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Abstract

We study the disturbance decoupled estimation problem for linear differential-algebraic systems which are not necessarily regular. We introduce the notion of partial state observers following a recent approach to observer design motivated by considerations for behavioral systems. In our framework, a partial state observer is itself a differential-algebraic system. We derive a characterization for existence of (asymptotic) partial state observers. Exploiting the freedom in the proposed observer design, we derive a solution of the disturbance decoupled estimation problem. The characterization of solvability is obtained via geometric conditions in terms of the generalized Wong sequences.

Keywords: Differential-algebraic systems; descriptor systems; disturbance decoupled estimation; observer design; Wong sequences.

1 Introduction

We study linear time-invariant systems (the *plant*) given by differential-algebraic equations (DAEs) of the form

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) &= Ax(t) + Bu(t)\,, \\ y(t) &= C_1x(t) + D_1u(t)\,, \\ z(t) &= C_2x(t) + D_2u(t)\,, \end{aligned} \tag{1}$$

where $E,A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$, $C_i \in \mathbb{R}^{p_i\times n}$, $D_i \in \mathbb{R}^{p_i\times m}$, $i = 1,2$. Systems of that type are also called *descriptor systems*. The set of DAE systems (1) is denoted by Σ_{l,n,m,p_1,p_2} and we write $[E,A,B,C_1,C_2,D_1,D_2] \in \Sigma_{l,n,m,p_1,p_2}$. Systems of the form (1) occur when modeling dynamical systems subject to algebraic constraints; for a further motivation see [5, 19, 31, 32, 38] and the references therein. In the present paper we do not assume that $sE-A$ is *regular*, which would mean that $l = n$ and $\det(sE-A) \in \mathbb{R}[s]\setminus\{0\}$.

The functions $u:\mathbb{R}\to\mathbb{R}^m$, $y:\mathbb{R}\to\mathbb{R}^{p_1}$ and $z:\mathbb{R}\to\mathbb{R}^{p_2}$ are called *input*, *measurement output* and *controlled output* of the system (1), resp. We call $x:\mathbb{R}\to\mathbb{R}^n$ the *state* although, strictly speaking, $x(t)$ is in general not a state in the sense that the free system (i.e., $u = 0$) can be arbitrarily initialized [28,

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Sec. 2.2]; however, $x(t)$ contains the full information about the system at time t. The tuple (x,u,y,z) : $\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{p_1}\times\mathbb{R}^{p_2}$ is said to be a *solution* of (1), if it belongs to the *behavior* of (1):

$$\mathscr{B}_{[E,A,B,C_1,C_2,D_1,D_2]} := \left\{ (x,u,y,z)\in\mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{p_1}\times\mathbb{R}^{p_2}) \,\middle|\, \begin{array}{l} Ex\in\mathscr{AC}(\mathbb{R}\to\mathbb{R}^l) \text{ and } (x,u,y,z) \\ \text{satisfies (1) for almost all } t\in\mathbb{R} \end{array} \right\}.$$

DAE control systems based on the above behavior have been studied in detail e.g. in [5]. For the analysis of DAE systems in Σ_{l,n,m,p_1,p_2} we assume that the states, inputs and outputs of the system are fixed a priori by the designer. This is different from other approaches based on the behavioral setting [16, 21].

Observer design for general differential-algebraic systems has been investigated recently in [9]. It is revealed that there is still a lot of freedom in the choice of the parameters which constitute an observer. This freedom may be exploited, for instance, for the construction of an observer such that the observation error does not depend on disturbances which might occur in the plant. In the present paper we consider this so called *disturbance decoupled estimation problem* (DDEP) for differential-algebraic systems. To this end, we consider disturbances of the plant (1), i.e., for given disturbance matrices $Q_1\in\mathbb{R}^{l\times q}, Q_2\in\mathbb{R}^{p_1\times q}, Q_3\in\mathbb{R}^{p_2\times q}$ we consider the system

$$\Sigma : \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t)+Bu(t)+Q_1d(t)\,, \\ \quad\;\; y(t) = C_1x(t)+D_1u(t)+Q_2d(t)\,, \\ \quad\;\; z(t) = C_2x(t)+D_2u(t)+Q_3d(t)\,, \end{cases} \tag{2}$$

where $d\in\mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^q)$ represents a smooth disturbance, which may be induced by noise, modeling or measuring errors, or by higher terms in linearization. A (partial state) observer is a dynamical system whose input is composed of the input u and the measurement output y of the plant. The output of the to-be-built dynamical system will be a variable $\hat{z}$ which approximates the controlled output z in a certain sense. We consider observers which are themselves DAEs given by

$$\Sigma_{\mathrm{obs}} : \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}E_ox_o(t) = A_ox_o(t)+B_o\begin{pmatrix} u(t) \\ y(t)\end{pmatrix}, \\ \qquad\;\; \hat{z}(t) = C_ox_o(t)+D_o\begin{pmatrix} u(t) \\ y(t)\end{pmatrix}, \end{cases} \tag{3}$$

with $[E_o,A_o,B_o,C_o,0,D_o,0]\in\Sigma_{l_o,n_o,m+p_1,p_2,0}$; the situation is depicted in Figure 1. For brevity we set $\Sigma_{l_o,n_o,m+p_1,p_2} := \Sigma_{l_o,n_o,m+p_1,p_2,0}$ and write $[E_o,A_o,B_o,C_o,D_o]\in\Sigma_{l_o,n_o,m+p_1,p_2}$ as well as $(x_o,\binom{u}{y},\hat{z})\in\mathscr{B}_{[E_o,A_o,B_o,C_o,D_o]}$ for a solution of (3).

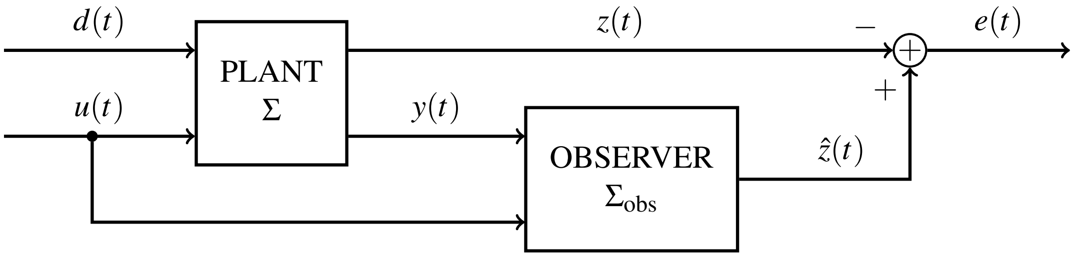

Figure 1: Interconnection of plant and observer

The idea of partial state observation goes back to Luenberger [34, 35], who showed that the state x of any linear system driven by the state w of a free linear system (i.e., $u=0$) is a linear combination of w, i.e., $x = Tw$ provided that $x(0) = Tw(0)$ for some matrix T. This idea has been picked up again in the context of disturbance decoupled estimation or, what is the same, unknown input observer design, see [17, 29, 25, 27]; this problem seems to have been first treated by Basile and Marro [2, 3]. In contrast to this, the idea of designing observers for a pre-defined linear combination of the state, the controlled output, has been first formulated in [1, 39, 43] and the corresponding DDEP has been solved by Willems and Commault [43]. Another version, the *almost DDEP*, has been considered by Willems [42]. In practice, disturbance decoupled observers are important e.g. in the design of fault detection and isolation observers [26].

Observer design for DAE systems is the topic of recent (survey) articles [9, 18, 20], see also the references therein. Partial state observers for DAEs have been first considered in [23], where they are called *Kx-observers* for the case $z = Kx$. The DDEP for DAEs using Luenberger observers in the case $z = x$ has been considered in [24, 30, 36, 46], where in [24, 36, 46] $sE - A$ is required to be regular and in [30] also singular $sE - A$ is allowed. The DDEP with DAE observers of the form (3) has been treated in [22, 23], where in [23] it is assumed that $sE - A$ and $sE_o - A_o$ are regular and $z = x$, and in [22] it is assumed that both $sE - A$ and $sE_o - A_o$ are regular and of index at most one. In the present paper we consider the general setting.

