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Matthias Beckmann and Armin Iske

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ERROR ESTIMATES AND CONVERGENCE RATES FOR FILTERED BACK PROJECTION

MATTHIAS BECKMANN AND ARMIN ISKE

Abstract. Computerized tomography allows us to reconstruct a bivariate function from Radon samples. The reconstruction is based on the *filtered back projection* (FBP) formula, which gives an analytical inversion of the Radon transform. However, the FBP formula is numerically unstable and suitable low-pass filters with a compactly supported window function and finite bandwidth are employed to make the reconstruction by FBP less sensitive to noise.

The objective of this paper is to analyse the intrinsic FBP reconstruction error which is incurred by the use of a low-pass filter. To this end, we prove L^2-error estimates on Sobolev spaces of fractional order. The obtained error bounds are affine-linear with respect to the distance between the filter's window function and the constant function 1 in the L^∞-norm. With assuming more regularity of the window function, we refine the error estimates to prove convergence for the FBP reconstruction in the L^2-norm as the filter's bandwidth goes to infinity. Further, we determine asymptotic convergence rates in terms of the bandwidth of the low-pass filter and the smoothness of the target function.

1. Introduction

The term *filtered back projection* (FBP) refers to a well-known and commonly used reconstruction technique in computerized tomography (CT), which deals with the generation of medical images. The classical reconstruction problem in CT consists in recovering the interior structure of a scanned object from given measurements of X-ray scans. This X-ray data can be interpreted as a finite set of line integrals of the (unknown) *attenuation function* of the scanned object which describes the amount of energy that is absorbed by the medium. Thus, the CT reconstruction problem requires the reconstruction of the scanned object's attenuation function from its line integrals.

In order to formulate this basic reconstruction problem mathematically, we regard for $f \in \mathrm{L}^1(\mathbb{R}^2)$ its *Radon transform*

$$\mathcal{R}f(t,\theta) = \int_{\{x\cos(\theta)+y\sin(\theta)=t\}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \qquad \text{for } (t,\theta) \in \mathbb{R}\times[0,\pi).$$

Here, the set $\{(x,y) \mid x\cos(\theta)+y\sin(\theta)=t\} \subset \mathbb{R}^2$ describes the straight line $\ell_{t,\theta}$ with distance t to the origin that is perpendicular to the unit vector $\mathbf{n}_\theta = (\cos(\theta),\sin(\theta))^T$. Note that the Radon transform $\mathcal{R}$ maps a bivariate function $f \equiv f(x,y)$ in Cartesian coordinates onto a bivariate function $\mathcal{R}f \equiv \mathcal{R}f(t,\theta)$ in polar coordinates.

Now the CT reconstruction problem reads as follows.

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Problem 1.1 (Basic reconstruction problem). *Let $\Omega \subset \mathbb{R}^2$ be bounded. Reconstruct a bivariate function $f \equiv f(x,y)$ with compact support* $\operatorname{supp}(f) \subseteq \Omega$ *from its Radon data*

$$\{\mathcal{R}f(t,\theta) \mid t \in \mathbb{R},\, \theta \in [0,\pi)\}\,.$$

Therefore, the basic reconstruction problem seeks for the inversion of the Radon transform $\mathcal{R}$. For a comprehensive mathematical treatment of the Radon transform and its inversion, we refer to the textbooks [3, 9].

The outline of this paper is as follows. In §2 we address the inversion of the Radon transform by the classical FBP formula. Since the FBP formula is highly sensitive with respect to noise, we also describe how to stabilize the reconstruction formula by using suitable low-pass filters with a compactly supported window function and finite bandwidth. This standard approach leads us to an approximate reconstruction formula, whose approximation quality strongly depends on the chosen low-pass filter.

The evaluation of the reconstruction quality requires a rigorous analysis of the approximation error, where error bounds depending on the low-pass filter's window function, on its bandwidth and on the regularity of the target function are of particular interest. To this end, we first recall in §3 an error estimate from our previous work [1] concerning the FBP reconstruction error in the L^2-norm for relevant cases of target functions from Sobolev spaces of fractional order.

That error estimate from [1] allows us to show convergence of the approximate reconstruction to the target function as the filter's bandwidth goes to infinity, but only under rather strong assumptions. In contrast, due to a result by Madych [4], convergence can be shown under much weaker assumptions. This has motivated us to investigate the refinement of our previous L^2-error estimate, as detailed in §4. On the basis of our refined error estimates we are able to prove convergence under much weaker conditions. Furthermore, this allows us to determine asymptotic convergence rates in terms of the bandwidth of the low-pass filter and the smoothness of the target function. In §5 and §6 we show that the convergence rate saturates with respect to the differentiability order of the filter's window function. Our theoretical results are supported by numerical simulations.

2. Filtered Back Projection

The inversion of the Radon transform $\mathcal{R}$ is well understood and involves the (continuous) *Fourier transform*, here taken as

$$\mathcal{F}g(S,\theta) = \int_{\mathbb{R}} g(t,\theta)\mathrm{e}^{-\mathrm{i}tS}\,\mathrm{d}t \qquad \text{for } (S,\theta) \in \mathbb{R} \times [0,\pi)$$

for $g \equiv g(t,\theta)$ in polar coordinates satisfying $g(\cdot,\theta) \in \mathrm{L}^1(\mathbb{R})$ for all $\theta \in [0,\pi)$, as well as the *back projection*

$$\mathcal{B}h(x,y) = \frac{1}{\pi}\int_0^\pi h(x\cos(\theta) + y\sin(\theta),\theta)\,\mathrm{d}\theta \qquad \text{for } (x,y) \in \mathbb{R}^2$$

for $h \in \mathrm{L}^1(\mathbb{R} \times [0,\pi))$. Note that the back projection $\mathcal{B}$ maps a bivariate function $h \equiv h(t,\theta)$ in polar coordinates onto a bivariate function $\mathcal{B}h \equiv \mathcal{B}h(x,y)$ in Cartesian coordinates.

Later in this work we also use the (continuous) *Fourier transform* on $\mathbb{R}^2$, defined as

$$\mathcal{F}f(X,Y) = \int_{\mathbb{R}^2} f(x,y)\mathrm{e}^{-\mathrm{i}(xX+yY)}\,\mathrm{d}x\,\mathrm{d}y \qquad \text{for } (X,Y) \in \mathbb{R}^2$$

for $f \equiv f(x,y)$ in Cartesian coordinates, where $f \in \mathrm{L}^1(\mathbb{R}^2)$.

Now the inversion of the Radon transform is given by the classical *filtered back projection formula* (see e.g. [2, Theorem 6.2.])

$$f(x,y) = \frac{1}{2}\,\mathcal{B}\big(\mathcal{F}^{-1}[|S|\mathcal{F}(\mathcal{R}f)(S,\theta)]\big)(x,y) \quad \forall\,(x,y) \in \mathbb{R}^2, \tag{2.1}$$

which holds for any function $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathcal{C}(\mathbb{R}^2)$.

We remark that the FBP formula is numerically *unstable*. Indeed, by applying the filter $|S|$ to the Fourier transform $\mathcal{F}(\mathcal{R}f)$ in (2.1), especially the high frequency components of $\mathcal{R}f$ are amplified by the magnitude of $|S|$. Therefore, the filtered back projection formula is in particular highly sensitive with respect to noise. Needless to say that this is critical in many relevant applications, where a reconstruction by FBP would lead to an undesired corruption of the image.

To reduce the sensitivity of the FBP formula with respect to noise, we follow a standard approach and replace the filter $|S|$ in (2.1) by a *low-pass filter* A_L of the form

$$A_L(S) = |S| W(S/L)$$

with finite *bandwidth* $L > 0$ and an even *window function* $W : \mathbb{R} \longrightarrow \mathbb{R}$ with compact support $\mathrm{supp}(W) \subseteq [-1,1]$. Further, we assume $W \in \mathrm{L}^\infty(\mathbb{R})$.

Therefore, the scaled window function $W_L(S) = W(S/L)$ is even and compactly supported with $\mathrm{supp}(W_L) \subseteq [-L,L]$. In particular, $W_L \in \mathrm{L}^1(\mathbb{R})$, and so, unlike $|S|$, any low-pass filter of the form $A_L(S) = |S| W_L(S)$ is in $\mathrm{L}^1(\mathbb{R})$. When replacing the filter $|S|$ in (2.1) by a low-pass filter $A_L(S)$, the reconstruction of f is no longer exact. However, we can simplify the resulting *approximate FBP formula* as

$$f_L = \frac{1}{2}\,\mathcal{B}\big(\mathcal{F}^{-1}A_L * \mathcal{R}f\big), \tag{2.2}$$

where $*$ denotes the usual convolution product. Relying on the standard relation

$$\mathcal{B}g * f = \mathcal{B}\big(g * \mathcal{R}f\big),$$

which holds for $f \in \mathrm{L}^1(\mathbb{R}^2)$ and $g \in \mathrm{L}^1(\mathbb{R} \times [0,\pi))$, see [9, Theorem II.1.3], we can rewrite the approximate FBP reconstruction f_L in terms of the target function f via

$$f_L = \frac{1}{2}\,\mathcal{B}\big(\mathcal{F}^{-1}A_L * \mathcal{R}f\big) = f * K_L,$$

where we define the *convolution kernel* $K_L : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as

$$K_L(x,y) = \frac{1}{2}\,\mathcal{B}\big(\mathcal{F}^{-1}A_L\big)(x,y) \quad \text{for } (x,y) \in \mathbb{R}^2.$$

For the sake of brevity, we call any application of the approximate FBP formula (2.2) an *FBP method*. Therefore, each FBP method provides one approximation f_L to f, $f_L \approx f$, whose quality depends on the choice of the low-pass filter A_L.

