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Abstract

We consider linear time-invariant differential-algebraic systems which are not necessarily regular. The following question is addressed: When does an (asymptotic) observer which is realized by an ODE system exist? In our main result we characterize the existence of such observers by means of a simple criterion on the system matrices. To be specific, we show that an ODE observer exists if, and only if, the completely controllable part of the system is impulse observable.

Keywords: Differential-algebraic equations; observers; controllability; observability; Kalman decomposition.

1. Introduction

In a recent work [6] we have considered observer design for linear systems governed by differential-algebraic equations (DAEs) which are not necessarily regular in the sense that they may be under- or overdetermined. This approach is based on the observer notions in [22], where equivalent criteria for the existence of (exact, asymptotic) observers have been presented. It has been shown in [6] that, with the same criteria as in [22], DAE systems admit (exact, asymptotic) observers which are again DAEs.

In the present article, we address the question when observers for DAE systems can be constructed which are described by ordinary differential equations (ODEs). This observer type is preferable from a practical point of view, since unconstrained observer dynamics do not involve derivatives of the inputs and outputs, which would lead to an ill-posed problem. Another advantage of ODE observers is that they can be initialized without any further restrictions.

We show, using a simple argument, that the existence of ODE observers is equivalent to the existence of observers with index at most one. The latter class has already been introduced in [18] for ODE systems, generalized to semi-explicit index-1 DAEs in [2], and considered for nonlinear descriptor systems in [16], see also the references therein. However, neither a characterization of existence of index-1 observers nor of ODE observers for DAE systems is available. In the present paper, we show that for general linear DAE systems, an ODE observer exists if, and only if, the completely controllable part of the system is impulse observable. The observer is moreover asymptotic if, and only if, the DAE system is additionally behaviorally detectable. In each case, we provide an explicit construction procedure for the observer.

Throughout this article, we use the following notation: For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $A \in \mathbb{K}^{n\times m}$, we use the symbols $\operatorname{im}_{\mathbb{K}} A$, $\ker_{\mathbb{K}} A$ and $\operatorname{rk}_{\mathbb{K}} A$ for the image, kernel and rank of A, resp. The subscripts are omitted when they are clear from context. The group of invertible real matrices of size $n \times n$ is denoted by $\mathbf{Gl}_n(\mathbb{R})$, and $\|x\|$ is the Euclidean norm of $x \in \mathbb{R}^n$. $\mathbb{N}$ is the the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The symbols $\mathbb{C}_+$ and $\overline{\mathbb{C}_+}$ denote the sets of complex numbers with positive and nonnegative real part, resp.

Further, $f|_I$ is the restriction of a function $f : \mathbb{R} \to \mathbb{R}^n$ to $I \subseteq \mathbb{R}$ and $\dot{f}$ ($f^{(i)}$) is the (i-th) weak derivative of f, see [1, Chap. 1]. We further use the following function spaces in this article:

$\mathscr{C}^\infty(\mathbb{R};\mathbb{R}^n)$ the set of infinitely-times continuously differentiable $\mathbb{R}^n$-valued functions

$\mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$ the set of locally Lebesgue integrable $\mathbb{R}^n$-valued functions

$\mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) := \left\{ f \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \;\middle|\; \dot{f} \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \right\}$

2. Preliminaries

We study linear time-invariant DAE systems

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

where $E, A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$. Systems of that type are also called *descriptor systems*. The set of systems (1) is denoted by $\Sigma_{l,n,m,p}$ and we write $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$. DAE systems of the form (1) naturally occur when modeling dynamical systems subject to algebraic constraints, e.g. chemical process systems [14], mechanical systems [21, 20], and modified nodal analysis models of electrical circuits [19]; see also the textbooks [15, 17]. In the present paper we do not assume that the matrix pencil $sE - A$ is *regular*, which would mean that $l = n$ and $\det(sE - A)$ is not the zero polynomial.

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The functions $u:\mathbb{R}\to\mathbb{R}^m$ and $y:\mathbb{R}\to\mathbb{R}^p$ are called *input* and *output* of the system, resp. A trajectory $(x,u,y):\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p$ is said to be a *solution* of (1), if $Ex\in\mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^l)$ and (x,u,y) solves (1) for almost all $t\in\mathbb{R}$. Recall that $Ex\in\mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^l)$ implies continuity of Ex (though x itself may be discontinuous). The *behavior* $\mathfrak{B}_{[E,A,B,C,D]}$ of (1) is defined as the set of all solutions $(x,u,y):\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p$ of (1). Based on this behavior, DAE systems have been studied in detail e.g. in [3]. For the analysis of DAE systems in $\Sigma_{l,n,m,p}$ we assume that the states, inputs and outputs of the system are fixed a priori by the designer. This is different from other approaches based on the behavioral setting [12, 13, 22].

We consider different notions of controllability and observability for DAE systems. For a rigorous time domain definition and a detailed discussion we refer to the surveys [5, 8]. In the following we state their algebraic characterizations.

Proposition 2.1. *A system* $[E,A,B,C,D]\in\Sigma_{l,n,m,p}$ *is*

(i) completely controllable *if, and only if,* $\mathrm{im}_{\mathbb{C}}(\lambda E-A)+\mathrm{im}_{\mathbb{C}}B=\mathrm{im}_{\mathbb{C}}E+\mathrm{im}_{\mathbb{C}}B=\mathrm{im}_{\mathbb{C}}E+\mathrm{im}_{\mathbb{C}}A+\mathrm{im}_{\mathbb{C}}B$ *for all* $\lambda\in\mathbb{C}$.

(ii) impulse observable *if, and only if,* $\ker E\cap A^{-1}(\mathrm{im}\,E)\cap\ker C=\{0\}$.

(iii) completely observable *if, and only if,* $\ker E\cap\ker C=\{0\}$ *and* $\ker_{\mathbb{C}}(\lambda E-A)\cap\ker_{\mathbb{C}}C=\{0\}$ *for all* $\lambda\in\mathbb{C}$.

(iv) behaviorally detectable *if, and only if,* $\ker_{\mathbb{C}}(\lambda E-A)\cap\ker_{\mathbb{C}}C=\{0\}$ *for all* $\lambda\in\overline{\mathbb{C}_+}$.

We also need the Kalman controllability decomposition derived in [11].