The contribution of this paper is twofold. A rigorous definition of partial state observers is not available in the literature, not even for ODE systems, hence we introduce this concept following the behavioral approach as in [9, 41]. Thereafter, we derive a characterization for existence of (asymptotic) partial state observers for DAE systems. The second contribution of this paper is the solution of the DDEP for DAEs. To this end, we introduce the notion of disturbance decoupled partial state observers and characterize existence. The aforementioned characterizations are completely geometric with which we follow the classical approach of geometric control theory, see e.g. the textbooks [4, 40, 45].

The present paper is organized as follows: In Section 2 we recall the generalized Wong sequences, which are the crucial geometric tool for the characterization of solvability of the DDEP. The notion of partial state observers for DAE systems is introduced and characterized in Section 3. In Section 4 we define when a partial state observer is called disturbance decoupled and show how feedthrough terms in the plant can be treated. Solvability of the DDEP for DAEs, i.e., existence of a disturbance decoupled partial state observer, is then characterized in Section 5. For completeness, we consider the problem of disturbance decoupled asymptotic estimation in Section 6.

Nomenclature

$\mathbb{N}$, $\mathbb{N}_0$		the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{C}_+ (\mathbb{C}_-)$		open set of complex numbers with positive (negative) real part, resp.
$\mathbb{R}[s]$		the ring of polynomials with coefficients in $\mathbb{R}$
$R^{n \times m}$		the set of $n \times m$ matrices with entries in a ring R
$\operatorname{im}_R A$, $\ker_R A$, $\operatorname{rk}_R A$		image, kernel and rank of the matrix $A \in R^{n \times m}$, resp.
$\|x\|$	$=$	$\sqrt{x^\top x}$, the Euclidean norm of $x \in \mathbb{R}^n$
$M\mathscr{S}$	$=$	$\{ x \in \mathbb{R}^l \mid x \in \mathscr{S} \}$, the image of $\mathscr{S} \subseteq \mathbb{R}^n$ under $M \in \mathbb{R}^{l \times n}$
$M^{-1}\mathscr{S}$	$=$	$\{ x \in \mathbb{R}^n \mid Mx \in \mathscr{S} \}$, the pre-image of the set $\mathscr{S} \subseteq \mathbb{R}^l$ under $M \in \mathbb{R}^{l \times n}$
$\mathscr{C}^\infty(\mathbb{R} \to \mathbb{R}^n)$		the set of infinitely-times continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}^n$

$\mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}\to\mathbb{R}^n)$	the set of locally Lebesgue integrable functions $f:\mathbb{R}\to\mathbb{R}^n$, where $\int_K\|f(t)\|\,\mathrm{d}t<\infty$ for all compact $K\subseteq\mathbb{R}$
$\mathscr{AC}(\mathbb{R}\to\mathbb{R}^n)$	the set of absolutely continuous functions $f:\mathbb{R}\to\mathbb{R}^n$
$f\overset{\text{a.e.}}{=}g$	the functions $f,g\in\mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$ are equal "almost everywhere", i.e., $f(t)=g(t)$ for almost all (a.a.) $t\in\mathbb{R}$
$\operatorname{ess\,sup}_I\|f\|$	the essential supremum of the measurable function $f:\mathbb{R}\to\mathbb{R}^n$ over $I\subseteq\mathbb{R}$

2 Generalized Wong sequences

In this section we introduce the crucial geometric tools for the solution of the DDEP. For $E,A\in\mathbb{R}^{l\times n},B\in\mathbb{R}^{l\times m},C\in\mathbb{R}^{p\times n}$ we define the sequences

$$\begin{aligned}
\mathscr{V}^0_{[E,A,B,C]}&=\ker C, & \mathscr{V}^{i+1}_{[E,A,B,C]}&=A^{-1}(E\mathscr{V}^i_{[E,A,B,C]}+\operatorname{im}B)\cap\ker C,\quad i\geq 0,\\
\mathscr{W}^0_{[E,A,B,C]}&=\{0\}, & \mathscr{W}^{i+1}_{[E,A,B,C]}&=E^{-1}(A\mathscr{W}^i_{[E,A,B,C]}+\operatorname{im}B)\cap\ker C,\quad i\geq 0.
\end{aligned}$$

The sequence $(\mathscr{V}^i_{[E,A,B,C]})_{i\in\mathbb{N}_0}$ is non-increasing and $(\mathscr{W}^i_{[E,A,B,C]})_{i\in\mathbb{N}_0}$ is non-decreasing and both sequences terminate after finitely many steps, thus we may set

$$\mathscr{V}^*_{[E,A,B,C]}=\bigcap_{i\in\mathbb{N}_0}\mathscr{V}^i_{[E,A,B,C]},\quad \mathscr{W}^*_{[E,A,B,C]}=\bigcup_{i\in\mathbb{N}_0}\mathscr{W}^i_{[E,A,B,C]}.$$

We call the sequences $(\mathscr{V}^i_{[E,A,B,C]})_{i\in\mathbb{N}_0}$ and $(\mathscr{W}^i_{[E,A,B,C]})_{i\in\mathbb{N}_0}$ *generalized Wong sequences*, as introduced in [6]. In [7, 13, 14] the Wong sequences for matrix pencils (i.e., $B=0$ and $C=0$) are investigated, the name chosen this way since Wong [44] was the first who used both sequences for the analysis of matrix pencils. In [8, 10, 15] the case $C=0$ is considered and the sequences $(\mathscr{V}^i_{[E,A,B,0]})_{i\in\mathbb{N}_0}$ and $(\mathscr{W}^i_{[E,A,B,0]})_{i\in\mathbb{N}_0}$ are called *augmented Wong sequences*. Similarly, in [12] the sequences $(\mathscr{V}^i_{[E,A,0,C]})_{i\in\mathbb{N}_0}$ and $(\mathscr{W}^i_{[E,A,0,C]})_{i\in\mathbb{N}_0}$ (i.e., $B=0$) are called *restricted Wong sequences*. For more details on these sequences see the surveys [8, 12] and the references therein.

Note that in geometric control theory for ODE systems (i.e., $E=I$), see e.g. [45], the sequence $(\mathscr{V}^i_{[I,A,B,C]})_{i\in\mathbb{N}_0}$ is called *invariant subspace algorithm*, and the sequence $(\mathscr{W}^i_{[I,A,B,C]})_{i\in\mathbb{N}_0}$ is called *controllability subspace algorithm*.

We need the following technical lemma.

Lemma 2.1. *Let $E,A\in\mathbb{R}^{l\times n}$, $B\in\mathbb{R}^{l\times m}$, $C_i\in\mathbb{R}^{p_i\times n}$, $i=1,2$. Then*

$$\mathscr{V}^*_{\left[\left[\begin{smallmatrix}E&0\\0&0\end{smallmatrix}\right],\left[\begin{smallmatrix}A&B\\C_1&0\end{smallmatrix}\right],0,0\right]}\cap\mathscr{W}^*_{\left[\left[\begin{smallmatrix}E&0\\0&0\end{smallmatrix}\right],\left[\begin{smallmatrix}A&B\\C_1&0\end{smallmatrix}\right],0,0\right]}\subseteq\ker[C_2,0]\quad\Longleftrightarrow\quad\mathscr{V}^*_{[E,A,B,C_1]}\cap\mathscr{W}^*_{[E,A,B,C_1]}\subseteq\ker C_2.$$

Proof. For convenience we define, for $i\in\mathbb{N}_0$,

$$\hat{\mathscr{V}}_i:=\mathscr{V}^i_{\left[\left[\begin{smallmatrix}E&0\\0&0\end{smallmatrix}\right],\left[\begin{smallmatrix}A&B\\C_1&0\end{smallmatrix}\right],0,0\right]},\quad \mathscr{V}_i:=\mathscr{V}^i_{[E,A,B,C_1]},\quad \hat{\mathscr{W}}_i:=\mathscr{W}^i_{\left[\left[\begin{smallmatrix}E&0\\0&0\end{smallmatrix}\right],\left[\begin{smallmatrix}A&B\\C_1&0\end{smallmatrix}\right],0,0\right]},\quad \mathscr{W}_i:=\mathscr{W}^i_{[E,A,B,C_1]}.$$

