The Funnel Observer

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Abstract

We introduce the funnel observer as a novel and simple adaptive observer of "high-gain type". We show that this observer is feasible for a large class of nonlinear systems described by functional differential equations which have a known strict relative degree, the internal dynamics map bounded signals to bounded signals, and the operators involved are sufficiently smooth. Apart from that the funnel observer does not need specific knowledge of the system parameters, and we show that it guarantees prescribed transient behavior of the observation error. We compare the funnel observer to existing (adaptive) high-gain observers and illustrate it by a simulation of a bioreactor model. As an application in feedback control, a cascade of funnel observers is exploited to obtain an artificial output with explicitly known derivatives which tracks the system output with prescribed transient behavior. In some important cases the interconnection of the system with the observer cascade is shown to have stable internal dynamics.

Keywords: nonlinear systems; funnel observer; observer design; high-gain observer; feedback control; internal dynamics.

Nomenclature:

$\mathbb{R}_{\geq 0}$	$= [0, \infty)$
$\mathbb{C}_-$	$= \{ \lambda \in \mathbb{C} \mid \operatorname{Re} \lambda < 0 \}$
$\mathbf{Gl}_n(\mathbb{R})$	the group of invertible matrices in $\mathbb{R}^{n\times n}$
$\sigma(A)$	the spectrum of $A \in \mathbb{R}^{n\times n}$
$\mathscr{L}^\infty_{\text{loc}}(I \to \mathbb{R}^n)$	the set of locally essentially bounded functions $f : I \to \mathbb{R}^n$, $I \subseteq \mathbb{R}$ an interval
$\mathscr{L}^\infty(I \to \mathbb{R}^n)$	the set of essentially bounded functions $f : I \to \mathbb{R}^n$ with norm
$\|f\|_\infty$	$= \text{ess sup}_{t\in I} \|f(t)\|$
$\mathscr{W}^{k,\infty}(I \to \mathbb{R}^n)$	the set of k-times weakly differentiable functions $f : I \to \mathbb{R}^n$ such that $f, \ldots, f^{(k)} \in \mathscr{L}^\infty(I \to \mathbb{R}^n)$
$\mathscr{C}^k(I \to \mathbb{R}^n)$	the set of k-times continuously differentiable functions $f : I \to \mathbb{R}^n$
$\mathscr{C}(I \to \mathbb{R}^n)$	$= \mathscr{C}^0(I \to \mathbb{R}^n)$
$f\|_J$	restriction of the function $f : I \to \mathbb{R}^n$ to $J \subseteq I$

1. Introduction

In the present paper we propose a novel and simple adaptive observer of "high-gain type", the *funnel observer*. The high-gain parameter is determined adaptively online such that the observer output error satisfies a prescribed transient behavior.

High-gain observers have been developed around 30 years ago in the works [7, 22, 24, 28], see also [1] and the recent survey [21]. Choosing the observer gain k large enough, the observer error can be made arbitrarily small, see e.g. [29]. The advantage of high-gain observers is that they can be used to estimate the system states without knowing the exact parameters (in contrast to observer synthesis, see e.g. [5, 6] and the references therein); only some structural assumptions, such as a known relative degree, are necessary. Furthermore, they are robust with respect to input noise. The drawback is that in most cases it is not known a priori how large k must be chosen and appropriate values must be identified by offline simulations. If k is chosen unnecessarily large, the sensitivity to measurement noise increases dramatically.

In order to resolve these problems, the constant high-gain parameter k has been replaced by an adaptation scheme in [3]. The gain $k(t)$ is determined by a differential equation depending on the observation error. This leads to a monotonically increasing $k(t)$ as long as the observation error lies outside a predefined λ-strip $[-\lambda, \lambda]$, and it stops increasing as soon as the error enters the strip. The advantage of this observer is that $k(t)$ is adapted online to the actual needed value, which also leads to lower high-gain parameters in general. However, $k(t)$ is monotonically non-decreasing and hence susceptible to unwarranted increase due to perturbations to the system. Furthermore, while convergence of the observation error to the λ-strip is guaranteed, its transient behavior cannot be influenced.

Another high-gain observer with gain adaptation law is proposed in [25]. Compared to [3] it resolves the drawback of monotonically non-decreasing gain, however a certain condition on the system is necessary which either requires exact knowledge of the high-gain parameter of the system or boundedness of the input $u(t)$. Furthermore, the adaptation law in [25] is not able to influence the transient behavior of the observation error, but only its mean value.

To resolve the above mentioned issues we introduce the fol-

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lowing funnel observer:

$$\boxed{\begin{aligned}\dot z_1(t) &= z_2(t) + \big(q_1 + p_1 k(t)\big)\big(y(t) - z_1(t)\big),\\ \dot z_2(t) &= z_3(t) + \big(q_2 + p_2 k(t)\big)\big(y(t) - z_1(t)\big),\\ &\ \ \vdots\\ \dot z_{r-1}(t) &= z_r(t) + \big(q_{r-1} + p_{r-1} k(t)\big)\big(y(t) - z_1(t)\big),\\ \dot z_r(t) &= \tilde\Gamma u(t) + \big(q_r + p_r k(t)\big)\big(y(t) - z_1(t)\big),\\ k(t) &= \frac{1}{1-\varphi(t)^2\|y(t)-z_1(t)\|^2},\end{aligned}} \tag{1}$$

where the design parameters $p_i > 0$, $q_i > 0$, $\tilde\Gamma \in \mathbb{R}^{m\times m}$ and the function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are explained in detail in Section 3.

We like to emphasize that:

- The proposed adaptation scheme for $k(t)$ is simple, non-dynamic, and non-monotone,
- it guarantees prescribed transient behavior of the observation error, and
- has typical advantages of high-gain observers like no specific knowledge of system parameters required and excellent robustness properties.

To illustrate the observer (1) we consider, as a prototype, the following minimum-phase linear time-invariant system

$$\begin{aligned}\dot x(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t)\end{aligned} \tag{2}$$

with strict relative degree $r \in \mathbb{N}$, i.e., $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$ with the properties:

(A1) $\mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda I - A & B \\ C & 0 \end{bmatrix} = n + m$ for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq 0$;

(A2) $CB = CAB = \ldots = CA^{r-2}B = 0$ and $CA^{r-1}B \in \mathbf{Gl}_m(\mathbb{R})$.

Condition (A1) characterizes the minimum-phase assumption and condition (A2) the strict relative degree.

We will show in Theorem 4.1 that for any solution (x,u,y) of the system (2) such that $y,\ldots,y^{(r-1)}$ are bounded, the funnel observer (1) has an absolutely continuous and bounded solution $(z_1,\ldots,z_r)$ such that k is bounded and

$$\exists\, \varepsilon > 0\ \forall\, t > 0 :\ \|y(t) - z_1(t)\| < \varphi(t)^{-1} - \varepsilon. \tag{3}$$

We stress that condition (3) means prescribed transient behavior of the observation error $e_1(t) := y(t) - z_1(t)$ in the sense that it is pointwise below a given funnel function $1/\varphi$, see Figure 1. To achieve this, the observer gain will be increased whenever $\|e_1(t)\|$ approaches the funnel boundary. High values of the gain function lead to a faster decay of the observation error.

The funnel observer is *not limited to linear systems* (2). We show that the funnel observer (1) is feasible for a large class of nonlinear systems described by functional differential equations which satisfy that

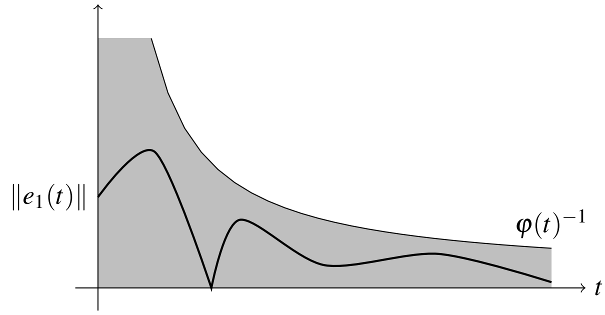

Figure 1: Observation error and funnel function

(i) the system has known strict relative degree r,

(ii) the internal dynamics map bounded signals to bounded signals,

(iii) the operators involved are sufficiently smooth to guarantee local maximal existence of solutions.

We exploit the funnel observer for feedback control. While a drawback of (1) is that the transient behavior of the derivatives of e_1 cannot be influenced, the derivative of z_1 is known explicitly. We show that an application of a cascade of funnel observers yields

- an estimate z for the output y with prescribed transient behavior and
- the derivatives $\dot z,\ldots,z^{(r-1)}$ are known explicitly.

Furthermore, we investigate the internal dynamics of the interconnection of the system with the observer cascade. We show that for a special class of systems with stable internal dynamics, this interconnection has again stable internal dynamics. However, this result is limited to systems with relative degree two or three; for higher relative degree it remains an open problem.

The present paper is organized as follows: In Section 2 we specify the considered system class and discuss several important subclasses. The funnel observer is introduced in Section 3 and feasibility is proved in Section 4. A simulation of the funnel observer for a bioreactor model is provided in Section 5 and the results are compared to the simulation in [3]. Applications in feedback control are discussed in Section 6. Some conclusions are given in Section 7.

2. System Class

In the present paper we consider a large class of nonlinear systems described by functional differential equations of the form

$$\begin{aligned} y^{(r)}(t) &= f\big(d(t), T(y,\dot y,\ldots,y^{(r-1)})(t)\big)\\ &\quad + \Gamma\big(d(t), T(y,\dot y,\ldots,y^{(r-1)})(t)\big)u(t),\\ y|_{[-h,0]} &= y^0 \in \mathscr{W}^{r-1,\infty}([-h,0]\to\mathbb{R}^m),\end{aligned} \tag{4}$$

where $h > 0$ is the "memory" of the system, $r \in \mathbb{N}$ is the strict relative degree, and

- $d \in \mathscr{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^p)$, $p \in \mathbb{N}$, is a disturbance;
- $f \in \mathscr{C}(\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^m)$, $q \in \mathbb{N}$;
- $\Gamma \in \mathscr{C}^1(\mathbb{R}^p \times \mathbb{R}^q \to \mathbf{Gl}_m(\mathbb{R}))$ is the high-frequency gain matrix function;
- $T : \mathscr{C}([-h,\infty) \to \mathbb{R}^m)^r \to \mathscr{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$ is an operator with the following properties:

  a) T maps bounded trajectories to bounded trajectories, i.e., for all $c_1 > 0$ there exists $c_2 > 0$ such that for all $\zeta \in \mathscr{C}([-h,\infty) \to \mathbb{R}^m)^r$:
  $$\sup_{t\in[-h,\infty)} \|\zeta(t)\| \leq c_1 \implies \sup_{t\in[0,\infty)} \|T(\zeta)(t)\| \leq c_2;$$

  b) T is causal, i.e., for all $t \geq 0$ and all $\zeta, \xi \in \mathscr{C}([-h,\infty) \to \mathbb{R}^m)^r$:
  $$\zeta|_{[-h,t)} = \xi|_{[-h,t)} \implies T(\zeta)|_{[0,t]} \overset{\text{a.e.}}{=} T(\xi)|_{[0,t]};$$

  c) T is "locally Lipschitz" continuous in the following sense: for all $t \geq 0$ there exist $\tau, \delta, c > 0$ such that for all $\zeta, \Delta\zeta \in \mathscr{C}([-h,\infty) \to \mathbb{R}^m)^r$ with $\Delta\zeta|_{[-h,t]} = 0$ and $\| \Delta\zeta|_{[t,t+\tau]} \|_\infty < \delta$ we have
  $$\left\| \big(T(\zeta + \Delta\zeta) - T(\zeta)\big)\big|_{[t,t+\tau]} \right\|_\infty \leq c \| \Delta\zeta|_{[t,t+\tau]} \|_\infty.$$

- for every bounded $\zeta \in \mathscr{C}([-h,\infty) \to \mathbb{R}^m)^r$ the map
  $$\psi : \mathbb{R}_{\geq 0} \to \mathbf{Gl}_m(\mathbb{R}),\ t \mapsto \Gamma\big(d(t), T(\zeta)(t)\big)^{-1} \tag{5}$$
  is continuously differentiable and $\frac{\mathrm{d}}{\mathrm{d}t}\psi$ is bounded.

The functions $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ and $y : [-h,\infty) \to \mathbb{R}^m$ are called *input* and *output* of the system (4), respectively. For fixed $u \in \mathscr{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ we call $y \in \mathscr{C}^{r-1}([-h,\omega) \to \mathbb{R}^m)$ a solution of (4) on $[-h,\omega)$, $\omega \in (0,\infty]$, if $y|_{[-h,0]} = y^0$ and $y^{(r-1)}|_{[0,\omega)}$ is absolutely continuous and satisfies the differential equation in (4) for almost all $t \in [0,\omega)$; y is called *maximal*, if it has no right extension that is also a solution. Existence of maximal solutions of (4) for every $y^0 \in \mathscr{W}^{r-1,\infty}([-h,0] \to \mathbb{R}^m)$ and every $u \in \mathscr{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ is guaranteed by [14, Thm. 5]; if $y, \dot{y}, \ldots, y^{(r-1)}$ are bounded, then $\omega = \infty$.