Theorem 2.2. *For any* $[E,A,B,C,D]\in\Sigma_{l,n,m,p}$ *there exist* $T\in\mathbf{Gl}_n(\mathbb{R})$, $S\in\mathbf{Gl}_l(\mathbb{R})$ *such that* $[SET,SAT,SB,CT,D]$ *is in* Kalman controllability decomposition (KCD)*, i.e.,*

$$S(sE-A)T=\begin{bmatrix} sE_{11}-A_{11} & sE_{12}-A_{12} & sE_{13}-A_{13}\\ 0 & sE_{22}-A_{22} & sE_{23}-A_{23}\\ 0 & 0 & sE_{33}-A_{33}\end{bmatrix},$$
$$SB=\begin{bmatrix}B_1\\0\\0\end{bmatrix},\quad CT=\begin{bmatrix}C_1 & C_2 & C_3\end{bmatrix},\tag{2}$$

where

(i) $[E_{11},A_{11},B_1,C_1,D]\in\Sigma_{l_1,n_1,m,p}$ *with* $l_1=\mathrm{rk}[E_{11},B_1]\le n_1+m$ *is completely controllable,*

(ii) $[E_{22},A_{22},0,C_2,D]\in\Sigma_{l_2,n_2,m,p}$ *with* $l_2=n_2$ *and* E_{22} *is invertible,*

(iii) $[E_{33},A_{33},0,C_3,D]\in\Sigma_{l_3,n_3,m,p}$ *with* $l_3\ge n_3$ *satisfies* $\mathrm{rk}_{\mathbb{C}}(\lambda E_{33}-A_{33})=n_3$ *for all* $\lambda\in\mathbb{C}$.

To introduce the concept of an (asymptotic) observer for a DAE system, we first need to define acceptors.

Definition 2.3. Consider a system $[E,A,B,C,D]\in\Sigma_{l,n,m,p}$. A system $[E_o,A_o,B_o,C_o,D_o]\in\Sigma_{l_o,n_o,m+p,p_o}$ is called an *acceptor for* $[E,A,B,C,D]$, if for all $(x,u,y)\in\mathfrak{B}_{[E,A,B,C,D]}$, there exist $x_o\in\mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{n_o})$, $z\in\mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{p_o})$ such that

$$(x_o,\left(\begin{smallmatrix}u\\y\end{smallmatrix}\right),z)\in\mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}.$$

The concept of an acceptor has been first introduced by Valcher and Willems [22] for behaviors. Loosely speaking, an acceptor absorbs the external signals of a given system without influencing the system. A special class of acceptors is that of observers. We use the definition of observers of DAE systems from [6]. Note that the following definition has also been stated for behavioral systems in [22].

Definition 2.4. Consider a system $[E,A,B,C,D]\in\Sigma_{l,n,m,p}$. Then a system $[E_o,A_o,B_o,C_o,D_o]\in\Sigma_{l_o,n_o,m+p,n}$ is called

a) an *observer for* $[E,A,B,C,D]$, if it is an acceptor for $[E,A,B,C,D]$, and

$$\begin{aligned}&\forall\,(x,u,y,x_o,z)\in\mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p\times\mathbb{R}^{n_o}\times\mathbb{R}^n):\\ &\begin{pmatrix}(x,u,y)\in\mathfrak{B}_{[E,A,B,C,D]}\ \wedge\ (x_o,\left(\begin{smallmatrix}u\\y\end{smallmatrix}\right),z)\in\mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}\\ \wedge\ Ez(0)=Ex(0)\end{pmatrix}\\ &\qquad\qquad\Longrightarrow\ z=x \text{ for almost all } t\in\mathbb{R}.\end{aligned}$$

b) an *asymptotic observer for* $[E,A,B,C,D]$, if it is an observer for $[E,A,B,C,D]$, and

$$\begin{aligned}&\forall\,(x,u,y,x_o,z)\in\mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p\times\mathbb{R}^{n_o}\times\mathbb{R}^n):\\ &\big((x,u,y)\in\mathfrak{B}_{[E,A,B,C,D]}\ \wedge\ (x_o,\left(\begin{smallmatrix}u\\y\end{smallmatrix}\right),z)\in\mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}\big)\\ &\Longrightarrow\ \lim_{t\to\infty}\operatorname{ess\,sup}_{[t,\infty)}\|z-x\|=0,\end{aligned}$$

where "ess sup" denotes the essential supremum.

The definition of an observer means that once the observer matches the state via $Ez(0)=Ex(0)$, it does not lose track, i.e., the whole trajectories have to coincide ($z=x$). Note that, by time-invariance, the condition $Ez(0)=Ex(0)$ may be replaced by the existence of some $t\in\mathbb{R}$ such that $Ez(t)=Ex(t)$.

In order to define index-1 observers we need to introduce the notion of the index: The index $\nu\in\mathbb{N}_0$ of a regular matrix pencil $sE-A$ is defined via its (quasi-)Weierstraß form [4, 15, 17]: if for some $S,T\in\mathbf{Gl}_n(\mathbb{R})$

$$S(sE-A)T=\begin{bmatrix}sI_r-J & 0\\ 0 & sN-I_{n-r}\end{bmatrix},\quad N \text{ nilpotent},$$

$$\text{then}\quad \nu:=\begin{cases}0, & \text{if } r=n,\\ \min\{\,k\in\mathbb{N}_0\mid N^k=0\,\}, & \text{if } r<n.\end{cases}$$

The index is independent of the choice of S,T and can be computed via the Wong sequences corresponding to $sE-A$ as shown in [4].

Definition 2.5. Let a system $[E,A,B,C,D]\in\Sigma_{l,n,m,p}$ be given and let $[E_o,A_o,B_o,C_o,D_o]\in\Sigma_{l_o,n_o,m+p,n}$ be an observer for $[E,A,B,C,D]$. Then we call $[E_o,A_o,B_o,C_o,D_o]$

a) *regular*, if $l_o = n_o$ and $sE_o - A_o$ is regular;

b) an *index-1 observer*, if it is regular and the index of $sE_o - A_o$ is at most one.

c) an *ODE observer*, if $E_o = I_{n_o}$.

Clearly, every ODE observer is an index-1 observer. The following result highlights some advantages of index-1 observers. Its proof is straightforward and therefore omitted.

Proposition 2.6. *Let a system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ be given and let $[E_o,A_o,B_o,C_o,D_o] \in \Sigma_{l_o,n_o,m+p,n}$ be an observer for $[E,A,B,C,D]$. Then the following two statements are equivalent:*

(i) $[E_o,A_o,B_o,C_o,D_o]$ is regular and freely initializable in the sense that for all $(x,u,y) \in \mathfrak{B}_{[E,A,B,C,D]}$ and $x_o^0 \in \mathbb{R}^{n_o}$ there exist $x_o \in \mathscr{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^{n_o})$, $z \in \mathscr{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$ such that

$$(x_o, \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), z) \in \mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]} \quad \text{and} \quad E_o x_o(0) = E_o x_o^0.$$

(ii) $[E_o,A_o,B_o,C_o,D_o]$ is an index-1 observer.

