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***Abstract*—We propose a wetting and drying SPH algorithm for the numerical solution of the shallow water equations, following the semi-implicit scheme of Casulli [10] and that of our previous work [3]. To this end, we derive a mildly nonlinear system for the discrete free surface elevation from the shallow water equations by taking into consideration a correct mass balance in wet regions and in transition regions, i.e., the regions from wet particles to dry particles and those from dry particles to wet particles. Unlike in other approaches, our algorithm does not place screens or threshold values at some points to deal with the treatment of wetting and drying. The proposed method of this paper is simple, efficient, conserves mass, guarantees the production of non-negative free surface water depths, and it permits large time step sizes. The algorithm is validated on an inviscid hydrostatic free surface flow for the shallow water equations.**

I. INTRODUCTION

This paper proposes a new *wetting and drying* semi-implicit SPH algorithm applied to the shallow water equations. We consider the inviscid hydrostatic free surface flows. Such flows are governed by the *shallow water equations* which we can derive by vertically or laterally averaging the fully three dimensional incompressible Navier-Stokes equations with the assumption of a hydrostatic pressure distribution (see [11], [12]).

Wetting and drying is a common phenomena in shallow water flows where water level rises, called *wetting*, and where water level recedes, called *drying*. This process can occur during events, such as inundation of coastal regions that are often due to storm surges and wave driven run-up on beaches, even more in biological processes i.e., during the drying phase on a tidal mud flat algal mats [17]. These processes occur on periodic time intervals. Since the shallow water equations are well defined in a fully wetted region in the domain, when water height recedes and goes to zero, this affects the numerical solution of the equations, where the arising problems may become ill-posed. Viable approaches to tackle such problems are essentially incorporating wetting and drying into the numerical scheme or a dynamic adaptivity in the computational domain as the water level moves. Pioneering work in wetting and drying on two-dimensional shallow water equations is due to Leendertse [20], whose approach makes use of an alternating direction implicit ADI method to discretize the governing equations. There is a considerable amount of work relying on finite volume and finite element schemes to treat wetting and drying, e.g. [1], [6], [9], [15], [18], [19], [23], [29], to mention but a few. All these techniques make use of mesh adaptation (by deforming domains and meshs) and mesh reduction. The latter is by putting 'screens' at velocity points of the flow configuration when the water height drops below a certain *drying threshold* and removing the screens when the water height rises above a *wetting threshold*. This approach is problem-dependent and the threshold parameters must be tuned, where the thin water layer technique uses a fixed mesh to maintain a thin layer of water in nominally dry elements. Vater, Beisiegel and Behrens [27] propose a limiter-based approach in the velocity and water height to prevent instabilities.

In explicit numerical methods, the major problem is their severe time step restriction, where the Courant-Friedrichs-Lewy (CFL) condition imposes the time step size in terms of the wave propagation speed and the mesh size. Hence, the major advantage of a semi-implicit approach is that stable schemes are obtained which allow large time step sizes at a reasonable computational cost. In a staggered mesh approach for finite differences and volumes, discrete variables are often defined at different (staggered) locations. The pressure term, which is the free surface elevation, is defined in the cell center, while the velocity components are defined at the cell interfaces. In the momentum equation, pressure terms that are due to the gradients in the free surface elevations and the velocity in the mass conservation are both discretized implicitly, whereas the nonlinear convective terms are discretized explicitly.

The treatment of wetting and drying in shallow water equations using a truly meshfree numerical method is a new approach. In fact, to the best of our knowledge, only [25], [26] solve the Thacker's test case [24] and flooding problem with a shallow water SPH model using a dynamic particle coalescing and splitting method.

This paper proposes a new wetting and drying semi-implicit *Smoothed Particle Hydrodynamics* (SPH) algorithm for the numerical solution of the shallow water equations, following the semi-implicit SPH scheme in [3]. The wetting and drying relies on the work of Casulli [10] for unstructured meshes, where the resulting numerical algorithm can directly be developed from the governing equations. In this way, a correct mass balance is assured in wet particle regions and in transition regions, i.e., particle regions, from wet to dry and from dry to wet regions, with maintaining a nonnegative water height. The approach taken boils down to solving a *mildly nonlinear* system. When wetting and drying is occurs, more iterations are need for the solution of the mildly nonlinear system.

The remainder of this paper is organized as follows: In Section II, the numerical model for the shallow water equations and the basic concept for the particle approximations are presented. In Section III, the key ideas of the wetting and drying algorithm are presented together with smoothed particle hydrodynamics (SPH) approximations. Finally, a model problem concerning an oscillating lake resulting in a parabolic basin is used in our numerical examples in Section IV to validate the proposed semi-implicit SPH algorithm. Concluding remarks, along with an outlook to future research are provided in Section V.

II. Problem Statement and Modeling

This section briefly introduces the utilized models and particle approximations. Vectors are defined by reference to Cartesian coordinates. Latin subscripts are used to identify particle locations, where subscript i refers to the focal particle and subscript j denotes the neighbour of particle i.