We stress that in (4) we consider systems with the same number of inputs and outputs. A generalization to $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^\mu$ with $\mu \leq m$ is possible, provided we require for $\Gamma \in \mathscr{C}^1(\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^{m\times\mu})$ that $\Gamma(d,w)$ has full column rank for all $d \in \mathbb{R}^p$ and all $w \in \mathbb{R}^q$. Then we may use the pseudoinverse of Γ instead of the inverse in the subsequent considerations.

In the case of relative degree one, i.e., $r = 1$, systems similar to (4) are well studied, see [13, 14, 17, 23]. For relative degree two systems see [10], and for higher relative degree see [16]. In the aforementioned references it is shown that the class of systems (4) encompasses linear and nonlinear systems with existing strict relative degree and exponentially stable internal dynamics (zero dynamics in the linear case) and the operator T allows for infinite-dimensional linear systems, systems with hysteretic effects or nonlinear delay elements, input-to-state stable systems, and combinations thereof. Compared to these works we have added the condition of boundedness of $\dot{\psi}$ as in (5) which ensures an input-independent formulation of the observer error dynamics.

Important subclasses of the systems (4) are linear systems (2) with (A1) and (A2) and infinite-dimensional linear systems (2), where for some real Hilbert space X, the linear operator $A : D(A) \subseteq X \to X$ is the generator of a strongly continuous semigroup, and $B : \mathbb{R}^m \to X$, $C : X \to \mathbb{R}^m$ are linear and bounded. In this case, we further need to assume:

- The *zero dynamics* of (2) are exponentially stable, that is, there exist $M, \omega > 0$ such that for all solutions of $\dot{x} = Ax + Bu$ with $Cx = 0$ we have $\|x(t)\|_X + \|u(t)\| \leq M\|x(0)\|_X e^{-\omega t}$ for all $t \geq 0$;
- $\operatorname{im} B \subseteq D(A^r)$, $\operatorname{im} C^* \subseteq D((A^*)^r)$, $CB = CAB = \ldots = CA^{r-2}B = 0$ and $CA^{r-1}B \in \mathbf{Gl}_m(\mathbb{R})$.

We note that, in the finite-dimensional case, exponential stability of the zero dynamics is equivalent to the system being minimum-phase. It was shown in [18] that this class allows the transformation into a Byrnes-Isidori form, and it can then be shown that it belongs to the class (4) by a straightforward argument.

We have a closer look at nonlinear systems: Consider the nonlinear input-affine system
$$\begin{aligned} \dot{x}(t) &= f\big(x(t)\big) + g\big(x(t)\big)u(t), \\ y(t) &= h\big(x(t)\big) \end{aligned} \tag{6}$$
with $f \in \mathscr{C}(\mathbb{R}^n \to \mathbb{R}^n)$, $g \in \mathscr{C}(\mathbb{R}^n \to \mathbb{R}^{n\times m})$ and $h \in \mathscr{C}(\mathbb{R}^n \to \mathbb{R}^m)$. We assume that there exists a global diffeomorphism $\chi : \mathbb{R}^n \to \mathbb{R}^n$ such that the coordinate transformation $[x_1(t)^\top, \ldots, x_r(t)^\top, \eta(t)^\top]^\top = \chi\big(x(t)\big)$ transforms (6) into *input-normalized Byrnes-Isidori form* (see e.g. [19]):
$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ &\ \vdots \\ \dot{x}_{r-1}(t) &= x_r(t), \\ \dot{x}_r(t) &= g_1(\hat{x}(t), \eta(t)) + g_2(\hat{x}(t), \eta(t))u(t), \\ \dot{\eta}(t) &= g_3(\hat{x}(t), \eta(t)), \\ y(t) &= x_1(t), \end{aligned}$$
with $\hat{x}(t) = [x_1(t)^\top, \ldots, x_r(t)^\top]^\top$, where $g_1 \in \mathscr{C}(\mathbb{R}^n \to \mathbb{R}^m)$, $g_3 \in \mathscr{C}^1(\mathbb{R}^n \to \mathbb{R}^{n-rm})$ and $g_2 \in \mathscr{C}^1(\mathbb{R}^n \to \mathbf{Gl}_m(\mathbb{R}))$; the latter means that the system has (global) strict relative degree r. We assume that
$$\frac{\partial g_2(\cdot)^{-1}}{\partial x_r}\, g_2(\cdot) = 0. \tag{7}$$
For fixed $\hat{x} \in \mathscr{C}(\mathbb{R}_{\geq 0} \to \mathbb{R}^{rm})$ and $\eta^0 \in \mathbb{R}^{n-rm}$ we denote the unique maximal solution of the initial value problem
$$\dot{\eta}(t) = g_3(\hat{x}(t), \eta(t)), \quad \eta(0) = \eta^0$$

by $\eta(\cdot;\eta^0,\hat{x}) : [0,\omega) \to \mathbb{R}^{n-rm}$, $\omega \in (0,\infty]$. Similar to [14] we assume that there exists $\kappa \in \mathscr{C}(\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0})$ and $c > 0$ such that for all $\hat{x} \in \mathscr{C}(\mathbb{R}_{\geq 0} \to \mathbb{R}^{rm})$ and all $t \in [0,\omega)$ we have

$$\|\eta(t;\eta^0,\hat{x})\| \leq c\left(1 + \max_{s\in[0,t]} \kappa(\|\hat{x}(s)\|)\right); \tag{8}$$

this condition in particular implies $\omega = \infty$. Condition (8) on the internal dynamics of (6) resembles Sontag's [27] input-to-state stability, but in fact it is weaker. To show that systems (6) satisfying the above properties belong to the class (4) we set

$$T(y,\dots,y^{(r-1)})(t) \\ := \left(y(t)^\top,\dots,y^{(r-1)}(t)^\top, \eta(t;\eta^0,y,\dots,y^{(r-1)})^\top\right)^\top$$

and calculate that

$$y^{(r)}(t) = g_1\big(T(y,\dots,y^{(r-1)})(t)\big) + g_2\big(T(y,\dots,y^{(r-1)})(t)\big)\,u(t),$$

which is of the form (4) with $f = g_1$ and $\Gamma = g_2$. The operator T is parameterized by η^0 and obviously causal and locally Lipschitz. Condition (8) implies the required bounded-input, bounded-output property of T, cf. also [14]. To show that $\dot{\psi}$ as in (5) is bounded we calculate

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t} T(\hat{x}) \\
&= \left(\dot{y}^\top,\dots,(y^{(r)})^\top, \dot{\eta}(t;\eta^0,\hat{x})^\top\right)^\top \\
&= \left(\dot{y}^\top,\dots,(y^{(r-1)})^\top, \big(g_1\big(T(\hat{x})\big) + g_2\big(T(\hat{x})\big)\,u\big)^\top, g_3\big(T(\hat{x})\big)^\top\right)^\top
\end{aligned}$$

and hence

$$\begin{aligned}
\dot{\psi}(t) &= \frac{\partial g_2(\cdot)^{-1}}{\partial T}\big(T(\hat{x})\big)\,\tfrac{\mathrm{d}}{\mathrm{d}t}T(\hat{x}) \\
&\overset{(7)}{=} \frac{\partial g_2(\cdot)^{-1}}{\partial x_1}\big(T(\hat{x})\big)\,\dot{y} + \ldots + \frac{\partial g_2(\cdot)^{-1}}{\partial x_{r-1}}\big(T(\hat{x})\big)\,y^{(r-1)} \\
&\quad + \frac{\partial g_2(\cdot)^{-1}}{\partial x_r}\big(T(\hat{x})\big)\,g_1\big(T(\hat{x})\big) + \frac{\partial g_2(\cdot)^{-1}}{\partial \eta}\big(T(\hat{x})\big)\,g_3\big(T(\hat{x})\big),
\end{aligned}$$

which proves the desired condition.

In the aforementioned classes of systems which can be transformed into a functional differential equation (4), the operator T is basically the solution operator of a differential equation. We can further consider systems which are of the form (4) with T being of some more involved nature: For instance, T may encompass time delays as well as hysteresis. For a detailed explanation of these classes we refer to [14].

Remark 2.1. It is possible to incorporate a more involved dependence on the input and its derivatives in the system class (4) by adding a term

$$g\big(d(t), T(y,\dot{y},\dots,y^{(r-1)})(t), u(t),\dots,u^{(k)}(t)\big) \tag{9}$$

to the right-hand side of the differential equation in (4), where $g \in \mathscr{C}(\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{(k+1)m} \to \mathbb{R}^m)$ and $u \in \mathscr{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ is k-times weakly differentiable. If u is fix and there exist $\tilde{g} \in \mathscr{C}(\mathbb{R}^{(k+1)m} \to \mathbb{R}^j)$ and $G \in \mathscr{C}(\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^j \to \mathbb{R}^m)$, where $\tilde{g}$ is bounded, such that

$$\begin{aligned}
&g\big(d(t), T(y,\dot{y},\dots,y^{(r-1)})(t), u(t),\dots,u^{(k)}(t)\big) \\
&\quad = G\Big(d(t), T(y,\dot{y},\dots,y^{(r-1)})(t), \tilde{g}\big(u(t),\dots,u^{(k)}(t)\big)\Big),
\end{aligned}$$

then $\tilde{g}\big(u(t),\dots,u^{(k)}(t)\big)$ can be rewritten as a bounded "disturbance" $\tilde{d}(t)$ and hence the system is again of type (4). If $u,\dots,u^{(k)}$ are bounded, then this is always possible.

3. Observer Design

In this section we consider the funnel observer (1) as a new adaptive high-gain observer. Following the methodology of funnel control, see [14, 12] and the references therein, it is our aim that the funnel observer (1) achieves that the error $e_1 = y - z_1$ evolves within a prescribed performance funnel

$$\mathscr{F}_\varphi := \{\, (t,e_1) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|e_1\| < 1 \,\}, \tag{10}$$

which is determined by a function φ belonging to

$$\Phi := \left\{ \varphi \in \mathscr{C}^1(\mathbb{R}_{\geq 0} \to \mathbb{R}) \;\middle|\; \begin{array}{l} \varphi, \dot{\varphi} \text{ are bounded,} \\ \varphi(s) > 0 \text{ for all } s > 0, \\ \text{and } \liminf_{s\to\infty} \varphi(s) > 0 \end{array} \right\}.$$

Note that the funnel boundary is given by the reciprocal of φ, see Figure 2. The case $\varphi(0) = 0$ is explicitly allowed and puts no restriction on the initial value since $\varphi(0)\|e_1(0)\| < 1$; in this case the funnel boundary $1/\varphi$ has a pole at $t = 0$.

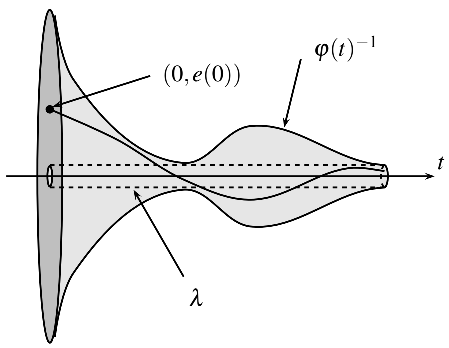

Figure 2: Error evolution in a funnel $\mathscr{F}_\varphi$ with boundary $\varphi(t)^{-1}$ for $t > 0$.

An important property of the funnel class Φ is that each performance funnel $\mathscr{F}_\varphi$ with $\varphi \in \Phi$ is bounded away from zero, i.e., due to boundedness of φ there exists $\lambda > 0$ such that $1/\varphi(t) \geq \lambda$ for all $t > 0$. The funnel boundary is not necessarily monotonically decreasing, while in most situations it is convenient to choose a monotone funnel. However, there are situations where widening the funnel over some later time interval might be beneficial, e.g., when the output signal changes strongly or the system is perturbed by some calibration so that a large observation error would enforce a large observer gain.

The objective is robust estimation of the output y of the system (4) and its derivatives $\dot{y},\dots,y^{(r-1)}$ so that the observation error $e_1 = y - z_1$ evolves within the funnel $\mathscr{F}_\varphi$ and all variables are bounded. To achieve this objective we consider the funnel observer (1) for system (4) with initial conditions

$$z_i(0) = z_i^0 \in \mathbb{R}^m, \quad i = 1, \dots, r, \tag{11}$$

where $\varphi \in \Phi$, $\tilde{\Gamma} \in \mathbb{R}^{m \times m}$ and $q_i > 0$, $p_i > 0$ for all $i = 1, \dots, r$. The functions $z_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, $i = 1, \dots, r$, are the observer states and $k : \mathbb{R}_{\geq 0} \to [1, \infty)$ is the observer gain. The constants $q_i > 0$ are such that the matrix

$$A = \begin{bmatrix} -q_1 & 1 & & \\ \vdots & & \ddots & \\ -q_{r-1} & & & 1 \\ -q_r & & & 0 \end{bmatrix} \in \mathbb{R}^{r \times r}$$

is Hurwitz, i.e., $\sigma(A) \subseteq \mathbb{C}_-$. The constants p_i depend on the choice of the q_i in the following way: Let $Q = Q^\top > 0$ and

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix}, \quad P_{11} \in \mathbb{R},\ P_{12} \in \mathbb{R}^{1 \times (r-1)},\ P_{22} \in \mathbb{R}^{(r-1) \times (r-1)}$$

be such that

$$A^\top P + PA + Q = 0, \quad P = P^\top > 0.$$

The matrix P depends only on the choice of the constants q_i and the matrix Q. The constants p_i must then satisfy

$$\begin{pmatrix} p_1 \\ \vdots \\ p_r \end{pmatrix} = P^{-1} \begin{pmatrix} P_{11} - P_{12} P_{22}^{-1} P_{12}^\top \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -P_{22}^{-1} P_{12}^\top \end{pmatrix}. \tag{12}$$

This condition guarantees that P defines a quadratic Lyapunov function for the observer error dynamics.

