = \{0\}$, thus $T_2 x = 0$ and hence $x = 0$ by full column rank of T_2. By Proposition 5, $[E, A, Q, C]$ is disturbance decoupled if, and only if, (4) holds. Applying the coordinate transformation $z = T^{-1}x$, (4) is equivalent to

$$\forall\, w \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^q)\ \exists\, z \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^n) : \\ CTz = 0\ \wedge\ ET\dot{z} = ATz + Qw.$$

Partitioning $z = (z_1^\top, z_2^\top)^\top$ with $z_1 \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^r), z_2 \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n-r})$ and invoking that $0 = CTz = C_2 z_2$ implies $z_2 = 0$, we find that, with $E_1 = ET_1, A_1 = AT_1$, $[E, A, Q, C]$ is disturbance decoupled if, and only if,

$$\forall\, w \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^q)\ \exists\, z_1 \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^r) : \\ E_1\dot{z}_1 = A_1 z_1 + Qw.$$

Step 2: We prove (5). Choose full rank matrices $P_1 \in \mathbb{R}^{n \times n_1}, R_1 \in \mathbb{R}^{n \times n_2}, P_2 \in \mathbb{R}^{l \times l_1}, R_2 \in \mathbb{R}^{l \times l_2}$ such that

$$\begin{aligned} \operatorname{im} P_1 &= \mathcal{V}^*_{[E_1,A_1,0,0]} + \mathcal{W}^*_{[E_1,A_1,0,0]}, \\ \operatorname{im} P_1 \oplus \operatorname{im} R_1 &= \mathbb{R}^n, \\ \operatorname{im} P_2 &= E_1\mathcal{V}^*_{[E_1,A_1,0,0]} + A_1\mathcal{W}^*_{[E_1,A_1,0,0]}, \\ \operatorname{im} P_2 \oplus \operatorname{im} R_2 &= \mathbb{R}^l. \end{aligned}$$

Then, by [15, Thm. 2.3], with $V = [P_1, R_1]$ and $W = [P_2, R_2]^{-1}$ we have

$$W(sE_1 - A_1)V = \begin{bmatrix} sE_{11} - A_{11} & sE_{12} - A_{12} \\ 0 & sE_{22} - A_{22} \end{bmatrix},$$

where

(i) $E_{11}, A_{11} \in \mathbb{R}^{l_1 \times n_1}$, $l_1 \leq n_1$, satisfy $\operatorname{rk}_{\mathbb{R}(s)}(sE_{11} - A_{11}) = l_1$,
(ii) $E_{22}, A_{22} \in \mathbb{R}^{l_2 \times n_2}$, $l_2 > n_2$ or $l_2 = n_2 = 0$, satisfy $\operatorname{rk} E_{22} = n_2$ and $\operatorname{rk}_{\mathbb{C}}(\lambda E_{22} - A_{22}) = n_2$ for all $\lambda \in \mathbb{C}$.

By [15, Lem. 3.1] there exists a unimodular matrix $\begin{bmatrix} M(s) \\ K(s) \end{bmatrix} \in \mathbb{R}[s]^{(n_2 + (l_2 - n_2)) \times l_2}$ (i.e., $\begin{bmatrix} M(s) \\ K(s) \end{bmatrix}$ is invertible over $\mathbb{R}[s]$) such that

$$\begin{bmatrix} M(s) \\ K(s) \end{bmatrix} (sE_{22} - A_{22}) = \begin{bmatrix} I_{n_2} \\ 0 \end{bmatrix}.$$

Then [15, Thm. 3.2] yields that for given $w \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^q)$ the DAE $E_1\dot{z}_1 = A_1 z_1 + Qw$ has a solution $z_1 \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^r)$ if, and only if,

$$K(\tfrac{\mathrm{d}}{\mathrm{d}t})(\tilde{w}) = 0, \quad \text{where } \tilde{w} = [0, I_{l_2}]WQw.$$

This implies that there exists a solution for all $w \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^q)$ if, and only if, $[0, I_{l_2}]WQ = 0$; sufficiency is clear and for necessity observe that $K(s)[0, I_{l_2}]WQ = 0$ implies that $\operatorname{im}[0, I_{l_2}]WQ \subseteq \ker[K_1^\top, \ldots, K_p^\top]^\top$, where $K(s) = K_1 + sK_2 + \ldots + s^{p-1}K_p$, and $\ker[K_1^\top, \ldots, K_p^\top]^\top = \{0\}$ as shown in Step 3a of the proof of [15, Lem. 4.17].
Finally, we have that $[E, A, Q, C]$ is disturbance decoupled if, and only if,

$$\begin{aligned} \operatorname{im} Q \subseteq \ker[0, I_{l_2}]W = W^{-1} \operatorname{im} \begin{bmatrix} I_{l_1} \\ 0 \end{bmatrix} = \operatorname{im} P_2 \\ = E_1\mathcal{V}^*_{[E_1,A_1,0,0]} + A_1\mathcal{W}^*_{[E_1,A_1,0,0]}. \end{aligned}$$

From Lemma 3 we may deduce that $E_1\mathcal{V}^*_{[E_1,A_1,0,0]} + A_1\mathcal{W}^*_{[E_1,A_1,0,0]} = E\mathcal{V}^*_{[E,A,0,C]} + A\mathcal{W}^*_{[E,A,0,C]}$ and this concludes the proof. □

As mentioned in the introduction, the solution of the DDP with proportional state feedback has been derived in [2] for DAEs, where it is shown that for $[E, A, B, C] \in \Sigma_{l,n,m,p}$ and disturbance matrix $Q \in \mathbb{R}^{l\times q}$ there exists $F \in \mathbb{R}^{m\times n}$ such that $[E, A + BF, Q, C]$ is disturbance decoupled if, and only if, conditions (D1) and (D2) as depicted in Figure 2 hold true. However, condition (D2) has no intuitive interpretation and is unsatisfactory from the point of view that something similar does not appear in the ODE case, see Figure 2.

In contrast to the approach in [2], we seek a feedback in the behavioral sense, i.e., a control $K = [K_1, K_2] \in \mathbb{R}^{k\times n} \times \mathbb{R}^{k\times m}$ such that the closed-loop system $[E^K, A^K, Q^K, C^K]$ as in (3) (cf. also Figure 1) is disturbance decoupled. In the undisturbed case $w = 0$, the control K has to be compatible with the system in a certain sense, cf. also [11] and the references therein. Here we introduce a slightly different notion of compatible control which uses smooth solutions only.

Definition 8 *Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$. A control matrix $K = [K_1, K_2] \in \mathbb{R}^{k\times n} \times \mathbb{R}^{k\times m}$ is called* compatible *for $[E, A, B, C]$, if*

$$\begin{aligned}&\forall\,(x,u,y) \in \mathfrak{B}_{[E,A,B,C]} \cap \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p)\\ &\exists\,(\tilde{x},\tilde{u}) \in \mathfrak{B}_{[E^K,A^K,0_{l\times 0},0_{0\times n}]} \cap \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^n \times \mathbb{R}^m) :\\ &\qquad\qquad Ex(0) = E\tilde{x}(0).\end{aligned}$$

The concept of a compatible control is important from a practical point of view. If we assume that the controller is switched on at time $t = 0$, then it must be guaranteed that there actually exists a closed-loop trajectory $(\tilde{x}, \tilde{u})$ such that the initial differential variables $E\tilde{x}(0)$ match those of the open-loop trajectory (x, u, y). Otherwise, a jump from $Ex(0)$ to $E\tilde{x}(0)$ would occur which must be avoided. Note that our concept of compatible control is a slight modification of the concept introduced in [25].

The following theorem is the analog of [2, Thm. 5.2] for the case of behavioral feedback. In this more general setting we can avoid the dimensionality condition (D2).