In the following we show that the existence of an index-1 observer is equivalent to the existence of an ODE observer.

Proposition 2.7. *Let a system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ be given. Then the following two statements are equivalent.*

(i) There exists an (asymptotic) index-1 observer for $[E,A,B,C,D]$.

(ii) There exists an (asymptotic) ODE observer for $[E,A,B,C,D]$.

Proof. It suffices to show (i)⇒(ii): Assume that $[E_o,A_o,B_o,C_o,D_o] \in \Sigma_{l_o,n_o,m+p,n}$ is an (asymptotic) index-1 observer for $[E,A,B,C,D]$. Then there exist $S \in \mathbf{Gl}_{l_o}(\mathbb{R})$, $T \in \mathbf{Gl}_{n_o}(\mathbb{R})$ such that $S(sE_o - A_o)T = \begin{bmatrix} sI_r - J & 0 \\ 0 & -I_{n_o-r} \end{bmatrix}$, $SB_o = \begin{bmatrix} B_{o,1} \\ B_{o,2} \end{bmatrix}$, $C_oT = [C_{o,1}, C_{o,1}]$, and hence $(x_o, \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), z) \in \mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}$ with $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = T^{-1}x_o$ if, and only if,

$$\begin{aligned} \dot{x}_1 &= Jx_1 + B_{o,1} \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), \\ 0 &= x_2 + B_{o,2} \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), \\ z &= C_{o,1}x_1 + C_{o,2}x_2 + D_o \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), \end{aligned}$$

which is equivalent to

$$\begin{aligned} \dot{x}_1 &= Jx_1 + B_{o,1} \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), \\ z &= C_{o,1}x_1 + (D_o - C_{o,2}B_{o,2}) \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right). \end{aligned} \tag{3}$$

Therefore, $[I_r, J, B_{o,1}, C_{o,1}, D_o - C_{o,2}B_{o,2}]$ is an (asymptotic) ODE observer for $[E,A,B,C,D]$. ☐

In order to geometrically characterize the existence of an (asymptotic) ODE observer we use the augmented Wong sequences (see [5, 7, 11] and the references therein) which are defined as follows for $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$:

$$\begin{aligned} &\mathscr{V}^0_{[E,A,B]} := \mathbb{R}^n, && \mathscr{V}^{i+1}_{[E,A,B]} := A^{-1}(E\mathscr{V}^i_{[E,A,B]} + \operatorname{im} B), \\ &\mathscr{W}^0_{[E,A,B]} := \{0\}, && \mathscr{W}^{i+1}_{[E,A,B]} := E^{-1}(A\mathscr{W}^i_{[E,A,B]} + \operatorname{im} B), \\ &\mathscr{V}^*_{[E,A,B]} := \bigcap_{i \in \mathbb{N}_0} \mathscr{V}^i_{[E,A,B]}, && \mathscr{W}^*_{[E,A,B]} := \bigcup_{i \in \mathbb{N}_0} \mathscr{W}^i_{[E,A,B]}. \end{aligned} \tag{4}$$

Recall that, for some matrix $M \in \mathbb{R}^{l \times n}$, $M\mathscr{S} = \{ x \in \mathbb{R}^l \mid x \in \mathscr{S} \}$ denotes the image of $\mathscr{S} \subseteq \mathbb{R}^n$ under M and $M^{-1}\mathscr{S} = \{ x \in \mathbb{R}^n \mid Mx \in \mathscr{S} \}$ denotes the preimage of $\mathscr{S} \subseteq \mathbb{R}^l$ under M.

The sequences $(\mathscr{V}^i_{[E,A,B]})_{i \in \mathbb{N}}$ and $(\mathscr{W}^i_{[E,A,B]})_{i \in \mathbb{N}}$ are called *augmented Wong sequences* since they are based on the Wong sequences ($B = 0$) used in [4, 9, 10] and which have their origin in Wong [23] who was the first using both sequences (with $B = 0$) for the analysis of matrix pencils.

As shown in [5] the augmented Wong sequences allow a characterization of complete controllability as follows.

Lemma 2.8. *$[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ is completely controllable if, and only if, $\mathscr{V}^*_{[E,A,B]} \cap \mathscr{W}^*_{[E,A,B]} = \mathbb{R}^n$.*

Remark 2.9. The augmented Wong sequences are related to the *reachable space* of a system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$, which is defined as

$$\begin{aligned} \mathscr{R}_{[E,A,B]} := \{ x_f \in \mathbb{R}^n \mid \exists t_f > 0 \ \exists (x,u,y) \in \mathfrak{B}_{[E,A,B,C,D]} : \\ x \in \mathscr{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^n) \wedge x(0) = 0 \wedge x(t_f) = x_f \}, \end{aligned}$$

cf. also [5]. In [5] it is shown that

$$\mathscr{R}_{[E,A,B]} = \mathscr{V}^*_{[E,A,B]} \cap \mathscr{W}^*_{[E,A,B]},$$

hence complete controllability can also be characterized by the intuitive condition $\mathscr{R}_{[E,A,B]} = \mathbb{R}^n$.

3. Main result

In this section we state and prove a characterization of existence of (asymptotic) ODE and index-1 observers. Before this result is shown, we advance to simple examples. We start with an example of a system for which there does not exist any ODE observer (and thus, by Proposition 2.7, there does neither exist any index-1 observer).

Example 3.1. Consider the system

$$[E,A,B,C,D] = \left[\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}, 0 \right] \in \Sigma_{2,2,1,1}.$$

Then we have $x_2 = y = -u$ and $x_1 = \dot{y} = -\dot{u}$. Consequently, an observer $[E_o,A_o,B_o,C_o,D_o]$ has to take derivatives of y or u in order to satisfy Definition 2.4 a). Hence, by Proposition 2.7, the construction of ODE or index-1 observers for $[E,A,B,C,D]$ is impossible.

The subsequent example shows that there exist systems with higher index which admit the construction of ODE observers.