The funnel observer (1) is different in its structure when compared to the high-gain observers in [29, 3], where the gain enters with power k^i into the equation for $\dot{z}_i$. Furthermore, the constants q_i are not present in [29, 3], but we show that they are important to ensure boundedness of the error dynamics even when $k(t)$ is small.

Although the observer (1) is a nonlinear and time-varying system, it is simple in its structure and its dimension depends only on the relative degree r of the system (4). Apart from the relative degree, no knowledge of the system (4) is required for the construction of the funnel observer (1); it only uses the input signal $u(t)$ and the output signal $y(t)$, see Figure 3. The bounded-input, bounded-output property of the operator T in (4) can be exploited for an inherent high-gain property of the system (4) and hence to maintain error evolution within the funnel: by the design of the observer (1), the gain $k(t)$ increases if the norm of the error $\|y(t) - z_1(t)\|$ approaches the funnel boundary $1/\varphi(t)$, and decreases if a high gain is not necessary.

For a sketch of the construction of the funnel observer (1) see also Figure 4.

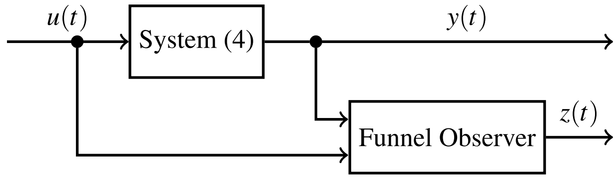

Figure 3: Interconnection of system (4) with the funnel observer (1).

4. Properties of the funnel observer

In this section we prove one of the main results of the present paper: The funnel observer (1), using $u(t)$ and $y(t)$, provides estimates for all bounded signals $y, \dot{y}, \dots, y^{(r-1)}$ of the system (4) such that $y - z_1$ evolves in a prescribed performance funnel $\mathscr{F}_\varphi$ and all signals are bounded; this is true for any disturbance d, i.e., the observer is robust. We only consider the relevant case of strict relative degree $r \geq 2$.

Theorem 4.1. *Consider a system* (4) *with* $r \geq 2$. *Let* $y^0 \in \mathscr{W}^{r-1,\infty}([-h, 0] \to \mathbb{R}^m)$, $u \in \mathscr{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ *and let* $y \in \mathscr{C}^{r-1}([-h, \infty) \to \mathbb{R}^m)$ *be a solution of* (4) *such that* $y, \dot{y}, \dots, y^{(r-1)}$ *are bounded. Consider the funnel observer* (1), (11) *with* $\varphi \in \Phi$ *such that*

$$\varphi(0) \|y(0) - z_1^0\| < 1,$$

$\tilde{\Gamma} \in \mathbb{R}^{m \times m}$ *and* $q_i > 0$, $p_i > 0$ *such that* (12) *is satisfied for corresponding matrices* A, P, Q.
Then (1), (11) *have an absolutely continuous solution* $z = (z_1, \dots, z_r) \in \mathscr{L}^\infty(\mathbb{R}_{\geq 0} \to (\mathbb{R}^m)^r)$ *with* $k \in \mathscr{L}^\infty(\mathbb{R}_{\geq 0} \to [1, \infty))$ *and*

$$\exists\, \varepsilon > 0\ \forall\, t > 0 : \ \|y(t) - z_1(t)\| < \varphi(t)^{-1} - \varepsilon. \tag{13}$$

Furthermore, using the constants M_1 *and* M_2 *in the estimates* (19) *and* (20), *resp., with* $M = \sqrt{M_1^2 + M_2^2}$ *we have*

$$\limsup_{t \to \infty} \|e(t)\| \leq \frac{4 M \lambda_{\max}(P)^2}{\lambda_{\min}(Q) \lambda_{\min}(P)}. \tag{14}$$

Here $\lambda_{\max}(P)$ *denotes the largest eigenvalue of the positive definite matrix* P, *and* $\lambda_{\min}(P)$ *denotes its smallest eigenvalue.*

Proof. We proceed in several steps.

Step 1: We show existence of a local solution of (1), (11). Set $\mathscr{D} := \{\ (t, e_1, \dots, e_r) \in \mathbb{R}_{\geq 0} \times (\mathbb{R}^m)^r \ \mid\ \varphi(t) \|e_1\| < 1\ \}$ and

$$\begin{aligned} Y &:= (y, \dot{y}, \dots, y^{(r-1)}), \\ F(t, Y) &:= \Gamma\big(d(t), T(Y)(t)\big)^{-1} f\big(d(t), T(Y)(t)\big) \\ &\quad + \Big(\tfrac{\mathrm{d}}{\mathrm{d}t} \Gamma\big(d(t), T(Y)(t)\big)^{-1}\Big) y^{(r-1)}(t), \\ G(t, Y) &:= \big(I - \tilde{\Gamma} \Gamma\big(d(t), T(Y)(t)\big)^{-1}\big). \end{aligned}$$

Defining

$$\begin{aligned} e_i &:= y^{(i-1)} - z_i, \quad i = 1, \dots, r-1 \\ e_r &:= \tilde{\Gamma} \Gamma\big(d, T(Y)\big)^{-1} y^{(r-1)} - z_r, \end{aligned} \tag{15}$$

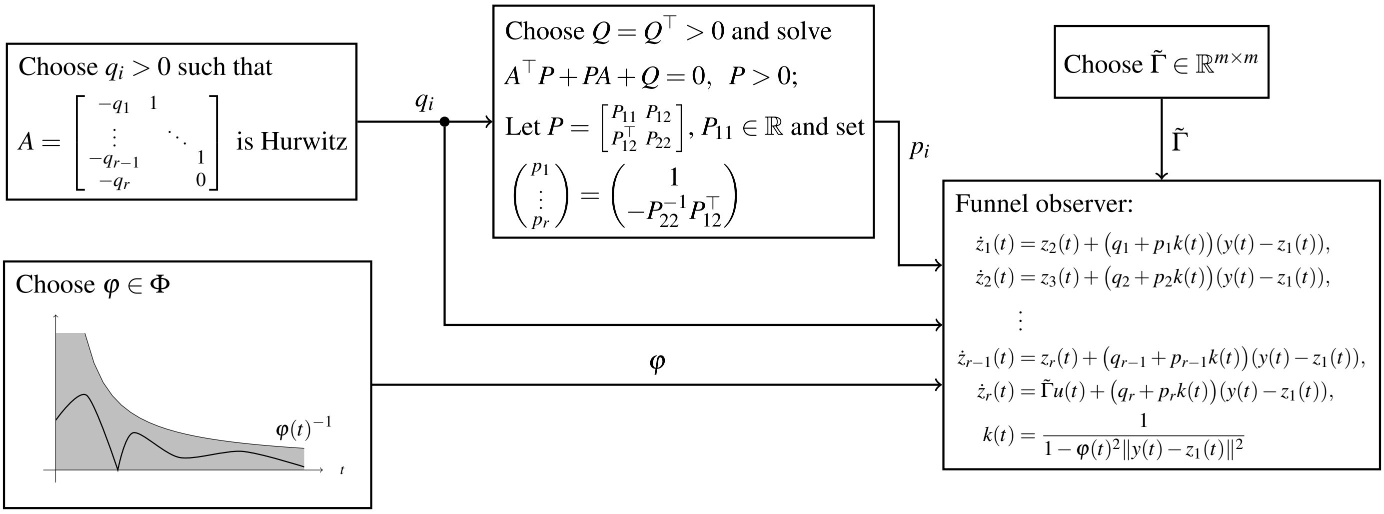

Figure 4: Construction of the funnel observer (1) depending on its design parameters.

and invoking $r \geq 2$ we find

$$
\begin{aligned}
\dot e_1(t) &= e_2(t) - \big(q_1 + p_1 k(t)\big) e_1(t),\\
&\;\;\vdots\\
\dot e_{r-2}(t) &= e_{r-1}(t) - \big(q_{r-2} + p_{r-2} k(t)\big) e_1(t),\\
\dot e_{r-1}(t) &= e_r(t) - \big(q_{r-1} + p_{r-1} k(t)\big) e_1(t) + G(t,Y) y^{(r-1)}(t),\\
\dot e_r(t) &= -\big(q_r + p_r k(t)\big) e_1(t) + \tilde{\Gamma} F(t,Y)
\end{aligned}
\tag{16a}
$$

for

$$
k(t) = \frac{1}{1 - \varphi(t)^2 \|e_1(t)\|^2}. \tag{16b}
$$

By the existence theorem for ordinary differential equations (see e.g. [30, § 10, Thm. VI]), there exists a maximal absolutely continuous solution $e = (e_1, \ldots, e_r) : [0, \omega) \to (\mathbb{R}^m)^r$, $\omega \in (0, \infty]$, of (16) satisfying the initial conditions

$$
\begin{aligned}
e_i(0) &= y^{(i-1)}(0) - z_i^0, \quad i = 1, \ldots, r,\\
e_r(0) &= \tilde{\Gamma}\Gamma\big(d(0), T(Y)(0)\big)^{-1} y^{(r-1)}(0) - z_r^0,
\end{aligned}
$$

and $(t, e(t)) \in \mathscr{D}$ for all $t \in [0, \omega)$. Furthermore, the closure of the graph of e, i.e., the set

$$
\overline{\text{graph}\, e} := \overline{\{\, (t, e(t)) \mid t \in [0, \omega) \,\}},
$$

is not a compact subset of $\mathscr{D}$. Thus, a local solution $(z_1, \ldots, z_r)$ of (1), (11) can be reconstructed.

Step 2: We show that $e \in \mathscr{L}^\infty([0, \omega) \to (\mathbb{R}^m)^r)$. Recalling that the Kronecker product of two matrices $V \in \mathbb{R}^{l\times n}$ and $W \in \mathbb{R}^{p\times q}$ is given by

$$
V \otimes W = \begin{bmatrix} v_{11} W & \cdots & v_{1n} W \\ \vdots & & \vdots \\ v_{l1} W & \cdots & v_{ln} W \end{bmatrix} \in \mathbb{R}^{lp\times nq}, \tag{17}
$$

let

$$
\hat{A} := A \otimes I_m = \begin{bmatrix} -q_1 I_m & I_m & & \\ \vdots & & \ddots & \\ -q_{r-1} I_m & & & I_m \\ -q_r I_m & & & 0 \end{bmatrix} \in \mathbb{R}^{rm\times rm},
$$

and

$$
\hat{P} := P \otimes I_m \in \mathbb{R}^{rm\times rm}, \quad \hat{Q} = Q \otimes I_m \in \mathbb{R}^{rm\times rm}.
$$

Since the Kronecker product (17) satisfies that, if $l = n$ and $p = q$, then

$$
\det(V \otimes W) = (\det V)^p \, (\det W)^l,
$$

we obtain that

$$
\sigma(\hat{A}) = \sigma(A), \quad \sigma(\hat{Q}) = \sigma(Q), \quad \sigma(\hat{P}) = \sigma(P). \tag{18}
$$

Then it follows from $A^\top P + PA + Q = 0$ that $\hat{P} = \hat{P}^\top > 0$, $\hat{Q} = \hat{Q}^\top > 0$ and

$$
\hat{A}^\top \hat{P} + \hat{P}\hat{A} + \hat{Q} = 0.
$$

Since $P_{12}^\top + P_{22} \begin{pmatrix} p_2 \\ \vdots \\ p_r \end{pmatrix} = 0$ we find

$$
\hat{P} \begin{bmatrix} p_1 I_m \\ \vdots \\ p_r I_m \end{bmatrix} = \begin{bmatrix} (P_{11} - P_{12} P_{22}^{-1} P_{12}^\top) I_m \\ 0 \\ \vdots \\ 0 \end{bmatrix},
$$

where $P_{11} - P_{12} P_{22}^{-1} P_{12}^\top > 0$. Observe that we may write (16) in the form

$$
\dot e(t) = \hat{A} e(t) - k(t) \begin{bmatrix} p_1 I_m \\ \vdots \\ p_r I_m \end{bmatrix} e_1(t) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ G(t,Y) y^{(r-1)}(t) \\ \tilde{\Gamma} F(t,Y) \end{pmatrix}.
$$