Theorem 9 *Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ and $Q \in \mathbb{R}^{l\times q}$. Then there exists a control $K = [K_1, K_2] \in \mathbb{R}^{k\times n} \times \mathbb{R}^{k\times m}$ compatible for $[E, A, B, C]$ such that $[E^K, A^K, Q^K, C^K]$ is disturbance decoupled if, and only if,*

$$\boxed{\operatorname{im} Q \subseteq E\mathcal{V}^*_{[E,A,B,C]} + A\mathcal{W}^*_{[E,A,B,C]} + \operatorname{im} B.}$$

PROOF. Let $T_1 \in \mathbb{R}^{n\times r}, T_2 \in \mathbb{R}^{n\times(n-r)}$ be such that $\operatorname{im} T_1 = \ker C$ and $T = [T_1, T_2]$ is invertible. As in Step 1 of the proof of Theorem 7 we may show that for some compatible control $K \in \mathbb{R}^{q\times(n+m)}$, the system $[E^K, A^K, Q^K, C^K]$ is disturbance decoupled if, and only if,

$$\begin{aligned}&\forall\, w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\ \exists\,(z,u) \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^r \times \mathbb{R}^m) :\\ &\quad ET_1\dot{z} = AT_1 z + Bu + Qw \ \wedge\ K_1T_1z + K_2u = 0.\end{aligned} \tag{6}$$

Writing

$$[E_1^K, A_1^K] = \left[\begin{bmatrix} ET_1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix} AT_1 & B\\ K_1T_1 & K_2\end{bmatrix}\right],$$

(6) is equivalent to

$$\begin{aligned}&\forall\, w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\ \exists\, v \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^{r+m}) :\\ &\qquad\qquad E_1^K\dot{v} = A_1^K v + \left[\begin{smallmatrix} Q\\ 0\end{smallmatrix}\right] w.\end{aligned}$$

$\Leftarrow$: Choosing $q = 0$ (i.e., $K = 0 \in \mathbb{R}^{0\times(n+m)}$) and invoking the above statement, $[E^K, A^K, Q^K, C^K]$ is disturbance decoupled if, and only if, $\big[[ET_1, 0], [AT_1, B], Q, 0\big]$ is disturbance decoupled. By Theorem 7 this is equivalent to

$$\begin{aligned}\operatorname{im} Q \subseteq\ & [ET_1, 0]\mathcal{V}^*_{\left[[ET_1,0],[AT_1,B],0,0\right]}\\ &+ [AT_1, B]\mathcal{W}^*_{\left[[ET_1,0],[AT_1,B],0,0\right]},\end{aligned} \tag{7}$$

and invoking Lemmas 3 and 4 we find that

$$\begin{aligned}&E\mathcal{V}^*_{[E,A,B,C]} = [ET_1, 0]\mathcal{V}^*_{\left[[ET_1,0],[AT_1,B],0,0\right]},\\ &A\mathcal{W}^*_{[E,A,B,C]} + \operatorname{im} B = [AT_1, B]\mathcal{W}^*_{\left[[ET_1,0],[AT_1,B],0,0\right]},\end{aligned} \tag{8}$$

which yields the claim.

$\Rightarrow$: If $K = [K_1, K_2] \in \mathbb{R}^{k\times n} \times \mathbb{R}^{k\times m}$ is a compatible control for $[E, A, B, C]$ such that the system $[E^K, A^K, Q^K, C^K]$ is disturbance decoupled, then (6) in particular implies

$$\begin{aligned}&\forall\, w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\ \exists\,(z,u) \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^r \times \mathbb{R}^m) :\\ &\qquad\qquad ET_1\dot{z} = AT_1z + Bu + Qw,\end{aligned}$$

which means that $\big[[ET_1, 0], [AT_1, B], Q, 0\big]$ is disturbance decoupled. Then (7) and (8) hold true and the claim follows. □

Remark 10 *We consider an ODE system $[I, A, B, C] \in \Sigma_{n,n,m,p}$ with disturbance matrix $Q \in \mathbb{R}^{n\times q}$ and compare the classical result in [41, Thm. 4.2] to Theorem 9, see also Figure 2. The main difference is that in [41, Thm. 4.2] proportional state feedback $u = Fx$ is considered to achieve disturbance decoupling, whereas we consider behavioral feedback $K_1x + K_2u = 0$. Roughly speaking, in the latter the input variables are not completely* *determined by the state variables in general, but are free variables in the closed-loop system. These input variables can be used to cancel the disturbances in the closed-loop system. Exemplary, we consider the case $C = I$ in more detail. In this case, the condition $\operatorname{im} Q \subseteq \mathcal{V}^*_{[I,A,B,C]}$ from [41, Thm. 4.2] implies $Q = 0$, which may also be verified by investigating the solutions of*

$$\dot{x} = (A + BF)x + Qw, \quad y = x$$

for some feedback matrix $F \in \mathbb{R}^{m\times n}$. The output admits the representation

$$y(t) = e^{(A+BF)t}x(0) + \int_0^t e^{(A+BF)(t-s)}Qw(s)\,\mathrm{d}s$$

for all $t \in \mathbb{R}$, and is independent of w if, and only if, $Q = 0$. If we consider a behavioral feedback instead and choose $K = [K_1, K_2] = 0 \in \mathbb{R}^{0\times(n+m)}$, then the output of the corresponding closed-loop system $[I^K, A^K, Q^K, C^K]$, namely

$$\dot{x} = Ax + Bu + Qw, \quad y = x,$$

reads

$$y(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}\big(Bu(s) + Qw(s)\big)\,\mathrm{d}s\,, \quad t \in \mathbb{R},$$

and it is independent of w if, and only if, for any w there exists u such that $Bu = -Qw$. This is equivalent to $\operatorname{im} Q \subseteq \operatorname{im} B$ or, what is the same, to the condition $\operatorname{im} Q \subseteq \mathcal{V}^*_{[I,A,B,C]} + A\mathcal{W}^*_{[I,A,B,C]} + \operatorname{im} B$ from Theorem 9.

4 Disturbance decoupling and zero dynamics

The proof of Theorem 9 does not exploit any freedom in choosing the control $[K_1, K_2]$. In view of Proposition 5 the closed-loop system $[E^K, A^K, Q^K, C^K]$ has, for every smooth "input" w, a solution which generates zero output. We show that an appropriate additional condition yields uniqueness of this solution in the sense

$$\begin{aligned}
&\forall\, w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\\
&\quad \forall\, (z_1, w, 0), (z_2, w, 0) \in \mathfrak{B}_{[E^K,A^K,Q^K,C^K]}:\\
&\qquad E^K z_1(0) = E^K z_2(0) \quad\Longrightarrow\quad z_1 \overset{\text{a.e.}}{=} z_2.
\end{aligned}$$

By linearity of the behavior the above statement is equivalent to

$$\begin{aligned}
&\forall\, (z, 0, 0) \in \mathfrak{B}_{[E^K,A^K,0,C^K]}:\\
&\qquad E^K z(0) = 0 \quad\Longrightarrow\quad z \overset{\text{a.e.}}{=} 0
\end{aligned}$$

and therefore independent of the disturbance matrix Q. In fact, the above property means that the zero dynamics of $[E^K, A^K, 0, C^K] \in \Sigma_{l+k,n+m,0,p}$ are autonomous. Loosely speaking, the zero dynamics are those dynamics which are not visible at the output. For ODE systems this concept has been introduced in [19]. The zero dynamics are, for $[E, A, B, C] \in \Sigma_{l,n,m,p}$, defined by

$$\mathcal{ZD}_{[E,A,B,C]} := \left\{ (x, u, y) \in \mathfrak{B}_{[E,A,B,C]} \;\middle|\; y \overset{\text{a.e.}}{=} 0 \right\}.$$

For linear DAE systems the zero dynamics have been well investigated, see [6–9]. The zero dynamics of (1) are called *autonomous*, if

$$\begin{aligned}
&\forall\, w \in \mathcal{ZD}_{[E,A,B,C]}\ \forall\, I \subseteq \mathbb{R} \text{ open interval}:\\
&\qquad w|_I \overset{\text{a.e.}}{=} 0 \quad\Longrightarrow\quad w \overset{\text{a.e.}}{=} 0\,.
\end{aligned}$$

The definition of autonomous zero dynamics is a special case of the definition of autonomy, as it has been introduced in [30, Sec. 3.2] for general behaviors.
Recall the following characterization of autonomous zero dynamics from [7].

Lemma 11 *For $[E, A, B, C] \in \Sigma_{l,n,m,p}$ we have*

$$\begin{aligned}
&\mathcal{ZD}_{[E,A,B,C]} \text{ are autonomous}\\
\Longleftrightarrow\quad &\operatorname{rk}_{\mathbb{R}[s]} \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} = n + m.
\end{aligned}$$

In the following we show that under the condition in Theorem 9 we may choose $[K_1, K_2]$ such that, additionally to the disturbance decoupling, we achieve autonomous zero dynamics of the undisturbed closed-loop system.