In the following, we analyse the intrinsic error of the FBP method which is incurred by the use of the low-pass filter A_L, i.e., we wish to analyse the reconstruction error

$$e_L = f - f_L \tag{2.3}$$

with respect to the filter's window function W and bandwidth L.

We remark at this point that pointwise and L^∞-error estimates on e_L were proven by Munshi in [5] and by Munshi et al. in [6]. Their theoretical results were further supported by numerical experiments in [7]. Error bounds on the L^p-norm of e_L, in terms of an L^p-modulus of continuity of f, were proven by Madych in [4].

In the following sections, we prove L^2-error estimates on e_L for target functions f from Sobolev spaces of fractional order. Here, the *Sobolev space* $\mathrm{H}^\alpha(\mathbb{R}^2)$ of order $\alpha \in \mathbb{R}$, defined as

$$\mathrm{H}^\alpha(\mathbb{R}^2) = \left\{ f \in \mathcal{S}'(\mathbb{R}^2) \mid \|f\|_\alpha < \infty \right\}, \tag{2.4}$$

is equipped with the norm $\|\cdot\|_\alpha$, where

$$\|f\|_\alpha^2 = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(1 + x^2 + y^2\right)^\alpha \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y,$$

and where $\mathcal{S}'$ in (2.4) denotes the Schwartz space of tempered distributions.

In relevant applications of (medical) image processing, Sobolev spaces of compactly supported functions,

$$\mathrm{H}_0^\alpha(\Omega) = \left\{ f \in \mathrm{H}^\alpha(\mathbb{R}^2) \mid \mathrm{supp}(f) \subseteq \overline{\Omega} \right\},$$

on an open and bounded domain $\Omega \subset \mathbb{R}^2$, and of fractional order $\alpha > 0$ play an important role (cf. [8]). In fact, the density function f of an image in $\Omega \subset \mathbb{R}^2$ has usually jumps along smooth curves, but is otherwise smooth off these curve singularities. Such functions belong to the Sobolev space $\mathrm{H}_0^\alpha(\mathbb{R}^2)$ for $\alpha < \frac{1}{2}$. Thus, we can consider the density of an image as a function in a Sobolev space $\mathrm{H}_0^\alpha(\Omega)$ whose order α is close to $\frac{1}{2}$.

We remark that the approach taken in this paper is essentially different from previous approaches, in particular different from that in [4].

3. Error Analysis

In this section we prove an L^2-error estimate for $e_L = f - f_L$, where the upper bound on the L^2-norm of e_L is split into two error terms, a first term depending on the filter's window function W and a second one depending on its bandwidth $L > 0$. Although the results of this section are already published in [1], it will be quite instructive for the following analysis in this paper to recall the details of our previous error estimates in [1].

Theorem 3.1 (L^2-error estimate, see [1, Theorem 1])**.** *Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, for some $\alpha > 0$, $W \in \mathrm{L}^\infty(\mathbb{R})$ and $K_L \in \mathrm{L}^1(\mathbb{R}^2)$. Then, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ is bounded above by*

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \|1 - W\|_{\infty,[-1,1]} \, \|f\|_{\mathrm{L}^2(\mathbb{R}^2)} + L^{-\alpha} \, \|f\|_\alpha. \tag{3.1}$$

Since we will use some parts of the proof for a refined error analysis, we recall the proof of the theorem for the reader's convenience.

Proof. For $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{L}^2(\mathbb{R}^2)$, we get, by using the Rayleigh–Plancherel theorem,

$$\begin{aligned} \|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)}^2 = \|f - f * K_L\|_{\mathrm{L}^2(\mathbb{R}^2)}^2 &= \frac{1}{2\pi} \, \|\mathcal{F}f - \mathcal{F}f \cdot \mathcal{F}K_L\|_{\mathrm{L}^2(\mathbb{R}^2;\mathbb{C})}^2 \\ &= \frac{1}{2\pi} \, \|\mathcal{F}f - W_L \cdot \mathcal{F}f\|_{\mathrm{L}^2(\mathbb{R}^2;\mathbb{C})}^2, \end{aligned}$$

since, by letting $W_L(x,y) := W_L(r(x,y))$ for $r(x,y) = \sqrt{x^2+y^2}$ and $(x,y) \in \mathbb{R}^2$, we have the identity

$$W_L(x,y) = \mathcal{F}K_L(x,y) \quad \forall \, (x,y) \in \mathbb{R}^2$$

for $K_L \in \mathrm{L}^1(\mathbb{R}^2)$ (in consequence of [9, Theorem II.1.4]).

We split the above representation of the L^2-error into a sum of two integrals,

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)}^2 = I_1 + I_2, \tag{3.2}$$

where we let

$$I_1 := \frac{1}{2\pi} \int_{r(x,y) \le L} |(\mathcal{F}f - W_L \cdot \mathcal{F}f)(x,y)|^2 \, \mathrm{d}(x,y), \tag{3.3}$$

$$I_2 := \frac{1}{2\pi} \int_{r(x,y) > L} |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y). \tag{3.4}$$

For $W \in \mathrm{L}^\infty(\mathbb{R})$, integral I_1 can be bounded above by

$$I_1 \le \frac{1}{2\pi} \|1 - W_L\|^2_{\infty,[-L,L]} \, \|\mathcal{F}f\|^2_{\mathrm{L}^2(\mathbb{R}^2;\mathbb{C})} = \|1 - W\|^2_{\infty,[-1,1]} \, \|f\|^2_{\mathrm{L}^2(\mathbb{R}^2)}$$

and, for $f \in \mathrm{H}^\alpha(\mathbb{R}^2)$, with $\alpha > 0$, integral I_2 can be bounded above by

$$I_2 \le \frac{1}{2\pi} \int_{r(x,y)>L} \left(1 + x^2 + y^2\right)^\alpha \, L^{-2\alpha} \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y) \le L^{-2\alpha} \, \|f\|^2_\alpha,$$

which completes the proof. □

The above theorem shows that the choices of both the window function W and the bandwidth L are of fundamental importance for the L^2-error of the FBP method. In fact, for fixed target function f and bandwidth L, the obtained error estimate is affine-linear with respect to the distance between the window function W and the constant function 1 in the L^∞-norm on the interval $[-1, 1]$. This behaviour has also been observed numerically in [1].

Moreover, the error term $\|1 - W\|_{\infty,[-1,1]}$ can be used to evaluate the quality of the window function W. Note that the window $W \equiv \chi_{[-1,1]}$ of the Ram–Lak filter is the unique minimizer of that quality indicator, so that the Ram–Lak filter is in this sense the *optimal* low-pass filter.

Finally, the smoothness of the target function f determines the decay rate of the second error term by

$$L^{-\alpha} \|f\|_\alpha = \mathcal{O}(L^{-\alpha}) \quad \text{for} \quad L \longrightarrow \infty.$$

However, the right hand side of our L^2-error estimate can only tend to zero if we choose the Ram–Lak filter, $W \equiv \chi_{[-1,1]}$, and let the bandwidth L go to ∞.

Nevertheless, the following theorem of Madych [4] shows that we get convergence of the FBP reconstruction f_L in the L^p-norm under weaker assumptions, for target functions $f \in \mathrm{L}^p(\mathbb{R}^2)$ with $1 \le p < \infty$.

Theorem 3.2 (Convergence in the L^p-norm, see [4, Proposition 5])**.** *Let the convolution kernel $K \equiv K_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfy $K \in \mathrm{L}^1(\mathbb{R}^2)$ with*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} K(x,y) \, \mathrm{d}x \, \mathrm{d}y = 1.$$

Then, for $f \in \mathrm{L}^p(\mathbb{R}^2)$, $1 \le p < \infty$,

$$\|e_L\|_{\mathrm{L}^p(\mathbb{R}^2)} \longrightarrow 0 \quad \textit{for} \quad L \longrightarrow \infty.$$

For the reader's convenience, we give a proof of the theorem, which relies on Lebegue's theorem on dominated convergence.

Proof. For $f \in \mathrm{L}^p(\mathbb{R}^2)$, $1 \le p < \infty$, and $(X, Y) \in \mathbb{R}^2$, we define

$$\Delta_f(X,Y) = \|f(\cdot - X, \cdot - Y) - f\|_{\mathrm{L}^p(\mathbb{R}^2)}.$$

Then, we have

$$\Delta_f(X,Y) \longrightarrow 0 \quad \text{for} \quad (X,Y) \longrightarrow (0,0),$$

since this holds for continuous functions f with compact support, i.e., $f \in \mathcal{C}_c(\mathbb{R}^2)$, and $\mathcal{C}_c(\mathbb{R}^2)$ is dense in $\mathrm{L}^p(\mathbb{R}^2)$ for $1 \leq p < \infty$.