A. The Kernel Function

We use a mollifying function W, a positive decreasing radially symmetric function with compact support, of the generic form

$$W(r,h) = \frac{1}{h^d} W\left(\frac{\|r\|}{h}\right) \quad \text{for} \quad r \in [0,\infty) \quad \text{and} \quad h > 0.$$

In our numerical examples, we work with the B-spline kernel of degree 3 [21], given as

$$W(r,h) = K \times \begin{cases} 1 - \frac{3}{2}\left(\frac{r}{h}\right)^2 + \frac{3}{4}\left(\frac{r}{h}\right)^3 & \text{for} \quad 0 \le \frac{r}{h} \le 1 \\ \frac{1}{4}\left(2 - \frac{r}{h}\right)^3 & \text{for} \quad 1 \le \frac{r}{h} \le 2 \\ 0 & \text{for} \quad \frac{r}{h} > 2 \end{cases}$$

where the normalisation coefficient K takes the value $2/3$ (for dimension $d = 1$), $10/(7\pi)$ (for $d = 2$), or $1/\pi$ (for $d = 3$) and $W(r,h) = W_{ij}$ henceforth. For the mollifyer $W \in W^{3,\infty}(\mathbb{R}^d)$, $h > 0$ is referred to as the *smoothing length*, being related to the particle spacing Δ_P by $h = 2\Delta_P$. The smoothing length h can vary locally according to

$$h_{ij} = \frac{1}{2}[h_i + h_j] \qquad \text{where } h_i = \sigma \sqrt[d]{\frac{m_j}{\rho_j}}. \tag{1}$$

In this study, we use the smoothing length in (1). Moreover, σ is in $[1.5, 2.0]$, which ensures approximately a constant number of particle neighbours of between $40 - 50$ in the compact

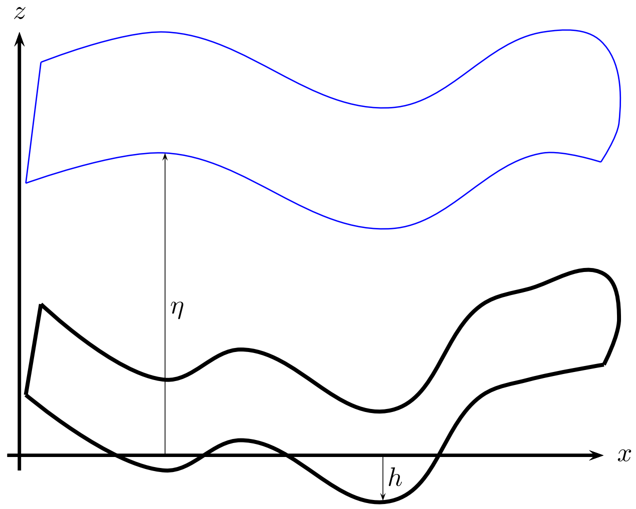

Fig. 1. Sketch of the free surface (light blue) and the bottom bathymetry (thick black)

support of each kernel. A popular approach for the kernel's normalisation is by Shepard interpolation [22], where

$$W'_{ij} = \frac{W_{ij}}{\sum_{j=1}^{N} \frac{m_j}{\rho_j} W_{ij}}.$$

Normalisation is of particular importance for particles close to free surfaces, since this will reduce numerical instabilities and other undesired effects near the boundary.

The gradient of the kernel function is corrected by using the formulation proposed by Belytschko et al [4]. For the sake of notational convenience, we will from now refer to the kernel function W'_{ij} as W_{ij} and to its gradient $\nabla W'_{ij}$ as ∇W_{ij}.

B. Governing Equations

The governing equations considered in this work are nonlinear hyperbolic conservation laws of the form

$$L_b(\mathbf{\Phi}) + \nabla \cdot (\boldsymbol{F}(\mathbf{\Phi}, \boldsymbol{x}, t)) = 0 \qquad \text{for } t \in \mathbb{R}^+, \mathbf{\Phi} \in \mathbb{R} \tag{2}$$

together with the initial condition

$$\mathbf{\Phi}(\boldsymbol{x}, 0) = \mathbf{\Phi}_0(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \Omega \subset \mathbb{R}^d, \mathbf{\Phi}_0 \in \mathbb{R}$$

where L_b is the transport operator given by

$$L_b(\mathbf{\Phi}) = \frac{\partial \mathbf{\Phi}}{\partial t} + \nabla \cdot ((b\mathbf{\Phi}))$$

and

$$\boldsymbol{x} = (x^1, ..., x^d), \quad \boldsymbol{F} = (F^1, ..., F^d), \quad \boldsymbol{b} = (b^1, ..., b^d),$$

where $\boldsymbol{b}$ is a regular vector field in $\mathbb{R}^d$, $\boldsymbol{F}$ is a flux vector in $\mathbb{R}^d$, and $\boldsymbol{x}$ is the position.

Fig. 1 gives a sketch of the flow domain, i.e., the free surface elevation and the bottom bathymetry. In this configuration, the vertical variation is much smaller than the horizontal variation, as typical for rivers flowing over long distances of e.g. hundreds or thousands of kilometres. We consider the frictionless, inviscid shallow water equations in Lagrangian derivatives, given as

$$\frac{D\eta}{Dt} + \nabla \cdot (H\boldsymbol{v}) = 0 \tag{3}$$

$$\frac{D\boldsymbol{v}}{Dt} + g\nabla\eta = 0 \tag{4}$$

$$\frac{D\boldsymbol{r}}{Dt} = \boldsymbol{v} \tag{5}$$

where $\eta = \eta(x, y, t)$ is the free surface location,

$$H(x, y, t) = h(x, y) + \eta(x, y, t)$$

is the total water depth with bottom bathymetry $h(x, y)$, and where $\boldsymbol{v} = v(x, y, t)$ is the particle velocity, $\boldsymbol{r} = r(x, y, t)$ the particle position, and g the gravity acceleration.