Theorem 12 *Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ and $Q \in \mathbb{R}^{l\times q}$. Then there exists a control $K = [K_1, K_2] \in \mathbb{R}^{k\times n} \times \mathbb{R}^{k\times m}$ compatible for $[E, A, B, C]$ such that $[E^K, A^K, Q^K, C^K]$ is disturbance decoupled and $\mathcal{ZD}_{[E^K,A^K,0,C^K]}$ are autonomous if, and only if,*

$$\operatorname{im} Q \subseteq E\mathcal{V}^*_{[E,A,B,C]} + A\mathcal{W}^*_{[E,A,B,C]} + \operatorname{im} B.$$

PROOF. $\Rightarrow$: Follows from Theorem 9.
$\Leftarrow$: *Step 1*: Let $T_1 \in \mathbb{R}^{n\times r}, T_2 \in \mathbb{R}^{n\times(n-r)}$ be such that $\operatorname{im} T_1 = \ker C$ and $T = [T_1, T_2]$ is invertible. As in the proof of Theorem 9 we may show that for any control $K = [K_1, K_2] \in \mathbb{R}^{k\times n} \times \mathbb{R}^{k\times m}$ compatible for $[E, A, B, C]$, the system $[E^K, A^K, Q^K, C^K]$ is disturbance decoupled if, and only if,

$$\begin{aligned}
&\forall\, w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\ \exists\, v \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^{r+m}):\\
&\qquad E_1^K \dot{v} = A_1^K v + \left[\begin{smallmatrix} Q \\ 0 \end{smallmatrix}\right] w, \quad (9)
\end{aligned}$$

where

$$[E_1^K, A_1^K] = \left[\begin{bmatrix} ET_1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} AT_1 & B \\ K_1T_1 & K_2 \end{bmatrix} \right].$$

In particular, this implies that

$$\forall\, w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\ \exists\,(x_1,u)\in\mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^r\times\mathbb{R}^m):\quad ET_1\dot{x}_1 = AT_1x_1 + Bu + Qw. \tag{10}$$

Furthermore, it is a straightforward calculation to see that for any $(z,u)\in\mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^m)$ with $\left(\begin{smallmatrix} z_1\\ z_2\end{smallmatrix}\right) := [T_1,T_2]^{-1}z$ we have

$$\begin{aligned} &\left(\left(\begin{smallmatrix} z\\ u\end{smallmatrix}\right),0,0\right)\in\mathcal{ZD}_{[E^K,A^K,0,C^K]}\\ \iff\ &\left(\left(\begin{smallmatrix} z_1\\ u\end{smallmatrix}\right),0,0\right)\in\mathcal{ZD}_{[E_1^K,A_1^K,0,0]}\ \wedge\ z_2=0.\end{aligned}$$

Therefore, it suffices to find a control $K = [K_1,K_2]\in\mathbb{R}^{k\times n}\times\mathbb{R}^{k\times m}$ compatible for $[E,A,B,C]$ such that $[E_1^K,A_1^K,Q^K,0]$ is disturbance decoupled and $\mathcal{ZD}_{[E_1^K,A_1^K,0,0]}$ are autonomous.

Step 2: In order to construct K, we consider a Kalman controllability decomposition of $[ET_1,AT_1,B]$ according to [17]. This means that there exists $W\in\mathbf{Gl}_l(\mathbb{R})$ and $V\in\mathbf{Gl}_r(\mathbb{R})$ such that

$$\begin{aligned}&[WET_1V, WAT_1V, WB]\\ &= \left[\begin{bmatrix} E_{11} & E_{12} & E_{13}\\ 0 & E_{22} & E_{23}\\ 0 & 0 & E_{33}\end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13}\\ 0 & A_{22} & A_{23}\\ 0 & 0 & A_{33}\end{bmatrix}, \begin{bmatrix} B_1\\ 0\\ 0\end{bmatrix}\right],\end{aligned} \tag{11}$$

where

(i) $[E_{11},A_{11},B_1]\in\Sigma_{l_1,n_1,m}$ with $l_1 = \operatorname{rk}[E_{11},B_1]\le n_1+m$ is completely controllable,
(ii) $[E_{22},A_{22},0]\in\Sigma_{l_2,n_2,m}$ with $l_2=n_2$ and E_{22} is invertible,
(iii) $[E_{33},A_{33},0]\in\Sigma_{l_3,n_3,m}$ with $l_3\ge n_3$ satisfies $\operatorname{rk}_{\mathbb{C}}(\lambda E_{33}-A_{33}) = n_3$ for all $\lambda\in\mathbb{C}$.

Recall that $[E_{11},A_{11},B_1]$ is completely controllable (see e.g. the survey [11]) if, and only if,

$$\mathcal{V}^*_{[E_{11},A_{11},B_1,0]}\cap\mathcal{W}^*_{[E_{11},A_{11},B_1,0]} = \mathbb{R}^{n_1}.$$

Partitioning $WQ = [Q_1^\top,Q_2^\top,Q_3^\top]^\top$ according to the block structure in (11), it follows from (10) that

$$\forall\, w\in\mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\ \exists\, z_3\in\mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^{n_3}):\quad E_{33}\dot{z}_3 = A_{33}z_3 + Q_3w. \tag{12}$$

For the investigation of $[E_{11},A_{11},B_1]$ we put this system into so called feedback form, see e.g. [11]. To this end we introduce the following notation: For $j\in\mathbb{N}$ let

$$K_j = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0\end{bmatrix},\quad L_j = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1\end{bmatrix}\in\mathbb{R}^{(j-1)\times j}$$

and, for some multi-index $\alpha = (\alpha_1,\dots,\alpha_j)\in\mathbb{N}^j$, we define

$$\begin{aligned} K_\alpha &= \operatorname{diag}(K_{\alpha_1},\dots,K_{\alpha_j}),\\ L_\alpha &= \operatorname{diag}(L_{\alpha_1},\dots,L_{\alpha_j})\in\mathbb{R}^{(|\alpha|-\ell(\alpha))\times|\alpha|},\end{aligned}$$

where $\ell(\alpha)=j$ and $|\alpha| = \sum_{i=1}^j\alpha_i$ are the length and absolute value of the multi-index α, resp. Now, by complete controllability of $[E_{11},A_{11},B_1]$ and $\operatorname{rk}[E_{11},B_1]=l_1$ it follows from [11, Thm. 3.3 & Cor. 3.4] that there exist $\tilde{W}\in\mathbf{Gl}_{l_1}(\mathbb{R}), \tilde{V}\in\mathbf{Gl}_{n_1}(\mathbb{R}), U\in\mathbf{Gl}_m(\mathbb{R}), F\in\mathbb{R}^{n_1\times m}$ such that

$$\begin{aligned}&[\tilde{W}E_{11}\tilde{V}, \tilde{W}A_{11}\tilde{V}+\tilde{W}B_1F, \tilde{W}B_1U]\\ &= \left[\begin{bmatrix} I_{n_{11}} & 0\\ 0 & K_\alpha\\ 0 & 0\end{bmatrix}, \begin{bmatrix}\tilde{A}_{11} & 0\\ 0 & L_\alpha\\ 0 & 0\end{bmatrix}, \begin{bmatrix} B_{11} & 0 & 0\\ 0 & 0 & 0\\ 0 & I_{m_2} & 0\end{bmatrix}\right],\end{aligned} \tag{13}$$

where $\tilde{A}_{11}\in\mathbb{R}^{n_{11}\times n_{11}}$, $B_{11}\in\mathbb{R}^{n_{11}\times m_1}$, $m = m_1+m_2+m_3$ and $\alpha\in N^{n_\alpha}$ is some multi-index. We may now observe that for

$$E = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0\end{bmatrix},\quad A = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1\end{bmatrix},\quad K = [0,\cdots,0,1],$$

the system

$$\begin{bmatrix} E\\ 0\end{bmatrix}\dot{x}(t) = \begin{bmatrix} A\\ K\end{bmatrix}x(t)$$

has a smooth solution $x(\cdot)$ with $Ex(0)=Ex^0$ for every initial condition x^0. Therefore, K is a compatible control for $[E,A,0,0]$. Define

$$E_\alpha = \operatorname{diag}(e_{\alpha_1}^{[\alpha_1]},\dots,e_{\alpha_{n_\alpha}}^{[\alpha_{n_\alpha}]})\in\mathbb{R}^{|\alpha|\times n_\alpha},$$

where $e_i^{[j]}\in\mathbb{R}^j$ is the ith canonical unit vector in $\mathbb{R}^j$. Blockwise application of this argument yields that the system

$$\begin{bmatrix} K_\alpha\\ 0\end{bmatrix}\dot{x}(t) = \begin{bmatrix} L_\alpha\\ E_\alpha^\top\end{bmatrix}x(t)$$

has a smooth solution $x(\cdot)$ with $K_\alpha x(0) = K_\alpha x^0$ for every initial condition x^0. Therefore, $E_\alpha^\top$ is a compatible control for $[K_\alpha,L_\alpha,0,0]$. We may now define