By boundedness of Y and the bounded-input, bounded-output property of T it follows that $T(Y)$ is bounded, and since d is bounded and $\Gamma(\cdot)^{-1}$ is continuous we further obtain boundedness of $\Gamma\big(d(\cdot),T(Y)(\cdot)\big)^{-1}$. Hence there exists $M_1>0$ such that

$$\text{for a.a. } t\in[0,\omega): \ \|G(t,Y)y^{(r-1)}(t)\|\le M_1. \tag{19}$$

Since f is continuous we have that $f\big(d(\cdot),T(Y)(\cdot)\big)$ is bounded on $[0,\omega)$. By boundedness of ψ as in (5) we then find $M_2>0$ such that

$$\text{for a.a. } t\in[0,\omega): \ \|\tilde\Gamma F(t,Y)\|\le M_2. \tag{20}$$

Let $M:=\sqrt{M_1^2+M_2^2}$. We may now calculate that, for almost all $t\in[0,\omega)$,

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}e(t)^\top\hat Pe(t)\\
&=e(t)^\top\hat A^\top\hat Pe(t)+e(t)^\top\hat P\hat Ae(t)-2k(t)e(t)^\top\hat P\begin{bmatrix}p_1I_m\\ \vdots\\ p_rI_m\end{bmatrix}e_1(t)\\
&\quad+2e(t)^\top\hat P\begin{pmatrix}0\\ \vdots\\ 0\\ G(t,Y)y^{(r-1)}(t)\\ \tilde\Gamma F(t,Y)\end{pmatrix}\\
&\le -e(t)^\top\hat Qe(t)-2k(t)(P_{11}-P_{12}P_{22}^{-1}P_{12}^\top)\|e_1(t)\|^2\\
&\quad+2M\|\hat P\|\|e(t)\|\\
&\le -\mu e(t)^\top\hat Pe(t)+2M\|\hat P\|\|e(t)\|,
\end{aligned}$$

where $\mu=\lambda_{\min}(\hat Q)/\lambda_{\max}(\hat P)$. Now let $\delta\in\big(0,\mu\lambda_{\min}(\hat P)\big)$ be arbitrary and

$$R=\frac{2M\|\hat P\|}{\delta}.$$

Then

$$2M\|\hat P\|\|e(t)\|\le\delta\|e(t)\|^2+2M\|\hat P\|R \tag{21}$$

provided that $\|e(t)\|\le R$, and if $\|e(t)\|>R$, then

$$2M\|\hat P\|\|e(t)\|-\delta\|e(t)\|^2\le\big(2M\|\hat P\|-\delta R\big)\|e(t)\|=0,$$

and hence (21) is also true in this case. Therefore,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}e(t)^\top\hat Pe(t)\le\left(-\mu+\frac{\delta}{\lambda_{\min}(\hat P)}\right)e(t)^\top\hat Pe(t)+2M\|\hat P\|R$$

for almost all $t\in[0,\omega)$. Gronwall's lemma now implies that, with $\nu=\mu-\frac{\delta}{\lambda_{\min}(\hat P)}>0$,

$$e(t)^\top\hat Pe(t)\le e(0)^\top\hat Pe(0)e^{-\nu t}+\frac{2M\|\hat P\|R}{\nu},$$

and hence

$$\|e(t)\|^2\le\frac{\lambda_{\max}(\hat P)}{\lambda_{\min}(\hat P)}e^{-\nu t}\|e(0)\|^2+\frac{2M\|\hat P\|R}{\nu\lambda_{\min}(\hat P)} \tag{22}$$

for all $t\in[0,\omega)$. Equation (22) in particular implies that $e\in\mathscr{L}^\infty([0,\omega)\to(\mathbb{R}^m)^r)$.

Step 3: We show that $k\in\mathscr{L}^\infty([0,\omega)\to\mathbb{R})$. Let $\kappa\in(0,\omega)$ be arbitrary but fixed and $\lambda:=\inf_{t\in(0,\omega)}\varphi(t)^{-1}>0$. Since $\dot\varphi$ is bounded and $\liminf_{t\to\infty}\varphi(t)>0$ we find that $\frac{\mathrm{d}}{\mathrm{d}t}\varphi|_{[\kappa,\infty)}(\cdot)^{-1}$ is bounded and hence there exists a Lipschitz bound $L>0$ of $\varphi|_{[\kappa,\infty)}(\cdot)^{-1}$. By Step 2, e_2 is bounded and we may choose $\varepsilon>0$ small enough so that

$$\varepsilon\le\min\left\{\frac{\lambda}{2},\inf_{t\in(0,\kappa]}(\varphi(t)^{-1}-\|e_1(t)\|)\right\}$$

and

$$L\le-\sup_{t\in[0,\omega)}\|e_2(t)\|-M_1+\frac{q_1\lambda}{2}+\frac{\lambda^2}{4\varepsilon}; \tag{23}$$

feasibility of this choice is guaranteed by $r\ge2$. We show that

$$\forall t\in(0,\omega): \ \varphi(t)^{-1}-\|e_1(t)\|\ge\varepsilon. \tag{24}$$

By definition of ε this holds on $(0,\kappa]$. Seeking a contradiction suppose that

$$\exists t_1\in[\kappa,\omega): \ \varphi(t_1)^{-1}-\|e_1(t_1)\|<\varepsilon.$$

Then for

$$t_0:=\max\left\{\,t\in[\kappa,t_1)\ \big|\ \varphi(t)^{-1}-\|e_1(t)\|=\varepsilon\,\right\}$$

we have for all $t\in[t_0,t_1]$ that

$$\varphi(t)^{-1}-\|e_1(t)\|\le\varepsilon,$$
$$\|e_1(t)\|\ge\varphi(t)^{-1}-\varepsilon\ge\lambda-\varepsilon\ge\frac{\lambda}{2}$$

and

$$k(t)=\frac{1}{1-\varphi(t)^2\|e_1(t)\|^2}\ge\frac{1}{2\varepsilon\varphi(t)}\ge\frac{\lambda}{2\varepsilon}.$$

Now we have, for all $t\in[t_0,t_1]$,

$$\begin{aligned}
&\frac12\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2=e_1(t)^\top\big(e_2(t)-\big(q_1+p_1k(t)\big)e_1(t)\\
&\quad+\underbrace{G(t,Y)\dot y(t)\big)}_{\text{if } r=2}\\
&\le\ -(q_1+p_1k(t))\|e_1(t)\|^2+\left(\sup_{t\in[0,\omega)}\|e_2(t)\|+M_1\right)\|e_1(t)\|\\
&\le\ -\left(\frac{q_1\lambda}{2}+\frac{\lambda^2}{4\varepsilon}\right)\|e_1(t)\|+\left(\sup_{t\in[0,\omega)}\|e_2(t)+M_1\|\right)\|e_1(t)\|\\
&\overset{(23)}{\le}\ -L\|e_1(t)\|.
\end{aligned}$$

Therefore, using

$$\frac12\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2=\|e_1(t)\|\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|,$$

and that $\|e_1(t)\| > 0$ for all $t \in [t_0, t_1]$, we find that

$$\begin{aligned}\|e_1(t_1)\| - \|e_1(t_0)\| &= \int_{t_0}^{t_1} \frac{1}{2}\|e_1(t)\|^{-1} \tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2 \,\mathrm{d}t \\ &\le -L(t_1-t_0) \le -|\varphi(t_1)^{-1} - \varphi(t_0)^{-1}| \\ &\le \varphi(t_1)^{-1} - \varphi(t_0)^{-1},\end{aligned}$$

and hence

$$\varepsilon = \varphi(t_0)^{-1} - \|e_1(t_0)\| \le \varphi(t_1)^{-1} - \|e_1(t_1)\| < \varepsilon,$$

a contradiction. Therefore, (24) holds and this implies boundedness of k.

Step 4: We show $\omega = \infty$. Assume that $\omega < \infty$. Then, since e and k are bounded by Steps 2 and 3, it follows that $\overline{\operatorname{graph} e}$ is a compact subset of $\mathscr{D}$, a contradiction. Therefore, $\omega = \infty$. In particular, Steps 3 and 4 imply (13).

Step 5: We show (14). Consider the estimate (22) and observe that by (18) we have $\lambda_{\min}(\hat{P}) = \lambda_{\min}(P)$, $\lambda_{\max}(\hat{P}) = \lambda_{\max}(P)$ and $\lambda_{\min}(\hat{Q}) = \lambda_{\min}(Q)$. Furthermore, since $\hat{P}$ is positive definite we have $\|\hat{P}\| = \lambda_{\max}(\hat{P}) = \lambda_{\max}(P)$. Then (22) gives

$$\limsup_{t\to\infty} \|e(t)\| \le \sqrt{\frac{2M\,\lambda_{\max}(P)\,R}{\nu\,\lambda_{\min}(P)}}.$$

A close look at the δ-dependent expression

$$\frac{R}{\nu} = \frac{2M\,\lambda_{\max}(P)}{\delta\left(\mu - \frac{\delta}{\lambda_{\min}(P)}\right)}$$

reveals that it is minimal for

$$\delta = \frac{\mu\,\lambda_{\min}(P)}{2}.$$

With this choice we obtain

$$\frac{R}{\nu} = \frac{8M\,\lambda_{\max}(P)}{\mu^2\,\lambda_{\min}(P)}$$

from which the assertion (14) follows. ☐

In [25, Thm. 2.2], using the adaptive high-gain observer proposed therein, bounds for the mean value of e_i are given; we stress that both the bounds in [25, (14)] and in (14) cannot be made arbitrarily small in general, they depend on the system data.

Remark 4.2. If the input u and its first k derivatives are bounded, then the funnel observer works for an even larger system class than (4) and strict relative degree is not required. Consider a system of the form

$$\begin{aligned} y^{(r)}(t) &= F\big(d_0(t), T(y,\dot{y},\ldots,y^{(r-1)})(t), u(t),\ldots,u^{(k)}(t)\big)\\ y|_{[-h,0]} &= y^0 \in \mathscr{W}^{r-1,\infty}([-h,0]\to\mathbb{R}^m),\end{aligned} \tag{25}$$

where $F \in \mathscr{C}(\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{(k+1)m} \to \mathbb{R}^m)$, $d_0 \in \mathscr{L}^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^p)$, $u \in \mathscr{W}^{k,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$ and $T : \mathscr{C}([-h,\infty) \to \mathbb{R}^m)^r \to \mathscr{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\ge 0} \to \mathbb{R}^q)$ is an operator with the properties as discussed in Section 2. It is then possible to reformulate (25) as a system of the form (4). To this end, let $d_1 := \big(u^\top,\ldots,(u^{(k)})^\top\big)^\top$, $d_2 := u$, $d := \big(d_0^\top, d_1^\top, d_2^\top\big)^\top \in \mathscr{L}^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^{(k+1)m})$ and

$$f : \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^{(k+1)m} \times \mathbb{R}^q,\ (d_0, d_1, d_2, T) \mapsto F(d_0, T, d_1) - d_2.$$

Then (25) is equivalent to

$$y^{(r)}(t) = f\big(d(t), T(y,\dot{y},\ldots,y^{(r-1)})(t)\big) + u(t),$$

i.e., it is of the form (4) with $\Gamma \equiv I_m$ and in particular the condition of bounded $\dot{\psi}$ as in(5) is always satisfied.

Furthermore, exact knowledge of the number r of derivatives of y involved in (25) (the relative degree in case of (4)) is not required for feasibility of the funnel observer. Only an upper bound $\rho \in \mathbb{N}$ is required, i.e., $r \le \rho$. If $y,\ldots,y^{(\rho)}$ are bounded, then the funnel observer (1) (with $r = \rho$ in (1)) works for (25) in the sense of Theorem 4.1. To see this, the proof of Theorem (4.1) has to be recapitulated with the new observation errors $e_i := y^{(i-1)} - z_i$ for $i = 1,\ldots,\rho$.

Remark 4.3. We consider two special cases for (4) and the funnel observer (1), and the resulting estimate (14).

(i) $\tilde{\Gamma} = 0$. A careful inspection of the proof of Theorem 4.1 reveals that in this case the condition of bounded $\dot{\psi}$ is superfluous. Furthermore, M_1 in (19) can be chosen as $M_1 = \|y^{(r-1)}\|_\infty$ and $M_2 = 0$ in (20). Therefore, we find that $M = \|y^{(r-1)}\|_\infty$ in (14). Note that the choice of $\tilde{\Gamma}$ is independent of (4).

(ii) $\tilde{\Gamma} = \Gamma \in \mathbf{Gl}_m(\mathbb{R})$ and $f = 0$. This means to assume that (4) is of the very special form $y^{(r)}(t) = \Gamma u(t)$ and we have exact knowledge of the invertible matrix Γ. Then $M_1 = M_2 = 0$ in (19) and (20), resp., and hence $M = 0$ in (14). In particular, this implies that $e(t) \to 0$ and $k(t) \to 1$ for $t \to \infty$.