C. Hydrostatic Approximation

In geophysical flows the vertical acceleration is often small when compared to the gravitational acceleration and to the pressure gradient in the vertical direction as in the case of our flow domain in Fig 1. For instance, if we consider tidal flows in the ocean the velocity in the horizontal direction is of the order of $1m/s$, while the velocity in the vertical direction is much smaller of the order of one meter per tidal cycle i.e., $10^{-5}m/s$. To this end, if the advective and viscous terms are neglected in the vertical momentum equation of the Navier-Stokes equation, we have the equation for pressure which reads

$$\frac{dp}{dz} = -g. \tag{6}$$

The pressure represents a normalised pressure, that is we mean the pressure is divided by constant density. The solution that satisfies (6) is given by the hydrostatic pressure

$$p(x, y, z, t) = p_0(x, y, t) + g[\eta(x, y, t) - z],$$

where $p_0(x, y, t)$ marks the atmospheric pressure at the free surface which without loss of generality is taken as a constant.

III. Wetting and Drying Methodology

This section introduces the methodology employed towards the construction of our proposed wetting and drying semi-implicit SPH algorithm. Fig. 2 depicts a simple hydraulic wetting and drying pattern. Below the free surface, the domain is fully wetted with a nonvanishing velocity i.e., $\boldsymbol{v} \neq 0, H > 0$ and at the dry region both velocity and total water depth is zero, $\boldsymbol{v} = 0, H = 0$.

A. Subparticle Modeling

When wetting and drying processes are modelled and simulated, the shallow water equations are defined on a time dependent domain $\Omega(t)$ as

$$\Omega(t) = \{(x, y) : H(x, y, t) > 0\} \tag{7}$$

where $\Omega(t)$ is intrinsically one of unknowns to be determined numerically. Also, since the fluid boundary is also moving and one can not determine the position *a priori*. To circumvent this difficulty, Casulli [10] defined a piecewise constant

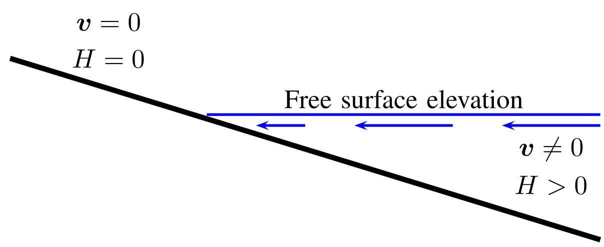

Fig. 2. Wetting and drying hydraulic pattern

function. For a specified bathymetry $h(x, y)$ we give a precise description of the flow by a function $a(x, y, z)$ defined by

$$a(x, y, z) = \begin{cases} 1 & \text{for} \quad h(x, y) + z > 0 \\ 0 & \text{otherwise} \end{cases}$$

for $(x, y) \in \Omega$ and $-\infty < z < \infty$. At $z = \eta_i^n$, the horizontal integral for each particle i given by

$$a_i(\eta_i^n) = \int_{\Omega_i} a(x, y, \eta_i^n) dxdy \tag{8}$$

represents the free-surface area. We can state clearly that when $a_i(\eta_i^n) = 0$, the ith particle is dry, when $a_i(\eta_i^n) = V_i$, the ith particle is wet and when $0 < a_i(\eta_i^n) < V_i$, the ith is partially wet respectively. The piecewise constant function defined by $a(x, y, z)$ means that $a_i(\eta_i^n)$ is nonnegative, nondecreasing and bounded. For each particle i, the total water depth is given by

$$\begin{aligned} H(x, y, \eta_i^n) &= \int_{-\infty}^{\eta_i^n} a(x, y, \eta_i^n) dz \\ &= \max\left[0, h(x, y) + \eta_i^n\right] \end{aligned} \tag{9}$$

so that $H(x, y, \eta_i^n) \geq 0$, and strict inequality identifies a wet particle. Hence, the wet region is given by

$$\Omega_i^n = \{(x, y) \in \Omega_i : H(x, y, \eta_i^n) > 0\} \tag{10}$$

The water volume for particle i is given by

$$V_i(\eta_i^n) = \int_{-\infty}^{\eta_i^n} a_i(z) dz = \int_{\Omega_i} H(x, y, \eta_i^n) dxdy \tag{11}$$

Because $a_i(z)$ is nonnegative and nondecreasing, we have $V_i(\eta_i^n) \geq 0$ and strict inequality necessarily implies $a_i(\eta_i^n) > 0$.