$$\tilde{K}_1 := \begin{bmatrix} 0 & 0\\ 0 & 0\\ 0 & E_\alpha^\top\end{bmatrix},\quad \tilde{K}_2 := \begin{bmatrix} I_{m_1} & 0 & 0\\ 0 & 0 & I_{m_3}\\ 0 & 0 & 0\end{bmatrix}$$

and conclude that $[\tilde{K}_1, \tilde{K}_2]$ is a compatible control for $[\tilde{W}E_{11}\tilde{V}, \tilde{W}A_{11}\tilde{V} + \tilde{W}B_1F, \tilde{W}B_1U, 0]$. Defining

$$\begin{aligned}
\hat{K}_1 &:= \tilde{K}_1\tilde{V}^{-1} + \tilde{K}_2U^{-1}F\tilde{V}^{-1},\\
K_1 &:= \left[[\hat{K}_1, 0_{k\times n_2}, 0_{k\times n_3}]V^{-1}, 0_{k\times(n-r)}\right][T_1, T_2]^{-1},\\
\hat{K}_2 &:= \tilde{K}_2U^{-1}, \quad K_2 := \hat{K}_2,
\end{aligned}$$

it is then easy to see that $[K_1, K_2]$ is compatible for $[E, A, B, C]$.

Step 3: It remains to show that $K = [K_1, K_2]$ satisfies the requirements. By Step 1 and Lemma 11 we find that $\mathcal{ZD}_{[E_1^K, A_1^K, 0, 0]}$ are autonomous if, and only if, $\mathrm{rk}_{\mathbb{R}[s]}(sE_1^K - A_1^K) = r + m$. The latter follows from the fact that by Step 2 we have

$$\begin{aligned}
&\mathrm{rk}_{\mathbb{R}[s]}\begin{bmatrix} sE_{11}-A_{11} & sE_{12}-A_{12} & sE_{13}-A_{13} & -B_1\\ 0 & sE_{22}-A_{22} & sE_{23}-A_{23} & 0\\ 0 & 0 & sE_{33}-A_{33} & 0\\ -\hat{K}_1 & 0 & 0 & -\hat{K}_2\end{bmatrix}\\
&= \mathrm{rk}_{\mathbb{R}[s]}\begin{bmatrix}
\begin{bmatrix} sI_{n_{11}}-\tilde{A}_{11} & 0\\ 0 & sK_\alpha - L_\alpha\\ 0 & 0\end{bmatrix} & \begin{bmatrix} sE_{12}-A_{12} & sE_{13}-A_{13}\end{bmatrix} & \begin{bmatrix} -B_{11} & 0 & 0\\ 0 & 0 & 0\\ 0 & -I_{m_2} & 0\end{bmatrix}\\
0 & \begin{bmatrix} sE_{22}-A_{22} & sE_{23}-A_{23}\\ 0 & sE_{33}-A_{33}\end{bmatrix} & 0\\
\begin{bmatrix} 0 & 0\\ 0 & 0\\ 0 & -E_\alpha^\top\end{bmatrix} & 0 & \begin{bmatrix} -I_{m_1} & 0 & 0\\ 0 & 0 & -I_{m_3}\\ 0 & 0 & 0\end{bmatrix}
\end{bmatrix} = r + m
\end{aligned}$$

since

$$\mathrm{rk}_{\mathbb{R}[s]}\begin{bmatrix} sE_{22}-A_{22} & sE_{23}-A_{23}\\ 0 & sE_{33}-A_{33}\end{bmatrix} = n_2 + n_3$$

$$\text{and}\quad \mathrm{rk}_{\mathbb{R}[s]}\begin{bmatrix} sK_\alpha - L_\alpha\\ -E_\alpha^\top\end{bmatrix} = |\alpha|.$$

We show that (9) holds. This is clearly equivalent to showing that for all $w \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^q)$ there exists $(z_1, z_2, z_3, u) \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3} \times \mathbb{R}^m)$ such that

$$\begin{aligned}
&\begin{bmatrix} E_{11} & E_{12} & E_{13}\\ 0 & E_{22} & E_{23}\\ 0 & 0 & E_{33}\\ 0 & 0 & 0\end{bmatrix}\begin{pmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\end{pmatrix}\\
&\quad = \begin{bmatrix} A_{11} & A_{12} & A_{13}\\ 0 & A_{22} & A_{23}\\ 0 & 0 & A_{33}\\ \hat{K}_1 & 0 & 0\end{bmatrix}\begin{pmatrix} z_1\\ z_2\\ z_3\end{pmatrix} + \begin{bmatrix} B_1\\ 0\\ 0\\ \hat{K}_2\end{bmatrix}u + \begin{bmatrix} Q_1\\ Q_2\\ Q_3\\ 0\end{bmatrix}w.
\end{aligned}$$

By (12) there exists z_3, and z_2 is then the solution of an ODE as E_{22} is invertible. In order to find (z_1, u) we need to find a solution to the equation

$$\begin{bmatrix} E_{11} & 0\\ 0 & 0\end{bmatrix}\begin{pmatrix}\dot{z}_1(t)\\ \dot{u}(t)\end{pmatrix} = \begin{bmatrix} A_{11} & B_1\\ \hat{K}_1 & \hat{K}_2\end{bmatrix}\begin{pmatrix} z_1(t)\\ u(t)\end{pmatrix} + \begin{pmatrix} f(t)\\ 0\end{pmatrix}$$

for any given inhomogeneity $f \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{l_1})$. In view of (13), this is equivalent to finding a solution $(v_1, v_2, u_1, u_2, u_3) \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n_{11}} \times \mathbb{R}^{|\alpha|} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3})$ of

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}I - \tilde{A}_{11} & 0 & -B_{11} & 0 & 0\\ 0 & \frac{\mathrm{d}}{\mathrm{d}t}K_\alpha - L_\alpha & 0 & 0 & 0\\ 0 & 0 & 0 & -I_{m_2} & 0\\ 0 & 0 & -I_{m_1} & 0 & 0\\ 0 & 0 & 0 & 0 & -I_{m_3}\\ 0 & -E_\alpha^\top & 0 & 0 & 0\end{bmatrix}\begin{pmatrix} v_1\\ v_2\\ u_1\\ u_2\\ u_3\end{pmatrix} = \begin{pmatrix} f_1\\ f_2\\ f_3\\ 0\\ 0\\ 0\end{pmatrix}$$

for $(f_1, f_2, f_3) \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n_{11}} \times \mathbb{R}^{|\alpha| - n_\alpha} \times \mathbb{R}^{m_2})$. Set $u_1 = 0, u_2 = f_3, u_3 = 0$ and choose v_1 such that $\dot{v}_1 = \tilde{A}_{11}v_1 + f_1$. It remains to find v_2 such that

$$\begin{bmatrix} K_\alpha\\ 0\end{bmatrix}\dot{v}_2 = \begin{bmatrix} L_\alpha\\ E_\alpha^\top\end{bmatrix}v_2 + \begin{pmatrix} f_2\\ 0\end{pmatrix}.$$

By a permutation of variables in v_2, the above system can equivalently be written as

$$\begin{bmatrix} I_{|\alpha|-n_\alpha} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot v_{21} \\ \dot v_{22} \end{pmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ 0 & I_{n_\alpha} \end{bmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} + \begin{pmatrix} f_2 \\ 0 \end{pmatrix}$$

and this equation obviously has a solution for every f_2. This completes the proof of the theorem. □

Note that as a consequence of Theorem 12, for any $[E, A, B, C] \in \Sigma_{l,n,m,p}$ there always exists a control $K \in \mathbb{R}^{k\times(n+m)}$ compatible for $[E, A, B, C]$ such that $\mathcal{ZD}_{[E^K,A^K,0,C^K]}$ are autonomous.

5 Lebret's open problem

Lebret [28] pointed out that solvability of the DDP with proportional state feedback does not guarantee disturbance rejection in general. This is still true when we consider behavioral feedback. The following example is taken from [28].

Example 13 *Consider the system*

$$[1, 0]\, \dot x(t) = [0, -1]\, x(t) + u(t) + w(t), \quad y(t) = [1, 0]x(t).$$

It is straightforward to check that the condition in Theorem 12 is satisfied and hence there exists a compatible control $[K_1, K_2]$ which achieves disturbance decoupling and autonomous zero dynamics of the closed-loop system. For the present example we may choose, e.g.,

$$K_1 = [0, 0], \quad K_2 = [1]$$

and hence the closed-loop system reads

$$\dot x_1(t) = -x_2(t) + w(t), \quad u(t) = 0, \quad y(t) = x_1(t).$$

However, y still depends on w as

$$y(t) = x_1(0) + \int_0^t w(s) - x_2(s)\ \mathrm{d}s\,, \quad t \in \mathbb{R},$$

but the disturbance is canceled by the free variable x_2 in the sense that two different disturbances are not distinguishable at the output. The dependence of y on the disturbance is therefore hidden.