Step 1: We show by induction that

$$\forall\,i\in\mathbb{N}_0:\ [I_n,0]\hat{\mathscr{V}}_i=\mathscr{V}_i.$$

For $i=0$ the statement is clearly true, so assume it is true for some $i\in\mathbb{N}_0$. Then

$$\begin{aligned}
[I_n,0]\hat{\mathcal{V}}_{i+1} &= [I_n,0]\left[\begin{smallmatrix} A & B\\ C_1 & 0\end{smallmatrix}\right]^{-1}\left(\left[\begin{smallmatrix} E & 0\\ 0 & 0\end{smallmatrix}\right]\hat{\mathcal{V}}_i\right)\\
&= [I_n,0]\left\{\ \left(\begin{smallmatrix} x_1\\ x_2\end{smallmatrix}\right)\in\mathbb{R}^n\times\mathbb{R}^m\ \middle|\ Ax_1+Bx_2\in E[I_n,0]\hat{\mathcal{V}}_i\ \wedge\ C_1x_1=0\ \right\}\\
&= \{\ x\in\mathbb{R}^n\ |\ \exists y\in\mathbb{R}^m,\ z\in\mathcal{V}_i:\ Ax=By+Ez\ \wedge\ C_1x=0\ \}\\
&= A^{-1}(E\mathcal{V}_i+\operatorname{im}B)\cap\ker C_1=\mathcal{V}_{i+1},
\end{aligned}$$

which proves the assertion.

Step 2: We show that

$$\forall i\geq 1:\ \hat{\mathcal{W}}_i=[I_n,0]\hat{\mathcal{W}}_i\times\mathbb{R}^m.$$

This is immediate from

$$\begin{aligned}
\hat{\mathcal{W}}_i &= \left[\begin{smallmatrix} E & 0\\ 0 & 0\end{smallmatrix}\right]^{-1}\left(\left[\begin{smallmatrix} A & B\\ C_1 & 0\end{smallmatrix}\right]\hat{\mathcal{W}}_{i-1}\right)\\
&= \left\{\ \left(\begin{smallmatrix} x_1\\ x_2\end{smallmatrix}\right)\in\mathbb{R}^n\times\mathbb{R}^m\ \middle|\ \exists\left(\begin{smallmatrix} y_1\\ y_2\end{smallmatrix}\right)\in\hat{\mathcal{W}}_{i-1}:\ Ex_1=Ay_1+By_2\ \wedge\ 0=C_1y_1\ \right\}\\
&= \left\{\ \left(\begin{smallmatrix} x_1\\ x_2\end{smallmatrix}\right)\in\mathbb{R}^n\times\mathbb{R}^m\ \middle|\ x_1\in[I_n,0]\hat{\mathcal{W}}_i\ \right\}
\end{aligned}$$

for $i\geq 1$.

Step 3: We show by induction that

$$\forall i\in\mathbb{N}_0:\ \mathcal{W}_i\subseteq[I_n,0]\hat{\mathcal{W}}_{i+1}\cap\ker C_1\subseteq\mathcal{W}_{i+1}.$$

For $i=0$ we have

$$\mathcal{W}_0=\{0\},\quad [I_n,0]\hat{\mathcal{W}}_1=\ker E\cap\ker C_1,\quad \mathcal{W}_1=E^{-1}(\operatorname{im}B)\cap\ker C_1,$$

and thus the assertion is true in this case. So we assume that it is true for some $i\in\mathbb{N}_0$. Then

$$\begin{aligned}
[I_n,0]\hat{\mathcal{W}}_{i+2}\cap\ker C_1 &= [I_n,0]\left[\begin{smallmatrix} E & 0\\ 0 & 0\end{smallmatrix}\right]^{-1}\left(\left[\begin{smallmatrix} A & B\\ C_1 & 0\end{smallmatrix}\right]\hat{\mathcal{W}}_{i+1}\right)\cap\ker C_1\\
&\overset{\text{Step 2}}{=} [I_n,0]\left\{\ \left(\begin{smallmatrix} x_1\\ x_2\end{smallmatrix}\right)\in\mathbb{R}^n\times\mathbb{R}^m\ \middle|\ Ex_1\in A([I_n,0]\hat{\mathcal{W}}_{i+1}\cap\ker C_1)+\operatorname{im}B\ \right\}\cap\ker C_1\\
&= E^{-1}\big(A([I_n,0]\hat{\mathcal{W}}_{i+1}\cap\ker C_1)+\operatorname{im}B\big)\cap\ker C_1\\
&\supseteq E^{-1}\big(A\mathcal{W}_i+\operatorname{im}B\big)\cap\ker C_1=\mathcal{W}_{i+1}.
\end{aligned}$$

and analogously, just using the opposite inclusion in the line above, we obtain $[I_n,0]\hat{\mathcal{W}}_{i+2}\cap\ker C_1\subseteq\mathcal{W}_{i+2}$.

Step 4: We show the statement of the lemma. It follows from Step 1 and Step 3 that

$$\mathcal{V}^*_{[E,A,B,C_1]}=[I_n,0]\mathcal{V}^*_{\left[\left[\begin{smallmatrix} E & 0\\ 0 & 0\end{smallmatrix}\right],\left[\begin{smallmatrix} A & B\\ C_1 & 0\end{smallmatrix}\right],0,0\right]},\quad \mathcal{W}^*_{[E,A,B,C_1]}=[I_n,0]\mathcal{W}^*_{\left[\left[\begin{smallmatrix} E & 0\\ 0 & 0\end{smallmatrix}\right],\left[\begin{smallmatrix} A & B\\ C_1 & 0\end{smallmatrix}\right],0,0\right]}\cap\ker C_1.$$

Since $\mathcal{V}^*_{[E,A,B,C_1]}\subseteq\ker C_1$ we have

$$\mathcal{V}^*_{[E,A,B,C_1]}\cap\mathcal{W}^*_{[E,A,B,C_1]}=[I_n,0]\mathcal{V}^*_{\left[\left[\begin{smallmatrix} E & 0\\ 0 & 0\end{smallmatrix}\right],\left[\begin{smallmatrix} A & B\\ C_1 & 0\end{smallmatrix}\right],0,0\right]}\cap[I_n,0]\mathcal{W}^*_{\left[\left[\begin{smallmatrix} E & 0\\ 0 & 0\end{smallmatrix}\right],\left[\begin{smallmatrix} A & B\\ C_1 & 0\end{smallmatrix}\right],0,0\right]}.$$

This already implies $\Leftarrow$ in the statement of the lemma. To show $\Rightarrow$ let $x\in\mathcal{V}^*_{[E,A,B,C_1]}\cap\mathcal{W}^*_{[E,A,B,C_1]}$. Then there exists $y\in\mathbb{R}^m$ such that $\left(\begin{smallmatrix} x\\ y\end{smallmatrix}\right)\in\mathcal{V}^*_{\left[\left[\begin{smallmatrix} E & 0\\ 0 & 0\end{smallmatrix}\right],\left[\begin{smallmatrix} A & B\\ C_1 & 0\end{smallmatrix}\right],0,0\right]}$. Furthermore, $x\in[I_n,0]\mathcal{W}^*_{\left[\left[\begin{smallmatrix} E & 0\\ 0 & 0\end{smallmatrix}\right],\left[\begin{smallmatrix} A & B\\ C_1 & 0\end{smallmatrix}\right],0,0\right]}$ and by Step 2 it follows $\left(\begin{smallmatrix} x\\ y\end{smallmatrix}\right)\in\mathcal{W}^*_{\left[\left[\begin{smallmatrix} E & 0\\ 0 & 0\end{smallmatrix}\right],\left[\begin{smallmatrix} A & B\\ C_1 & 0\end{smallmatrix}\right],0,0\right]}$, hence $x\in\ker C_2$ which concludes the proof. $\square$

In order to consider (generalized) eigenspaces of a DAE we need to introduce the following modification of the second Wong sequence. As in [7, 10, 14], for $E,A \in \mathbb{R}^{l\times n}$ and $\lambda \in \mathbb{C}$ we define the sequence of complex subspaces

$$\mathscr{W}^0_{[E,A],\lambda} := \{0\}, \quad \mathscr{W}^{i+1}_{[E,A],\lambda} := (A-\lambda E)^{-1}(E\mathscr{W}^i_{[E,A],\lambda}) \subseteq \mathbb{C}^n, \quad i \geq 0.$$

This sequence is non-decreasing and terminates after finitely many steps, hence we may set

$$\mathscr{W}^*_{[E,A],\lambda} := \bigcup_{i\in\mathbb{N}_0} \mathscr{W}^i_{[E,A],\lambda}.$$

It is a straightforward calculation, see also [14], that

$$\operatorname{span}_{\mathbb{C}}\left(\mathscr{V}^*_{[E,A,0,0]} \cap \mathscr{W}^*_{[E,A,0,0]}\right) \subseteq \mathscr{W}^*_{[E,A],\lambda} \tag{4}$$

for all $\lambda \in \mathbb{C}$, where $\operatorname{span}_{\mathbb{C}}(\mathscr{V})$ denotes the complex span of all the vectors in $\mathscr{V}$.