Relying on the scaling property

$$K_L(x,y) = L^2\, K(Lx, Ly) \quad \forall\, (x,y) \in \mathbb{R}^2 \tag{3.5}$$

we get

$$\int_{\mathbb{R}} \int_{\mathbb{R}} K_L(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}} \int_{\mathbb{R}} K(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1,$$

and can rewrite the pointwise error

$$e_L(x,y) = (f - f_L)(x,y) \quad \text{for } (x,y) \in \mathbb{R}^2$$

as

$$e_L(x,y) = (f - f * K_L)(x,y) = \int_{\mathbb{R}} \int_{\mathbb{R}} [f(x,y) - f(x-X, y-Y)]\, K_L(X,Y)\,\mathrm{d}X\,\mathrm{d}Y.$$

Using Minkowski's integral inequality we can estimate the L^p-norm of e_L by

$$\begin{aligned}
\|e_L\|_{\mathrm{L}^p(\mathbb{R}^2)} &= \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} [f(x-X, y-Y) - f(x,y)]\, K_L(X,Y)\,\mathrm{d}X\,\mathrm{d}Y \right|^p \mathrm{d}x\,\mathrm{d}y \right)^{1/p} \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-X, y-Y) - f(x,y)|^p\, |K_L(X,Y)|^p\,\mathrm{d}x\,\mathrm{d}y \right)^{1/p} \mathrm{d}X\,\mathrm{d}Y \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-X, y-Y) - f(x,y)|^p\,\mathrm{d}x\,\mathrm{d}y \right)^{1/p} |K_L(X,Y)|\,\mathrm{d}X\,\mathrm{d}Y \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \Delta_f(X,Y)\, |K_L(X,Y)|\,\mathrm{d}X\,\mathrm{d}Y.
\end{aligned}$$

Again, by using the scaling property (3.5), we get

$$\|e_L\|_{\mathrm{L}^p(\mathbb{R}^2)} \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \Delta_f(X/L, Y/L)\; |K(X,Y)|\,\mathrm{d}X\,\mathrm{d}Y.$$

Since

$$|\Delta_f(X/L, Y/L)|\; |K(X,Y)| \leq 2\, \|f\|_{\mathrm{L}^p(\mathbb{R}^2)}\, |K(X,Y)|$$

and, by assumption,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |K(X,Y)|\,\mathrm{d}X\,\mathrm{d}Y < \infty,$$

in combination with

$$\Delta_f(X/L, Y/L) \longrightarrow 0 \quad \text{for} \quad L \longrightarrow \infty,$$

we finally obtain

$$\|e_L\|_{\mathrm{L}^p(\mathbb{R}^2)} \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \Delta_f(X/L, Y/L)\; |K(X,Y)|\,\mathrm{d}X\,\mathrm{d}Y \longrightarrow 0 \quad \text{for} \quad L \longrightarrow \infty$$

by Lebesgue's theorem on dominated convergence. $\square$

4. Refined Error Analysis

According to Theorem 3.2, the L^2-norm of the FBP reconstruction error $f - f_L$ tends to zero as L goes to ∞. On the grounds of our error estimate in (3.1), however, convergence follows only for the Ram–Lak filter, where $W \equiv \chi_{[-1,1]}$. To obtain convergence under weaker conditions, we need to refine our error estimate.

As in Theorem 3.1 we assume $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, for $\alpha > 0$, $W \in \mathrm{L}^\infty(\mathbb{R})$ and $K_L \in \mathrm{L}^1(\mathbb{R}^2)$. For the sake of brevity, we set $r(x,y) = \sqrt{x^2+y^2}$ for $(x,y) \in \mathbb{R}^2$. Recall the representation of the FBP reconstruction error $e_L = f - f_L$ with respect to the L^2-norm in (3.2), by the sum of two integrals, I_1 in (3.3) and I_2 in (3.4), where integral I_2 can be bounded above by

$$I_2 \leq L^{-2\alpha} \, \|f\|_\alpha^2. \tag{4.1}$$

In Theorem 3.1 we derived an upper bound for integral I_1 in terms of the L^2-norm of the target function f. To obtain convergence for a larger class of window functions, we bound I_1 from above, now also with respect to the H^α-norm of f. Indeed, for $f \in \mathrm{H}^\alpha(\mathbb{R}^2)$, with $\alpha > 0$, we can estimate integral I_1 in (3.3) by

$$\begin{aligned} I_1 &= \frac{1}{2\pi} \int_{r(x,y) \leq L} |1 - W_L(x,y)|^2 \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y) \\ &= \frac{1}{2\pi} \int_{r(x,y) \leq L} \frac{|1 - W_L(x,y)|^2}{(1+x^2+y^2)^\alpha} \left(1+x^2+y^2\right)^\alpha \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y) \\ &\leq \left(\sup_{S \in [-L,L]} \frac{(1 - W_L(S))^2}{(1+S^2)^\alpha} \right) \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(1+x^2+y^2\right)^\alpha \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y. \end{aligned}$$

Now note that

$$\sup_{S \in [-L,L]} \frac{(1 - W_L(S))^2}{(1+S^2)^\alpha} = \sup_{S \in [-L,L]} \frac{(1 - W(S/L))^2}{(1+S^2)^\alpha} = \sup_{S \in [-1,1]} \frac{(1 - W(S))^2}{(1+L^2S^2)^\alpha}.$$

Therefore, with letting

$$\Phi_{\alpha,W}(L) = \sup_{S \in [-1,1]} \frac{(1 - W(S))^2}{(1+L^2S^2)^\alpha} \quad \text{for } L > 0$$

we can express the above bound on I_1 as

$$I_1 \leq \left(\sup_{S \in [-1,1]} \frac{(1 - W(S))^2}{(1+L^2S^2)^\alpha} \right) \|f\|_\alpha^2 = \Phi_{\alpha,W}(L) \, \|f\|_\alpha^2.$$

Combining our bounds for integrals I_1 and I_2, this finally leads us to the L^2-error estimate

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)}^2 \leq \left(\sup_{S \in [-1,1]} \frac{(1 - W(S))^2}{(1+L^2S^2)^\alpha} + L^{-2\alpha} \right) \|f\|_\alpha^2 = \left(\Phi_{\alpha,W}(L) + L^{-2\alpha} \right) \|f\|_\alpha^2.$$

In summary, we have just established the following result.

Theorem 4.1 (Refined L^2-error estimate)**.** *Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, for $\alpha > 0$, $W \in \mathrm{L}^\infty(\mathbb{R})$, and $K_L \in \mathrm{L}^1(\mathbb{R}^2)$. Then, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ is bounded above by*

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(\Phi_{\alpha,W}^{1/2}(L) + L^{-\alpha} \right) \|f\|_\alpha. \tag{4.2}$$

Our next result shows that, under suitable assumptions on the window W, the function $\Phi_{\alpha,W}(L)$ tends to zero as L goes to ∞.

Theorem 4.2 (Convergence of $\Phi_{\alpha,W}$)**.** *Let the window W be continuous on $[-1,1]$ and $W(0)=1$. Then, for any $\alpha>0$,*

$$\Phi_{\alpha,W}(L) = \max_{S\in[-1,1]} \frac{(1-W(S))^2}{(1+L^2S^2)^\alpha} \longrightarrow 0 \quad \textit{for} \quad L \longrightarrow \infty.$$

Proof. For the sake of brevity, we define the function $\Phi_{\alpha,W,L} : [-1,1] \longrightarrow \mathbb{R}$ via

$$\Phi_{\alpha,W,L}(S) = \frac{(1-W(S))^2}{(1+L^2S^2)^\alpha} \quad \text{for } S\in[-1,1].$$

Because W is continuous on $[-1,1]$ and even, $\Phi_{\alpha,W,L}$ attains a maximum on $[-1,1]$, and we have

$$\Phi_{\alpha,W}(L) = \sup_{S\in[-1,1]} \Phi_{\alpha,W,L}(S) = \max_{S\in[-1,1]} \Phi_{\alpha,W,L}(S) = \max_{S\in[0,1]} \Phi_{\alpha,W,L}(S).$$

In the following, let $S^*_{\alpha,W,L}\in[0,1]$ be the smallest maximizer of the even function $\Phi_{\alpha,W,L}$ on $[0,1]$.

<u>Case 1:</u> $S^*_{\alpha,W,L}$ is uniformly bounded away from 0, i.e.,

$$\exists\, c \equiv c(\alpha,W) > 0\; \forall\, L>0 :\; S^*_{\alpha,W,L} \geq c,$$

in which case we get

$$0 \leq \Phi_{\alpha,W,L}\big(S^*_{\alpha,W,L}\big) = \frac{\big(1-W(S^*_{\alpha,W,L})\big)^2}{\big(1+L^2(S^*_{\alpha,W,L})^2\big)^\alpha} \leq \frac{\|1-W\|^2_{\infty,[-1,1]}}{(1+L^2c^2)^\alpha} \xrightarrow{L\to\infty} 0.$$

<u>Case 2:</u> $S^*_{\alpha,W,L}$ tends to 0 as L goes to ∞, i.e.,

$$S^*_{\alpha,W,L} \longrightarrow 0 \quad \text{for} \quad L\longrightarrow\infty.$$

Because W is continuous on $[-1,1]$ and satisfies $W(0)=1$, we have

$$W(S^*_{\alpha,W,L}) \longrightarrow W(0) = 1 \quad \text{for} \quad L \longrightarrow \infty$$

and, consequently,

$$0 \leq \Phi_{\alpha,W,L}\big(S^*_{\alpha,W,L}\big) = \frac{\big(1-W(S^*_{\alpha,W,L})\big)^2}{\big(1+L^2(S^*_{\alpha,W,L})^2\big)^\alpha} \leq \big(1-W(S^*_{\alpha,W,L})\big)^2 \xrightarrow{L\to\infty} 0.$$

Hence, in both cases we have

$$\Phi_{\alpha,W}(L) = \Phi_{\alpha,W,L}\big(S^*_{\alpha,W,L}\big) \longrightarrow 0 \quad \text{for} \quad L\longrightarrow\infty,$$

which completes our proof. □

By combining Theorems 4.1 and 4.2, we can now conclude convergence of the FBP reconstruction f_L in the L^2-norm for a larger class of window functions W.