To exclude a hidden dependence on the disturbance an additional assumption for disturbance decoupling suggested by Lebret [28] is uniqueness of the output of the closed-loop system. This justifies the following definition.

Definition 14 *Let $[E, A, 0, C] \in \Sigma_{l,n,0,p}$ and $Q \in \mathbb{R}^{l\times q}$. Then we call $[E, A, Q, C]$* disturbance decoupled with output uniqueness (DDOU)*, if $[E, A, Q, C]$ is disturbance decoupled and*

$$\begin{aligned} &\forall\, w \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\ \forall\, (x_1, w, y_1), (x_2, w, y_2) \\ &\qquad \in \mathfrak{B}_{[E,A,Q,C]} \cap \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^p) \\ &\forall\, I \subseteq \mathbb{R} \text{ open interval}: \quad y_1|_I = y_2|_I \implies y_1 = y_2. \end{aligned}$$

Note that by linearity, $[E, A, Q, C]$ is DDOU if, and only if, $[E, A, Q, C]$ is disturbance decoupled and

$$\begin{aligned} &\forall\, (x, y) \in \mathfrak{B}_{[E,A,0_{l\times 0},C]} \cap \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^n \times \mathbb{R}^p) \\ &\qquad \forall\, I \subseteq \mathbb{R} \text{ open interval}: \quad y|_I = 0 \implies y = 0. \end{aligned}$$

Therefore, compared to disturbance decoupling, the additional condition of output uniqueness in the property DDOU is independent of Q.
In the context of feedback in the behavioral sense, and using the notation from Theorem 12, we may now seek for a compatible control K such that $[E^K, A^K, Q^K, C^K]$ is DDOU and $\mathcal{ZD}_{[E^K,A^K,0,C^K]}$ are autonomous. Lebret [28] conjectures a characterization of this problem (without the additional property of the zero dynamics) where proportional state feedback $u = Fx$ is considered – a proof or counterexample to this conjecture has not been found so far. In Subsections 5.1 and 5.2 we derive two different solutions using the more general behavioral feedback.
First, we show that the problem of achieving a unique output is the same as the problem of achieving a unique state. The latter means that the underlying DAE is autonomous. To define this we consider the set of homogeneous DAEs

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t), \tag{14}$$

where $E, A \in \mathbb{R}^{l\times n}$, which is denoted by $\Sigma_{l,n}$ and we write $[E, A] \in \Sigma_{l,n}$. The behavior of $[E, A] \in \Sigma_{l,n}$ is given by

$$\begin{aligned} \mathfrak{B}_{[E,A]} := \{ x \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}\to\mathbb{R}^n)\ \mid Ex \in \mathcal{AC}(\mathbb{R}\to\mathbb{R}^l), \\ x \text{ satisfies (14) for almost all } t \in \mathbb{R}\ \}, \end{aligned}$$

Similar to autonomous zero dynamics, a DAE $[E, A] \in \Sigma_{l,n}$ is called *autonomous*, if

$$\begin{aligned} &\forall\, x_1, x_2 \in \mathfrak{B}_{[E,A]}\ \forall\, I \subseteq \mathbb{R} \text{ open interval}: \\ &\qquad x_1|_I \overset{\text{a.e.}}{=} x_2|_I \implies x_1 \overset{\text{a.e.}}{=} x_2. \end{aligned}$$

For characterizations of autonomy see also [13]. Here we need the following algebraic characterization which is an immediate consequence of [11, Cor. 5.2].

Lemma 15 *A DAE $[E, A] \in \Sigma_{l,n}$ is autonomous if, and only if, $\mathrm{rk}_{\mathbb{R}[s]}(sE - A) = n$.*

In the following result we show that a system is DDOU with autonomous zero dynamics if, and only if, it is disturbance decoupled and autonomous.

Proposition 16 *Let $[E, A, 0, C] \in \Sigma_{l,n,0,p}$ and $Q \in \mathbb{R}^{l\times q}$. Then $[E, A, Q, C]$ is DDOU and $\mathcal{ZD}_{[E,A,0,C]}$ are autonomous if, and only if, $[E, A, Q, C]$ is disturbance decoupled and $[E, A]$ is autonomous.*

PROOF. $\Leftarrow$: First, by autonomy of $[E, A]$ and Lemma 15 we have

$$\mathrm{rk}_{\mathbb{R}[s]}(sE - A) = n, \quad \text{thus} \quad \mathrm{rk}_{\mathbb{R}[s]}\begin{bmatrix} sE - A \\ -C \end{bmatrix} = n,$$

which, by Lemma 11 is equivalent to $\mathcal{ZD}_{[E,A,0,C]}$ being autonomous. Choose $V \in \mathbf{Gl}_p(\mathbb{R})$ such that $VC = \left[\begin{smallmatrix} C_1 \\ 0 \end{smallmatrix}\right]$ where $C_1 \in \mathbb{R}^{p_1\times n}$ has full row rank. Then, invoking that $[E, A, Q, C]$ is disturbance decoupled, $[E, A, Q, C]$ is DDOU, if

$$\begin{aligned}&\forall\, (x, y) \in \mathfrak{B}_{[E,A,0,C_1]} \cap \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^n \times \mathbb{R}^{p_1})\\ &\qquad \forall\, I \subseteq \mathbb{R} \text{ open interval}: \quad y|_I = 0 \;\Rightarrow\; y = 0.\end{aligned}$$

Clearly, $\mathcal{ZD}_{[E,A,0,C_1]}$ are autonomous as well, so we may apply [7, Thm. 4.3] to find $S \in \mathbf{Gl}_l(\mathbb{R})$, $T \in \mathbf{Gl}_n(\mathbb{R})$ such that

$$SET = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & E_{32} & N \\ 0 & E_{42} & E_{43} \end{bmatrix}, \; SAT = \begin{bmatrix} A_{11} & A_{12} & 0 \\ 0 & 0 & I_{n_3} \\ 0 & A_{42} & 0 \end{bmatrix},$$
$$C_1T = [0, I_{p_1}, 0], \quad (15)$$

where $N \in \mathbb{R}^{n_3\times n_3}$ is nilpotent with $N^\nu = 0$, $N^{\nu-1} \neq 0$. Seeking a contradiction, assume that there exists an open interval $I = (a, b) \subseteq \mathbb{R}$ and $(x, y) \in \mathfrak{B}_{[E,A,0,C_1]} \cap \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^{p_1})$ such that $y|_I = 0$ and $y \neq 0$. Let $(x_1^\top, y^\top, x_3^\top)^\top = T^{-1}x$, then

$$x_3 = \sum_{i=0}^{\nu-1} E_{32}N^i y^{(i+1)} \quad \text{and}$$
$$x_1(t) = e^{A_{11}(t-a)}x_1(a) + \int_a^t e^{A_{11}(t-s)}A_{12}y(s)\,\mathrm{d}s$$

for all $t \in \mathbb{R}$, and hence $x_3|_I = 0$. Since $\tilde{x}_1(t) := e^{A_{11}(t-a)}x_1(a)$, $t \in \mathbb{R}$, and $\tilde{x} := T(x_1^\top, 0, 0)^\top$ satisfy $(\tilde{x}, 0) \in \mathfrak{B}_{[E,A,0,C_1]}$, by linearity of the behavior we find $(x - \tilde{x}, y) \in \mathfrak{B}_{[E,A,0,C_1]}$ and we have $(x - \tilde{x})|_I = 0$. Autonomy of $[E, A]$ now implies $x - \tilde{x} = 0$, thus, in particular, $y = 0$.