3 Partial state observers

In this section we introduce the concept of (asymptotic) partial state observers for DAE systems and derive characterizations for their existence. A partial state observer is a dynamical system which aims to reconstruct the controlled output. It should be able to process the signals of the plant without influencing the plant itself. This is subject of the following definition which follows the approach in [9, 41].

Definition 3.1 (Acceptor)**.** Consider a system $[E,A,B,C_1,C_2,D_1,D_2] \in \Sigma_{l,n,m,p_1,p_2}$. Then $[E_o,A_o,B_o,C_o,D_o] \in \Sigma_{l_o,n_o,m+p_1,p_o}$ is called an *acceptor for* $[E,A,B,C_1,C_2,D_1,D_2]$, if for all $(x,u,y,z) \in \mathscr{B}_{[E,A,B,C_1,C_2,D_1,D_2]}$, there exist $x_0 \in \mathscr{L}^1_{\text{loc}}(\mathbb{R}\to\mathbb{R}^{n_o})$, $\hat{z} \in \mathscr{L}^1_{\text{loc}}(\mathbb{R}\to\mathbb{R}^{p_o})$ such that

$$(x_o, \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), \hat{z}) \in \mathscr{B}_{[E_o,A_o,B_o,C_o,D_o]}.$$

The above definition means that there is a one-directed signal flow from the plant to its acceptor via input and measurement output, see Fig. 2. That is, $[E,A,B,C_1,C_2,D_1,D_2]$ may influence $[E_o,A_o,B_o,C_o,D_o]$ but not vice-versa.

The following definition of a partial state observer is a modification of the observer definition in [9].

Definition 3.2 (Partial state observer)**.** Consider a system $[E,A,B,C_1,C_2,D_1,D_2] \in \Sigma_{l,n,m,p_1,p_2}$. Then $[E_o,A_o,B_o,C_o,D_o] \in \Sigma_{l_o,n_o,m+p_1,p_2}$ is called

a) a *partial state observer for* $[E,A,B,C_1,C_2,D_1,D_2]$, if it is an acceptor for $[E,A,B,C_1,C_2,D_1,D_2]$, and

$$\begin{array}{l}\forall\,(x,u,y,z,x_o,\hat{z}) \in \mathscr{L}^1_{\text{loc}}(\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{p_1}\times\mathbb{R}^{p_2}\times\mathbb{R}^{n_o}\times\mathbb{R}^{p_2}): \\ \begin{pmatrix} (x,u,y,z) \in \mathscr{B}_{[E,A,B,C_1,C_2,D_1,D_2]} \quad \wedge \quad (x_o, \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), \hat{z}) \in \\ \mathscr{B}_{[E_o,A_o,B_o,C_o,D_o]} \\ \wedge\, \hat{z}(0) = z(0) \end{pmatrix} \implies \hat{z} \overset{\text{a.e.}}{=} z.\end{array}$$

b) an *asymptotic partial state observer for* $[E,A,B,C_1,C_2,D_1,D_2]$, if it is a partial state observer for $[E,A,B,C_1,C_2,D_1,D_2]$, and

$$\begin{array}{l}\forall\,(x,u,y,z,x_o,\hat{z}) \in \mathscr{L}^1_{\text{loc}}(\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{p_1}\times\mathbb{R}^{p_2}\times\mathbb{R}^{n_o}\times\mathbb{R}^{p_2}): \\ \left((x,u,y,z) \in \mathscr{B}_{[E,A,B,C_1,C_2,D_1,D_2]} \quad \wedge \quad (x_o, \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), \hat{z}) \in \mathscr{B}_{[E_o,A_o,B_o,C_o,D_o]}\right) \\ \implies \quad \lim\limits_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} \|\hat{z} - z\| = 0.\end{array}$$

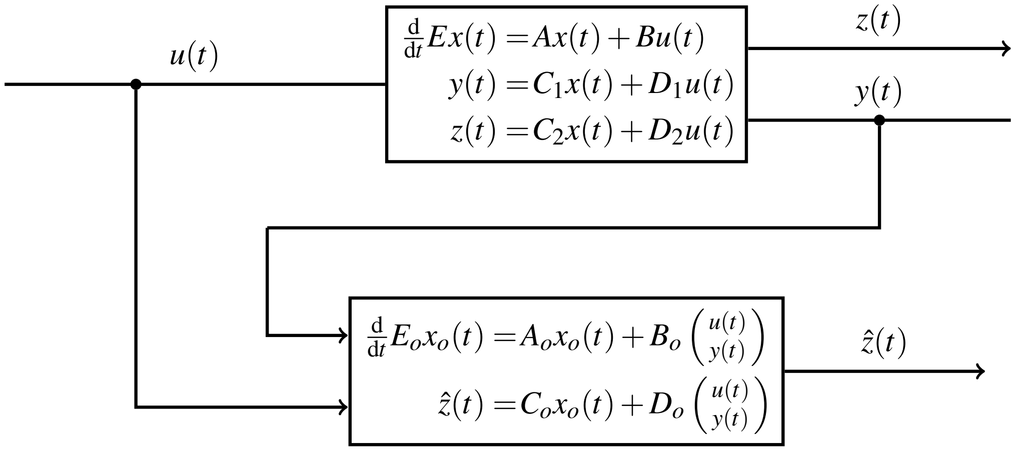

Figure 2: Interconnection with an acceptor

Partial state observers have been discussed in [17, 27, 29, 43] for ODE systems and in [22, 23] for DAE systems. However, mostly observers of Luenberger type are considered and a characterization of existence of partial state observers of the form (3) is still not available in the literature. This is the topic of the remainder of this section.

We propose a partial state observer similar to the observer design introduced in [9]. Given a plant $[E,A,B,C_1,C_2,D_1,D_2] \in \Sigma_{l,n,m,p_1,p_2}$, let $k \in \mathbb{N}_0$, $L_x \in \mathbb{R}^{l\times k}$, $L_y \in \mathbb{R}^{p_1\times k}$ and $L_z \in \mathbb{R}^{p_2\times k}$, and consider the following observer design,

$$\underbrace{\begin{array}{rcl}\frac{\mathrm{d}}{\mathrm{d}t}Ex_o(t) &=& Ax_o(t) + Bu(t)\\ y(t) &=& C_1x_o(t) + D_1u(t)\\ \hat{z}(t) &=& C_2x_o(t) + D_2u(t)\end{array}}_{\text{internal model}}\ \underbrace{\begin{array}{l}+L_xw(t),\\ +L_yw(t),\\ +L_zw(t),\end{array}}_{\text{innovations}} \tag{5}$$

where $\hat{z}$ is the observer output. In other words, we consider a partial state observer of the form (3) with

$$[E_o,A_o,B_o,C_o,D_o] = \left[\left[\begin{smallmatrix}E&0\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}A&L_x\\C_1&L_y\end{smallmatrix}\right], \left[\begin{smallmatrix}B&0\\D_1&-I_m\end{smallmatrix}\right], \left[C_2\ L_z\right], \left[D_2\ 0\right]\right] \in \Sigma_{l+p_1,n+k,m+p_1,p_2}. \tag{6}$$

As depicted in (5), the partial state observer consists of an internal model of the plant driven by *innovations*. This design goes back to Polderman and Willems [37, p. 351] and the innovations "express how far the actual observed output differs from what we would have expected to observe".