Corollary 4.3. *Let $f\in \mathrm{L}^1(\mathbb{R}^2)\cap \mathrm{H}^\alpha(\mathbb{R}^2)$, for some $\alpha>0$, let $K_L\in \mathrm{L}^1(\mathbb{R}^2)$, and $W\in\mathcal{C}([-1,1])$ with $W(0)=1$. Then, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ satisfies*

$$\|e_L\|^2_{\mathrm{L}^2(\mathbb{R}^2)} \leq \big(\Phi_{\alpha,W}(L) + L^{-2\alpha}\big)\, \|f\|^2_\alpha \longrightarrow 0 \quad \textit{for} \quad L\longrightarrow\infty.$$

In particular,

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} = o(1) \quad \textit{for} \quad L\longrightarrow\infty.$$

We are now interested in the rate of convergence for the FBP reconstruction error $\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)}$ as L goes to ∞. Thus, we need to determine the decay rate of $\Phi_{\alpha,W}(L)$. To this end, let $S^*_{\alpha,W,L} \in [0,1]$ again denote the smallest maximizer in $[0,1]$ of the even function

$$\Phi_{\alpha,W,L}(S) = \frac{(1-W(S))^2}{(1+L^2S^2)^\alpha} \qquad \text{for } S \in [-1,1].$$

In the following analysis, we rely on the following assumption.

Assumption 4.4. *$S^*_{\alpha,W,L}$ is uniformly bounded away from 0, i.e., there exists a constant $c_{\alpha,W} > 0$, such that*

$$S^*_{\alpha,W,L} \geq c_{\alpha,W} \quad \forall\, L > 0.$$

Under this assumption, we can conclude

$$\Phi_{\alpha,W}(L) = \Phi_{\alpha,W,L}\big(S^*_{\alpha,W,L}\big) \leq \frac{\|1-W\|^2_{\infty,[-1,1]}}{\big(1+L^2\,c^2_{\alpha,W}\big)^\alpha} \leq c^{-2\alpha}_{\alpha,W}\,\|1-W\|^2_{\infty,[-1,1]}\,L^{-2\alpha},$$

in which case we obtain

$$\|e_L\|^2_{\mathrm{L}^2(\mathbb{R}^2)} \leq \Big(c^{-2\alpha}_{\alpha,W}\,\|1-W\|^2_{\infty,[-1,1]} + 1\Big)\,L^{-2\alpha}\,\|f\|^2_\alpha,$$

i.e.,

$$\|e_L\|^2_{\mathrm{L}^2(\mathbb{R}^2)} = \mathcal{O}(L^{-2\alpha}) \quad \text{for} \quad L \longrightarrow \infty.$$

In summary, we can, under the above assumption, establish asymptotic L^2-error estimates for the FBP reconstruction with convergence rates as follows.

Theorem 4.5 (Rate of convergence)**.** *Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, for some $\alpha > 0$, $K_L \in \mathrm{L}^1(\mathbb{R}^2)$, and $W \in \mathcal{C}([-1,1])$ with $W(0) = 1$. Further, let Assumption 4.4 be satisfied. Then, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ is bounded above by*

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \Big(c^{-\alpha}_{\alpha,W}\,\|1-W\|_{\infty,[-1,1]} + 1\Big)\,L^{-\alpha}\,\|f\|_\alpha, \tag{4.3}$$

i.e.,

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} = \mathcal{O}(L^{-\alpha}) \quad \textit{for} \quad L \longrightarrow \infty.$$

Note that the decay rate of the L^2-error in (4.3) is determined by the smoothness α of the target f.

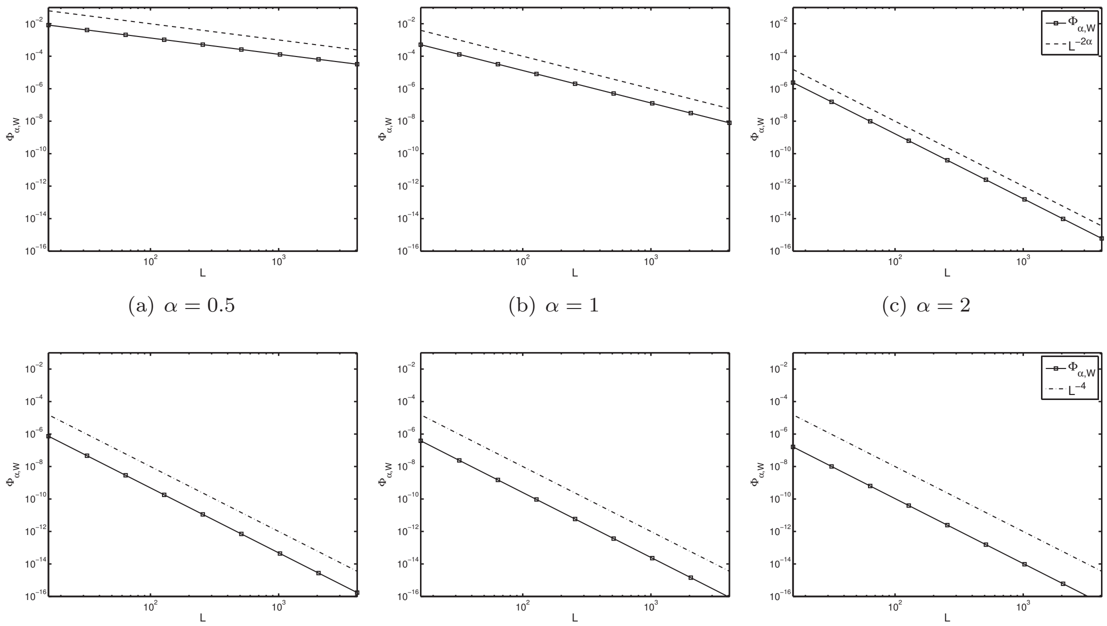
We remark that Assumption 4.4 is satisfied for a large class of window functions. For example, let the window function $W \in \mathcal{C}([-1,1])$ satisfy

$$W(S) = 1 \quad \forall\, S \in [-\varepsilon,\varepsilon]$$

for $\varepsilon > 0$ and

$$\exists\, R \in [0,1] :\; W(R) \neq 1.$$

Then, Assumption 4.4 is fulfilled with $c_{\alpha,W} = \varepsilon$.



(a) $\alpha = 0.5$ (b) $\alpha = 1$ (c) $\alpha = 2$

(d) $\alpha = 2.5$ (e) $\alpha = 3$ (f) $\alpha = 4$

FIGURE 1. Decay rate of $\Phi_{\alpha,W}$ for the Shepp–Logan filter.

Numerical Observations. We investigate the behaviour of $S^*_{\alpha,W,L}$ and $\Phi_{\alpha,W}$ numerically for the following commonly used choices of the filter function $A_L(S) = |S|\, W(S/L)$:

Name	$W(S)$ for $|S| \leq 1$	Parameter
Shepp–Logan	$\operatorname{sinc}(\pi S/2)$	-
Cosine	$\cos(\pi S/2)$	-
Hamming	$\beta + (1-\beta)\cos(\pi S)$	$\beta \in [1/2, 1]$
Gaussian	$\exp\big(-(\pi S/\beta)^2\big)$	$\beta > 1$

Note that each of these window functions W is compactly supported with $\operatorname{supp}(W) = [-1, 1]$.

In our numerical experiments, we calculated $S^*_{\alpha,W,L}$ and $\Phi_{\alpha,W}(L)$ as a function of the bandwidth $L > 0$ for the above mentioned window functions W and for different parameters $\alpha > 0$, reflecting the smoothness of the target function $f \in \mathrm{H}^\alpha(\mathbb{R}^2)$. Figure 1 shows the behaviour of $\Phi_{\alpha,W}$ in log-log scale for the Shepp–Logan filter and for smoothness parameters $\alpha \in \{0.5, 1, 2, 2.5, 3, 4\}$. For $\alpha \in \{0.5, 1, 2\}$ we observe that $\Phi_{\alpha,W}(L)$ behaves exactly as $L^{-2\alpha}$, see Figure 1(a)–(c), whereas for $\alpha \in \{2.5, 3, 4\}$ the behaviour of $\Phi_{\alpha,W}(L)$ corresponds to L^{-4}, see Figure 1(d)–(f). In the latter case, however, $\Phi_{\alpha,W}(L)$ decreases at increasing values $\alpha > 2$. We remark that the same behaviour was observed in our numerical experiments for the other window functions W mentioned above.

We summarize our numerical experiments (for all windows W listed above) as follows.

For $\alpha < 2$, we see that Assumption 4.4, i.e.,

$$\exists\, c_{\alpha,W} > 0 \,\forall\, L > 0: \; S^*_{\alpha,W,L} \geq c_{\alpha,W},$$

is fulfilled, where in particular,

$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-2\alpha}) \quad \text{for} \quad L \longrightarrow \infty.$$

For $\alpha \geq 2$, we have

$$S^*_{\alpha,W,L} \longrightarrow 0 \quad \text{for} \quad L \longrightarrow \infty$$

and the convergence rate of $\Phi_{\alpha,W}$ stagnates at

$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-4}) \quad \text{for} \quad L \longrightarrow \infty.$$

5. Error Analysis for $\mathcal{C}^2$-Windows

Note that all window functions W mentioned above are in $\mathcal{C}^2([-1,1])$. Therefore, in the following analysis we consider even window functions W with compact support in $[-1,1]$ that additionally satisfy $W \in \mathcal{C}^2([-1,1])$ and $W(0) = 1$. As a first result, we obtain the following convergence rate.