Example 3.2. Consider the system

$$[E,A,B,C,D] = \left[\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix},\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix},\begin{bmatrix}1&0&0\end{bmatrix},0\right] \in \Sigma_{3,3,1,1}.$$

We show that the ODE system

$$\left[\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&-1\\1&-2\end{bmatrix},\begin{bmatrix}-1&1\\-2&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix}\right],$$

i.e.,

$$\begin{aligned}\dot{x}_{o1} &= -x_{o2} - u + y,\\ \dot{x}_{o2} &= x_{o1} - 2x_{o2} - 2u,\\ z_1 &= y,\\ z_2 &= x_{o1},\\ z_3 &= x_{o2}\end{aligned}$$

is an asymptotic observer for $[E,A,B,C,D]$. Denoting the states of $[E,A,B,C,D]$ by x_1, x_2 and x_3, we obtain $z_1 = y = x_1$ and, for $e_2 = z_2 - x_2$ and $e_3 = z_3 - x_3$,

$$\begin{aligned}\dot{e}_2 &= -x_{o2} - u + y - x_1 = -z_3 - u = -e_3,\\ \dot{e}_3 &= x_{o1} - 2x_{o2} - 2u - x_2 = e_2 - 2e_3.\end{aligned}$$

Therefore, we have

$$\begin{pmatrix}\dot{e}_2\\ \dot{e}_3\end{pmatrix} = \begin{bmatrix}0&-1\\1&-2\end{bmatrix}\begin{pmatrix}e_2\\ e_3\end{pmatrix}$$

and since the matrix $\left[\begin{smallmatrix}0&-1\\1&-2\end{smallmatrix}\right]$ only has the eigenvalue -1, it follows that $e_2(t) \to 0$ and $e_3(t) \to 0$ for $t \to \infty$. This shows that Definition 2.4 b) is satisfied.

Now we present the main result of the present paper.

Theorem 3.3. *Let a system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ be given. Then the following statements are equivalent:*

1) There exists an ODE observer for $[E,A,B,C,D]$.

2) There exists an index-1 observer for $[E,A,B,C,D]$.

3) For some (and hence any) KCD (2) the completely controllable part $[E_{11},A_{11},B_1,C_1,D]$ of $[E,A,B,C,D]$ is impulse observable.

4) The Wong sequences in (4) satisfy

$$\mathcal{V}^*_{[E,A,B]} \cap \mathcal{W}^*_{[E,A,B]} \cap \ker E \cap A^{-1}(\operatorname{im} E) \cap \ker C = \{0\}. \quad (5)$$

Furthermore, the following statements are equivalent:

1') There exists an asymptotic ODE observer for $[E,A,B,C,D]$.

2') There exists an asymptotic index-1 observer for $[E,A,B,C,D]$.

3') $[E,A,B,C,D]$ is behaviorally detectable and for some (and hence any) KCD (2) we have that $[E_{11},A_{11},B_1,C_1,D]$ is impulse observable.

4') $[E,A,B,C,D]$ is behaviorally detectable and the Wong sequences in (4) satisfy (5).

Proof. By Proposition 2.7 we have 1) $\Leftrightarrow$ 2).

3) $\Rightarrow$ 2): Since $[E_{11},A_{11},B_1,C_1,D]$ is impulse observable and E_{22} in (2) is invertible it follows that

$$[\tilde{E},\tilde{A},\tilde{B},\tilde{C},D] := \left[\begin{bmatrix}E_{11}&E_{12}\\0&E_{22}\end{bmatrix},\begin{bmatrix}A_{11}&A_{12}\\0&A_{22}\end{bmatrix},\begin{bmatrix}B_1\\0\end{bmatrix},[C_1,C_2],D\right] \in \Sigma_{l_1+l_2,n_1+n_2,m,p}$$

is impulse observable. Then [6, Thm. 3.8] implies that there exists an index-1 observer $[E_o^1,A_o^1,B_o^1,C_o^1,D_o^1] \in \Sigma_{l_o,n_o,m+p,n_1+n_2}$ for $[\tilde{E},\tilde{A},\tilde{B},\tilde{C},D]$. Define

$$[E_o,A_o,B_o,C_o,D_o] := \left[E_o^1,A_o^1,B_o^1,\begin{bmatrix}C_o^1\\0\end{bmatrix},\begin{bmatrix}D_o^1\\0\end{bmatrix}\right] \in \Sigma_{l_o,n_o,m+p,n}.$$

Since the DAE $\frac{\mathrm{d}}{\mathrm{d}t}E_{33}x_3 = A_{33}x_3$ does only have the trivial solution, cf. [9], it follows that $[E_o,A_o,B_o,C_o,D_o]$ is an observer for $[E,A,B,C,D]$. Since $sE_o^1 - A_o^1$ has index at most one, it follows that $[E_o,A_o,B_o,C_o,D_o]$ is an index-1 observer.

2) $\Rightarrow$ 3): Without loss of generality we may assume that $[E,A,B,C,D]$ is in KCD (2). Consider the completely controllable part $[E_{11},A_{11},B_1,C_1,D]$ of $[E,A,B,C,D]$. By the Kalman observability decomposition, see [8, Thm. 8.3], there exist $\tilde{S} \in \mathbf{Gl}_{l_1}(\mathbb{R})$ and $\tilde{T} \in \mathbf{Gl}_{n_1}(\mathbb{R})$ such that

$$[\tilde{S}E_{11}\tilde{T},\tilde{S}A_{11}\tilde{T},\tilde{S}B_1,C_1\tilde{T}] = \left[\begin{bmatrix}\tilde{E}_{11}&\tilde{E}_{12}&\tilde{E}_{13}\\0&\tilde{E}_{22}&\tilde{E}_{23}\\0&0&\tilde{E}_{33}\end{bmatrix},\begin{bmatrix}\tilde{A}_{11}&\tilde{A}_{12}&\tilde{A}_{13}\\0&\tilde{A}_{22}&\tilde{A}_{23}\\0&0&\tilde{A}_{33}\end{bmatrix},\begin{bmatrix}\tilde{B}_1\\ \tilde{B}_2\\ \tilde{B}_3\end{bmatrix},[0,0,\tilde{C}_3]\right], \quad (6)$$

where $\tilde{E}_{ij},\tilde{A}_{ij} \in \mathbb{R}^{r_i\times q_j}$, $\tilde{B}_i \in \mathbb{R}^{r_i\times m}$ for $i,j = 1,\ldots,3$, $\tilde{C}_3 \in \mathbb{R}^{p\times q_3}$ with

a) $r_1 \le q_1$ and $\operatorname{rk}_{\mathbb{C}}(\lambda\tilde{E}_{11} - \tilde{A}_{11}) = r_1$ for all $\lambda \in \mathbb{C}$,

b) $r_2 = q_2$ and $\tilde{E}_{22}$ is invertible,

c) $[\tilde{E}_{33},\tilde{A}_{33},\tilde{B}_3,\tilde{C}_3,0]$ is completely observable.

Using the existence of an index-1 observer $[E_o,A_o,B_o,C_o,D_o] \in \Sigma_{l_o,n_o,m+p,n}$ for $[E,A,B,C,D]$ we derive some consequences for the form (6) and proceed in several steps.