The interconnection of $[E,A,B,C_1,C_2,D_1,D_2]$ and $[E_o,A_o,B_o,C_o,D_o]$ is described by the control system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}E&0&0&0\\0&0&0&0\\0&0&E&0\\0&0&0&0\end{bmatrix}\begin{pmatrix}x(t)\\y(t)\\x_o(t)\\w(t)\end{pmatrix} = \begin{bmatrix}A&0&0&0\\C_1&-I&0&0\\0&0&A&L_x\\0&-I&C_1&L_y\end{bmatrix}\begin{pmatrix}x(t)\\y(t)\\x_o(t)\\w(t)\end{pmatrix} + \begin{bmatrix}B\\D_1\\B\\D_1\end{bmatrix}u(t). \tag{7}$$

Introducing the difference $v(t) = x_o(t) - x(t)$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}E&0&0&0\\0&0&0&0\\0&0&E&0\\0&0&0&0\end{bmatrix}\begin{pmatrix}x(t)\\y(t)\\v(t)\\w(t)\end{pmatrix} = \begin{bmatrix}A&0&0&0\\C_1&-I&0&0\\0&0&A&L_x\\0&0&C_1&L_y\end{bmatrix}\begin{pmatrix}x(t)\\y(t)\\v(t)\\w(t)\end{pmatrix} + \begin{bmatrix}B\\D_1\\0\\0\end{bmatrix}u(t) \tag{8}$$

and the observation error satisfies

$$e(t) = \hat{z}(t) - z(t) = C_2 x_o(t) + D_2 u(t) + L_z w(t) - C_2 x(t) - D_2 u(t) = C_2 v(t) + L_z w(t). \tag{9}$$

In particular, the error is the output of the DAE system

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} &= \begin{bmatrix} A & L_x \\ C_1 & L_y \end{bmatrix} \begin{pmatrix} v(t) \\ w(t) \end{pmatrix}, \\ e(t) &= \begin{bmatrix} C_2 & L_z \end{bmatrix} \begin{pmatrix} v(t) \\ w(t) \end{pmatrix}. \end{aligned} \tag{10}$$

Theorem 3.3. *Consider the system* $[E,A,B,C_1,C_2,D_1,D_2] \in \Sigma_{l,n,m,p_1,p_2}$ *and let* $k \in \mathbb{N}_0$, $L_x \in \mathbb{R}^{l \times k}$, $L_y \in \mathbb{R}^{p_1 \times k}$ *and* $L_z \in \mathbb{R}^{p_2 \times k}$*. Then we have the following for the system* $[E_o,A_o,B_o,C_o,D_o]$ *as in* (6)*:*

(i) $[E_o,A_o,B_o,C_o,D_o]$ *is an acceptor for* $[E,A,B,C_1,C_2,D_1,D_2]$*.*

(ii) $[E_o,A_o,B_o,C_o,D_o]$ *is a partial state observer for* $[E,A,B,C_1,C_2,D_1,D_2]$ *if, and only if,*

a) $\mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & L_x \\ C_1 & L_y \end{smallmatrix}\right], 0, [C_2, L_z]\right]} = \mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & L_x \\ C_1 & L_y \end{smallmatrix}\right], 0, 0\right]} \cap \ker[C_2, L_z]$,

b) $\mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & L_x \\ C_1 & L_y \end{smallmatrix}\right], 0, 0\right]} \cap \mathscr{W}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & L_x \\ C_1 & L_y \end{smallmatrix}\right], 0, 0\right]} \subseteq \ker[C_2, L_z]$.

(iii) $[E_o,A_o,B_o,C_o,D_o]$ *is an asymptotic partial state observer for* $[E,A,B,C_1,C_2,D_1,D_2]$ *if, and only if, a) in (ii) holds true and*

$$\forall \lambda \in \overline{\mathbb{C}_+}: \ \mathscr{W}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & L_x \\ C_1 & L_y \end{smallmatrix}\right]\right], \lambda} \subseteq \ker_{\mathbb{C}}[C_2, L_z]. \tag{11}$$

Proof.

(i) The system $[E_o,A_o,B_o,C_o,D_o]$ is an acceptor for $[E,A,B,C_1,C_2,D_1,D_2]$, since for all $(x,u,y,z) \in \mathscr{B}_{[E,A,B,C_1,C_2,D_1,D_2]}$ we have

$$\left(\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), z\right) \in \mathscr{B}_{[E_o,A_o,B_o,C_o,D_o]}.$$

(ii) $\Rightarrow$: Suppose that $[E_o,A_o,B_o,C_o,D_o]$ is a partial state observer for $[E,A,B,C_1,C_2,D_1,D_2]$. Consider a solution (v,w,e) of (10) with $e(0) = 0$. By (7), (8) and (9) we have

$$(0,0,0) \in \mathscr{B}_{[E,A,B,C_1,C_2,D_1,D_2]} \quad \wedge \quad \left(\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), e\right) \in \mathscr{B}_{[E_o,A_o,B_o,C_o,D_o]}. \tag{12}$$

The definition of a partial state observer implies $e \overset{\text{a.e.}}{=} 0$. The statement then follows from Lemma A.2.

$\Leftarrow$: Assume that a) and b) are satisfied and consider $(x,u,y,z) \in \mathscr{B}_{[E,A,B,C_1,C_2,D_1,D_2]}$ and $\left(\left(\begin{smallmatrix} x_0 \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), \hat{z}\right) \in \mathscr{B}_{[E_o,A_o,B_o,C_o,D_o]}$ with $z(0) = \hat{z}(0)$. Then $(v = x_o - x, w, e = \hat{z} - z)$ solves (10) with $e(0) = 0$ and it follows from Lemma A.2 that $e \overset{\text{a.e.}}{=} 0$. This means that $[E_o,A_o,B_o,C_o,D_o]$ in (6) is a partial state observer for $[E,A,B,C_1,C_2,D_1,D_2]$.

(iii) $\Rightarrow$: Assume that $[E_o,A_o,B_o,C_o,D_o]$ in (6) is an asymptotic partial state observer for $[E,A,B,C_1,C_2,D_1,D_2]$. Then a) in (ii) holds true. To show (11), consider a solution (v,w,e) of (10). Then the relations in (12) hold true and the definition of an asymptotic partial state observer gives $\lim_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} \|e\| = 0$. Hence, for all solutions (v,w,e) of (10) we have

$$\lim_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} \left\| [C_2, L_z] \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \right\| = 0.$$

Then Lemma A.4 implies (11).

$\Leftarrow$: Now assume that a) in (ii) and (11) are satisfied. It is a consequence of (4) that (11) implies b) in (ii) and hence $[E_o,A_o,B_o,C_o,D_o]$ in (6) is a partial state observer for $[E,A,B,C_1,C_2,D_1,D_2]$. Now, consider $(x,u,y,z) \in \mathscr{B}_{[E,A,B,C_1,C_2,D_1,D_2]}$ and $\left(\left(\begin{smallmatrix} x_o \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), \hat{z}\right) \in \mathscr{B}_{[E_o,A_o,B_o,C_o,D_o]}$. Then $(v = x_o - x, w, e = \hat{z} - z)$ solves (10) and by condition (11) and Lemma A.4 we find

$$\lim_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} \|e(t)\| = \lim_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} \left\| [C_2, L_z] \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \right\| = 0.$$

The system $[E_o,A_o,B_o,C_o,D_o]$ in (6) is therefore an asymptotic partial state observer for $[E,A,B,C_1,C_2,D_1,D_2]$. $\square$

Remark 3.4. Note that in the partial state observer (6) in almost all cases we have $E_o \neq I$, i.e., the observer is not governed by an ODE, even if the plant is an ODE ($E = I$). However, if the system $[E,A,B,C_1,D_1]$ is impulse observable (for a definition, see e.g. the survey [12]), then $sE_o - A_o$ in (6) can be constructed to have index at most one as shown in [9]. In this case there exists an equivalent observer where $E_o = I$, see also [11].

Next we show that the partial state observer $[E_o,A_o,B_o,C_o,D_o] \in \Sigma_{l+p_1,n+k,m+p_1,p_2}$ in (6) has a universal property in a certain sense: If an (asymptotic) partial state observer exists, then it can be constructed to be of the form (6).