Remark 4.4. If the output of the system (4) is subject to measurement noise, i.e., the funnel observer (1) receives $y + n$ instead of y, where $n \in \mathscr{C}^r([-h,\infty) \to \mathbb{R}^m)$ and its first r derivatives are bounded, then the funnel observer achieves that

$$\forall\, t > 0 :\ \varphi(t)\|y(t) + n(t) - z_1(t)\| < 1,$$

which implies

$$\forall\, t > 0 :\ \frac{\varphi(t)}{1 + \varphi(t)\|n(t)\|}\|y(t) - z_1(t)\| < 1,$$

i.e., $y - z_1$ evolves in the funnel $\mathscr{F}_\psi$, where $\psi = \frac{\varphi(t)}{1+\varphi(t)\|n(t)\|}$. If an upper bound for n is known, say $\|n(t)\| \le \nu$ for all $t \ge 0$, then

$$\forall\, t > 0 :\ \|y(t) - z_1(t)\| < \varphi(t)^{-1} + \nu.$$

Hence, the actual error remains in the wider funnel obtained by adding the corresponding bound of the noise to the funnel bounds used for the observer. The bound in (14) changes as follows: Modify M_1 to $\tilde{M}_1$ such that

$$\text{for a.a. } t \in [0,\infty): \; \|G(t,Y)(y+n)^{(r-1)}(t)\| \le \tilde{M}_1$$

and modify M_2 to $\tilde{M}_2$ such that

$$\Big\|\tilde{\Gamma}F(t,Y)+\tilde{\Gamma}\Big(\Gamma\big(d(t),T(Y)(t)\big)^{-1}n^{(r)}(t) + \Big(\tfrac{\mathrm{d}}{\mathrm{d}t}\Gamma\big(d(t),T(Y)(t)\big)^{-1}\Big)n^{(r-1)}(t)\Big)\Big\| \le \tilde{M}_2.$$

for a.a. $t \in [0,\infty)$. Then, with $\tilde{M} := \sqrt{\tilde{M}_1^2+\tilde{M}_2^2}$, we have that

$$\limsup_{t\to\infty}\|e(t)\| \le \frac{4\tilde{M}\,\lambda_{\max}(P)^2}{\lambda_{\min}(Q)\,\lambda_{\min}(P)} + \Big\|\Big(n,\dot{n},\ldots,n^{(r-2)},\tilde{\Gamma}\Gamma\big(d,T(Y)\big)^{-1}n^{(r-1)}\Big)\Big\|_\infty.$$

If the input of the system (4) is subject to noise before the funnel observer receives it, i.e., u enters system (4) and $u+v$ enters the observer (1), where $v \in \mathscr{L}^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$, then the statement of Theorem 4.1 remains the same (the funnel observer still works) and the proof only changes slightly: on the right hand side of the equation for $\dot{e}_r$ in (16) the term $-\tilde{\Gamma}v(t)$ has to be added. Due to boundedness of v, the remaining calculations stay the same and only the constant M_2 possibly needs to be increased.

5. Simulations

We illustrate the funnel observer by comparing it to the simulations of the λ-strip observer for a bioreactor model in [3]. We consider the generic model as in [3], cf. also [8]:

$$\begin{aligned}
\dot{m}(t) &= \frac{a_1 m(t)s(t)}{a_2 m(t)+s(t)} - m(t)u(t),\\
\dot{s}(t) &= -\frac{a_1 a_3 m(t)s(t)}{a_2 m(t)+s(t)} + \big(a_4 - s(t)\big)u(t),\\
y(t) &= m(t),
\end{aligned} \tag{26}$$

where $m(t)$ and $s(t)$ denote the concentrations of the microorganism and the substrate, resp., and $u(t)$ is the substrate inflow rate. All state variables are strictly positive and the parameters are $a_1 = a_2 = a_3 = 1$, $a_4 = 0.1$, $m(0) = 0.075$, and $s(0) = 0.03$. For the simulation we choose the following substrate inflow rate:

$$u(t) = \begin{cases} 0.08, & t \in [0,30-\varepsilon] \\ 0.02, & t \in [30+\varepsilon,50-\varepsilon] \\ 0.08, & t \ge 50+\varepsilon, \end{cases}$$

where $\varepsilon \ll 1$ is some positive constant and on the intervals $(30-\varepsilon,30+\varepsilon)$ and $(50-\varepsilon,50+\varepsilon)$ the function u is chosen such that it is continuously differentiable on $\mathbb{R}_{\ge 0}$. This setup for the bioreactor coincides with that considered in [3], where it is also explained that (26) can be reformulated in the form

$$\ddot{y}(t) = \Phi\big(y(t),\dot{y}(t),u(t),\dot{u}(t)\big).$$

Therefore, invoking Remark 4.2, system (26) belongs to the class (4) with $r = 2$ and $\Gamma \equiv I_m$. Theorem 4.1 thus implies that the funnel observer works for (26). We note that we applied the funnel observer to the original system (26) in the simulation and not to the reformulated system as above.

As design parameters for the funnel observer (1) (see also Figure 4) we choose $\tilde{\Gamma} = 0$, $q_1 = 1$, $q_2 = 0.2$ and

$$\varphi : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0},\; t \mapsto \tfrac{1}{2}te^{-t} + \tfrac{100}{\pi}\arctan t.$$

Note that this prescribes an exponentially (exponent 1) decaying funnel in the transient phase $[0,T]$, where $T \approx 3$, and a tracking accuracy quantified by $\lambda = 0.02$ thereafter. The solution of the Lyapunov equation $A^\top P + PA + I_2 = 0$ is given by

$$P = \begin{bmatrix} 0.6 & -0.5 \\ -0.5 & 5.5 \end{bmatrix}$$

and hence $p_1 = 1$ and $p_2 = \frac{1}{11}$. A numerical computation yields that the eigenvalues of P are given by $\lambda_1 \approx 0.5495$ and $\lambda_2 \approx 5.5505$. Therefore, the estimate (14) becomes

$$\limsup_{t\to\infty}\|e(t)\| \le \frac{4M\lambda_2^2}{\lambda_1} \approx 224.26M. \tag{27}$$

Since no knowledge of the initial values for (26) is assumed we set the observer initial values to $z_1^0 = z_2^0 = 0$.

The simulation has been performed in MATLAB (solver: ode15s, relative tolerance: 10^{-14}, absolute tolerance: 10^{-10}). In Figure 5 the simulation of the funnel observer (1) for the bioreactor model (26) with the above stated parameters is depicted. Figure 5a shows the output m and its estimate m_e, while Figure 5b show the concentration of the substrate s and its estimate s_e. In fact, the estimate is much better than the bound (27) guarantees. An action of the gain function k in Figure 5c is required only if the error $|m(t) - m_e(t)|$ is close to the funnel boundary $1/\varphi(t)$. It can be seen that initially the error is very close to the funnel boundary and hence the gain rises rather sharply by about 0.25. After this initial error correction the gain is nearly equal to 1 for most of the time; only slight corrections are necessary when the input $u(t)$ changes its value at $t = 30$ and $t = 50$. This in particular shows that the gain function k is non-monotone.

Compared to the simulation in [3] we see that the funnel observer achieves better estimation results for m and s, while the gain function is much smaller (k is equal to its minimal value 1 most of the time). The main reason for this is that the funnel observer is able to influence the transient behavior of the observation error.

6. Application in feedback control

While Theorem 4.1 shows that the funnel observer is able to achieve prescribed transient behavior of the observation error $e_1 = y - z_1$ and that the errors $e_2,\ldots,e_r$ as in (15) converge to a certain strip, we like to stress that no transient behavior can be prescribed for $e_2,\ldots,e_r$ since $\dot{y},\ldots,y^{(r-1)}$ are not known.

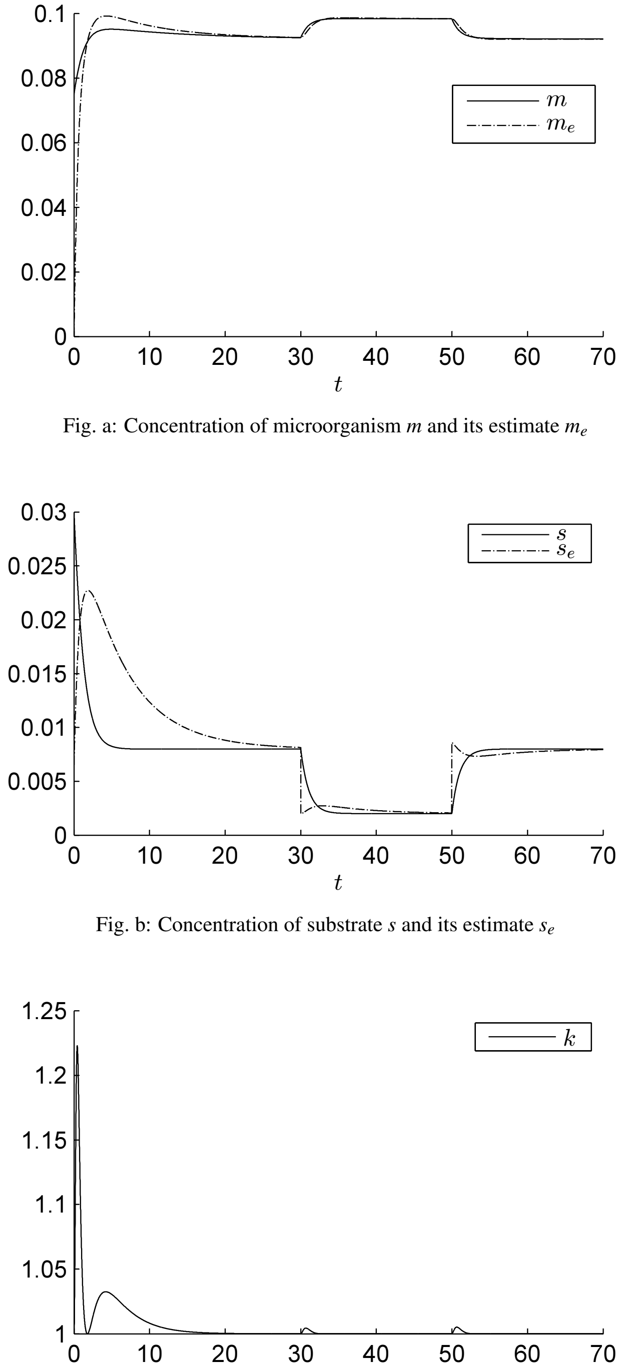

Fig. a: Concentration of microorganism m and its estimate m_e

Fig. b: Concentration of substrate s and its estimate s_e

Fig. c: Observer gain k

Figure 5: Simulation of the funnel observer (1) for the bioreactor model (26).

Therefore, $z_2, \ldots, z_r$ from the funnel observer cannot be viewed as estimates for $\dot{y}, \ldots, y^{(r-1)}$. Nevertheless, an advantage of the funnel observer is that the derivative of z_1 is known explicitly. In Subsection 6.1 we show that the successive application of the funnel observer to the observer-plant system with output z_1 results in a cascade of observers which yields

- an estimate z for the output y with prescribed transient behavior (i.e., $(t, y(t) - z(t)) \in \mathscr{F}_\varphi$) and
- the derivatives $\dot{z}, \ldots, z^{(r-1)}$ are known explicitly.

Furthermore, the high-frequency gain matrix $\tilde{\Gamma}$ of the observer-plant system may be prescribed. This allows for the application of different feedback control techniques, which would usually need the first $r-1$ derivatives of the output of system (4) (see e.g. [2, 10, 11, 12]), by just applying the controller to the artificial output z produced by the cascade of funnel observers.

When tracking problems for systems (4) are considered, a crucial condition is the stability of the internal dynamics (the minimum phase property in case of linear systems, see also Section 2), cf. [4, 12, 26]. This condition is modelled by the property a) of the operator T in (4). When tracking controllers are to be applied to the interconnection of the system (4) with a cascade of funnel observers, it is thus desirable that this interconnection again has stable internal dynamics in the sense that it can be described by an appropriate functional differential equation where the involved operator has property a). In Subsection 6.2 we show that this can be achieved for a special class of systems which are linear up to the influence of an operator T and have relative degree two or three. For relative degree larger than three this remains an open problem; we show explicitly where our proof does not work in this case.

6.1. The observer cascade

We introduce a cascade of funnel observers as follows:

$$\begin{aligned}
\dot{z}_{i,1}(t) &= z_{i,2}(t) + \big(q_{i,1} + p_{i,1}k_i(t)\big)\big(z_{i-1,1}(t) - z_{i,1}(t)\big),\\
\dot{z}_{i,2}(t) &= z_{i,3}(t) + \big(q_{i,2} + p_{i,2}k_i(t)\big)\big(z_{i-1,1}(t) - z_{i,1}(t)\big),\\
&\ \ \vdots\\
\dot{z}_{i,r-1}(t) &= z_{i,r}(t) + \big(q_{i,r-1} + p_{i,r-1}k_i(t)\big)\big(z_{i-1,1}(t) - z_{i,1}(t)\big),\\
\dot{z}_{i,r}(t) &= \tilde{\Gamma}_i u(t) + \big(q_{i,r} + p_{i,r}k_i(t)\big)\big(z_{i-1,1}(t) - z_{i,1}(t)\big),\\
k_i(t) &= \frac{1}{1 - \varphi_i(t)^2 \|z_{i-1,1}(t) - z_{i,1}(t)\|^2},
\end{aligned} \tag{28}$$

for $i = 1, \ldots, r-1$, where $z_{0,1} := y$, $\tilde{\Gamma}_i \in \mathbb{R}^{m\times m}$,

$$\varphi_i \in \Phi_r := \Phi \cap \Big\{ \varphi \in \mathscr{C}^{r-1}(\mathbb{R}_{\geq 0} \to \mathbb{R}) \,\Big|\, \dot{\varphi}, \ldots, \varphi^{(r-1)} \text{ bounded} \Big\}$$

and $q_{i,j} > 0$, $p_{i,j} > 0$ are such that (12) is satisfied for corresponding matrices A_i, P_i, Q_i for $i = 1, \ldots, r-1$. We consider initial values

$$z_{i,j}(0) = z^0_{i,j} \in \mathbb{R}^m, \quad i, j = 1, \ldots, r-1. \tag{29}$$

The situation is illustrated in Figure 6.