Step 1: We consider the subsystem $[\tilde{E}_{11},\tilde{A}_{11},\tilde{B}_1,0,0]$ with property a). Using the quasi-Kronecker form [10, Cor. 2.3] we find $V \in \mathbf{Gl}_{r_1}(\mathbb{R}), W \in \mathbf{Gl}_{q_1}(\mathbb{R})$ such that

$$V(s\tilde{E}_{11} - \tilde{A}_{11})W = \begin{bmatrix}sE_P - A_P & 0\\ 0 & sN - I_k\end{bmatrix},$$

where $E_P,A_P \in \mathbb{R}^{l_P\times n_P}, l_P < n_P$ (or $l_P = n_P = 0$), such that $\operatorname{rk}_{\mathbb{C}}(\lambda E_P - A_P) = l_P$, $\operatorname{rk} E_P = l_P$, and $N \in \mathbb{R}^{k\times k}$ is nilpotent. If $n_P > 0$, then [9, Thm. 3.2] implies existence of $x_P \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^{n_P})$, $x_P \neq 0$, with $x_P(0)=0$ such that $E_P\dot{x}_P(t) = A_P x_P(t)$ for all $t\in\mathbb{R}$. Then

$$x := T\left(\tilde{T}\begin{pmatrix} Wx_P \\ 0\end{pmatrix} \atop 0\right) \quad \text{satisfies } (x,0,0)\in\mathfrak{B}_{[E,A,B,C,D]}.$$

Since $(0,0,0)\in\mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}$ and $[E_o,A_o,B_o,C_o,D_o]$ is an observer for $[E,A,B,C,D]$ it follows that $x=0$, whence $x_P=0$, a contradiction. Therefore, $n_P=0$ and we may without loss of generality assume that

$$s\tilde{E}_{11}-\tilde{A}_{11} = sN - I_{r_1}, \quad N \text{ nilpotent.} \tag{7}$$

Step 2: We consider the subsystem $[\tilde{E}_{33},\tilde{A}_{33},\tilde{B}_3,\tilde{C}_3,0]$ with property c). Choose $\tilde{W}\in\mathbf{Gl}_{r_3}(\mathbb{R}), \tilde{V}\in\mathbf{Gl}_{q_3}(\mathbb{R})$ such that

$$[\tilde{W}\tilde{E}_{33}\tilde{V},\tilde{W}\tilde{A}_{33}\tilde{V},\tilde{C}_3\tilde{V}] = \left[\begin{bmatrix} I_k & 0\\ 0 & 0\end{bmatrix},\begin{bmatrix} J_{11} & J_{12}\\ J_{21} & J_{22}\end{bmatrix},[\tilde{C}_{31},\tilde{C}_{32}]\right],$$

where $k=\operatorname{rk}\tilde{E}_{33}$, $J_{11}\in\mathbb{R}^{k\times k}, J_{12}\in\mathbb{R}^{k\times(q_3-k)}, J_{21}\in\mathbb{R}^{(r_3-k)\times k}, J_{22}\in\mathbb{R}^{(r_3-k)\times(q_3-k)}, \tilde{C}_{31}\in\mathbb{R}^{p\times k}$ and $\tilde{C}_{32}\in\mathbb{R}^{p\times(q_3-k)}$. By complete observability due to c) we have $\ker\tilde{E}_{33}\cap\ker\tilde{C}_3=\{0\}$, which gives

$$\ker\begin{bmatrix} I_k & 0\\ \tilde{C}_{31} & \tilde{C}_{32}\end{bmatrix} = \{0\},$$

thus

$$\operatorname{rk}\tilde{C}_{32} = q_3-k.$$

Henceforth, without loss of generality we may assume that $\tilde{W}=I_{r_3}$ and $\tilde{V}=I_{q_3}$.
Step 3: We show that $N=0$. If $r_1=0$, then there is nothing to show. Assume that $r_1>0$ and let $\nu\in\mathbb{N}$ be such that $N^\nu=0$ and $N^{\nu-1}\neq 0$. Since $\tilde{E}_{22}$ is invertible and (7) holds, we may without loss of generality assume that $\tilde{E}_{22}=I_{r_2}$ and $\tilde{E}_{12}=\tilde{A}_{12}=0$. If the latter is not satisfied it can always be achieved by a straightforward transformation. Therefore, (6) takes the form

$$\begin{aligned}&[\tilde{S}E_{11}\tilde{T},\tilde{S}A_{11}\tilde{T},\tilde{S}B_1,C_1\tilde{T}] = \\ &\left[\begin{bmatrix} N & 0 & \tilde{E}_{13} & \tilde{E}_{14}\\ 0 & I_{r_2} & \tilde{E}_{23} & \tilde{E}_{24}\\ 0 & 0 & I_k & 0\\ 0&0&0&0\end{bmatrix},\begin{bmatrix} I_{r_1} & 0 & \tilde{A}_{13} & \tilde{A}_{14}\\ 0 & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24}\\ 0&0&J_{11}&J_{12}\\ 0&0&J_{21}&J_{22}\end{bmatrix},\right.\\ &\qquad\left.\begin{bmatrix}\tilde{B}_1\\ \tilde{B}_2\\ \tilde{B}_3\\ \tilde{B}_4\end{bmatrix},[0,0,\tilde{C}_{31},\tilde{C}_{32}]\right].\end{aligned} \tag{8}$$

Step 3a: Since the observer $[E_o,A_o,B_o,C_o,D_o]$ is index-1, we find that it does not differentiate the input and output of the system $[E,A,B,C,D]$, cf. also (3), that is

$$\begin{aligned}&\forall T>0\ \exists C(T)>0\ \forall t\in[0,T]\\ &\quad \forall (x_o,\left(\begin{smallmatrix} u\\ y\end{smallmatrix}\right),z)\in\mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}\cap\mathscr{C}^\infty(\mathbb{R};\mathbb{R}^{n_o+m+p+n}):\\ &\qquad \|z(t)\|\le C(T)\left(\|x_o(0)\| + \max_{0\le s\le t}\left\|\left(\begin{smallmatrix} u(s)\\ y(s)\end{smallmatrix}\right)\right\|\right).\end{aligned} \tag{9}$$

In the following we relate the solutions of the system to those of the observer to show that the solutions cannot contain derivatives of the input. We consider $(x,u,y)\in\mathfrak{B}_{[E,A,B,C,D]}$ with the following properties:

(i) $(x,u,y)\in\mathscr{C}^\infty(\mathbb{R};\mathbb{R}^{n+m+p})$,

(ii) $\left[\begin{smallmatrix}\tilde{T}^{-1} & 0\\ 0 & I_{n_2+n_3}\end{smallmatrix}\right]T^{-1}x = \begin{pmatrix}\tilde{x}\\ 0\end{pmatrix}$ and $\tilde{x}=(x_1^\top,\dots,x_4^\top)^\top$ according to the partitioning in (8),

(iii) $\tilde{x}(0)=0$ and $u(0)=0$.