$\Rightarrow$: We only need to show that $[E, A]$ is autonomous. Again we assume that (15) holds for some invertible S and T. Then, invoking Lemma 15, $[E, A]$ is autonomous if, and only if,

$$\mathrm{rk}_{\mathbb{R}[s]}\begin{bmatrix} sI - A_{11} & -A_{12} & 0 \\ 0 & sE_{32} & sN - I \\ 0 & sE_{42} - A_{42} & sE_{43} \end{bmatrix} = n$$
$$\iff \mathrm{rk}_{\mathbb{R}[s]}\begin{bmatrix} sE_{32} & sN - I \\ sE_{42} - A_{42} & sE_{43} \end{bmatrix} = p_1 + n_3.$$

Seeking a contradiction, assume that $\mathrm{rk}_{\mathbb{R}[s]}\left[\begin{smallmatrix} sE_{32} & sN-I \\ sE_{42}-A_{42} & sE_{43} \end{smallmatrix}\right] < p_1 + n_3$ and hence the DAE $\left[\left[\begin{smallmatrix} E_{32} & N \\ E_{42} & E_{43} \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & I \\ A_{42} & 0 \end{smallmatrix}\right]\right]$ is not autonomous. Therefore, there exists $(y, x_3) \in \mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^{p_1} \times \mathbb{R}^{n_3})$ and an open interval $I \subseteq \mathbb{R}$ such that $(y, x_3)|_I = 0$ and $(y, x_3) \neq 0$. If $y = 0$, then $x_3 = \sum_{i=0}^{\nu-1} E_{32}N^i y^{(i+1)} = 0$ which cannot be, thus $y \neq 0$. Chosse $x_1 \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n_1})$ such that $\dot{x}_1 = A_{11}x_1 + A_{12}y$, then, with $x := T(x_1^\top, y^\top, x_3^\top)^\top$ we have $(x, y) \in \mathfrak{B}_{[E,A,0,C_1]} \cap \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^{p_1})$. Since $y|_I = 0$ and $y \neq 0$ this contradicts the assumption of unique output. $\square$

Using state feedback $u = Fx$, the problem of finding F such that the closed-loop system is disturbance decoupled and the output is unique has been called DDP in [28] and the problem of finding F such that the closed-loop system is autonomous and disturbance decoupled has been called DDPU in [2,28]. Proposition 16 shows that in the case of behavioral feedback the DDP (where we can always achieve autonomous zero dynamics) is equivalent to the DDPU.

In the remainder of this section we present two different solutions to the DDP with unique output. In the first approach we seek for an additional condition on $[E, A, B, C]$ and Q compared to that in Theorem 12 that ensures autonomy of the closed-loop system. In the second approach, we keep the condition from Theorem 12 and relax the assumption on the behavioral control K. Instead of requiring K to be compatible for $[E, A, B, C]$, we require K to be compatible for the system which consists of those trajectories which produce zero output.

5.1 Solution by additional assumption

We start with a characterization of all compatible controls K which render $[E^K, A^K]$ autonomous.

Proposition 17 *Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ and let $K \in \mathbb{R}^{k\times(n+m)}$. Choose, according to [16, Cor. 2.3], $S \in \mathbf{Gl}_l(\mathbb{R})$ and $T \in \mathbf{Gl}_{n+m}(\mathbb{R})$ such that*

$$S[sE - A, -B]T = \begin{bmatrix} sI_{n_1} - A_{11} & -A_{12} & 0 \\ 0 & 0 & sE_{22} - A_{22} \end{bmatrix}, \quad (16)$$

where $A_{11} \in \mathbb{R}^{n_1\times n_1}, A_{12} \in \mathbb{R}^{n_1\times n_2}, E_{22}, A_{22} \in \mathbb{R}^{(l-n_1)\times n_3}$ with $\mathrm{rk}_{\mathbb{C}}(\lambda E_{22} - A_{22}) = n_3$ for all $\lambda \in$

$\mathbb{C}$, *and let* $KT = [K_1, K_2, K_3]$ *according to the partitioning in* (16). *Then* K *is a compatible control for* $[E, A, B, C]$ *such that* $[E^K, A^K]$ *is autonomous if, and only if,* $\operatorname{im} K_1 \subseteq \operatorname{im} K_2$ *and* K_2 *has full column rank* n_2.

PROOF. $\Rightarrow$: Let $x_1 \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n_1})$ be such that $\dot{x}_1 = A_{11} x_1$. Then $(x, u) := T(x_1^\top, 0, 0)^\top$ satisfies $(x, u, Cx) \in \mathfrak{B}_{[E,A,B,C]}$. Since K is compatible for $[E, A, B, C]$ there exists $(\tilde{x}, \tilde{u}) \in \mathfrak{B}_{[E^K,A^K,0,0]} \cap \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m)$ such that $Ex(0) = E\tilde{x}(0)$. Write $(\tilde{x}_1^\top, \tilde{x}_2^\top, \tilde{x}_3^\top)^\top = T^{-1}\tilde{x}$ according to the decomposition (16). If follows from the condition $\operatorname{rk}_\mathbb{C}(\lambda E_{22} - A_{22}) = n_3$ for all $\lambda \in \mathbb{C}$ that $x_3 = 0$, see e.g. [15, Thm. 3.2]. Then $Ex(0) = E\tilde{x}(0)$ if, and only if, $x_1(0) = \tilde{x}_1(0)$. As furthermore

$$K_1 \tilde{x}_1(0) + K_2 \tilde{x}_2(0) = 0$$

and $x_1(0)$ is arbitrary it follows that $\operatorname{im} K_1 \subseteq \operatorname{im} K_2$. Then, in particular, there exists $Z \in \mathbb{R}^{n_2 \times n_1}$ such that $K_1 = K_2 Z$.

In order to show that K_2 has full column rank, we assume that $\operatorname{rk} K_2 < n_2$. Note that we have

$$\begin{bmatrix} S & 0 \\ 0 & I_k \end{bmatrix} (sE^K - A^K) T = \begin{bmatrix} sI_{n_1} - A_{11} & -A_{12} & 0 \\ 0 & 0 & sE_{22} - A_{22} \\ -K_2 Z & -K_2 & -K_3 \end{bmatrix}.$$

If $\operatorname{rk} \left[\begin{smallmatrix} A_{12} \\ K_2 \end{smallmatrix}\right] < n_2$, then there exists $y \in \mathbb{R}^{n_2}$ such that $A_{12} y = 0$ and $K_2 y = 0$. Therefore,

$$\begin{bmatrix} sI_{n_1} - A_{11} & -A_{12} & 0 \\ 0 & 0 & sE_{22} - A_{22} \\ -K_2 Z & -K_2 & -K_3 \end{bmatrix} \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} = 0,$$

which contradicts the fact that $\operatorname{rk}_{\mathbb{R}[s]}(sE^K - A^K) = n + m$ by autonomy of $[E^K, A^K]$ and Lemma 15. Assume that $\operatorname{rk} \left[\begin{smallmatrix} A_{12} \\ K_2 \end{smallmatrix}\right] = n_2$ and let $S = \left[\begin{smallmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{smallmatrix}\right] \in \mathbf{Gl}_{l+k}(\mathbb{R})$, where $S_{11} \in \mathbb{R}^{n_2 \times l}, S_{22} \in \mathbb{R}^{(l+k-n_2) \times k}$ and S_{12}, S_{21} are of appropriate sizes, be such that

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} A_{12} \\ K_2 \end{bmatrix} = \begin{bmatrix} I_{n_2} \\ 0 \end{bmatrix}.$$

We show that $\ker S_{21} \neq \{0\}$ by contradiction. So assume that $\ker S_{21} = \{0\}$. Then $S_{21} A_{12} + S_{22} K_2 = 0$ implies

$$A_{12} = -(S_{21}^\top S_{21})^{-1} S_{21}^\top S_{22} K_2,$$

and, since $\ker K_2 \neq \{0\}$, we arrive at

$$\ker \begin{bmatrix} A_{12} \\ K_2 \end{bmatrix} = \ker \begin{bmatrix} -(S_{21}^\top S_{21})^{-1} S_{21}^\top S_{22} K_2 \\ K_2 \end{bmatrix} \neq \{0\},$$

a contradiction. Now let $v \in \ker S_{21} \setminus \{0\}$ and set $p_1(s) := \big(sI - (A_{11} - A_{12} Z)\big)^{-1} v \in \mathbb{R}(s)^{n_1} \setminus \{0\}$. Then

$$\begin{aligned} & S_{21}\big(sI - (A_{11} - A_{12}Z)\big) p_1(s) = 0 \\ \iff \quad & S_{21}(sI - A_{11}) p_1(s) + S_{21} A_{12} Z p_1(s) = 0 \\ \overset{S_{21}A_{21} + S_{22}K_2 = 0}{\iff} \quad & S_{21}(sI - A_{11}) p_1(s) - S_{22} K_2 Z p_1(s) = 0 \\ \iff \quad & [S_{21}, S_{22}] \begin{bmatrix} sI - A_{11} \\ -K_2 Z \end{bmatrix} p_1(s) = 0. \end{aligned}$$

Set $p_2(s) := [S_{11}, S_{12}] \left[\begin{smallmatrix} sI - A_{11} \\ -K_2 Z \end{smallmatrix}\right] p_1(s) \in \mathbb{R}(s)^{n_2}$ and $p_3(s) := 0 \in \mathbb{R}(s)^{n_3}$. Then

$$\begin{aligned} & \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} sI - A_{11} \\ -K_2 Z \end{bmatrix} p_1(s) - \begin{bmatrix} I_{n_2} \\ 0 \end{bmatrix} p_2(s) = 0 \\ \iff \quad & \begin{bmatrix} sI - A_{11} \\ -K_2 Z \end{bmatrix} p_1(s) - \begin{bmatrix} A_{12} \\ K_2 \end{bmatrix} p_2(s) = 0 \\ \iff \quad & \begin{bmatrix} sI - A_{11} & -A_{12} & 0 \\ 0 & 0 & sE_{22} - A_{22} \\ -K_2 Z & -K_2 & -K_3 \end{bmatrix} \begin{pmatrix} p_1(s) \\ p_2(s) \\ p_3(s) \end{pmatrix} = 0, \end{aligned}$$

and hence $\operatorname{rk}_{\mathbb{R}[s]}(sE^K - A^K) = \operatorname{rk}_{\mathbb{R}(s)}(sE^K - A^K) < n + m$, a contradiction.