Theorem 3.5. *For $[E,A,B,C_1,C_2,D_1,D_2] \in \Sigma_{l,n,m,p_1,p_2}$ the following holds true:*

(i) *There exists a partial state observer for $[E,A,B,C_1,C_2,D_1,D_2]$ if, and only if,*

a) $\mathscr{V}^*_{\left[\left[\begin{smallmatrix} E \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A \\ C_1 \end{smallmatrix}\right], 0, C_2\right]} = \mathscr{V}^*_{\left[\left[\begin{smallmatrix} E \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A \\ C_1 \end{smallmatrix}\right], 0, 0\right]} \cap \ker C_2,$

b) $\mathscr{V}^*_{\left[\left[\begin{smallmatrix} E \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A \\ C_1 \end{smallmatrix}\right], 0, 0\right]} \cap \mathscr{W}^*_{\left[\left[\begin{smallmatrix} E \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A \\ C_1 \end{smallmatrix}\right], 0, 0\right]} \subseteq \ker C_2.$

(ii) *There exists an asymptotic partial state observer for $[E,A,B,C_1,C_2,D_1,D_2]$ if, and only if, a) in (i) holds true and*

$$\forall \lambda \in \overline{\mathbb{C}_+} : \ \mathscr{W}^*_{\left[\left[\begin{smallmatrix} E \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A \\ C_1 \end{smallmatrix}\right]\right], \lambda} \subseteq \ker_{\mathbb{C}} C_2. \tag{13}$$

Proof. We start with proving "$\Leftarrow$" for (i) and (ii) together: Consider the acceptor $[E_o,A_o,B_o,C_o,D_o] \in \Sigma_{l+p_1,n+k,m+p_1,p_2}$ in (6) with $k = 0$, $L_x = 0_{l,0}$, $L_y = 0_{p_1,0}$ and $L_z = 0_{p_2,0}$. Then, by Theorem 3.3 (ii) (resp. (iii)), $[E_o,A_o,B_o,C_o,D_o]$ is an (asymptotic) partial state observer, if the conditions in a) and b) (resp. a) and (13)) hold true.

It remains to prove "$\Rightarrow$" for (i) and (ii):

(i) Suppose that $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p_1,p_2}$ is a partial state observer for $[E,A,B,C_1,C_2,D_1,D_2]$. Consider $x \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R} \to \mathbb{R}^n)$ with $Ex \in \mathscr{AC}(\mathbb{R} \to \mathbb{R}^l)$ and

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} E \\ 0 \end{bmatrix} x \overset{\text{a.e.}}{=} \begin{bmatrix} A \\ C_1 \end{bmatrix} x \tag{14}$$

such that $C_2 x(0) = 0$. Then $(x, 0, 0, C_2 x) \in \mathscr{B}_{[E,A,B,C_1,C_2,D_1,D_2]}$ and

$$\left(0, \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), 0\right) \in \mathscr{B}_{[E_o,A_o,B_o,C_o,D_o]}. \tag{15}$$

Since $[E_o, A_o, B_o, C_o, D_o]$ is a partial state observer for $[E,A,B,C_1,C_2,D_1,D_2]$ we obtain $C_2 x \overset{\text{a.e.}}{=} 0$. By Lemma A.2 this implies a) and b).

(ii) Suppose that $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p_1,p_2}$ is an asymptotic partial state observer for $[E,A,B,C_1,C_2,D_1,D_2]$. Then a) in (i) holds true. To show (13) consider $x \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R} \to \mathbb{R}^n)$ with $Ex \in \mathscr{AC}(\mathbb{R} \to \mathbb{R}^l)$ which satisfies (14). Then $(x, 0, 0, C_2 x) \in \mathscr{B}_{[E,A,B,C_1,C_2,D_1,D_2]}$. Again consider the trivial trajectory (15) of the observer. The assumption that $[E_o, A_o, B_o, C_o, D_o]$ is an asymptotic partial state observer leads to

$$\lim_{t \to \infty} \operatorname{ess\,sup}_{[t,\infty)} \|C_2 x(t)\| = 0.$$

By Lemma A.4 this implies (13). $\square$

Remark 3.6. We consider some special cases using the notation from Theorem 3.5.

(i) Assume that $C_2 = I_n$. Then condition a) in Theorem 3.5 (i) is always satisfied and hence Theorem 3.5 is equivalent to [9, Thm. 3.5], i.e., the partial state observer is a full state observer in this case.

(ii) Assume that $sE - A$ is regular. By Lemma 2.1, condition b) in Theorem 3.5 (i) is equivalent to

$$\mathscr{V}^*_{[E,A,0,C_1]} \cap \mathscr{W}^*_{[E,A,0,C_1]} \subseteq \ker C_2.$$

It is a simple calculation that

$$\mathscr{V}^*_{[E,A,0,C_1]} \cap \mathscr{W}^*_{[E,A,0,C_1]} \subseteq \mathscr{V}^*_{[E,A,0,0]} \cap \mathscr{W}^*_{[E,A,0,0]} = \{0\},$$

where the last equality follows from regularity of $sE - A$, see [7, Prop. 2.4]. Therefore, condition b) is always satisfied in this case.

(iii) A combination of (i) and (ii) shows that if $sE - A$ is regular and $C_2 = I$, then a (partial state) observer always exists, i.e., full state reconstruction is always possible for regular DAE systems. This was already shown in [9].

(iv) Assume that $E = I_n$. Using the notation $\mathscr{O}(A, C) := [C^\top, A^\top C^\top, \ldots, (A^{n-1})^\top C^\top]^\top$ for some $C \in \mathbb{R}^{p \times n}$ it is a straightforward calculation that condition a) in Theorem 3.5 (i) is equivalent to

$$\mathscr{O}\left(A, \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}\right) = \mathscr{O}(A, C_1) \cap \ker C_2.$$

Therefore, invoking (ii), the above condition characterizes existence of a partial state observer for an ODE system.

$$
\begin{tikzcd}
\boxed{\operatorname{rk}_{\mathbb{R}[s]} \begin{bmatrix} sE-A \\ C_1 \end{bmatrix} = n} \arrow[r, Leftrightarrow, "{[9, \text{Thm. } 3.5]}"] \arrow[d, Leftrightarrow, "C_2 = I"] & \boxed{\begin{array}{l} \exists \text{ partial state observer} \\ \text{for } [E,A,B,C_1,I_n,D_1,D_2] \end{array}} \\
\boxed{\begin{array}{l} \text{a) } \mathscr{V}^*_{\left[\left[\begin{smallmatrix} E \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A \\ C_1 \end{smallmatrix}\right], 0, C_2\right]} = \mathscr{V}^*_{\left[\left[\begin{smallmatrix} E \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A \\ C_1 \end{smallmatrix}\right], 0, 0\right]} \cap \ker C_2, \\ \text{b) } \mathscr{V}^*_{[E,A,0,C_1]} \cap \mathscr{W}^*_{[E,A,0,C_1]} \subseteq \ker C_2 \end{array}} \arrow[r, Leftrightarrow, "\text{Thm. } 3.5"] \arrow[d, Leftrightarrow, "E = I"] & \boxed{\begin{array}{l} \exists \text{ partial state observer} \\ \text{for } [E,A,B,C_1,C_2,D_1,D_2] \end{array}} \\
\boxed{\mathscr{O}\left(A, \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}\right) = \mathscr{O}(A, C_1) \cap \ker C_2} \arrow[r, Leftrightarrow, "\text{Thm. } 3.5"] & \boxed{\begin{array}{l} \exists \text{ partial state observer} \\ \text{for } [I_n,A,B,C_1,C_2,D_1,D_2] \end{array}}
\end{tikzcd}
$$

Figure 3: Simplified conditions for the existence of a partial state observer in the cases $C_2 = I_n$ and $E = I_n$, and comparison with [9].

(v) The relations derived in (i)–(iv) are depicted in Figure 3.

(vi) We like to note that it is hard to compare Theorem 3.5 to the results derived e.g. in [43], since the notion of a partial state observer is different there. Roughly speaking, in [43] a system $[I, A_o, B_o, C_o, D_o]$ is called a partial state observer for a given plant, if zero initial values of the states, i.e., $x(0) = 0$ and $x_o(0) = 0$ (using the notation in (2), (3)) implies $z(t) = \hat{z}(t)$ for all $t \geq 0$. This is different from our framework, where $z(0) = \hat{z}(0)$ implies $z(t) = \hat{z}(t)$ for all $t \geq 0$.