We define the following nonempty subset of $\mathfrak{B}_{[E,A,B,C,D]}$,

$$\overline{\mathfrak{B}}_{[E,A,B,C,D]} := \left\{\ (x,u,y)\in\mathfrak{B}_{[E,A,B,C,D]}\ \middle|\ \text{(i)–(iii)}\ \right\}.$$

Let $(x,u,y)\in\overline{\mathfrak{B}}_{[E,A,B,C,D]}$. By Proposition 2.6 there exist $x_o\in\mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{n_o})$ and $z\in\mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$ such that

$$(x_o,\left(\begin{smallmatrix} u\\ y\end{smallmatrix}\right),z)\in\mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]} \quad\text{and}\quad E_ox_o(0)=0.$$

As sE_o-A_o has index at most one it further follows that x_o and z are smooth. By $x(0)=0$ and $u(0)=0$ it follows that $y(0)=0$, thus $x_o(0)=0$ (by the index-1 property) and $z(0)=0$. We may now conclude from the definition of an observer that $z=x$. Then using $\dot{x}_3=J_{11}x_3+J_{12}x_4+\tilde{B}_3u$, $y=\tilde{C}_{31}x_3+\tilde{C}_{32}x_4+Du$ and (9) we find that

$$\begin{aligned}&\forall T>0\ \exists C(T)>0\ \forall (x,u,y)\in\overline{\mathfrak{B}}_{[E,A,B,C,D]}\ \forall t\in[0,T]:\\ &\qquad \|x(t)\|\le C(T)\max_{0\le s\le t}\left\|\left(\begin{smallmatrix} x_4(s)\\ u(s)\end{smallmatrix}\right)\right\|.\end{aligned} \tag{10}$$

Step 3b: We show that x_4 and u can be chosen freely in a certain sense. Observe that the subsystem

$$\left[\begin{bmatrix} I_k & 0\\ 0 & 0\end{bmatrix},\begin{bmatrix} J_{11} & J_{12}\\ J_{21} & J_{22}\end{bmatrix},\begin{bmatrix}\tilde{B}_3\\ \tilde{B}_4\end{bmatrix},[\tilde{C}_{31},\tilde{C}_{32}],D\right]\in\Sigma_{r_3,q_3,m,p} \tag{11}$$

is completely controllable since $[E_{11},A_{11},B_1,C_1,D]$ is completely controllable. Choose $\hat{W}\in\mathbf{Gl}_{r_3-k}(\mathbb{R})$ such that

$$\hat{W}[J_{21},J_{22},\tilde{B}_4] = \begin{bmatrix}\tilde{J}_{21} & \tilde{J}_{22} & \tilde{B}_{41}\\ 0&0&0\end{bmatrix}$$

with $\tilde{J}_{21}\in\mathbb{R}^{k_2\times k}, \tilde{J}_{22}\in\mathbb{R}^{k_2\times(q_3-k)}, \tilde{B}_{41}\in\mathbb{R}^{k_2\times m}$ and $\operatorname{rk}[\tilde{J}_{21},\tilde{J}_{22},\tilde{B}_{41}]=k_2$. Then complete controllability yields

$$\operatorname{rk}\begin{bmatrix} I_k & \tilde{B}_3\\ 0 & \tilde{B}_{41}\\ 0&0\end{bmatrix} = \operatorname{rk}\begin{bmatrix} I_k & J_{11} & J_{12} & \tilde{B}_3\\ 0 & \tilde{J}_{21} & \tilde{J}_{22} & \tilde{B}_{41}\\ 0&0&0&0\end{bmatrix} = k+k_2,$$

and hence

$$\operatorname{rk}\tilde{B}_{41} = k_2.$$

Then there exist $F_1 \in \mathbb{R}^{m\times k}, F_2 \in \mathbb{R}^{m\times(q_3-k)}$ such that

$$[\tilde{J}_{21}, \tilde{J}_{22}] = \tilde{B}_{41}[F_1, F_2].$$

Therefore, applying the feedback

$$u(t) = -F_1 x_3(t) - F_2(t) x_4(t) + v(t), \tag{12}$$

where $v \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^m)$ to the DAE associated with system (11), we obtain

$$\begin{aligned} \dot{x}_3(t) &= (J_{11} - \tilde{B}_3 F_1)x_3(t) + (J_{12} - \tilde{B}_3 F_2)x_4(t) + v(t) \\ 0 &= \tilde{B}_{41} v(t). \end{aligned} \tag{13}$$

This proves the following statement:
For all $x_4 \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^{q_3-k})$, $v \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^m)$ with $x_4(0) = 0$, $v(0) = 0$, $\tilde{B}_{41} v = 0$ and the unique solution (x_1, x_2, x_3) of

$$\begin{bmatrix} N & 0 & \tilde{E}_{13} \\ 0 & I_{r_2} & \tilde{E}_{23} \\ 0 & 0 & I_k \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} I_{r_1} & 0 & \tilde{A}_{13} \\ 0 & \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & 0 & J_{11} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} \tilde{E}_{14} & \tilde{A}_{14} & \tilde{B}_1 \\ \tilde{E}_{24} & \tilde{A}_{24} & \tilde{B}_2 \\ 0 & J_{12} & \tilde{B}_3 \end{bmatrix} \begin{pmatrix} \dot{x}_4 \\ x_4 \\ u \end{pmatrix}$$

with $x_1(0) = 0$, $x_2(0) = 0$, $x_3(0) = 0$ and $u = -F_1 x_3 - F_2 x_4 + v$ as well as $y = \tilde{C}_{31} x_3 + \tilde{C}_{32} x_4 + Du$ we have, using the notation in (ii), $(x,u,y) \in \overline{\mathfrak{B}}_{[E,A,B,C,D]}$. Note that the condition $x_1(0) = 0$ may restrict the initial values of the derivatives of x_4 and u, which is only a slight restriction of their free choice.