We show that the cascade (28) applied to (4) yields an interconnection with new output $z = z_{r-1,1}$ such that $y - z$ has prescribed transient behavior and $\dot{z}, \ldots, z^{(r-1)}$ are known explicitly. In order to derive the dependence of $\dot{z}, \ldots, z^{(r-1)}$ on the states of the individual observers in (28) we define the following func-

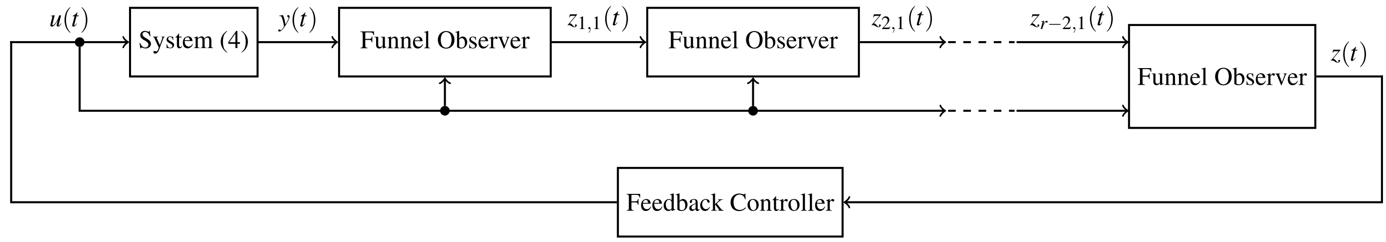

Figure 6: Cascade of funnel observers (1) applied to a system (4) in conjunction with a feedback controller.

tions in a recursive way:

$$\begin{aligned}
&P_0^{a,b}(k,\varphi_0,e_0) := (q_{a,b}+p_{a,b}k)e_0,\\
&P_{i+i}^{a,b}(k,\varphi_0,\dots,\varphi_{i+1},e_0,\dots,e_{i+1})\\
&:= \frac{\partial P_i^{a,b}}{\partial k}\Big(2k^2(\varphi_0\varphi_1 e_0^\top e_0+\varphi_0^2 e_0^\top e_1)\Big)+\frac{\partial P_i^{a,b}}{\partial \varphi_0}\varphi_1+\dots\\
&\quad \dots+\frac{\partial P_i^{a,b}}{\partial \varphi_i}\varphi_{i+1}+\frac{\partial P_i^{a,b}}{\partial e_0}e_1+\dots+\frac{\partial P_i^{a,b}}{\partial e_i}e_{i+1}
\end{aligned}$$

for $a,b\in\{1,\dots,r-1\}$ and $i\geq 0$, where $k,\varphi_i\in\mathbb{R}$ and $e_i\in\mathbb{R}^m$ for each $i\geq 0$. Further define, using (28),

$$\begin{aligned}
\tilde{P}_j^i(t) := \sum_{l=0}^{j-1} P_l^{i,j-l}\Big(&k_i(t),\varphi_i(t),\dots,\varphi_i^{(l)}(t),\\
&z_{i-1,1}(t)-z_{i,1}(t),\dots,z_{i-1,1}^{(l)}(t)-z_{i,1}^{(l)}(t)\Big)
\end{aligned}$$

for $i=1,\dots,r-1$ and $j=0,\dots,r-1$. We will show that

$$z_{i,1}^{(j)}(t)=z_{i,j+1}(t)+\tilde{P}_j^i(t),\qquad i=1,\dots,r-1,\ \ j=0,\dots,r-1. \tag{30}$$

Theorem 6.1. *Consider a system* (4) *with* $r\geq 2$. *Let* $y^0\in\mathscr{W}^{r-1,\infty}([-h,0]\to\mathbb{R}^m)$, $u\in\mathscr{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)$ *and let* $y\in\mathscr{C}^{r-1}([-h,\infty)\to\mathbb{R}^m)$ *be a solution of* (4) *such that* $y,\dot{y},\dots,y^{(r-1)}$ *are bounded. Consider the cascade of funnel observers* (28), (29) *with* $\varphi_i\in\Phi_r$ *such that*

$$\varphi_i(0)\|z_{i-1,1}(0)-z_{i,1}^0\|<1,$$

where $z_{0,1}:=y$, $\tilde{\Gamma}\in\mathbb{R}^{m\times m}$ *and* $q_{i,j}>0$, $p_{i,j}>0$ *are such that* (12) *is satisfied for corresponding matrices* A_i,P_i,Q_i *for all* $i=1,\dots,r-1$.
Then (28), (29) *has absolutely continuous solutions* $z_{i,j}\in\mathscr{L}^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)$ *with* $k_i\in\mathscr{L}^\infty(\mathbb{R}_{\geq 0}\to[1,\infty))$ *for* $i=1,\dots,r-1$, $j=1,\dots,r$ *and*

$$\begin{aligned}
&\forall\, i\in\{1,\dots,r-1\}\ \exists\,\varepsilon_i>0\ \forall\, t>0:\\
&\qquad\qquad \|z_{i-1,1}(t)-z_{i,1}(t)\|<\varphi_i(t)^{-1}-\varepsilon_i.
\end{aligned}\tag{31}$$

Furthermore, for $z:=z_{r-1,1}$ *we have that*

$$\forall\, t>0:\ \|y(t)-z(t)\|<\sum_{i=1}^{r-1}\varphi_i(t)^{-1}-\varepsilon_i \tag{32}$$

and $\dot{z},\dots,z^{(r-1)}$ *are known explicitly in the sense that* (30) *holds.*

Proof. Step 1: We show existence of bounded absolutely continuous solutions for each observer in (28) and the property (31) by induction. For $i=1$ we have $z_{0,1}=y$ and hence the existence of bounded global solutions follows from Theorem 4.1. We may calculate that

$$\begin{aligned}
z_{i,1}^{(j)}(t)&=z_{i,j+1}(t)\\
&+\sum_{l=0}^{j-1}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^l\big(q_{i,j-l}+p_{i,j-l}k_i(t)\big)\big(z_{i-1,1}(t)-z_{i,1}(t)\big)
\end{aligned}\tag{33}$$

for $i=1,\dots,r-1$ and $j=0,\dots,r-1$. With $w_i(t):=z_{i-1,1}(t)-z_{i,1}(t)$ we calculate

$$\dot{k}_i(t)=2k_i(t)^2\Big(\varphi_i(t)\dot{\varphi}_i(t)w_i(t)^\top w_i(t)+\varphi_i(t)^2 w_i(t)^\top\dot{w}_i(t)\Big) \tag{34}$$

for all $i=1,\dots,r-1$. In particular, for $i=1$ we obtain that $\dot{z}_{1,1},\dots,z_{1,1}^{(r-1)}$ are bounded since $y,\dots,y^{(r-1)},\varphi_1,\dots,\varphi_1^{(r-1)}$ are bounded and $z_{1,1},\dots,z_{1,r}$, and k_1 are bounded by Theorem 4.1. Now assume that the statement is true for $i\in\{1,\dots,r-2\}$ such that $\dot{z}_{i,1},\dots,z_{i,1}^{(r-1)}$ are bounded. Then an application of Theorem 4.1 again yields existence of bounded global solutions such that k_{i+1} is bounded. Again invoking (33) yields boundedness of $\dot{z}_{i+1,1},\dots,z_{i+1,1}^{(r-1)}$.

Step 2: Property (32) is obvious, so it remains to show (30). First observe that it follows from (34) and a simple induction that

$$\begin{aligned}
&\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^l\big(q_{i,j-l}+p_{i,j-l}k_i(t)\big)w_i(t)\\
&=P_l^{i,j-l}\big(k_i(t),\varphi_i(t),\dot{\varphi}_i(t),\dots,\varphi_i^{(l)}(t),w_i(t),\dot{w}_i(t),\dots,w_i^{(l)}(t)\big)
\end{aligned}$$

for $i=1,\dots,r-1$, $j=0,\dots,r-1$ and $l=0,\dots,j-1$. Then (33) immediately implies (30) and this finishes the proof. ☐

The derivatives of $z=z_{r-1,1}$ are given by (30) for $i=r-1$ as

$$z^{(j)}(t)=z_{r-1,j+1}(t)+\tilde{P}_j^{r-1}(t),\qquad j=0,\dots,r-1.$$

By definition, $\tilde{P}_j^{r-1}(t)$ depends on the derivatives of $z_{r-2,1}$ and $z_{r-1,1}=z$ up to order $j-1$. The dependencies on $\dot{z},\dots,z^{(j-1)}$ may be immediately resolved by applying the same formula again, thus $z^{(j)}$ depends on $z_{r-1,1},\dots,z_{r-1,j+1}$ and on $z_{r-2,1},\dot z_{r-2,1},\dots,z_{r-2,1}^{(j-1)}$. Applying (30) in a recursive way to $\dot z_{r-2,1},\dots,z_{r-2,1}^{(j-1)}$ we obtain dependencies as depicted in Figure 7.

6.2. Internal dynamics

In the following we restrict ourselves to a subclass of systems (4) which are linear up to the influence of an operator T which may enter nonlinearly, but is bounded whenever y is bounded:

$$
\begin{aligned}
y^{(r)}(t) &= \sum_{i=1}^{r} R_i y^{(i-1)}(t) + f\big(d(t),T(y,\dot y,\dots,y^{(r-1)})(t)\big) + \Gamma u(t),\\
y|_{[-h,0]} &= y^0 \in \mathscr{W}^{r-1,\infty}([-h,0]\to\mathbb{R}^m),
\end{aligned}
\tag{35}
$$

where $h>0$, $f\in\mathscr{C}(\mathbb{R}^p\times\mathbb{R}^q\to\mathbb{R}^m)$, $d\in\mathscr{L}^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^p)$ is a disturbance, $\Gamma\in\mathbf{Gl}_m(\mathbb{R})$ is the high-frequency gain matrix and $T:\mathscr{C}([-h,\infty)\to\mathbb{R}^m)^r\to\mathscr{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}\to\mathbb{R}^q)$ is an operator with the properties b) and c) described in Section 2, and the following (stronger) replacement of a):

a') for all $c_1>0$ there exists $c_2>0$ such that for all $\zeta_1,\dots,\zeta_r\in\mathscr{C}([-h,\infty)\to\mathbb{R}^m)$:

$$\sup_{t\in[-h,\infty)}\|\zeta_1(t)\|\leq c_1 \implies \sup_{t\in[0,\infty)}\|T(\zeta_1,\dots,\zeta_r)(t)\|\leq c_2.$$

We like to stress that the class (35) includes finite- and infinite-dimensional linear systems as well as nonlinear systems, as discussed in Section 2, provided that the latter satisfy that g_1 is linear, g_2 is constant and κ in (8) depends only on $\|y(s)\|$. In particular, it contains the system classes discussed in [9, 15, 16] and the nonlinear systems in [20] provided that the internal dynamics are input-to-state stable.

We show that, if $r=2$ or $r=3$, the composition of (35) with the cascade of funnel observers, where $\tilde\Gamma_i=\tilde\Gamma$ is invertible, has again relative degree r and stable internal dynamics in the sense that it can be rewritten as

$$z^{(r)}(t) = F\big(\tilde d(t),\tilde T(z,\dot z,\dots,z^{(r-1)})(t)\big)+\tilde\Gamma u(t),$$

where $\tilde T$ is an operator with the properties a)–c).

Theorem 6.2. *Consider a system* (35) *with* $r\in\{2,3\}$, $y^0\in\mathscr{W}^{r-1,\infty}([-h,0]\to\mathbb{R}^m)$ *and assume that* $\Gamma>0$. *Further consider the cascade of funnel observers* (28), (29) *with* $\varphi_i\in\Phi_r$ *such that*

$$\varphi_i(0)\|z_{i-1,1}(0)-z_{i,1}^0\|<1,$$

where $z_{0,1}:=y$ *and* $q_{i,j}=q_j>0$, $p_{i,j}=p_j>0$ *are such that* (12) *is satisfied for corresponding matrices* A,P,Q *for all* $i=1,\dots,r-1$, $j=1,\dots,r$. *Moreover, assume that* $\tilde\Gamma_i=\tilde\Gamma\in\mathbb{R}^{m\times m}$, $i=1,\dots,r-1$, *such that* $\tilde\Gamma>0$ *and,*

$$\text{if } r=3, \text{ then } I-\Gamma\tilde\Gamma^{-1}=\big(I-\Gamma\tilde\Gamma^{-1}\big)^\top>0. \tag{36}$$

Then the conjunction of (35) *and* (28) *with input* u *and output* $z:=z_{r-1,1}$ *can be equivalently written as*

$$z^{(r)}(t)=F\big(\tilde d(t),\tilde T(z,\dot z,\dots,z^{(r-1)})(t)\big)+\tilde\Gamma u(t),\quad z(0)=z_{r-1,1}^0, \tag{37}$$

for $\tilde d(t):=\big(\varphi_{r-1}(t),\dot\varphi_{r-1}(t),\dots,\varphi_{r-1}^{(r-1)}(t)\big)^\top\in\mathscr{L}^\infty(\mathbb{R}_{\geq0}\to\mathbb{R}^r)$, *some* $F\in\mathscr{C}(\mathbb{R}^r\times\mathbb{R}^{\tilde q}\to\mathbb{R}^m)$ *and an operator* $\tilde T:\mathscr{C}([-h,\infty)\to\mathbb{R}^m)^r\to\mathscr{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq0}\to\mathbb{R}^{\tilde q})$ *which satisfies the properties a)–c) as in Section 2. Furthermore, for any solution of* (28), (35) *we have* (32) *and the derivatives of the observer states satisfy* (30).