Theorem 5.1 (Convergence rate of $\Phi_{\alpha,W}$ for $\mathcal{C}^2$-windows)**.** *Let the window function W satisfy $W \in \mathcal{C}^2([-1,1])$ with $W(0) = 1$. Moreover, let $\alpha > 0$. Then, we have*

$$\Phi_{\alpha,W}(L) \leq \begin{cases} C_\alpha \, \|W''\|^2_{\infty,[-1,1]} \, L^{-4} & \text{for } \alpha > 2 \wedge L \geq \frac{\sqrt{2}}{\sqrt{\alpha-2}} \\ \frac{1}{4} \, \|W''\|^2_{\infty,[-1,1]} \, L^{-2\alpha} & \text{for } \alpha \leq 2 \vee \left(\alpha > 2 \wedge L < \frac{\sqrt{2}}{\sqrt{\alpha-2}}\right) \end{cases} \quad \forall\, L > 0,$$

i.e.,

$$\Phi_{\alpha,W}(L) = \mathcal{O}\Big(L^{-\min\{4,2\alpha\}}\Big) \quad \text{for} \quad L \longrightarrow \infty,$$

where the constant

$$C_\alpha = \frac{(\alpha-2)^{\alpha-2}}{\alpha^\alpha}$$

is strictly monotonically decreasing in $\alpha > 2$.

Proof. Since the window function W is assumed to be continuous on $[-1,1]$, we have

$$\Phi_{\alpha,W}(L) = \max_{S \in [-1,1]} \frac{(1 - W(S))^2}{(1 + L^2 S^2)^\alpha} = \max_{S \in [-1,1]} \Phi_{\alpha,W,L}(S).$$

Let $S \in [-1,1]$ be fixed. By assumption, W satisfies $W \in \mathcal{C}^2([-1,1])$ with $W(0) = 1$. Thus, we can apply Taylor's theorem and obtain

$$W(S) = W(0) + W'(0)\, S + \frac{1}{2}\, W''(\xi)\, S^2 = 1 + \frac{1}{2}\, W''(\xi)\, S^2$$

for some ξ between 0 and S, where we use that the window W is even and, consequently, $W'(0) = 0$. This leads to

$$\Phi_{\alpha,W,L}(S) = \frac{(W''(\xi))^2}{4} \, \frac{S^4}{(1 + L^2\, S^2)^\alpha} \leq \frac{\|W''\|^2_{\infty,[-1,1]}}{4} \, \frac{S^4}{(1 + L^2\, S^2)^\alpha}.$$

Hence,

$$\Phi_{\alpha,W}(L) \leq \frac{\|W''\|^2_{\infty,[-1,1]}}{4} \max_{S \in [-1,1]} \frac{S^4}{(1 + L^2\, S^2)^\alpha} = \frac{\|W''\|^2_{\infty,[-1,1]}}{4} \max_{S \in [-1,1]} \phi_{\alpha,L}(S).$$

We now need to analyse the function

$$\phi_{\alpha,L}(S) = \frac{S^4}{(1 + L^2\,S^2)^{\alpha}} \qquad \text{for } S \in [-1,1],$$

which is independent of the window function W. Since $\phi_{\alpha,L}$ is an even function, we have

$$\max_{S\in[-1,1]} \phi_{\alpha,L}(S) = \max_{S\in[0,1]} \phi_{\alpha,L}(S)$$

and so it suffices to consider $S \in [0,1]$. A necessary condition for a maximum of $\phi_{\alpha,L}$ on $(0,1)$ is

$$\phi'_{\alpha,L}(S) = 0.$$

From the first derivative

$$\phi'_{\alpha,L}(S) = \frac{2\,S^3\left(2 + (2-\alpha)L^2\,S^2\right)}{(1 + L^2\,S^2)^{\alpha+1}}$$

it follows that $\phi'_{\alpha,L}$ can vanish only for $S = 0$ or for $(\alpha - 2)L^2\,S^2 = 2$.

Now since $\phi_{\alpha,L}(0) = 0$ and $\phi_{\alpha,L}(S) > 0$, for all $S > 0$, it follows that $S = 0$ is the unique global minimizer of $\phi_{\alpha,L}$ on $[0,1]$.

Case 1: For $0 \leq \alpha \leq 2$ the equation

$$(\alpha - 2)L^2\,S^2 = 2$$

has no solution in $[0,1]$ and, moreover,

$$\phi'_{\alpha,L}(S) > 0 \quad \forall\, S \in (0,1].$$

This means that $\phi_{\alpha,L}$ is strictly monotonically increasing on $(0,1]$ and, thus, it is maximal on $[0,1]$ for $S^* = 1$, i.e.,

$$\max_{S\in[0,1]} \phi_{\alpha,L}(S) = \phi_{\alpha,L}(1) = \frac{1}{(1 + L^2)^{\alpha}} \leq L^{-2\alpha}.$$

Case 2: For $\alpha > 2$ the unique positive solution of the equation

$$(\alpha - 2)L^2\,S^2 = 2$$

is given by

$$S^* = \frac{\sqrt{2}}{L\sqrt{\alpha - 2}},$$

where

$$S^* \in [0,1] \qquad \Longleftrightarrow \qquad L \geq \frac{\sqrt{2}}{\sqrt{\alpha - 2}}.$$

For convenience, we define the function $g_{\alpha,L} : \mathbb{R} \longrightarrow \mathbb{R}$ via

$$g_{\alpha,L}(S) = 2 + (2 - \alpha)L^2\,S^2.$$

Then, $g_{\alpha,L}$ is a down open parabola with vertex in $S = 0$ and we obtain

$$g_{\alpha,L}(S_1) > g_{\alpha,L}(S_2) \quad \forall\, 0 \leq S_1 < S_2.$$

In particular, we have

$$g_{\alpha,L}(S_2) < g_{\alpha,L}(S^*) = 0 < g_{\alpha,L}(S_1) \quad \forall\, 0 < S_1 < S^* < S_2$$

and, consequently,

$$\phi'_{\alpha,L}(S_2) < \phi'_{\alpha,L}(S^*) = 0 < \phi'_{\alpha,L}(S_1) \quad \forall\, 0 < S_1 < S^* < S_2.$$

Thus, $\phi_{\alpha,L}$ is strictly monotonically increasing on $(0, S^*)$ and strictly monotonically decreasing on (S^*, ∞). Therefore, S^* is the unique maximizer of $\phi_{\alpha,L}$ and it follows that

$$\operatorname*{arg\,max}_{S \in [0,1]} \phi_{\alpha,L}(S) = \begin{cases} 1 & \text{for } L < \frac{\sqrt{2}}{\sqrt{\alpha-2}} \\ \frac{\sqrt{2}}{L\sqrt{\alpha-2}} & \text{for } L \geq \frac{\sqrt{2}}{\sqrt{\alpha-2}}. \end{cases}$$

Since

$$\phi_{\alpha,L}(S^*) = \frac{\left(\frac{\sqrt{2}}{L\sqrt{\alpha-2}}\right)^4}{\left(1 + L^2 \left(\frac{\sqrt{2}}{L\sqrt{\alpha-2}}\right)^2\right)^\alpha} = 4\, \frac{(\alpha-2)^{2-\alpha}}{\alpha^\alpha}\, L^{-4}$$

we finally obtain (for $\alpha > 2$)

$$\max_{S \in [0,1]} \phi_{\alpha,L}(S) = \begin{cases} \phi_{\alpha,L}(1) & \text{for } L < \frac{\sqrt{2}}{\sqrt{\alpha-2}} \\ \phi_{\alpha,L}(S^*) & \text{for } L \geq \frac{\sqrt{2}}{\sqrt{\alpha-2}} \end{cases} \leq \begin{cases} L^{-2\alpha} & \text{for } L < \frac{\sqrt{2}}{\sqrt{\alpha-2}} \\ 4\, \frac{(\alpha-2)^{2-\alpha}}{\alpha^\alpha}\, L^{-4} & \text{for } L \geq \frac{\sqrt{2}}{\sqrt{\alpha-2}}. \end{cases}$$

Combining our results yields

$$\begin{aligned} \Phi_{\alpha,W}(L) &\leq \frac{1}{4}\, \|W''\|^2_{\infty,[-1,1]} \max_{S \in [0,1]} \phi_{\alpha,L}(S) \\ &\leq \frac{1}{4}\, \|W''\|^2_{\infty,[-1,1]} \begin{cases} 4\, \frac{(\alpha-2)^{2-\alpha}}{\alpha^\alpha}\, L^{-4} & \text{for } \alpha > 2 \wedge L \geq \frac{\sqrt{2}}{\sqrt{\alpha-2}} \\ L^{-2\alpha} & \text{for } \alpha \leq 2 \vee \left(\alpha > 2 \wedge L < \frac{\sqrt{2}}{\sqrt{\alpha-2}}\right) \end{cases} \\ &= \begin{cases} \frac{(\alpha-2)^{2-\alpha}}{\alpha^\alpha}\, \|W''\|^2_{\infty,[-1,1]}\, L^{-4} & \text{for } \alpha > 2 \wedge L \geq \frac{\sqrt{2}}{\sqrt{\alpha-2}} \\ \frac{1}{4}\, \|W''\|^2_{\infty,[-1,1]}\, L^{-2\alpha} & \text{for } \alpha \leq 2 \vee \left(\alpha > 2 \wedge L < \frac{\sqrt{2}}{\sqrt{\alpha-2}}\right), \end{cases} \end{aligned}$$

as stated.