$\Leftarrow$: Let $Z \in \mathbb{R}^{n_2 \times n_1}$ be such that $K_1 = K_2 Z$. First we show that K is compatible for $[E, A, B, C]$. Let $(x_1, x_2) \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ be such that $\dot{x}_1 = A_{11} x_1 + A_{12} x_2$. Then, for the solution $\tilde{x}_1$ of

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \tilde{x}_1 = (A_{11} - A_{12} Z) \tilde{x}_1, \quad \tilde{x}_1(0) = x_1(0),$$

and $\tilde{x}_2 := -Z x_1$ we find

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \tilde{x}_1 = A_{11} \tilde{x}_1 + A_{12} \tilde{x}_2, \quad K_2 Z \tilde{x}_1 + K_2 \tilde{x}_2 = 0,$$

and, furthermore,

$$[I_{n_1}, 0] \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = [I_{n_1}, 0] \begin{pmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{pmatrix},$$

which proves that $[K_1, K_2, K_3]$ is compatible for $[E, A, B, C]$. For autonomy, by Lemma 15 it remains to show that

$$\operatorname{rk}_{\mathbb{R}[s]} \begin{bmatrix} sI - A_{11} & -A_{12} & 0 \\ 0 & 0 & sE_{22} - A_{22} \\ -K_2 Z & -K_2 & -K_3 \end{bmatrix} = n + m$$

or, what is the same because of the rank property of $sE_{22} - A_{22}$, that

$$\ker_{\mathbb{R}[s]} \begin{bmatrix} sI - A_{11} & -A_{12} \\ -K_2 Z & -K_2 \end{bmatrix} = \{0\}.$$

Let $p_1(s) \in \mathbb{R}[s]^{n_1}, p_2(s) \in \mathbb{R}[s]^{n_2}$ be such that

$$\begin{bmatrix} sI - A_{11} & -A_{12} \\ -K_2 Z & -K_2 \end{bmatrix} \begin{pmatrix} p_1(s) \\ p_2(s) \end{pmatrix} = 0.$$

Since K_2 has full column rank it follows $p_2(s) = -Zp_1(s)$. Therefore,

$$0 = (sI - A_{11})p_1(s) + A_{12}p_2(s) = \big(sI - (A_{11} - A_{12}Z)\big)p_1(s)$$

and hence $p_1(s) = 0$ and $p_2(s) = 0$. This completes the proof. $\square$

For the proof of the main result of this section we need some preliminaries on disturbance decoupling for ODE systems with a feedthrough term, i.e., systems of the form

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}x(t) &= Ax(t) + Bu(t) + Qw(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{17}$$

We seek a state feedback $u = Fx$ such that the closed-loop system $[I, A + BF, Q, C + DF]$ is disturbance decoupled. This problem has been treated in [32] and in a more general form in [33]. To find a geometric characterization of solvability of this problem, we need to introduce the following modification of the generalized Wong sequences which incorporates the feedthrough term. Define

$$\begin{aligned} \mathcal{U}^0_{[A,B,C,D]} &:= \mathbb{R}^n, \\ \mathcal{U}^{i+1}_{[A,B,C,D]} &:= \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left(\left(\mathcal{U}^i_{[A,B,C,D]} \times \{0\} \right) + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix} \right) \end{aligned}$$

and, since this is a non-increasing sequence of subspaces which terminates after finitely many steps, we may set

$$\mathcal{U}^*_{[A,B,C,D]} := \bigcap_{i \in \mathbb{N}_0} \mathcal{U}^i_{[A,B,C,D]}.$$

By [32, Lem. 2.5], $\mathcal{U}^*_{[A,B,C,D]}$ is the largest subspace $\mathcal{U} \subseteq \mathbb{R}^n$ such that there exists $F \in \mathbb{R}^{m \times n}$ which satisfies

$$(A + BF)\mathcal{U} \subseteq \mathcal{U} \quad \text{and} \quad (C + DF)\mathcal{U} = \{0\}.$$

The following result is a simple modification of [34, Thm. 4.8] or [32, Thm. 2.17], resp.

Lemma 18 *Let $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m}$ and $Q \in \mathbb{R}^{n \times q}$. Then there exists $F \in \mathbb{R}^{m \times n}$ such that $[I, A + BF, Q, C + DF] \in \Sigma_{n,n,q,p}$ is disturbance decoupled if, and only if,* $\operatorname{im} Q \subseteq \mathcal{U}^*_{[A,B,C,D]}$.

Theorem 19 *Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ and $Q \in \mathbb{R}^{l \times q}$. Choose $S \in \mathbf{Gl}_l(\mathbb{R})$, $T \in \mathbf{Gl}_{n+m}(\mathbb{R})$ such that (16) holds and let, accordingly, $SQ = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$ and $[C, 0]T = [C_1, C_2, C_3]$. Then there exists a control $K \in \mathbb{R}^{k \times (n+m)}$ compatible for $[E, A, B, C]$ such that $[E^K, A^K, Q^K, C^K]$ is disturbance decoupled and $[E^K, A^K]$ is autonomous if, and only if,*

(i) $\operatorname{im} Q \subseteq E\mathcal{V}^*_{[E,A,B,C]} + A\mathcal{W}^*_{[E,A,B,C]} + \operatorname{im} B$,

(ii) $\operatorname{im} Q_1 \subseteq \mathcal{U}^*_{[A_{11}, A_{12}, C_1, C_2]}$.

PROOF. $\Rightarrow$: (i) follows from Theorem 9. Let $KT = [K_1, K_2, K_3]$ according to the decomposition (16). Then Proposition 17 gives that $\operatorname{im} K_1 \subseteq \operatorname{im} K_2$ and K_2 has full column rank. In particular, $K_1 = K_2 Z$ for some $Z \in \mathbb{R}^{n_2 \times n_1}$. From Proposition 5 we may deduce that

$$\begin{aligned} &\forall\, w \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^q)\ \exists\, (x_1, x_2) \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) : \\ &\quad C_1 x_1 + C_2 x_2 = 0 \ \wedge\ \dot{x}_1 = A_{11}x_1 + A_{12}x_2 + Q_1 w \\ &\qquad \wedge\ K_2 Z x_1 + K_2 x_2 = 0. \end{aligned}$$

By $\operatorname{rk} K_2 = n_2$ we find $x_2 = -Zx_1$ and hence

$$\begin{aligned} &\forall\, w \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^q)\ \exists\, x_1 \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n_1}) : \\ &\quad (C_1 - C_2 Z)x_1 = 0 \ \wedge\ \dot{x}_1 = (A_{11} - A_{12}Z)x_1 + Q_1 w. \end{aligned} \tag{18}$$

This means that $[I, A_{11} - A_{12}Z, Q_1, C_1 - C_2 Z]$ is disturbance decoupled. Then Lemma 18 implies (ii).