Step 3c: In order to exploit the inequality (10) we consider arbitrary $(x,u,y) \in \overline{\mathfrak{B}}_{[E,A,B,C,D]}$. Using the partitioning in (ii) and equation (8) it follows that

$$\begin{aligned} N\dot{x}_1 + \tilde{E}_{13}\dot{x}_3 + \tilde{E}_{14}\dot{x}_4 &= x_1 + \tilde{A}_{13}x_3 + \tilde{A}_{14}x_4 + \tilde{B}_1 u, \\ \dot{x}_3 &= J_{11}x_3 + J_{12}x_4 + \tilde{B}_3 u, \\ 0 &= J_{21}x_3 + J_{22}x_4 + \tilde{B}_4 u. \end{aligned}$$

We ignore the equation for x_2 since it is always solvable provided the above equations are solvable. Since $x_3(0) = 0$ we calculate

$$x_3^{(j)}(t) = \int_0^t J_{11}^j e^{J_{11}(t-s)} \psi(s)\,\mathrm{d}s + \sum_{i=1}^{j} J_{11}^{i-1} \psi^{(j-i)}(t)$$

for all $j \in \mathbb{N}_0$ and all $t \in \mathbb{R}$, where

$$\psi(t) = J_{12}x_4(t) + \tilde{B}_3 u(t), \quad t \in \mathbb{R}.$$

Therefore,

$$\begin{aligned} x_1(t) = &-\sum_{k=0}^{\nu-1} \left(N \tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k \Big((\tilde{A}_{13} - \tilde{E}_{13}J_{11})x_3(t) + (\tilde{B}_1 - \tilde{E}_{13}\tilde{B}_3)u(t) \\ &+ (\tilde{A}_{14} - \tilde{E}_{13}J_{12})x_4(t) - \tilde{E}_{14}\dot{x}_4(t) \Big) \\ = &-\sum_{k=0}^{\nu-1} N^k \Big((\tilde{A}_{14} - \tilde{E}_{13}J_{12})x_4^{(k)}(t) - \tilde{E}_{14}x_4^{(k+1)}(t) \\ &+ (\tilde{B}_1 - \tilde{E}_{13}\tilde{B}_3)u^{(k)}(t) \\ &+ (\tilde{A}_{13} - \tilde{E}_{13}J_{11}) \sum_{i=1}^{k} J_{11}^{i-1} \big(J_{12}x_4^{(k-i)}(t) + \tilde{B}_3 u^{(k-i)}(t) \big) \Big) \\ &- \int_0^t \sum_{k=0}^{\nu-1} N^k (\tilde{A}_{13} - \tilde{E}_{13}J_{11}) J_{11}^k e^{J_{11}(t-s)} \big(J_{12}x_4(s) \\ &\qquad + \tilde{B}_3 u(s) \big)\,\mathrm{d}s \end{aligned}$$

for all $t \in \mathbb{R}$. We may now show that $\tilde{E}_{14} = 0$. If $\tilde{E}_{14} \neq 0$ (and $q_3 - k > 0$), then, due to the free choice of x_4 as proved in Step 3b, we may choose a sequence $(x^k, u^k, y^k) \in \overline{\mathfrak{B}}_{[E,A,B,C,D]}$ such that $\|x_4^k\|_\infty \leq 1$ and $\sup_{t\in[0,1]} \|\tilde{E}_{14}\dot{x}_4^k(t)\| \to \infty$ for $k \to \infty$. Furthermore, by (12) and (13) and choosing v^k such that $\|v^k\|_\infty \leq 1$, we can guarantee that $\sup_{k\in\mathbb{N}} \sup_{t\in[0,1]} \|u^k(t)\| < \infty$. This contradicts (10) and hence $\tilde{E}_{14} = 0$.

Step 3d: Using a similar argument as above, we can show that x_1 cannot depend on derivatives of x_4. Furthermore, according to Step 3b, using the free choice of v^k under the condition $\tilde{B}_{41}v^k = 0$ in the sequence $(x^k, u^k, y^k) \in \overline{\mathfrak{B}}_{[E,A,B,C,D]}$ we can show that x_1 cannot depend on derivatives of u as well. Note that if $\tilde{B}_{41}$ has full column rank (i.e., $v = 0$), then $u = -F_1 x_3 - F_2 x_4$ and thus derivatives of u involve derivatives of x_4 which have already been excluded. Hence, x_1 must be of the form

$$\begin{aligned} x_1(t) = &(\tilde{E}_{13}\tilde{B}_3 - \tilde{B}_1)u(t) + (\tilde{E}_{13}J_{12} - \tilde{A}_{14})x_4(t) \\ &+ \sum_{k=1}^{\nu-1} N^k (\tilde{E}_{13}J_{11} - \tilde{A}_{13}) J_{11}^{k-1} \big(J_{12}x_4(t) + \tilde{B}_3 u(t) \big) \\ &+ \sum_{k=0}^{\nu-1} N^k (\tilde{E}_{13}J_{11} - \tilde{A}_{13}) J_{11}^k x_3(t). \end{aligned} \tag{14}$$

Step 3e: By complete controllability of $[E_{11}, A_{11}, B_1, C_1, D]$ we have

$$\left\{ \tilde{x}(t) \;\middle|\; (x,u,y) \in \overline{\mathfrak{B}}_{[E,A,B,C,D]} \right\} = \mathbb{R}^{n_1} \tag{15}$$

for all $t > 0$. Note that in $\overline{\mathfrak{B}}_{[E,A,B,C,D]}$ only inputs with $u(0) = 0$ are considered. However, reachability of every state is still guaranteed which can be seen as follows. For controllable ODE systems this is a simple exercise.[1] For completely controllable !!! DAE systems this can then be concluded from the feedback form [5, Thm. 3.3]. Multiplying (14) by $N^{\nu-1}$ from the left

[1] Tatsächlich folgt das aus einer Übungsaufgabe die ich immer in der Systemtheorie mache, habe momentan aber noch keine vernünftige Quelle.

it follows that for all $(x,u,y) \in \overline{\mathfrak{B}}_{[E,A,B,C,D]}$ we have

$$M_1 \begin{pmatrix} x_1(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = M_2 u(t)$$

for all $t \in \mathbb{R}$, where

$$\begin{aligned} M_1 &:= \left[N^{\nu-1}, N^{\nu-1}(\tilde{E}_{13}J_{11} - \tilde{A}_{13}), N^{\nu-1}(\tilde{E}_{13}J_{12} - \tilde{A}_{14})\right], \\ M_2 &:= N^{\nu-1}(\tilde{E}_{13}\tilde{B}_3 - \tilde{B}_1). \end{aligned}$$

Then it follows from (15) that

$$\operatorname{im} M_1 \subseteq \operatorname{im} M_2,$$

and it is a simple calculation that

$$\operatorname{im} N^{\nu-1} = \operatorname{im} M_1 \subseteq \operatorname{im} M_2 \subseteq \operatorname{im} N^{\nu-1},$$

thus

$$\operatorname{im} M_2 = \operatorname{im} N^{\nu-1}.$$

Since $N^{\nu-1} \neq 0$ this implies $M_2 \neq 0$. By Step 3b, and invoking (12) and (13), it is then possible to find a sequence $(x^k,u^k,y^k) \in \overline{\mathfrak{B}}_{[E,A,B,C,D]}$ with $\sup_{k\in\mathbb{N}} \sup_{t\in[0,1]} \|u^k(t)\| < \infty$ and $\sup_{t\in[0,1]} \|N^{\nu-1}(\tilde{E}_{13}\tilde{B}_3 - \tilde{B}_1)(\frac{\mathrm{d}}{\mathrm{d}t})^{\nu-1}u^k(t)\| \to \infty$ for $k \to \infty$. If $u = -F_1x_3 - F_2x_4$, then we may choose x_4 accordingly. This contradicts (10) and proves that $N = 0$.