Let us finally regard the constant

$$C_\alpha = C(\alpha) = \frac{(\alpha-2)^{\alpha-2}}{\alpha^\alpha}$$

as a function of $\alpha > 2$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\, C(\alpha) = \frac{(\alpha-2)^{\alpha-2}}{\alpha^\alpha}\, \log\!\left(1 - \frac{2}{\alpha}\right) < 0 \quad \forall\, \alpha > 2$$

and, consequently, C_α is strictly monotonically decreasing in $\alpha > 2$. □

We remark that the results of Theorem 5.1 comply with our numerical observations from the previous section. We have in particular observed saturation of the convergence rate of $\Phi_{\alpha,W}$ for $\alpha > 2$ at

$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-4}) \quad \text{for} \quad L \longrightarrow \infty$$

through our numerical experiments. Therefore, our numerical results show that the proven order of convergence for $\Phi_{\alpha,W}$ is optimal for $\mathcal{C}^2$-windows.

By combining Theorems 4.1 and 5.1, we finally get the following result for the convergence order of FBP reconstruction with $\mathcal{C}^2$-windows.

Corollary 5.2 (L^2-error estimate for $\mathcal{C}^2$-windows)**.** *Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, for some $\alpha > 0$, $K_L \in \mathrm{L}^1(\mathbb{R}^2)$, and $W \in \mathcal{C}^2([-1,1])$ with $W(0) = 1$. Then, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ is bounded above by*

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \begin{cases} \left(\frac{c_{\alpha,2}}{2} \|W''\|_{\infty,[-1,1]} L^{-2} + L^{-\alpha}\right)\|f\|_\alpha & \text{for } \alpha > 2 \wedge L \geq L^* \\ \left(\frac{1}{2} \|W''\|_{\infty,[-1,1]} L^{-\alpha} + L^{-\alpha}\right)\|f\|_\alpha & \text{for } \alpha \leq 2 \vee (\alpha > 2 \wedge L < L^*) \end{cases}$$

with the critical bandwidth $L^* = \frac{\sqrt{2}}{\sqrt{\alpha-2}}$, for $\alpha > 2$. Moreover, the constant

$$c_{\alpha,2} = \frac{2}{\alpha-2}\left(\frac{\alpha-2}{\alpha}\right)^{\alpha/2}$$

is strictly monotonically decreasing in $\alpha > 2$. In particular,

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(c \|W''\|_{\infty,[-1,1]} L^{-\min\{2,\alpha\}} + L^{-\alpha}\right)\|f\|_\alpha = \mathcal{O}\left(L^{-\min\{2,\alpha\}}\right).$$

We close this section by the following two remarks.

Firstly, note that the bound on the inherent FBP reconstruction error in Corollary 5.2 is affine-linear with respect to $\|W''\|_{\infty,[-1,1]}$. Therefore, the quantity in the upper bound can be used to evaluate the approximation quality of the chosen $\mathcal{C}^2$-window function W.

Secondly, for $\alpha \leq 2$ the convergence order of the approximate reconstruction f_L is given by the smoothness of the target function f. But for $\alpha > 2$ the convergence rate of the error bound saturates at $\mathcal{O}(L^{-2})$. Nevertheless, the FBP reconstruction error continues to decrease at increasing $\alpha > 2$, since the involved constant $c_{\alpha,2}$ is strictly monotonically decreasing in $\alpha > 2$. This matches our perceptions, as the approximation error should be smaller for target functions of higher regularity.

6. Error Analysis for $\mathcal{C}^k$-Windows

In this section, we generalize our results from the previous section to $\mathcal{C}^k$-windows whose first $k-1$ derivatives vanish at the origin. Therefore, we now consider even window functions W with compact support in $[-1,1]$ that additionally satisfy $W \in \mathcal{C}^k([-1,1])$ for some $k \geq 2$ and

$$W(0) = 1 \quad \text{and} \quad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

According to Theorem 4.2, $\Phi_{\alpha,W}(L)$ tends to zero for $L \to \infty$. In Theorem 5.1 we obtained convergence rates for $\Phi_{\alpha,W}$ with $\mathcal{C}^2$-windows W. We can prove convergence rates for $\mathcal{C}^k$-windows by following along the lines of the presented proofs for $k = 2$, see Theorem 5.1 and Corollary 5.2. We formulate our results for $k \geq 2$ as follows.

Theorem 6.1 (Convergence rate of $\Phi_{\alpha,W}$ for $\mathcal{C}^k$-windows)**.** *Let the window function W satisfy $W \in \mathcal{C}^k([-1,1])$, for $k \geq 2$, with*

$$W(0) = 1 \quad \text{and} \quad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

Moreover, let $\alpha > 0$. Then, $\Phi_{\alpha,W}(L)$ can be bounded above by

$$\Phi_{\alpha,W}(L) \leq \begin{cases} \frac{c_{\alpha,k}^2}{(k!)^2} \|W^{(k)}\|_{\infty,[-1,1]}^2 L^{-2k} & \text{for } \alpha > k \wedge L \geq L^* \\ \frac{1}{(k!)^2} \|W^{(k)}\|_{\infty,[-1,1]}^2 L^{-2\alpha} & \text{for } \alpha \leq k \vee (\alpha > k \wedge L < L^*) \end{cases}$$

with the critical bandwidth $L^* = \frac{\sqrt{k}}{\sqrt{\alpha-k}}$, for $\alpha > k$, and the strictly increasing constant

$$c_{\alpha,k} = \Big(\frac{k}{\alpha - k}\Big)^{k/2} \Big(\frac{\alpha - k}{\alpha}\Big)^{\alpha/2} \qquad \text{for } \alpha > k.$$

In particular,

$$\Phi_{\alpha,W}(L) = \mathcal{O}\Big(L^{-2\min\{k,\alpha\}}\Big) \qquad \text{for} \quad L \longrightarrow \infty.$$

Combining Theorems 4.1 and 6.1, we obtain the following result concerning the convergence order of the FBP reconstruction with $\mathcal{C}^k$-windows.

Corollary 6.2 (L^2-error estimate for $\mathcal{C}^k$-windows)**.** *Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, for $\alpha > 0$, and $K_L \in \mathrm{L}^1(\mathbb{R}^2)$. Moreover, let $W \in \mathcal{C}^k([-1,1])$, for $k \geq 2$, with*

$$W(0) = 1 \quad \text{and} \quad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

Then, the L^2-norm of the inherent FBP reconstruction error $e_L = f - f_L$ is bounded above by

$$\|e_L\|_{\mathrm{L}^2} \leq \begin{cases} \Big(\frac{c_{\alpha,k}}{k!}\, \|W^{(k)}\|_{\infty,[-1,1]}\, L^{-k} + L^{-\alpha}\Big) \|f\|_\alpha & \text{for } \alpha > k \wedge L \geq L^* \\ \Big(\frac{1}{k!}\, \|W^{(k)}\|_{\infty,[-1,1]}\, L^{-\alpha} + L^{-\alpha}\Big) \|f\|_\alpha & \text{for } \alpha \leq k \vee (\alpha > k \wedge L < L^*)\,. \end{cases}$$

In particular,

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \Big(c\, \|W^{(k)}\|_{\infty,[-1,1]}\, L^{-\min\{k,\alpha\}} + L^{-\alpha}\Big)\, \|f\|_\alpha = \mathcal{O}\Big(L^{-\min\{k,\alpha\}}\Big)\,.$$

Note that our concluding remarks after Corollary 5.2 concerning the approximation order of the FBP reconstruction f_L continue to apply in the situation of $\mathcal{C}^k$-windows W. Indeed, the convergence order in Corollary 6.2, for $\alpha \leq k$, is determined by the smoothness of the target function f, whereas for $\alpha > k$ the convergence rate saturates at $\mathcal{O}(L^{-k})$. But in this case the error bound decreases at increasing α, since the involved constant $c_{\alpha,k}$ is strictly monotonically decreasing in $\alpha > k$. Thus, a smoother target function allows for a better approximation, as expected. Nevertheless, the attainable convergence rate is limited by the differentiability order k of the filter's $\mathcal{C}^k$-window W.

Finally, note that the bound on the inherent FBP reconstruction error in Corollary 6.2 is affine-linear with respect to $\|W^{(k)}\|_{\infty,[-1,1]}$ and this quantity can be used to evaluate the approximation quality of the chosen $\mathcal{C}^k$-window function W.