4 Disturbance decoupled partial state observers

In this section we define disturbance decoupling using the intuitive approach introduced in [6]. To this end, we consider a system $[E, A, 0, C, 0] \in \Sigma_{l,n,0,p}$ and disturbance matrix $Q = \left[\begin{smallmatrix} Q_1 \\ Q_2 \end{smallmatrix}\right] \in \mathbb{R}^{(l+p)\times q}$; the corresponding DAE is of the form (2) with $m = 0$ and $p_2 = 0$. We may treat the disturbance d as the input of this system and define disturbance decoupling in terms of the set-valued input-output map of the system $[E, A, Q_1, C, Q_2] \in \Sigma_{l,n,q,p}$.

Definition 4.1. For a system $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$ we call the set-valued map

$$
\begin{aligned}
\Phi_{[E,A,B,C,D]} : \mathscr{C}^\infty(\mathbb{R} \to \mathbb{R}^m) &\to \mathscr{P}\big(\mathscr{C}^\infty(\mathbb{R} \to \mathbb{R}^p)\big), \\
u &\mapsto \left\{ y \in \mathscr{C}^\infty(\mathbb{R} \to \mathbb{R}^p) \;\middle|\; \exists x \in \mathscr{C}^\infty(\mathbb{R} \to \mathbb{R}^n) : \ (x, u, y) \in \mathscr{B}_{[E,A,B,C,D]} \right\},
\end{aligned}
$$

the *input-output map* of $[E, A, B, C, D]$. Here, $\mathscr{P}(\mathscr{M})$ denotes the power set of a set $\mathscr{M}$.

Definition 4.2. Let $[E, A, 0, C, 0] \in \Sigma_{l,n,0,p}$ and $Q = \left[\begin{smallmatrix} Q_1 \\ Q_2 \end{smallmatrix}\right] \in \mathbb{R}^{(l+p)\times q}$. Then we call $[E, A, Q_1, C, Q_2]$ *disturbance decoupled*, if

$$
\forall d_1, d_2 \in \mathscr{C}^\infty(\mathbb{R} \to \mathbb{R}^q) : \ \Phi_{[E,A,Q_1,C,Q_2]}(d_1) = \Phi_{[E,A,Q_1,C,Q_2]}(d_2).
$$

Roughly speaking, $[E,A,Q_1,C,Q_2]$ is disturbance decoupled, if any two disturbances cannot be distinguished using knowledge of the output. The above definition generalizes the definition given in [6] to the case of output disturbances. The following characterization is a straightforward modification of [6, Prop. 7] to that case.

Lemma 4.3. *Let* $[E,A,0,C,0]\in\Sigma_{l,n,0,p}$ *and* $Q=\begin{bmatrix}Q_1\\Q_2\end{bmatrix}\in\mathbb{R}^{(l+p)\times q}$*. Then* $[E,A,Q_1,C,Q_2]$ *is disturbance decoupled if, and only if,*

$$\forall d\in\mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\ \exists x\in\mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^n):\ Cx+Q_2d=0\ \wedge\ E\dot{x}=Ax+Q_1d\,. \tag{16}$$

We record a straightforward consequence of Lemma 4.3.

Corollary 4.4. *Let* $[E,A,0,C,0]\in\Sigma_{l,n,0,p}$ *and* $Q=\begin{bmatrix}Q_1\\Q_2\end{bmatrix}\in\mathbb{R}^{(l+p)\times q}$*. Then* $[E,A,Q_1,C,Q_2]\in\Sigma_{l,n,q,p}$ *is disturbance decoupled if, and only if,* $\left[\begin{bmatrix}E\\0\end{bmatrix},\begin{bmatrix}A\\C\end{bmatrix},\begin{bmatrix}Q_1\\Q_2\end{bmatrix},0,0\right]\in\Sigma_{l+p,n,q,0}$ *is disturbance decoupled.*

In the following we give the definition for a disturbance decoupled partial state observer. Classically, an observer has this property, if in the closed-loop system the observation error is independent of the input and the disturbance. We give the precise definition using our framework.

Consider a system $[E,A,B,C_1,C_2,D_1,D_2]\in\Sigma_{l,n,m,p_1,p_2}$ with disturbance matrix $Q=\begin{bmatrix}Q_1\\Q_2\\Q_3\end{bmatrix}\in\mathbb{R}^{(l+p_1+p_2)\times q}$ as in (2) and a partial state observer $[E_o,A_o,B_o,C_o,D_o]\in\Sigma_{l_o,n_o,m+p_1,p_2}$ as in (3), where we write

$$B_o=[B_{o,1},B_{o,2}]\in\mathbb{R}^{l_o\times(m+p_1)}\quad\text{and}\quad D_o=[D_{o,1},D_{o,2}]\in\mathbb{R}^{p_2\times(m+p_1)}. \tag{17}$$

The interconnection is depicted in Figure 1 and, after some straightforward manipulations, we may write the resulting system as

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}E&0\\0&E_o\end{bmatrix}\begin{pmatrix}x\\x_o\end{pmatrix}&=\begin{bmatrix}A&0\\B_{o,2}C_1&A_o\end{bmatrix}\begin{pmatrix}x\\x_o\end{pmatrix}+\begin{bmatrix}B&Q_1\\B_{o,1}+B_{o,2}D_1&B_{o,2}Q_2\end{bmatrix}\begin{pmatrix}u\\d\end{pmatrix},\\ e=\hat{z}-z&=\begin{bmatrix}D_{o,2}C_1-C_2&C_o\end{bmatrix}\begin{pmatrix}x\\x_o\end{pmatrix}+\begin{bmatrix}D_{o,1}-D_{o,2}D_1-D_2&D_{o,2}Q_2-Q_3\end{bmatrix}\begin{pmatrix}u\\d\end{pmatrix},\end{aligned} \tag{18}$$

that is

$$\begin{aligned}[\tilde{E},\tilde{A},\tilde{B},\tilde{C},\tilde{D}]:=\Bigg[&\begin{bmatrix}E&0\\0&E_o\end{bmatrix},\begin{bmatrix}A&0\\B_{o,2}C_1&A_o\end{bmatrix},\begin{bmatrix}B&Q_1\\B_{o,1}+B_{o,2}D_1&B_{o,2}Q_2\end{bmatrix},\\ &\begin{bmatrix}D_{o,2}C_1-C_2&C_o\end{bmatrix},\begin{bmatrix}D_{o,1}-D_{o,2}D_1-D_2&D_{o,2}Q_2-Q_3\end{bmatrix}\Bigg]\in\Sigma_{l+l_o,n+n_o,m+q,p_2}.\end{aligned} \tag{19}$$

Definition 4.5. We call a partial state observer $[E_o,A_o,B_o,C_o,D_o]\in\Sigma_{l_o,n_o,m+p_1,p_2}$ *disturbance decoupled* for a system $[E,A,B,C_1,C_2,D_1,D_2]\in\Sigma_{l,n,m,p_1,p_2}$ and disturbance matrix $Q=\begin{bmatrix}Q_1\\Q_2\\Q_3\end{bmatrix}\in\mathbb{R}^{(l+p_1+p_2)\times q}$, if $[\tilde{E},\tilde{A},\tilde{B},\tilde{C},\tilde{D}]$ in (19) is disturbance decoupled.