B. Classical SPH Formulation

The standard SPH formulation discretizes the computational domain $\Omega(t)$ by a finite set of N particles, with positions $\boldsymbol{r}_i$. According to Gingold and Monaghan [16], the SPH discretization of the shallow water equations (3)-(5) are given as

$$\frac{\eta_i^{n+1}-\eta_i^n}{\Delta t}+\sum_{j=1}^{N}\frac{m_j}{\rho_j}H_{ij}\boldsymbol{v}_j\nabla W_{ij}=\mathbf{0} \tag{12}$$

$$\frac{\boldsymbol{v}_i^{n+1}-\boldsymbol{v}_i^n}{\Delta t}+g\sum_{j=1}^{N}\frac{m_j}{\rho_j}\eta_j\nabla W_{ij}=\mathbf{0} \tag{13}$$

$$\frac{D\boldsymbol{r}_i}{Dt}=\boldsymbol{v}_i \tag{14}$$

where the particles are advected by (14), with Δt being the time step size, m_j the particle mass, ρ_j the particle density, and ∇W_{ij} is the gradient of kernel W_{ij} w.r.t. x_i. In the scheme [16], [21] of Gingold & Monaghan, $\nabla\cdot(H\boldsymbol{v})$ and $\nabla\eta$ are explicitly computed. We remark that eqns. (12)-(14) follow from a substitution of the flow variable with corresponding derivatives, using integration by parts, and the divergence theorem.

C. SPH Formulation of Vila and Ben Moussa

In the construction of our proposed semi-implicit SPH scheme, we use the concept of Vila & Ben Moussa ([5], [28]), whose basic idea is to replace the centred approximation

$$(F(v_i,x_i,t)+F(v_j,x_j,t))\cdot n_{ij}$$

of (2) by a numerical flux $G(n_{ij},v_i,v_j)$, from a conservative finite difference scheme, satisfying

$$G(n(x),v,v)=F(v,x,t)\cdot n(x)$$
$$G(n,v,u)=-G(-n,u,v).$$

With using this formalism, the SPH discretization of equations (12)-(13) becomes

$$\begin{aligned}\frac{\eta_i^{n+1}-\eta_i^n}{\Delta t}+\sum_{j=1}^{N}\frac{m_j}{\rho_j}2H_{ij}\boldsymbol{v}_{ij}\nabla W_{ij} &= \mathbf{0},\\ \frac{\boldsymbol{v}_i^{n+1}-\boldsymbol{v}_i^n}{\Delta t}+g\sum_{j=1}^{N}\frac{m_j}{\rho_j}2\eta_{ij}\nabla W_{ij} &= \mathbf{0}.\end{aligned}$$

In this way, we define for a pair of particles, i and j, the free surface elevation η_i, η_j and the velocity $\boldsymbol{v}_i$, $\boldsymbol{v}_j$, respectively (see Fig. 3). In our approach, we, moreover, use a staggered velocity $\boldsymbol{v}_{ij}$ between two interacting particles i and j as

$$\boldsymbol{v}_{ij}=\frac{1}{2}(\boldsymbol{v}_i+\boldsymbol{v}_j)\cdot\boldsymbol{n}_{ij}$$

in the normal direction $\boldsymbol{n}_{ij}^{d=1,2}$ at the midpoint of the two interacting particles, where

$$n_{ij}^1=\frac{x_j-x_i}{\|x_j-x_i\|} \qquad \text{and} \qquad n_{ij}^2=\frac{y_j-y_i}{\|y_j-y_i\|}$$

for the two components of vector $\boldsymbol{n}_{ij}$. Moreover,

$$\delta_{ij}^1=\|x_j-x_i\| \qquad \text{and} \qquad \delta_{ij}^2=\|y_j-y_i\|$$

gives the distance between particles i and j. Since the velocities at the particles' midpoint are known, we can use kernel summation for velocity updates.

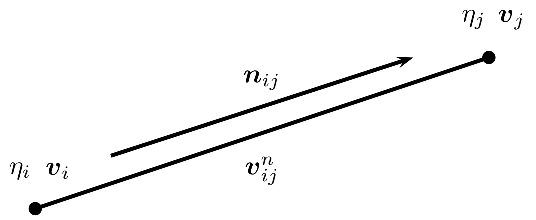

Fig. 3. Staggered velocity defined at the midpoint of two pair of interacting particles i and j.

D. Semi-implicit SPH Scheme

For the full derivation of the semi-implicit SPH scheme, we refer the reader to our previous work (see [2], [3]). Let us consider the continuity equation in the original conservative form given as

$$\eta_t^n+\nabla\cdot(H^n\boldsymbol{v}^{n+1})=0. \tag{15}$$

The velocity $\mathbf{v}$ will be discretized implicitly, the total water depth H is discretized explicitly. For the sake of notation, by implicitly and explicitly we mean $n+1$ and n in the superscript, respectively:

$$\boldsymbol{v}_t^n+g\cdot\nabla\eta^{n+1}=0$$

$$\eta_t^n+\nabla\cdot(H^n\boldsymbol{v}^{n+1})=0.$$

Furthermore, we have discretized the particle velocities and free surface elevation in time by the theta method for the sake of time accuracy and computational efficiency i.e $n+1=n+\Theta$. So we have

$$\boldsymbol{v}_t^n+g\cdot\nabla\eta^{n+\Theta}=0 \tag{16}$$

$$\eta_t^n+\nabla\cdot(H^n\boldsymbol{v}^{n+\Theta})=0, \tag{17}$$

where the theta method notation reads:

$$\eta^{n+\Theta}=\Theta\eta^{n+1}+(1-\Theta)\eta^n$$

$$\boldsymbol{v}^{n+\Theta}=\Theta\boldsymbol{v}^{n+1}+(1-\Theta)\boldsymbol{v}^n.$$

The factor Θ is called the implicitness factor which should be taken from $\left[\frac{1}{2},1\right]$ see Casulli and Cattani [11] for details.