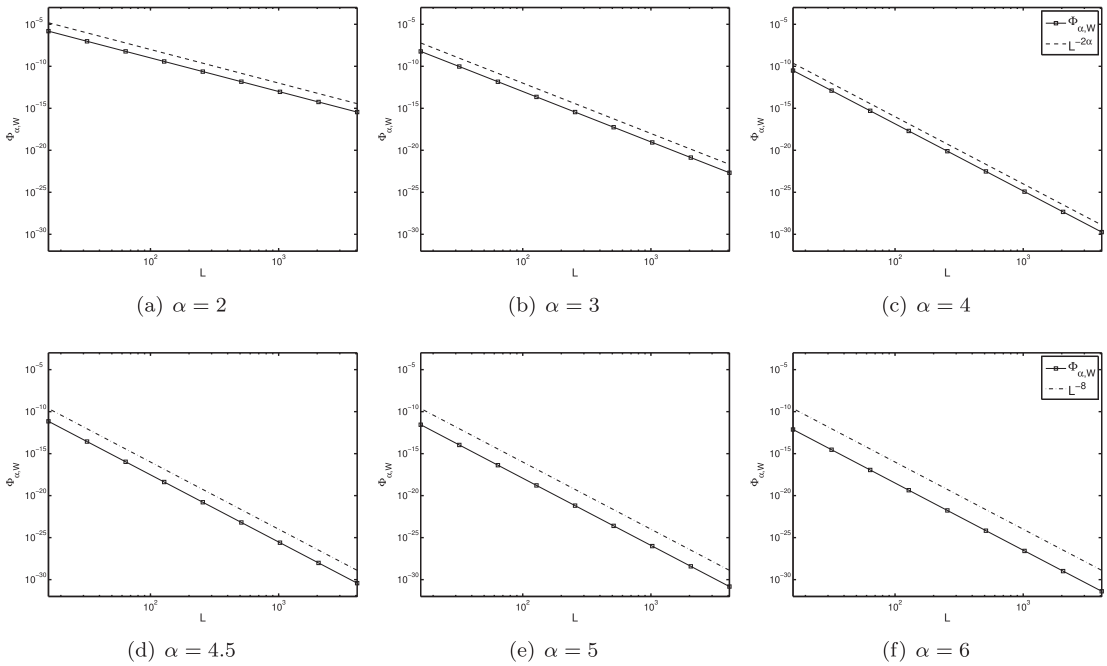
Numerical Experiments. We investigate the behaviour of $\Phi_{\alpha,W}$ numerically for the generalized Gaussian filter $A_L(S) = |S|\, W(S/L)$ with the window function

$$W(S) = \exp\left(-\left(\frac{\pi S}{\beta}\right)^k\right) \qquad \text{for } S \in [-1,1]$$

for $k \in \mathbb{N}_{\geq 2}$ and $\beta > 1$. In this case, $W \in \mathcal{C}^k([-1,1])$ is even and compactly supported in $[-1,1]$. Moreover,

$$W(0) = 1 \quad \text{and} \quad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1 \quad \text{and} \quad W^{(k)}(0) = -k!\left(\frac{\pi}{\beta}\right)^k \neq 0.$$

In our numerical experiments, we evaluated $\Phi_{\alpha,W}(L)$ as a function of the bandwidth $L > 0$ for the Gaussian's window W, using various combinations of parameters $k \in \mathbb{N}_{\geq 2}$, $\beta > 1$, and $\alpha > 0$. Figure 2 shows the behaviour of $\Phi_{\alpha,W}$ in log-log scale for the generalized Gaussian filter with $k = 4$ and $\beta = 4$, for the smoothness parameters $\alpha \in \{2, 3, 4, 4.5, 5, 6\}$. For $\alpha \in \{2, 3, 4\}$ we observe that $\Phi_{\alpha,W}(L)$ behaves as $L^{-2\alpha}$, see Figure 2(a)–(c), whereas for $\alpha \in \{4.5, 5, 6\}$ the behaviour of $\Phi_{\alpha,W}(L)$ corresponds to L^{-8}, see Figure 2(d)–(f). But $\Phi_{\alpha,W}(L)$ continues to decrease at increasing $\alpha > k$.



(a) $\alpha = 2$ (b) $\alpha = 3$ (c) $\alpha = 4$

(d) $\alpha = 4.5$ (e) $\alpha = 5$ (f) $\alpha = 6$

FIGURE 2. Decay rate of $\Phi_{\alpha,W}$ for the generalized Gaussian filter with $k = 4$, $\beta = 4$.

We can summarize the results of our numerical experiments as follows. For $\alpha < k$, we observe

$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-2\alpha}) \quad \text{for} \quad L \longrightarrow \infty.$$

For $\alpha \geq k$, the convergence rate of $\Phi_{\alpha,W}$ saturates at

$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-2k}) \quad \text{for} \quad L \longrightarrow \infty.$$

Note that the results of Theorem 6.1 entirely comply with our numerical observations (for the generalized Gaussian filters). So have we, in particular, observed the saturation of the convergence rate of $\Phi_{\alpha,W}$ for $\alpha > k$ at

$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-2k}) \quad \text{for} \quad L \longrightarrow \infty.$$

Our numerical results show that the proven convergence order of $\Phi_{\alpha,W}$ is optimal for $\mathcal{C}^k$-windows.

Asymptotic Error Estimates. In this subsection, we take a different approach to prove asymptotic error estimates for the proposed FBP reconstruction method with window functions which are k-times differentiable only at the origin. To this end, we now consider an even window function $W \in \mathrm{L}^\infty(\mathbb{R})$, with compact support on $[-1, 1]$. Moreover, W is required to have k derivatives at zero, for some $k \geq 2$, with

$$W(0) = 1 \quad \text{and} \quad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

As in the previous sections, we consider target functions $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, for some $\alpha > 0$, and assume $K_L \in \mathrm{L}^1(\mathbb{R}^2)$. For the sake of brevity, we again set $r(x, y) = \sqrt{x^2 + y^2}$ for $(x, y) \in \mathbb{R}^2$.

Recall the representation of the FBP reconstruction error $e_L = f - f_L$ with respect to the L^2-norm in (3.2), by the sum of two integrals, I_1 in (3.3) and I_2 in (3.4), where integral I_2 can be bounded above by (4.1).

As regards integral I_1, we have

$$\begin{aligned} I_1 &= \frac{1}{2\pi} \int_{r(x,y) \leq L} |1 - W_L(r(x,y))|^2 \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y) \\ &= \frac{1}{2\pi} \int_{r(x,y) \leq L} \left|1 - W\left(\frac{r(x,y)}{L}\right)\right|^2 \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y). \end{aligned}$$

Because $W : \mathbb{R} \longrightarrow \mathbb{R}$ is k-times differentiable at zero, we can apply Taylor's theorem and, thus, there exists a function $h_k : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying

$$W(S) = \sum_{j=0}^{k} \frac{W^{(j)}(0)}{j!} \, S^j + h_k(S) \, S^k \quad \forall \, S \in \mathbb{R}$$

and

$$\lim_{S \to 0} h_k(S) = 0.$$

By assumption, W satisfies

$$W(0) = 1 \quad \text{ and } \quad W^{(j)}(0) = 0 \quad \forall \, 1 \leq j \leq k-1.$$

Hence, for $(x,y) \in \mathbb{R}^2$ and $L > 0$ follows that

$$1 - W\left(\frac{r(x,y)}{L}\right) = -\left(\frac{W^{(k)}(0)}{k!} \, \left(\frac{r(x,y)}{L}\right)^k + h_k\left(\frac{r(x,y)}{L}\right) \, \left(\frac{r(x,y)}{L}\right)^k\right),$$

so that we obtain the representation

$$I_1 = \frac{1}{2\pi} \int_{r(x,y) \leq L} \left(\frac{W^{(k)}(0)}{k!} + h_k\left(\frac{r(x,y)}{L}\right)\right)^2 \, \left(\frac{r(x,y)}{L}\right)^{2k} \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y).$$

For convenience, we define

$$\phi^*_{\alpha,L,k} = \max_{r(x,y) \leq L} \frac{\left(\frac{r(x,y)}{L}\right)^{2k}}{(1 + r(x,y)^2)^\alpha} = \max_{S \in [0,1]} \frac{S^{2k}}{(1 + L^2 \, S^2)^\alpha}.$$

Then, I_1 can be bounded above by

$$I_1 \leq \phi^*_{\alpha,L,k} \, \frac{1}{2\pi} \int_{r(x,y) \leq L} \left(\frac{W^{(k)}(0)}{k!} + h_k\left(\frac{r(x,y)}{L}\right)\right)^2 \, \left(1 + r(x,y)^2\right)^\alpha \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y).$$

We now regard the integral

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left(h_k\left(\frac{r(x,y)}{L}\right)\right)^2 \, \left(1 + r(x,y)^2\right)^\alpha \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y.$$

For $S \neq 0$, the function h_k can be written as

$$h_k(S) = (W(S) - 1) \, S^{-2k} - \frac{W^{(k)}(0)}{k!}.$$

Since the window W is even and has compact support in $[-1,1]$, h_k is also even and satisfies

$$h_k(S) = -S^{-2k} - \frac{W^{(k)}(0)}{k!} \quad \forall\, |S| > 1,$$

which implies

$$h_k(S) \longrightarrow -\frac{W^{(k)}(0)}{k!} \quad \text{for} \quad S \longrightarrow \pm\infty.$$

From $W \in \mathrm{L}^\infty(\mathbb{R})$ and

$$h_k(S) \longrightarrow 0 \quad \text{for} \quad S \longrightarrow 0$$

it follows that h_k is bounded on $\mathbb{R}$, so that there exists some constant $M > 0$ satisfying

$$\left| h_k\left(\frac{r(x,y)}{L}\right) \right|^2 \leq M \quad \forall\, (x,y) \in \mathbb{R}^2,\, L > 0.$$