Proof. Step 1: We start with several transformations of the error dynamics between two successive systems.

Step 1a: Define $v_{i,j}:=z_{i-1,j}-z_{i,j}$ for $i=2,\dots,r-1$ and $j=1,\dots,r$. Then

$$
\begin{aligned}
\dot v_{i,1}(t) &= v_{i,2}(t)-\big(q_1+p_1k_i(t)\big)v_{i,1}(t)\\
&\quad+\big(q_1+p_1k_{i-1}(t)\big)v_{i-1,1}(t),\\
&\ \ \vdots\\
\dot v_{i,r-1}(t) &= v_{i,r}(t)-\big(q_{r-1}+p_{r-1}k_i(t)\big)v_{i,1}(t)\\
&\quad+\big(q_{r-1}+p_{r-1}k_{i-1}(t)\big)v_{i-1,1}(t),\\
\dot v_{i,r}(t) &= -\big(q_r+p_rk_i(t)\big)v_{i,1}(t)+\big(q_r+p_rk_{i-1}(t)\big)v_{i-1,1}(t).
\end{aligned}
$$

Step 1b: Defining $e_{1,j}(t):=y^{(j-1)}(t)-z_{1,j}(t)$ for $j=1,\dots,r-1$ and $e_{1,r}(t):=y^{(r-1)}(t)-\Gamma\tilde\Gamma^{-1}z_{1,r}(t)$ we obtain

$$
\begin{aligned}
\dot e_{1,1}(t) &= e_{1,2}(t)-\big(q_1+p_1k_1(t)\big)e_{1,1}(t),\\
&\ \ \vdots\\
\dot e_{1,r-2}(t) &= e_{1,r-1}(t)-\big(q_{r-2}+p_{r-2}k_1(t)\big)e_{1,1}(t),\\
\dot e_{1,r-1}(t) &= e_{1,r}(t)-\big(q_{r-1}+p_{r-1}k_1(t)\big)e_{1,1}(t)\\
&\quad+(\Gamma\tilde\Gamma^{-1}-I)z_{1,r}(t),\\
\dot e_{1,r}(t) &= -\Gamma\tilde\Gamma^{-1}\big(q_r+p_rk_1(t)\big)e_{1,1}(t)\\
&\quad+\sum_{i=1}^{r}R_iy^{(i-1)}(t)+f\big(d(t),T(y,\dot y,\dots,y^{(r-1)})(t)\big).
\end{aligned}
$$

Set $v_{1,1}(t):=e_{1,1}(t)$ and $\tilde v(t):=\sum_{i=1}^{r-1}v_{i,1}(t)$, then we may define $v_{1,j}(t):=e_{1,j}(t)-\sum_{k=1}^{j-1}R_{r-j+k+1}\tilde v^{(k-1)}(t)$ and obtain

$$
\begin{aligned}
\dot v_{1,1}(t) &= v_{1,2}(t)-\big(q_1+p_1k_1(t)\big)v_{1,1}(t)+R_r\tilde v(t),\\
\dot v_{1,2}(t) &= v_{1,3}(t)-\big(q_2+p_2k_1(t)\big)v_{1,1}(t)+R_{r-1}\tilde v(t),\\
&\ \ \vdots\\
\dot v_{1,r-2}(t) &= v_{1,r-1}(t)-\big(q_{r-2}+p_{r-2}k_1(t)\big)v_{1,1}(t)+R_3\tilde v(t),\\
\dot v_{1,r-1}(t) &= v_{1,r}(t)-\big(q_{r-1}+p_{r-1}k_1(t)\big)v_{1,1}(t)+R_2\tilde v(t)\\
&\quad+(\Gamma\tilde\Gamma^{-1}-I)z_{1,r}(t),
\end{aligned}
$$

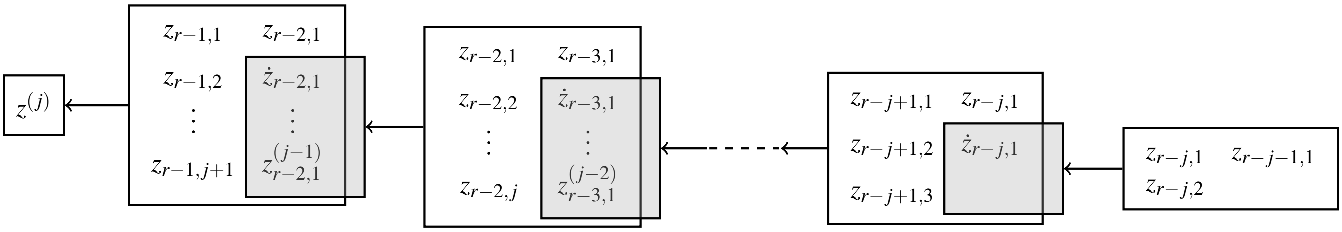

Figure 7: Dependency of $z^{(j)}$ on the observer states. Note that $z_{r-j-1,1} = z_{0,1} = y$ for $j = r-1$.

$$\begin{aligned}\dot v_{1,r}(t) &= -\Gamma\tilde\Gamma^{-1}\big(q_r + p_r k_1(t)\big)v_{1,1}(t) + R_1\tilde v(t)\\ &\quad + \sum_{i=1}^{r} R_i\left(y^{(i-1)}(t) - \tilde v^{(i-1)}(t)\right)\\ &\quad + f\big(d(t), T(y,\dot y,\dots,y^{(r-1)})(t)\big).\end{aligned}$$

Now we observe that

$$\begin{aligned}y(t)-\tilde v(t) &= y(t) - v_{1,1}(t) - v_{2,1}(t) - \ldots - v_{r-1,1}(t)\\ &= y(t) - \big(y(t) - z_{1,1}(t)\big) - \big(z_{1,1}(t) - z_{2,1}(t)\big)\\ &\quad - \ldots - \big(z_{r-2,1}(t) - z_{r-1,1}(t)\big) = z_{r-1,1}(t) = z(t).\end{aligned}$$

Furthermore,

$$z_{1,r}(t) = z_{1,1}^{(r-1)}(t) - \sum_{i=0}^{r-2}\left(\tfrac{\mathrm d}{\mathrm dt}\right)^i\left[\big(q_{r-i-1} + p_{r-i-1}k_1(t)\big)v_{1,1}(t)\right]$$

and

$$z_{1,1}(t) = y(t) - v_{1,1}(t) = z(t) + \tilde v(t) - v_{1,1}(t) = z(t) + \sum_{i=2}^{r-1} v_{i,1}(t),$$

hence

$$\begin{aligned}z_{1,r}(t) &= z^{(r-1)}(t) + \sum_{i=2}^{r-1} v_{i,1}(t)\\ &\quad - \sum_{i=0}^{r-2}\left(\tfrac{\mathrm d}{\mathrm dt}\right)^i\left[\big(q_{r-i-1} + p_{r-i-1}k_1(t)\big)v_{1,1}(t)\right].\end{aligned}$$

Step 1c: Define $w_{i,j}(t) := v_{i,j}(t)$ for $i = 2,\ldots,r-1$ and $j = 1,\ldots,r$ and

$$\begin{aligned}w_{1,r}(t) &:= v_{1,r}(t),\\ w_{1,r-j}(t) &:= v_{1,r-j}(t) + G\left[\sum_{i=2}^{r-1} v_{i,1}^{(r-j-1)}(t)\right.\\ &\quad \left. - \sum_{i=j}^{r-2}\left(\tfrac{\mathrm d}{\mathrm dt}\right)^{i-j}\left[\big(q_{r-i-1} + p_{r-i-1}k_1(t)\big)v_{1,1}(t)\right]\right]\end{aligned}$$

for $j = 1,\ldots,r-1$, where $G := (I - \Gamma\tilde\Gamma^{-1})$; in particular we have

$$w_{1,1}(t) = v_{1,1}(t) + G\sum_{i=2}^{r-1} v_{i,1}(t).$$

With $\tilde w(t) := \sum_{i=2}^{r-1} w_{i,1}(t)$ we find

$$\begin{aligned}\dot w_{1,1}(t) &= w_{1,2}(t) - \Gamma\tilde\Gamma^{-1}\big(q_1 + p_1 k_1(t)\big)\big(w_{1,1}(t) - G\tilde w(t)\big)\\ &\quad + R_r w_{1,1}(t) + R_r\Gamma\tilde\Gamma^{-1}\tilde w(t),\\ \dot w_{1,2}(t) &= w_{1,3}(t) - \Gamma\tilde\Gamma^{-1}\big(q_2 + p_2 k_1(t)\big)\big(w_{1,1}(t) - G\tilde w(t)\big)\\ &\quad + R_{r-1} w_{1,1}(t) + R_{r-1}\Gamma\tilde\Gamma^{-1}\tilde w(t),\\ &\ \ \vdots\\ \dot w_{1,r-2}(t) &= w_{1,r-1}(t) - \Gamma\tilde\Gamma^{-1}\big(q_{r-2} + p_{r-2} k_1(t)\big)\big(w_{1,1}(t) - G\tilde w(t)\big)\\ &\quad + R_3 w_{1,1}(t) + R_3\Gamma\tilde\Gamma^{-1}\tilde w(t),\\ \dot w_{1,r-1}(t) &= w_{1,r}(t) - \Gamma\tilde\Gamma^{-1}\big(q_{r-1} + p_{r-1} k_1(t)\big)\big(w_{1,1}(t) - G\tilde w(t)\big)\\ &\quad + R_2 w_{1,1}(t) + R_2\Gamma\tilde\Gamma^{-1}\tilde w(t) - Gz^{(r-1)}(t),\\ \dot w_{1,r}(t) &= -\Gamma\tilde\Gamma^{-1}\big(q_r + p_r k_1(t)\big)\big(w_{1,1}(t) - G\tilde w(t)\big)\\ &\quad + R_1 w_{1,1}(t) + R_1\Gamma\tilde\Gamma^{-1}\tilde w(t)\\ &\quad + \sum_{i=1}^{r} R_i z^{(i-1)}(t) + f\big(d(t), T(y,\dot y,\ldots,y^{(r-1)})(t)\big),\\ k_1(t) &= \frac{1}{1 - \varphi_1(t)^2\|w_{1,1}(t) - G\tilde w(t)\|^2}.\end{aligned} \tag{38a}$$

and

$$\begin{aligned}\dot w_{i,1}(t) &= w_{i,2}(t) - \big(q_1 + p_1 k_i(t)\big)w_{i,1}(t)\\ &\quad + \big(q_1 + p_1 k_{i-1}(t)\big)\underbrace{w_{i-1,1}(t)}_{=\left(w_{1,1}(t) - G\tilde w(t)\right)\ \text{if } i=2},\\ &\ \ \vdots\\ \dot w_{i,r-1}(t) &= w_{i,r}(t) - \big(q_{r-1} + p_{r-1} k_i(t)\big)w_{i,1}(t)\\ &\quad + \big(q_{r-1} + p_{r-1} k_{i-1}(t)\big)\underbrace{w_{i-1,1}(t)}_{=\left(w_{1,1}(t) - G\tilde w(t)\right)\ \text{if } i=2},\\ \dot w_{i,r}(t) &= -\big(q_r + p_r k_i(t)\big)w_{i,1}(t)\\ &\quad + \big(q_r + p_r k_{i-1}(t)\big)\underbrace{w_{i-1,1}(t)}_{=\left(w_{1,1}(t) - G\tilde w(t)\right)\ \text{if } i=2},\\ k_i(t) &= \frac{1}{1 - \varphi_i(t)^2\|w_{i,1}(t)\|^2}.\end{aligned} \tag{38b}$$

for $i = 2,\ldots,r-1$.