$\Leftarrow$: By (ii) and Lemma 18 there exists $Z \in \mathbb{R}^{n_2 \times n_1}$ such that (18) holds. By (i) and the decoupling of the systems in (16) we find that, using the same argument as in Step 1 of the proof of Theorem 12,

$$\begin{aligned} &\forall\, w \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^q)\ \exists\, x_3 \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n_3}) : \\ &\qquad E_{22}\dot{x}_3 = A_{22}x_3 + Q_2 w. \end{aligned} \tag{19}$$

Choosing $K_2 = I_{n_2}$, $K_1 = Z$ and $K_3 = 0$ we obtain that, invoking (19),

$$\begin{aligned} &\forall\, w \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^q) \\ &\quad \exists\, (x_1, x_2, x_3) \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}) : \end{aligned}$$

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}I - A_{11} & -A_{12} & 0 \\ 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}t}E_{22} - A_{22} \\ -Z & -I_{n_2} & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ 0 \end{bmatrix} w.$$

This shows that $[E^K, A^K, Q^K, C^K]$ is disturbance decoupled. Compatibility of K and autonomy of $[E^K, A^K]$

is an immediate consequence of the choice of K and Proposition 17. □

5.2 Solution by relaxing compatibility

In this subsection we present a different solution for DDOU where condition (ii) in Theorem 19 is avoided. However, the drawback is that the control is not compatible in general. This is a trade-off between requirements on the data and properties of the control. For a motivation we revisit Example 13.

Example 20 *Use the notation from Example 13. By implementing the condition $y = 0$ as an additional constraint in the system itself, i.e., extending $[K_1, K_2]$ to*

$$K_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

we achieve the closed-loop system

$$0 = x_2(t) + w(t), \quad x_1(t) = u(t) = 0, \quad y(t) = 0,$$

where now the output is independent of the disturbance. Furthermore, we still have existence of a solution for every $w \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R})$. However, the control $[K_1, K_2]$ is not compatible anymore for the system $[E, A, B, C]$, but it is compatible for the system $[[\begin{smallmatrix} E \\ 0 \end{smallmatrix}], [\begin{smallmatrix} A \\ C \end{smallmatrix}], [\begin{smallmatrix} B \\ 0 \end{smallmatrix}], 0]$.

Example 20 motivates to relax the assumption of compatibility of the control K. This may be justified by the fact that for disturbance decoupling only solutions (x, u) of the disturbed system with $Cx = 0$ are considered, cf. Proposition 5. Therefore, it is sometimes sufficient to restrict the compatibility of K to those solution trajectories, i.e., only require K to be compatible for $[[\begin{smallmatrix} E \\ 0 \end{smallmatrix}], [\begin{smallmatrix} A \\ C \end{smallmatrix}], [\begin{smallmatrix} B \\ 0 \end{smallmatrix}], 0]$. In other words, this means that K is a compatible control for the zero dynamics $\mathcal{ZD}_{[E,A,B,C]}$. The above motivation justifies the following ansatz for K,

$$K = \begin{bmatrix} C & 0 \\ Z_1 & Z_2 \end{bmatrix},$$

for $Z_1 \in \mathbb{R}^{k \times n}, Z_2 \in \mathbb{R}^{k \times m}$. The idea is that putting C into the control K does not change solvability of the DDP since the constraint $Cx = 0$ is present anyway. Furthermore, this "superfluous" constraint makes it easier rather than harder to find Z_1 and Z_2 such that $[E^K, A^K]$ is autonomous. This structure of K allows to derive some crucial connections.

Proposition 21 *Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ and $Q \in \mathbb{R}^{l \times q}$. There exists a control $K \in \mathbb{R}^{k \times (n+m)}$ compatible for $[[\begin{smallmatrix} E \\ 0 \end{smallmatrix}], [\begin{smallmatrix} A \\ C \end{smallmatrix}], [\begin{smallmatrix} B \\ 0 \end{smallmatrix}], 0]$ such that $[E^K, A^K, Q^K, C^K]$ is disturbance decoupled and $[E^K, A^K]$ is autonomous if, and only if, there exists a control $[Z_1, Z_2] \in \mathbb{R}^{k \times n} \times \mathbb{R}^{k \times m}$ compatible for $[[\begin{smallmatrix} E \\ 0 \end{smallmatrix}], [\begin{smallmatrix} A \\ C \end{smallmatrix}], [\begin{smallmatrix} B \\ 0 \end{smallmatrix}], 0]$ such that, with $\tilde{K} = [\begin{smallmatrix} C & 0 \\ Z_1 & Z_2 \end{smallmatrix}]$, $[E^{\tilde{K}}, A^{\tilde{K}}, Q^{\tilde{K}}, 0]$ is disturbance decoupled and $[E^{\tilde{K}}, A^{\tilde{K}}]$ is autonomous.*

PROOF. For $\Rightarrow$ set $[Z_1, Z_2] = K$ and for $\Leftarrow$ set $K = [\begin{smallmatrix} C & 0 \\ Z_1 & Z_2 \end{smallmatrix}]$. The remaining calculations are simple and straightforward. □

Under this relaxed compatibility assumption on K the DDP with output uniqueness is solvable if, and only if, the DDP is solvable. That is, using this larger class of controls, the output uniqueness (in fact, the state uniqueness by Proposition 16) can always be satisfied when disturbance decoupling can be achieved. In this sense, the disturbance decoupling problem (called IDDP in [28]), the disturbance decoupling problem with state uniqueness (called DDPU in [28]) and the disturbance decoupling problem with output uniqueness (called DDP in [28]) are all equally hard problems.

Theorem 22 *Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ and $Q \in \mathbb{R}^{l \times q}$. Then there exists a control $K \in \mathbb{R}^{k \times (n+m)}$ compatible for $[[\begin{smallmatrix} E \\ 0 \end{smallmatrix}], [\begin{smallmatrix} A \\ C \end{smallmatrix}], [\begin{smallmatrix} B \\ 0 \end{smallmatrix}], 0]$ such that $[E^K, A^K, Q^K, C^K]$ is disturbance decoupled and $[E^K, A^K]$ is autonomous if, and only if,*

$$\operatorname{im} Q \subseteq E\mathcal{V}^*_{[E,A,B,C]} + A\mathcal{W}^*_{[E,A,B,C]} + \operatorname{im} B.$$

PROOF. By Proposition 21 the problem of finding K is equivalent to finding a control $Z = [Z_1, Z_2] \in \mathbb{R}^{k \times n} \times \mathbb{R}^{k \times m}$ compatible for $[[\begin{smallmatrix} E \\ 0 \end{smallmatrix}], [\begin{smallmatrix} A \\ C \end{smallmatrix}], [\begin{smallmatrix} B \\ 0 \end{smallmatrix}], 0]$ such that, with $\tilde{K} = [\begin{smallmatrix} C & 0 \\ Z_1 & Z_2 \end{smallmatrix}]$, $[E^{\tilde{K}}, A^{\tilde{K}}, Q^{\tilde{K}}, 0]$ is disturbance decoupled and $[E^{\tilde{K}}, A^{\tilde{K}}]$ is autonomous. Observe that $[E^{\tilde{K}}, A^{\tilde{K}}, Q^{\tilde{K}}, 0]$ is disturbance decoupled if, and only if, $[E^Z, A^Z, Q^Z, C^Z]$ is disturbance decoupled. Furthermore, $[E^{\tilde{K}}, A^{\tilde{K}}]$ is autonomous if, and only if,

$$\operatorname{rk}_{\mathbb{R}[s]} \begin{bmatrix} sE - A & -B \\ -C & 0 \\ -Z_1 & -Z_2 \end{bmatrix} = n + m$$

and this is equivalent to $\mathcal{ZD}_{[E^Z, A^Z, 0, C^Z]}$ being autonomous. Therefore, the problem of finding K is equivalent to finding a compatible Z such that $[E^Z, A^Z, Q^Z, C^Z]$ is disturbance decoupled and $\mathcal{ZD}_{[E^Z, A^Z, 0, C^Z]}$ are autonomous. The assertion now follows from Theorem 12. □

6 Conclusions

In the present paper we have derived a geometric characterization for solvability of the DDP by behavioral feedback. It turns out that behavioral feedback achieves disturbance decoupling for a larger class of systems than proportional state feedback. Exploiting the freedom in the choice of the behavioral feedback we have shown that whenever disturbance decoupling can be achieved we may additionally achieve autonomous zero dynamics. Furthermore, the behavioral feedback approach allowed us to solve Lebret's twenty year old open problem [28] of disturbance decoupling with output uniqueness.

The behavioral feedback approach to disturbance decoupling presented in this paper opens the door for the study of various related problems and extensions, among them disturbance decoupled state estimation and disturbance decoupling by dynamic feedback controllers using behavioral feedback as well as almost disturbance decoupling by behavioral feedback. One may consider cases where the disturbances influence the measurement and controlled output, resp., and study the additional stabilization of the closed-loop system. In the absence of disturbances some of these problems have already been treated using the framework of behavioral feedback, see [6,11,12]. In the present paper we took the first step at incorporating disturbance decoupling.

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