Step 4: We show that $[E_{11}, A_{11}, B_1, C_1, D]$ is impulse observable. Since impulse observability is invariant under equivalence transformations it is sufficient to show that the system in (8) is impulse observable. We calculate that

$$\ker(\tilde{S}E_{11}\tilde{T})^\top = \operatorname{im} \begin{bmatrix} I_{r_1} & 0 \\ 0 & 0 \\ -\tilde{E}_{13}^\top & 0 \\ 0 & I_{r_3-k} \end{bmatrix} =: \operatorname{im} Z.$$

Then

$$\begin{bmatrix} \tilde{S}E_{11}\tilde{T} \\ Z^\top(\tilde{S}A_{11}\tilde{T}) \\ C_1\tilde{T} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \tilde{E}_{13} & 0 \\ 0 & I_{r_2} & \tilde{E}_{23} & \tilde{E}_{24} \\ 0 & 0 & I_k & 0 \\ 0 & 0 & 0 & 0 \\ I_{r_1} & 0 & \tilde{A}_{13} - \tilde{E}_{13}J_{11} & \tilde{A}_{14} - \tilde{E}_{13}J_{12} \\ 0 & 0 & J_{21} & J_{22} \\ 0 & 0 & \tilde{C}_{31} & \tilde{C}_{32} \end{bmatrix}$$

and the latter matrix has full column rank since $\tilde{C}_{32}$ has full column rank. Then impulse observability follows from [8, Cor. 6.6] and this finishes the proof of 2) $\Rightarrow$ 3).

3) $\Leftrightarrow$ 4): For T as in (2) we have, see [11],

$$\operatorname{im} T \begin{bmatrix} I_{n_1} \\ 0 \\ 0 \end{bmatrix} = \mathscr{V}^*_{[E,A,B]} \cap \mathscr{W}^*_{[E,A,B]}.$$

Let $T_1 := T \begin{bmatrix} I_{n_1} \\ 0 \\ 0 \end{bmatrix}$. Then

$$\begin{aligned} \ker C \cap \operatorname{im} T_1 &= \{\, T_1x \mid CT_1x = 0 \,\} = T_1 \ker C_1, \\ \ker E \cap \operatorname{im} T_1 &= \{\, T_1x \mid ET_1x = 0 \,\} = T_1 \{\, x \mid SET_1x = 0 \,\} \\ &= T_1 \ker \begin{bmatrix} E_{11} \\ 0 \\ 0 \end{bmatrix} = T_1 \ker E_{11}, \end{aligned}$$

and

$$\begin{aligned} &A^{-1}(\operatorname{im} E) \cap \operatorname{im} T_1 = \{\, T_1x \mid \exists y \in \mathbb{R}^n : \ AT_1x = Ey \,\} \\ &= T_1 \left\{\, x \;\middle|\; \exists y \in \mathbb{R}^n : \ SAT_1x = SETT^{-1}y \,\right\} \\ &= T_1 \left\{\, x \;\middle|\; \exists \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = T^{-1}y : \ \begin{pmatrix} A_{11}x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} E_{11}y_1 + E_{12}y_2 + E_{13}y_3 \\ E_{22}y_2 + E_{23}y_3 \\ E_{33}y_3 \end{pmatrix} \,\right\} \\ &\overset{(*)}{=} T_1 \{\, x \mid \exists y_1 \in \mathbb{R}^{n_1} : \ A_{11}x = E_{11}y_1 \,\} = T_1A_{11}^{-1}(\operatorname{im} E_{11}), \end{aligned}$$

where equality in $(*)$ follows from the fact that by Theorem 2.2 $E_{33}y_3 = 0$ implies $y_3 = 0$ and $E_{22}y_2 = 0$ implies $y_2 = 0$. Therefore, we find

$$\begin{aligned} &\mathscr{V}^*_{[E,A,B]} \cap \mathscr{W}^*_{[E,A,B]} \cap \ker E \cap A^{-1}(\operatorname{im} E) \cap \ker C \\ &= T_1 \ker E_{11} \cap T_1A_{11}^{-1}(\operatorname{im} E_{11}) \cap T_1 \ker C_1 \\ &= T_1 \left(\ker E_{11} \cap A_{11}^{-1}(\operatorname{im} E_{11}) \cap \ker C_1\right). \end{aligned}$$

Now, $[E_{11}, A_{11}, B_1, C_1, D]$ is impulse observable if, and only if,

$$\ker E_{11} \cap A_{11}^{-1}(\operatorname{im} E_{11}) \cap \ker C_1 = \{0\}$$

and the statement follows from full column rank of T_1.

1') $\Leftrightarrow$ 2') follows from Proposition 2.7 and 3') $\Leftrightarrow$ 4') is a consequence of 3) $\Leftrightarrow$ 4). It remains to show 2') $\Leftrightarrow$ 3').

3') $\Rightarrow$ 2'): We modify the proof of "3) $\Rightarrow$ 2)". Since $[E,A,B,C,D]$ is behaviorally detectable, it follows that $[\tilde{E},\tilde{A},\tilde{B},\tilde{C},D]$ is behaviorally detectable as well. Then, by [6, Thm. 3.8], the index-1 observer $[E_o^1, A_o^1, B_o^1, C_o^1, D_o^1]$ can be chosen to be asymptotic, and hence $[E_o, A_o, B_o, C_o, D_o]$ is an asymptotic index-1 observer for $[E,A,B,C,D]$.

2') $\Rightarrow$ 3'): By 2) $\Leftrightarrow$ 3) the completely controllable part $[E_{11}, A_{11}, B_1, C_1, D]$ is impulse observable. Furthermore, since there exists an asymptotic observer for $[E,A,B,C,D]$, it follows from [6, Thm. 3.5] that $[E,A,B,C,D]$ is behaviorally detectable. □

4. Conclusion

In the present paper we have considered the observer design approach to DAE systems introduced in [6]. We have shown that a necessary and sufficient condition for the existence of an ODE observer is that the completely controllable part of the system is impulse observable; and that the observer is moreover asymptotic if, and only if, the system is additionally behaviorally detectable. Our proof is constructive, and together with the proof presented in [6, Thm. 3.8] an algorithm for the construction of an (asymptotic) ODE observer may be provided.

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