The general semi-implicit SPH discretization of (16) - (17) assumes the form

$$\begin{aligned}\frac{\boldsymbol{v}_{ij}^{n+1}-\boldsymbol{F}\boldsymbol{v}_{ij}^n}{\Delta t}&+\frac{g}{\delta_{ij}}\Theta(\eta_j^{n+1}-\eta_i^{n+1})\\&+\frac{g}{\delta_{ij}}(1-\Theta)(\eta_j^n-\eta_i^n)\\&=0,\end{aligned} \tag{18}$$

$$\begin{aligned}\frac{\eta_i^{n+1}-\eta_i^n}{\Delta t} &+ \Theta\sum_{j=1}^{N}\frac{m_j}{\rho_j}(2H_{ij}^n\boldsymbol{v}_{ij}^{n+1})\nabla\boldsymbol{W}_{ij}\cdot\boldsymbol{n}_{ij} \\ &+ (1-\Theta)\sum_{j=1}^{N}\frac{m_j}{\rho_j}(2H_{ij}^n\boldsymbol{v}_{ij}^{n})\nabla\boldsymbol{W}_{ij}\cdot\boldsymbol{n}_{ij} \\ &= 0,\end{aligned} \tag{19}$$

where

$$H_{ij}^n = \max(0, h_{ij}^n + \eta_i^n, h_{ij}^n + \eta_j^n). \tag{20}$$

In a Lagrangian formalism, the explicit operator $\boldsymbol{Fv}_{ij}^n$ takes the simple form in (18)

$$\mathbf{Fv}_{ij}^n = \frac{1}{2}(\mathbf{v}_i + \mathbf{v}_j), \tag{21}$$

where $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ denotes the velocity of particles i and j at time t^n. The new velocity is computed through simple kernel summation:

$$\mathbf{v}_i^{n+1} = \mathbf{v}_i^n + \sum_{j=1}^{N}\frac{m_j}{\rho_j}(\mathbf{v}_{ij}^{n+1} - \mathbf{v}_i^n)W_{ij}. \tag{22}$$

We should note that in (18) we have not used the gradient of the kernel function for the discretization of the gradient of η. We rather used a finite difference discretization for the pressure gradient. This increases the accuracy, in (18) $\boldsymbol{F}$ corresponds to an explicit spatial discretization of the advective terms. Since SPH is a Lagrangian scheme, the nonlinear convective term is discretized automatically, using the Lagrangian (material) derivative contained in the particle motion in Eqn. (14). Equation (21) is used to interpolate the particle velocities from the particle location to the staggered velocity location.

E. The Free Surface Equation and Mass Conservation

Substituting the discrete momentum equation into the discrete continuity equation. The model is reduced into a smaller model in η_i^{n+1} as the only unknown.

Multiplying (19) by ω_i and inserting (18) into (19) we obtain

$$\begin{aligned}V(\eta_i^{n+1}) - g\Theta^2\frac{\Delta t^2}{\delta_{ij}}\sum_{j=1}^{N}2\omega_i\omega_j\left[H_{ij}^n(\eta_j^{n+1}-\eta_i^{n+1})\nabla\boldsymbol{W}_{ij}\cdot\boldsymbol{n}_{ij}\right] \\ = \boldsymbol{b}_i^n,\end{aligned} \tag{23}$$

where the right hand side $\boldsymbol{b}_i^n$ represents the known values at time level t^n given as

$$\begin{aligned}\boldsymbol{b}_i^n = V(\eta_i^n) &- \Delta t\sum_{j=1}^{N}2\omega_i\omega_j H_{ij}^n\boldsymbol{Fv}_{ij}^{n+\Theta}\nabla\boldsymbol{W}_{ij}\cdot\boldsymbol{n}_{ij} \\ &+ g\Theta(1-\Theta)\frac{\Delta t^2}{\delta_{ij}}\sum_{j=1}^{N}2\omega_i\omega_j\left[H_{ij}^n(\eta_j^n-\eta_i^n)\nabla\boldsymbol{W}_{ij}\cdot\boldsymbol{n}_{ij}\right],\end{aligned} \tag{24}$$

where $V(\eta_i^{n+1})$ is the water volume where the nonlinearity resides, $\boldsymbol{Fv}_{ij}^{n+\Theta} = \Theta\boldsymbol{Fv}_{ij}^n + (1-\Theta)\boldsymbol{v}_{ij}^n$. Since H_{ij}^n, ω_i, ω_j are non-negative numbers, equations (23) - (24) constitute a nonlinear system of N equations for η_i^{n+1} unknowns due to the piecewise constant water volumes.

Having computed the free surface and water velocity, the new total depth H_{ij}^{n+1} has to be updated. Since, the bathymetry h_{ij} are specified at the locations. A negative value for H is physically meaningless, then our discrete total depth H_{ij} at the next time are defined as

$$H_{ij}^{n+1} = \max(0, h_{ij}^{n+1} + \eta_i^{n+1}, h_{ij}^{n+1} + \eta_j^{n+1}) \tag{25}$$

where we note that $H_{ij} = H_{ji}$.

But a zero value for H simply means a particle is dry which can be later on wetted when the total water depth H becomes positive. So, if H is positive, the particle is wet and the vertical variation of the particle will be non zero whereas when H is zero, the particle is dry and the particle's vertical variation will be zero.