Step 2: We define the operator $\tilde T : \mathscr{C}([-h,\infty) \to \mathbb{R}^m)^r \to \mathscr{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\ge 0} \to \mathbb{R}^{\tilde q})$, where $\tilde q = (r-1)rm + r$, (essentially) as the solution operator of (38), i.e., for $\zeta_1,\dots,\zeta_r \in \mathscr{C}([-h,\infty)\to\mathbb{R}^m)$ let $w_{ij}:[0,\beta)\to\mathbb{R}^m$, $\beta\in(0,\infty]$, be the unique maximal solution of (38) for $z=\zeta_1, \dot z=\zeta_2,\dots,z^{(r-1)}=\zeta_r$ with appropriate initial values according to the transformation which leads to (38), and define

$$\begin{aligned}\tilde T(\zeta_1,\dots,\zeta_r)(t) := \big(&w_{1,1}(t),\dots,w_{1,r}(t),w_{2,1}(t),\\ &\dots,w_{r-1,r}(t),k_1(t),\dots,k_r(t)\big)^\top,\quad t\in[0,\beta).\end{aligned}$$

We stress that $y,\dot y,\dots,y^{(r-1)}$ in (38a)can be replaced by $w_{i,j}$ and $z,\dot z,\dots,z^{(r-1)}$ using $y^{(i)}=z^{(i)}+w_{1,1}^{(i)}+\Gamma\tilde\Gamma^{-1}\tilde w^{(i)}$ and the differential equations (38). Furthermore, the operator $\tilde T$ depends on the disturbance d and several initial values. In the following we show that $\tilde T$ is well-defined, i.e., $\beta=\infty$, and has the properties a)–c) as defined in Section 2. Note that $\big(t,w_{1,1}(t),\dots,w_{1,r}(t),w_{2,1}(t),\dots,w_{r-1,r}(t)\big)\in\mathscr{D}$ for all $t\in[0,\beta)$, with $\mathscr{D}$ as defined in (39), and the closure of the graph of the solution $\big(w_{1,1},\dots,w_{1,r},w_{2,1},\dots,w_{r-1,r}\big)$ is not a compact subset of $\mathscr{D}$.

Step 2a: First assume that $\zeta_1,\dots,\zeta_r$ are bounded on $[0,\beta)$. We show that $w_{i,j}$ and k_i are bounded as well. As the solution evolves in $\mathscr{D}$, it is clear that $w_{1,1}-G\tilde w$, $w_{2,1},\dots,w_{r-1,1}$ are bounded, and thus also $w_{1,1}$ is bounded. Since $y=z+w_{1,1}+\Gamma\tilde\Gamma^{-1}\tilde w$, it follows that y is bounded and hence $T(y,\dot y,\dots,y^{(r-1)})$ is bounded by property a'). Boundedness of d and continuity of f then imply that $f\big(d(\cdot),T(y,\dot y,\dots,y^{(r-1)})(\cdot)\big)$ is bounded.

Now let $w_i:=(w_{i,1}^\top,\dots,w_{i,r}^\top)^\top$, then it follows from (38) that

$$\begin{aligned}\dot w_1(t)&=\hat A w_1(t)-k_1(t)\bar P\Gamma\tilde\Gamma^{-1}\big(w_{1,1}(t)-G\tilde w(t)\big)+B_1(t),\\ \dot w_2(t)&=\hat A w_2(t)-k_2(t)\bar P w_{2,1}(t)\\ &\qquad +k_1(t)\bar P\big(w_{1,1}(t)-G\tilde w(t)\big)+B_2(t),\\ \dot w_i(t)&=\hat A w_i(t)-k_i(t)\bar P w_{i,1}(t)+k_{i-1}(t)\bar P w_{i-1,1}(t)+B_i(t)\end{aligned}\tag{40}$$

for $i=3,\dots,r-1$, where $\hat A$ is as in the proof of Theorem 4.1, B_i is some suitable bounded function and

$$\bar P := \begin{bmatrix} p_1 I_m\\ \vdots\\ p_r I_m\end{bmatrix}.$$

Recall that $\hat A^\top\hat P+\hat P\hat A+\hat Q=0$, where $\hat P>0$ and $\hat Q>0$, and that

$$\bar P^\top\hat P=[\tilde p I_m,0,\dots,0],\quad \tilde p:=(P_{11}-P_{12}P_{22}^{-1}P_{12}^\top)>0.$$

We consider the cases $r=2$ and $r=3$ separately.

Step 2b: Assume that $r=2$. Then (40) reads

$$\dot w_1(t)=\hat A w_1(t)-k_1(t)\bar P\Gamma\tilde\Gamma^{-1}w_{1,1}(t)+B_1(t).$$

Using the Lyapunov function $V(w_1)=w_1^\top\hat P w_1$ one can then show, as in the proof of Theorem 4.1, that w_1 and k_1 are bounded on $[0,\beta)$.

Step 2c: Assume that $r=3$. Then (40) reads

$$\begin{aligned}\dot w_1(t)&=\hat A w_1(t)-k_1(t)\bar P\Gamma\tilde\Gamma^{-1}\big(w_{1,1}(t)-Gw_{2,1}(t)\big)+B_1(t),\\ \dot w_2(t)&=\hat A w_2(t)-k_2(t)\bar P w_{2,1}(t)\\ &\qquad +k_1(t)\bar P\big(w_{1,1}(t)-Gw_{2,1}(t)\big)+B_2(t).\end{aligned}$$

From condition (36) we obtain that $G=G^\top>0$, hence $G\Gamma\tilde\Gamma^{-1}>0$ has a unique matrix square root. Let $K:=I_m\otimes\big(G\Gamma\tilde\Gamma^{-1}\big)^{\frac12}>0$ (recall the Kronecker product $\otimes$ from the proof of Theorem 4.1) and define the Lyapunov function $V(w_1,w_2):=w_1^\top\hat P w_1+w_2^\top K^\top\hat P K w_2$ for $w_1,w_2\in\mathbb{R}^{3m}$. Then, for all $t\in[0,\beta)$,

$$\begin{aligned}&\tfrac{\mathrm d}{\mathrm dt}V\big(w_1(t),w_2(t)\big)\\ &=w_1(t)^\top\big(\hat A^\top\hat P+\hat P\hat A\big)w_1(t)\\ &\quad -2k_1(t)w_1(t)^\top\hat P\bar P\Gamma\tilde\Gamma^{-1}\big(w_{1,1}(t)-G\tilde w(t)\big)\\ &\quad +2w_1(t)^\top B_1(t)+w_2(t)^\top\big(\hat A^\top K^\top\hat P K+K^\top\hat P K\hat A\big)w_2(t)\\ &\quad -2k_2(t)w_2(t)^\top K^\top\hat P K\bar P w_{2,1}(t)+2w_2(t)^\top K^\top\hat P K B_2(t)\\ &\quad +2k_1(t)w_2(t)^\top K^\top\hat P K\bar P\big(w_{1,1}(t)-Gw_{2,1}(t)\big),\end{aligned}$$

and since it is easy to see that $\hat A$ and K commute and $K^\top\hat P K\bar P=\tilde p[I_m,0,\dots,0]^\top G\Gamma\tilde\Gamma^{-1}$, it follows that, for some positive $\alpha_1,\alpha_2,M_1,M_2$,

$$\begin{aligned}&\tfrac{\mathrm d}{\mathrm dt}V\big(w_1(t),w_2(t)\big)\\ &\le-\alpha_1\|w_1(t)\|^2-\alpha_2\|w_2(t)\|^2\\ &\quad -2k_1(t)\left(\tilde p w_{1,1}^\top\Gamma\tilde\Gamma^{-1}-\tilde p w_{2,1}^\top G\Gamma\tilde\Gamma^{-1}\right)\big(w_{1,1}(t)-Gw_{2,1}(t)\big)\\ &\quad +M_1\|w_1(t)\|+M_2\|w_2(t)\|\\ &=-\alpha_1\|w_1(t)\|^2-\alpha_2\|w_2(t)\|^2+M_1\|w_1(t)\|+M_2\|w_2(t)\|\\ &\quad -2\tilde p k_1(t)\,(w_{1,1}-Gw_{2,1})^\top\Gamma\tilde\Gamma^{-1}\big(w_{1,1}(t)-Gw_{2,1}(t)\big)\\ &\le-\alpha_1\|w_1(t)\|^2-\alpha_2\|w_2(t)\|^2+M_1\|w_1(t)\|+M_2\|w_2(t)\|.\end{aligned}$$

As in the proof of Theorem 4.1 we may now show that w_1 and w_2 are bounded and that k_1 and k_2 are bounded as well on $[0,\beta)$.

Step 2d: We show $\beta=\infty$ (not assuming boundedness of $\zeta_1,\dots,\zeta_r$). Assume that $\beta<\infty$. Then $\zeta_1,\dots,\zeta_r$ are bounded on $[0,\beta)$ and hence $w_{i,j}$ and k_i are bounded by Steps 4a–4c. Therefore, it follows that the closure of the graph of the solution $\big(w_{1,1},\dots,w_{1,r},w_{2,1},\dots,w_{r-1,r}\big)$ is a compact subset of $\mathscr{D}$, a contradiction, thus $\beta=\infty$.

Step 2e: It remains to show that $\tilde T$ has the properties a)–c). Properties b) and c) are clear and property a) is an immediate consequence of Steps 4a–4c.

Step 3: By Step 2 we may write the conjunction of (35) and (28) with input u and output $z=z_{r-1,1}$ in the form

$$z^{(r)}(t)=\tilde\Gamma u(t)+\sum_{j=0}^{r-1}\left(\tfrac{\mathrm d}{\mathrm dt}\right)^j\left[\big(q_{r-j}+p_{r-j}k_{r-1}(t)\big)w_{r-1,1}(t)\right]$$

$$\mathscr{D} := \left\{ \left(t, w_{1,1}, \dots, w_{1,r}, w_{2,1}, \dots, w_{r-1,r}\right) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{rm} \;\middle|\; \varphi_1(t) \left\| w_{1,1} - G \sum_{i=2}^{r-1} w_{i,1} \right\| < 1,\ \varphi_i(t) \|w_{i,1}\| < 1,\ i = 2, \dots, r-1 \right\} \tag{39}$$

and hence

$$z^{(r)}(t) = F\left(\tilde{d}(t), \tilde{T}(z, \dot{z}, \dots, z^{(r-1)})(t)\right) + \tilde{\Gamma} u(t)$$

for $\tilde{d}(t) := \left(\varphi_{r-1}(t), \dot{\varphi}_{r-1}(t), \dots, \varphi_{r-1}^{(r-1)}(t)\right)^\top \in \mathscr{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^r)$, some $F \in \mathscr{C}(\mathbb{R}^r \times \mathbb{R}^{\tilde{q}} \to \mathbb{R}^m)$ and the operator $\tilde{T} : \mathscr{C}([-h, \infty) \to \mathbb{R}^m)^r \to \mathscr{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^{\tilde{q}})$ which satisfies the properties a)–c). It is clear that any solution of (28), (35) satisfies the properties (30) and (32). □

Remark 6.3. A careful inspection of the proof of Theorem 6.2 reveals that in order for Theorem 6.2 to holds true for $r \geq 4$ we would need to show that (40) has bounded solutions. However, we were only able to find suitable Lyapunov functions in the cases $r = 2$ and $r = 3$, thus the proof for $r \geq 4$ remains an open problem.

An immediate application of Theorem 6.2 is the following: Trajectory tracking with prescribed transient behavior of the tracking error for single-input, single-output systems (35) (i.e., $m = 1$) of relative degree $r = 2$ is possible without having to calculate the derivative of the output. To achieve this we may apply the funnel controller introduced in [10] in conjunction with the funnel observer (1) (the cascade consists only of one observer in this case). Using the assumptions in Theorem 6.2 we obtain that the assumptions of [10, Thm. 3.1] are satisfied when applied to the observer-plant system with output z and hence we obtain tracking of any $y_{\mathrm{ref}} \in \mathscr{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ such that, for given $\varphi \in \Phi$,

$$\exists\, \varepsilon > 0\ \forall\, t > 0 : \ \|z(t) - y_{\mathrm{ref}}\| < \varphi(t)^{-1} - \varepsilon.$$

Combining this with (32) we obtain

$$\|y(t) - y_{\mathrm{ref}}\| < \varphi(t)^{-1} + \varphi_1(t)^{-1} - \varepsilon - \varepsilon_1.$$

7. Conclusion

In the present paper we have introduced the funnel observer as a novel and simple adaptive high-gain observer. We showed that the funnel observer is feasible for a large class of nonlinear systems described by functional differential equations which have a known strict relative degree, the internal dynamics map bounded signals to bounded signals, and the operators involved are sufficiently smooth to guarantee local maximal existence of solutions. The proposed adaptation scheme for the observer gain is simple and non-monotone, and we showed that it guarantees prescribed transient behavior of the observation error. Using a cascade of funnel observers, we proved that it is possible to obtain an artificial output with explicitly known derivatives which tracks the system output with prescribed transient behavior. Furthermore, the interconnection of the system with the observer cascade is shown to have stable internal dynamics provided the relative degree does not exceed three.

The results that we obtained in Section 6 suggest that the funnel observer is a suitable tool for resolving the problem of higher relative degree in stabilization and tracking problems. If a system has a higher relative degree and derivatives of the output are not available, then a filter or observer is frequently used to obtain approximations of the output derivatives, see the survey [12] and the references therein. As explained there, the concept of funnel control is usually combined with a back-stepping procedure to overcome the higher relative degree, which however complicates the feedback structure. However, in the last sentence of [12, Sec. 6] it is conjectured that the combination of a high-gain observer with a funnel-type controller might be beneficial for tracking of higher relative degree systems. In Section 6 we have shown that the funnel observer may be used to achieve this for systems with relative degree two. Systems of higher relative degree are the topic of future research.

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