Hence, for all $L > 0$, the integrand

$$h_{k,L}(x,y) = \left(h_k\left(\frac{r(x,y)}{L}\right) \right)^2 \left(1 + r(x,y)^2\right)^\alpha |\mathcal{F}f(x,y)|^2$$

is bounded on $\mathbb{R}^2$ by the function

$$\Phi(x,y) = M \, \left(1 + r(x,y)^2\right)^\alpha |\mathcal{F}f(x,y)|^2,$$

which is integrable over $\mathbb{R}^2$ due to the assumption $f \in \mathrm{H}^\alpha(\mathbb{R}^2)$. Moreover, we have

$$h_k\left(\frac{r(x,y)}{L}\right) \longrightarrow 0 \quad \text{for} \quad \frac{r(x,y)}{L} \longrightarrow 0,$$

which implies that, for any $(x,y) \in \mathbb{R}^2$, $h_{k,L}(x,y)$ tends to zero as L goes to ∞. Thus, we can apply Lebesgue's theorem on dominated convergence to get

$$\lim_{L \to \infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(h_k\left(\frac{r(x,y)}{L}\right) \right)^2 \left(1 + r(x,y)^2\right)^\alpha |\mathcal{F}f(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y = 0,$$

i.e.,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left(h_k\left(\frac{r(x,y)}{L}\right) \right)^2 \left(1 + r(x,y)^2\right)^\alpha |\mathcal{F}f(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y = o(1) \quad \text{for} \quad L \longrightarrow \infty.$$

This leads us to the estimate

$$\begin{aligned}
I_1 &\leq \phi^*_{\alpha,L,k} \, \frac{1}{2\pi} \int_{r(x,y) \leq L} \underbrace{\left(\frac{W^{(k)}(0)}{k!} + h_k\left(\frac{r(x,y)}{L}\right) \right)^2}_{\leq 2 \left(\frac{W^{(k)}(0)}{k!}\right)^2 + 2 \left(h_k\left(\frac{r(x,y)}{L}\right)\right)^2} \left(1 + r(x,y)^2\right)^\alpha |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y) \\
&\leq 2\, \phi^*_{\alpha,L,k} \, \frac{1}{2\pi} \int_{r(x,y) \leq L} \left(\frac{W^{(k)}(0)}{k!} \right)^2 \left(1 + r(x,y)^2\right)^\alpha |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y) \\
&\quad + \phi^*_{\alpha,L,k} \, \frac{1}{\pi} \int_{r(x,y) \leq L} \left(h_k\left(\frac{r(x,y)}{L}\right) \right)^2 \left(1 + r(x,y)^2\right)^\alpha |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y) \\
&\leq 2\, \phi^*_{\alpha,L,k} \left(\frac{W^{(k)}(0)}{k!} \right)^2 \|f\|_\alpha^2 + \phi^*_{\alpha,L,k} \, o(1).
\end{aligned}$$

Using the same technique as in the proof of Theorem 5.1, we can bound $\phi^*_{\alpha,L,k}$ by

$$\phi^*_{\alpha,L,k} \leq \begin{cases} \big(\frac{k}{\alpha-k}\big)^k \big(\frac{\alpha-k}{\alpha}\big)^\alpha L^{-2k} & \text{for } \alpha > k \wedge L \geq L^* \\ L^{-2\alpha} & \text{for } \alpha \leq k \vee (\alpha > k \wedge L < L^*) \end{cases} = \mathcal{O}\Big(L^{-2\min\{k,\alpha\}}\Big)$$

with the critical bandwidth $L^* = \frac{\sqrt{k}}{\sqrt{\alpha-k}}$ for $\alpha > k$. Thus, it follows that

$$I_1 \leq \begin{cases} \frac{2}{(k!)^2}\, c^2_{\alpha,k}\, |W^{(k)}(0)|^2\, L^{-2k}\, \|f\|^2_\alpha + o\big(L^{-2k}\big) & \text{for } \alpha > k \wedge L \geq L^* \\ \frac{2}{(k!)^2}\, |W^{(k)}(0)|^2\, L^{-2\alpha}\, \|f\|^2_\alpha + o\big(L^{-2\alpha}\big) & \text{for } \alpha \leq k \vee (\alpha > k \wedge L < L^*)\,, \end{cases}$$

where the constant

$$c_{\alpha,k} = \Big(\frac{k}{\alpha-k}\Big)^{k/2} \Big(\frac{\alpha-k}{\alpha}\Big)^{\alpha/2} \qquad \text{for } \alpha > k$$

is strictly monotonically decreasing in $\alpha > k$ (cf. Theorem 6.1).

By combining our derived bounds for the integrals I_1 and I_2, we finally get the L^2-error estimate

$$\|e_L\|^2_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(2\left(C_{\alpha,k}\,|W^{(k)}(0)|\right)^2 L^{-2\min\{k,\alpha\}} + L^{-2\alpha}\right) \|f\|^2_\alpha + o\Big(L^{-2\min\{k,\alpha\}}\Big).$$

In conclusion, we have proven the following error theorem for the FBP reconstruction method.

Theorem 6.3 (Asymptotic L^2-error estimate)**.** *Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, for some $\alpha > 0$, and $K_L \in \mathrm{L}^1(\mathbb{R}^2)$. Moreover, let $W \in \mathrm{L}^\infty(\mathbb{R})$ be even, with* $\mathrm{supp}(W) \subseteq [-1,1]$*, and k-times differentiable at the origin, $k \geq 2$, with*

$$W(0) = 1 \qquad \textit{and} \qquad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

Then, for $\alpha \leq k$, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ is bounded above by

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(\frac{\sqrt{2}}{k!}\,|W^{(k)}(0)|\, L^{-\alpha} + L^{-\alpha}\right) \|f\|_\alpha + o(L^{-\alpha}). \tag{6.1}$$

If $\alpha > k$, the L^2-norm of e_L can be bounded above by

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \begin{cases} \Big(\frac{\sqrt{2}}{k!}\, c_{\alpha,k} |W^{(k)}(0)|\, L^{-k} + L^{-\alpha}\Big) \|f\|_\alpha + o(L^{-k}) & \textit{for } L \geq L^* \\ \Big(\frac{\sqrt{2}}{k!}\, |W^{(k)}(0)|\, L^{-\alpha} + L^{-\alpha}\Big) \|f\|_\alpha + o(L^{-\alpha}) & \textit{for } L < L^* \end{cases} \tag{6.2}$$

with the critical bandwidth $L^* = \frac{\sqrt{k}}{\sqrt{\alpha-k}}$ and the strictly monotonically decreasing constant

$$c_{\alpha,k} = \Big(\frac{k}{\alpha-k}\Big)^{k/2} \Big(\frac{\alpha-k}{\alpha}\Big)^{\alpha/2} \qquad \textit{for } \alpha > k.$$

In particular,

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \Big(c\,|W^{(k)}(0)|\, L^{-\min\{k,\alpha\}} + L^{-\alpha}\Big) \|f\|_\alpha + o\Big(L^{-\min\{k,\alpha\}}\Big).$$

We wish to draw the following conclusions from Theorem 6.3.

Firstly, the *flatness* of the filter's window function W determines the convergence rate of the error bounds (6.1), (6.2) for the inherent FBP reconstruction error. Indeed, if W is k-times differentiable at the origin such that the first $k-1$ derivatives of W vanish at zero, then the convergence rate in (6.1) is given by the smoothness α of the target function f as long as $\alpha \leq k$. But for $\alpha > k$ the order of convergence in (6.2) saturates at $\mathcal{O}(L^{-k})$.

Secondly, the quantity $|W^{(k)}(0)|$, i.e., the k-th derivative of W at the origin, dominates the error bound in both (6.1) and (6.2). Therefore, the value $|W^{(k)}(0)|$ can be used as an indicator to predict the approximation quality of the proposed FBP reconstruction method.

To conclude our discussion, we finally consider the following special case. Let the window function W fulfil the assumptions of Theorem 6.3 with $k \geq 2$ and let the smoothness α of $f \in \mathrm{H}^\alpha(\mathbb{R}^2)$ satisfy

$$\alpha > k.$$

Then, the asymptotic L^2-error estimate of the FBP method reduces to

$$\|f - f_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \sqrt{2}\, c_{\alpha,k}\, |W^{(k)}(0)|\, L^{-k}\, \|f\|_\alpha + o(L^{-k}).$$

Consequently, the intrinsic FBP reconstruction error is proportional to $|W^{(k)}(0)|$, if we neglect the higher order terms. For $k = 2$, this observation complies with the results of Munshi [5] and Munshi et al. [5, 6], where they assumed certain moment conditions on the convolution kernel K and differentiability of the target function f.

7. Conclusion

We have analysed the inherent FBP reconstruction error which is incurred by the use of a low-pass filter with a compactly supported window W and finite bandwidth L. We refined our L^2-error estimate from [1] to prove, under reasonable assumptions, convergence of the FBP reconstruction f_L to the target function f as the bandwidth L goes to infinity. Moreover, we developed asymptotic convergence rates in terms of the bandwidth L and the smoothness α of the target function f.

By deriving an asymptotic error estimate, we observed that the flatness of the filter's window function is of fundamental importance. Indeed, if the window W is k-times differentiable at the origin, such that the first $k-1$ derivatives vanish at zero, then the convergence rate of the obtained error bound saturates at $\mathcal{O}(L^{-k})$, and the quantity $|W^{(k)}(0)|$ determines the approximation quality of the chosen low-pass filter.

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Department of Mathematics, University of Hamburg, Bundesstr. 55, D-20146 Hamburg, Germany
E-mail address: {matthias.beckmann,armin.iske}@uni-hamburg.de