In this numerical model, considering Equation (23) we can inspect clearly that the resulting semi-implicit SPH equation for the free surface equation accurately accounts for the treatment of positive and zero values for the total water depth H. We can further see that the treatment of wetting and drying is naturally present in the present study without taking into account special treatment. And this formulation guarantees mass conservation while accounting for wetting and drying fronts. When the total water depth of a particle is zero, this implies a no mass flux or a zero velocity until at a later time when H becomes positive. In Equation (23), if we set H to be zero, the free surface equation becomes that the water volume at time level $n+1$ equals water level at time level n. This means there is no variation in the free surface elevation for a dry particle. On a dry particle the velocity equations are replaced by $\mathbf{v}_{ij}^{n+1} = 0$, so when wetting and drying of particles occurs, we still solve the same SPH equations having satisfied the condition of no mass flux.

In the entire particle configuration, when the total water depth is zero, $H_{ij}^n = 0$, the free surface equation (23) trivially implies

$$V(\eta_i^{n+1}) = V(\eta_i^n), \tag{26}$$

hence we can assume

$$\eta_i^{n+1} = \eta_i^n. \tag{27}$$

In this scenario, equation (23) does not form part of the system to be constructed. The remaining set of the free surface equation i.e., where there exist at least one H_{ij}^n that is nonzero the system is assembled into a *mildly nonlinear* sparse system for η_i^{n+1}. Brugnano and Casulli have presented convergent iterative schemes to solve this system even for piecewise polynomials for the definition of the water volume $V(\eta)$, (see [7], [8]) for details.

F. Mildly Nonlinear System

We hereby write system (23) in a compact vector notation:

$$\mathbf{V}(\eta) + \mathbf{T}\eta = \mathbf{b} \tag{28}$$

where

$$\mathbf{V}(\eta) = \begin{pmatrix} V_1(\eta_1) \\ V_2(\eta_2) \\ \vdots \\ V_N(\eta_N) \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1^{n+1} \\ \eta_2^{n+1} \\ \vdots \\ \eta_N^{n+1} \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} b_1^n \\ b_2^n \\ \vdots \\ b_N^n \end{pmatrix},$$

where $\mathbf{T}$ is a sparse and symmetric $N_\eta \times N_\eta$ matrix which comes from the second and third term in the left hand side of equation (23), matrix $\boldsymbol{T}$ is positive definite, then its inverse is also positive definite, $\mathbf{b}$ is a vector of N_η components from the right hand side of equation (23). Let us assume that the matrix $\mathbf{T}$ is irreducible. From Equation (23) we write the coefficient of η_i^{n+1} the ith main diagonal element of the matrix $\mathbf{T}$ given as

$$t_{i,i} = g\Theta^2 \frac{\Delta t^2}{\delta_{ij}} \sum_{j=1}^{N} 2\omega_i\omega_j H_{ij}^n \nabla \mathbf{W}_{ij} \cdot \mathbf{n}_{ij} \tag{29}$$

In the same vein, if we consider the non-zero off diagonal elements in each row of the matrix $\mathbf{T}$ which represents the coefficients of η_j^{n+1} in Equation (23), we have

$$t_{i,j} = -g\Theta^2 \frac{\Delta t^2}{\delta_{ij}} \sum_{j=1}^{N} 2\omega_i\omega_j H_{ij}^n \nabla \mathbf{W}_{ij} \cdot \mathbf{n}_{ij} \tag{30}$$

From the assumption that $\mathbf{T}$ to be irreducible we have that $t_{i,j} \leq 0$ for all particles $i = 1, 2. \cdots N$, so we have atleast one of $t_{i,j}$ is nonzero. From the above, such that $t_{i,i} > 0$ for each particle i and $t_{i,j} \leq 0$ whenever particle i is different from particle j, $i \neq j$, then we conclude that the $\mathbf{T}$ is an irreducible symmetric and positive semidefinite matrix. We can say that $\sum_{j=1}^{N} t_{i,j} = 0$ for $i = 1, 2. \cdots N$. If we denote any nonzero diagonal matrix by $\mathbf{P}$, from the above considerations, we have $\mathbf{P} \geq 0$, then we have that $\mathbf{P}+\mathbf{T}$ is an irreducible symmetric M-matrix. Therefore, $\mathbf{P}+\mathbf{T}$ is positive definite and consequently $(\mathbf{P}+\mathbf{T})^{-1} > 0$. From a physical point of view, the contribution of matrix $\mathbf{T}$ denotes the mass fluxes between pair of interacting particles.

For clarity, we define the water volumes and its corresponding gradient as

$$\mathbf{V}(\eta) = \begin{cases} \eta + h & \text{if} \quad \eta + h > 0 \quad \text{for} \quad \text{wet case} \\ 0 & \text{if} \quad \eta + h \leq 0 \quad \text{for} \quad \text{dry case} \end{cases} \tag{31}$$

The matrix $\mathbf{P}$ evaluated at η_i corresponds to the diagonal entries

$$\mathbf{P}_{ii} = \text{diag}\left(\frac{\partial \mathbf{V}}{\partial \eta}\right) \tag{32}$$

The gradient of the water volumes is given as

$$\frac{\partial \mathbf{V}}{\partial \eta} = \begin{cases} 1 & \text{if} \quad \eta + h > 0 \quad \text{for} \quad \text{wet case} \\ 0 & \text{if} \quad \eta + h \leq 0 \quad \text{for} \quad \text{dry case} \end{cases} \tag{33}$$

From the definition of the water volumes and in Fig. 4, we see that the function is not differentiable at the black dot (red broken lines).

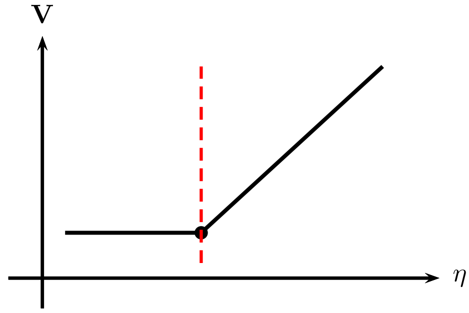

Fig. 4. Non-differentiability of water volume

G. A Newton Method

We arrive at the piecewise system which is strongly diagonally dominant, symmetric and positive definite. Hence, a unique solution can be efficiently obtained by a matrix free version of the conjugate gradient method and solved exactly in a Newton-type iteration. A nested Newton-type method can be see in the work of Casulli and Zanolli (see [13], [14]).

Initializing the free surface elevation $\boldsymbol{\eta}$, for all $k = 1, 2, \cdots$ where $\eta^{k,0} = \eta^{k-1}$ a sequence of iterates η^μ is obtained from Equ. (28). Linearising $\boldsymbol{V}(\eta)$ as follows we have,

$$\left[\mathbf{V}(\eta^{k,\mu-1}) + \mathbf{P}(\eta^{k,\mu-1})(\eta^{k,\mu} - \eta^{k,\mu-1})\right] + \mathbf{T}\eta^{k,\mu} = \mathbf{b}^{k-1}, \tag{34}$$

we obtain the iterates from the linear systems

$$(\mathbf{P}^{k,\mu-1} + \mathbf{T})\eta^{k,\mu} = \mathbf{g}^{k,\mu-1}, \qquad \mu = 1, 2, \cdots \tag{35}$$

where $\mathbf{P}^{k,\mu-1} = \mathbf{P}(\eta^{k,\mu-1})$ and

$$\mathbf{g}^{k,\mu-1} = \mathbf{b}^{k-1} - \mathbf{V}^{k,\mu-1} + \mathbf{P}^{k,\mu-1}\eta^{k,\mu-1}$$

The (k, μ)th residual $\boldsymbol{r}$ is given as

$$\boldsymbol{r}^{k,\mu} = \mathbf{V}(\eta^{k,\mu}) + \mathbf{T}\eta^{k,\mu} - \mathbf{b}, \tag{36}$$

and a stopping criterion for the iterates is given as $\|\boldsymbol{r}^{k,\mu}\| < \epsilon$ where ϵ is a sufficiently small tolerance value. The nonlinear problem to solve reads:

$$\eta^{k+1} = \eta^k - \left[\mathbf{P}(\eta^k) + \mathbf{T}\right]^{-1} \left[\mathbf{V}(\eta^k) + \mathbf{T}\eta^k - \mathbf{b}\right], \quad k = 0, 1, \cdots \tag{37}$$

where k denotes the iteration index, $\mathbf{P}(\eta^k)$ is a diagonal matrix. The iterative scheme in (37) is hereby summarized into Algorithm 1.

Once the free surface location $\boldsymbol{\eta}_i$ is computed. Equation (18) constitute a linear system for $\boldsymbol{v}_i^{n+1}$, the systems are independent of each other and are symmetric and positive definite. This is conveniently solved to determine $\boldsymbol{v}_i^{n+1}$ throughout the particle configurations and the particle positions can be subsequently updated. Following our mildly nonlinear construction in equation (23), a correct mass balance is always achieved in all particle regions irrespective of the specified bottom bathymetry. Nonnegative water volumes and water heights are assured.

Algorithm 1 Calculate η

Input: **V**, **P**, **T**, **b**, and ϵ
Do $k = 1, 2, \cdots$
  Set $\boldsymbol{P}^{k,0} = \boldsymbol{I}$
  Do $\mu = 1, 2, \cdots$
    Solve $[\mathbf{P}(\eta^{k,\mu-1} + \mathbf{T})]\eta^{k,\mu} = \mathbf{b} - \mathbf{P}(\eta^{k,\mu-1})$
    If $\|\boldsymbol{r}^{k,\mu}\| < \epsilon$
      set $\eta^k = \eta^{k,\mu}$ and Exit
    End If
  End Do
End Do
Output η

IV. Numerical Example

In this section, the wetting and drying semi-implicit SPH algorithm that has been proposed and derived in Section III will be validated on simple test problem for the shallow water equations. An academic numerical example of an oscillating lake will be presented. This test case is analogous to the Thacker's periodic flow with a planar free surface, a test case that is extremely difficult for numerical models to handle. In the subsequent problems, the acceleration due to gravity constant g is set to $g = 9.81$.

A. An Oscillating Lake in a Parabolic Basin

In this one dimensional example, we consider an oscillating lake inside a parabolic basin denoting the bathymetric bottom. We consider the initial value problem

$$\eta(x, 0) = 0.1x,$$

$$u(x, 0) = 0,$$

in the parabolic basin with bottom bathymetry

$$h(x) = 1 - 0.1x^2$$

in the domain $\Omega = [-5, 5]$. The oscillating lake is restricted below by a fixed bottom boundary h and bounded above by a moving free surface elevation η. We have discretized the computational domain Ω with 400 particles. The final simulation time $t = 7.2s$ is used and a time step is chosen to be $\Delta t = 0.01$. An implicitness factor $\Theta = 0.85$ is used. A varying smoothing length is taken as $l_i = \alpha(\omega_i)^{\frac{1}{d}}$, where $\alpha = [1.5, 2]$ and $d = 1$, a tolerance value of $\epsilon = 10^{-14}$ has been used in our Newton iteration. The numerical solution is shown in Fig. 5 at times $t = 0.0s, 1.8s, 3.6s, 5.4s, 7.2s$. Because, we do not have the exact solution for this particular problem, we obtain a reference solution by solving the shallow water equation with a finite difference approach of Casulli [12]. The comparison between our numerical results obtained with wetting and drying semi-implicit SPH and reference solution is shown. A very good agreement between the two solutions is observed in Fig. 5 even at the transition regions.

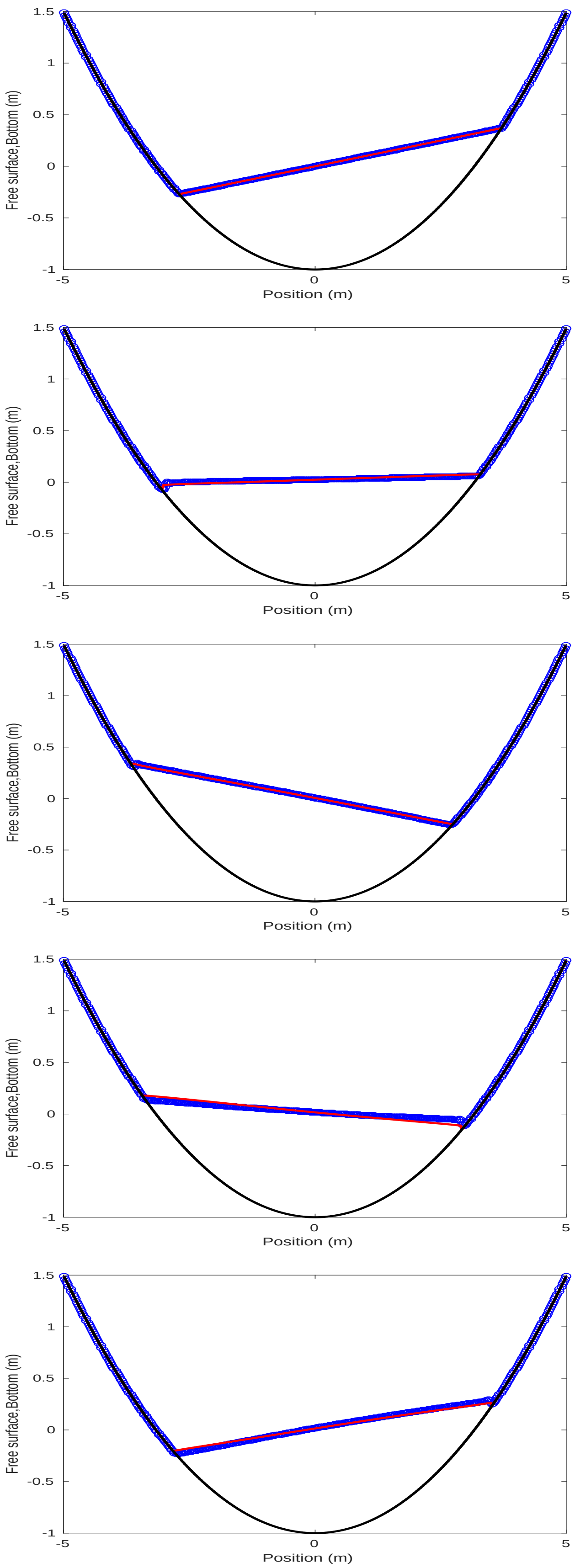

Fig. 5. Wetting and drying semi-implicit solution (blue dots) versus reference solution (red - solid line), the bottom bathymetry (black - solid line): Free surface inside parabolic basin at times $t = 0.0s, 1.8s, 3.6s, 5.4s, 7.2s$.

V. Conclusion

This paper proposes a new wetting and drying SPH algorithm that is based on a novel semi-implicit SPH discretization. The semi-implicit algorithm applied to the shallow water equations has been derived and discussed. The momentum equation is discretized by a finite difference approximation for the gradient of the free surface elevation and SPH approximation for the mass conservation equation.

Because we substituted the discrete momentum equations into the discrete mass conservation equations and since we define the water particle volume as a piecewise constant function, we arrive at a mildly nonlinear sparse system for the free surface elevation. We thereby solve some Newton-type iterations when wetting and drying is encountered. We conveniently solve this mildly nonlinear system with the matrix-free version of the conjugate gradient (CG) algorithm.

The key features of the proposed wetting and drying algorithm are as follows. The method achieves a correct mass balance in wet regions and in transition regions, it is simple and efficient, and it guarantees non-negative water depths. Finally, the method's time step is not restricted by stability conditions that are dictated by the surface wave speed, thereby allowing large time steps.

Future research will be devoted to the method's extension to real life river flooding and drying scenarios.

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