In the following we show that it is sufficient to restrict ourselves to the case of vanishing feedthrough matrices, i.e., $D_1=0,D_2=0$ and $Q_2=0,Q_3=0$. To this end, we augment the state space of the system (2), that is we consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\begin{bmatrix} E & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}}_{=:\hat{E}}\begin{pmatrix} x\\ v_1\\ v_2\\ v_3\\ v_4\end{pmatrix} = \underbrace{\begin{bmatrix} A & 0 & 0 & 0 & 0\\ 0 & -I_{p_1} & 0 & 0 & 0\\ 0 & 0 & -I_{p_1} & 0 & 0\\ 0 & 0 & 0 & -I_{p_2} & 0\\ 0 & 0 & 0 & 0 & -I_{p_2}\end{bmatrix}}_{=:\hat{A}}\begin{pmatrix} x\\ v_1\\ v_2\\ v_3\\ v_4\end{pmatrix} + \underbrace{\begin{bmatrix} B\\ D_1\\ 0\\ D_2\\ 0\end{bmatrix}}_{=:\hat{B}}u + \underbrace{\begin{bmatrix} Q_1\\ 0\\ Q_2\\ 0\\ Q_3\end{bmatrix}}_{=:\hat{Q}}d,$$

$$y = \underbrace{\begin{bmatrix} C_1 & I_{p_1} & I_{p_1} & 0 & 0\end{bmatrix}}_{=:\hat{C}_1}\begin{pmatrix} x\\ v_1\\ v_2\\ v_3\\ v_4\end{pmatrix}, \tag{20}$$

$$z = \underbrace{\begin{bmatrix} C_2 & 0 & 0 & I_{p_2} & I_{p_2}\end{bmatrix}}_{=:\hat{C}_2}\begin{pmatrix} x\\ v_1\\ v_2\\ v_3\\ v_4\end{pmatrix}.$$

Lemma 4.6. *A partial state observer $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p_1,p_2}$ is disturbance decoupled for a system $[E,A,B,C_1,C_2,D_1,D_2] \in \Sigma_{l,n,m,p_1,p_2}$ and disturbance matrix $Q = \left[\begin{smallmatrix} Q_1\\ Q_2\\ Q_3\end{smallmatrix}\right] \in \mathbb{R}^{(l+p_1+p_2)\times q}$ if, and only if, using the notation from (20), $[E_o, A_o, B_o, C_o, D_o]$ is disturbance decoupled for the system $[\hat{E}, \hat{A}, \hat{B}, \hat{C}_1, \hat{C}_2, 0, 0] \in \Sigma_{l+2p_1+2p_2,n+2p_1+2p_2,m,p_1,p_2}$ and disturbance matrix $\left[\begin{smallmatrix} \hat{Q}\\ 0\\ 0\end{smallmatrix}\right]$.*

Proof. The statement follows from a straightforward calculation. □

5 Disturbance decoupled estimation

In this section we consider the disturbance decoupled estimation problem. In view of Lemma 4.6 we first consider the case of zero feedthrough matrices ($D_1 = 0$ and $D_2 = 0$) and undisturbed output equations ($Q_2 = 0$ and $Q_3 = 0$). That is, for a given system $[E,A,B,C_1,C_2,0,0] \in \Sigma_{l,n,m,p_1,p_2}$ and disturbance matrix $Q \in \mathbb{R}^{l\times q}$ we seek a partial state observer $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p_1,p_2}$ which is disturbance decoupled for $[E,A,B,C_1,C_2,0,0]$ and $\left[\begin{smallmatrix} Q\\ 0\\ 0\end{smallmatrix}\right]$ or, equivalently, invoking Definition 4.5 and Corollary 4.4,

$$\left[\begin{bmatrix} E & 0\\ 0 & E_o\\ 0 & 0\end{bmatrix}, \begin{bmatrix} A & 0\\ B_{o,2}C_1 & A_o\\ D_{o,2}C_1 - C_2 & C_o\end{bmatrix}, \begin{bmatrix} B & Q\\ B_{o,1} & 0\\ D_{o,1} & 0\end{bmatrix}, 0, 0\right] \quad \text{is disturbance decoupled,} \tag{21}$$

where we use the partitioning (17).

Theorem 5.1. *Let $[E,A,B,C_1,C_2,0,0] \in \Sigma_{l,n,m,p_1,p_2}$ and $Q \in \mathbb{R}^{l\times q}$. There exists a partial state observer $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p_1,p_2}$ such that* (21) *is satisfied if, and only if, the following statements hold:*

(i) $\operatorname{im}[B, Q] \subseteq E\mathscr{V}^*_{[E,A,0,0]} + A\mathscr{W}^*_{[E,A,0,0]}$,

(ii) $\mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0\\ 0 & 0\end{smallmatrix}\right],\left[\begin{smallmatrix} A & Q\\ C_1 & 0\end{smallmatrix}\right],0,[C_2,0]\right]} = \mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0\\ 0 & 0\end{smallmatrix}\right],\left[\begin{smallmatrix} A & Q\\ C_1 & 0\end{smallmatrix}\right],0,0\right]} \cap \ker[C_2,0]$,

(iii) $\mathscr{V}^*_{[E,A,Q,C_1]} \cap \mathscr{W}^*_{[E,A,Q,C_1]} \subseteq \ker C_2$.

Proof. $\Leftarrow$: We construct a disturbance decoupled partial state observer $[E_o,A_o,B_o,C_o,D_o] \in \Sigma_{l_o,n_o,m+p_1,p_2}$ of the form (6) with $k \in \mathbb{N}_0$, $L_x \in \mathbb{R}^{l\times k}, L_y \in \mathbb{R}^{p_1\times k}$ and $L_z \in \mathbb{R}^{p_2\times k}$. If we assume this form, then Lemma 4.3 yields that (21) is equivalent to

$$\begin{aligned} &\forall (u,d) \in \mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^m\times\mathbb{R}^q)\ \exists (x_1,x_2,x_3) \in \mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^k):\\ &\qquad E\dot{x}_1 = Ax_1 + Bu + Qd,\\ &\qquad E\dot{x}_2 = Ax_2 + L_x x_3 + Bu,\\ &\qquad 0 = C_1(x_2 - x_1) + L_y x_3,\\ &\qquad 0 = C_2(x_2 - x_1) + L_z x_3. \end{aligned}$$

Introducing the new variable $v = x_2 - x_1$ we obtain the equivalent decoupled conditions

$$\forall (u,d) \in \mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^m\times\mathbb{R}^q)\ \exists x_1 \in \mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^n):\ E\dot{x}_1 = Ax_1 + Bu + Qd,$$

which is always satisfied by statement (i), and

$$\begin{aligned} &\forall d \in \mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^q)\ \exists (v,x_3) \in \mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^k):\\ &\qquad E\dot{v} = Av + L_x x_3 - Qd,\\ &\qquad 0 = C_1 v + L_y x_3,\\ &\qquad 0 = C_2 v + L_z x_3. \end{aligned} \tag{22}$$

It remains to find L_x, L_y, L_z such that (22) is satisfied and $[E_o,A_o,B_o,C_o,D_o]$ is a partial state observer, i.e., conditions a) and b) in Theorem 3.3 (ii) are satisfied. We choose $k = q$, $L_x = Q$, $L_y = 0$ and $L_z = 0$. Then (22) is obviously satisfied, since for any $d \in \mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^q)$ we may choose $v = 0$ and $x_3 = d$. Condition a) in Theorem 3.3 (ii) is equivalent to (ii) and condition b) in Theorem 3.3 (ii) is equivalent to

$$\mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0\\ 0 & 0\end{smallmatrix}\right],\left[\begin{smallmatrix} A & Q\\ C_1 & 0\end{smallmatrix}\right],0,0\right]} \cap \mathscr{W}^*_{\left[\left[\begin{smallmatrix} E & 0\\ 0 & 0\end{smallmatrix}\right],\left[\begin{smallmatrix} A & Q\\ C_1 & 0\end{smallmatrix}\right],0,0\right]} \subseteq \ker[C_2,0] \tag{23}$$

which, invoking Lemma 2.1, follows from statement (iii).

$\Rightarrow$: Let a partial state observer $[E_o,A_o,B_o,C_o,D_o] \in \Sigma_{l_o,n_o,m+p_1,p_2}$ such that (21) is satisfied be given. From Lemma 4.3 it is clear that (i) must be satisfied. For (ii) and (iii) we show that

$$\begin{aligned} &\forall (x,u,d,y,z,x_o,\hat{z}) \in \mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^q\times\mathbb{R}^{p_1}\times\mathbb{R}^{p_2}\times\mathbb{R}^{n_o}\times\mathbb{R}^{p_2}):\\ &\begin{pmatrix} (x,\left(\begin{smallmatrix} u\\ d\end{smallmatrix}\right),y,z) \in \mathscr{B}_{[E,A,[B,Q],C_1,C_2,0,0]} \ \wedge\ (x_o,\left(\begin{smallmatrix} u\\ y\end{smallmatrix}\right),\hat{z}) \in \\ \mathscr{B}_{[E_o,A_o,B_o,C_o,D_o]}\\ \wedge\ \hat{z}(0) = z(0) \end{pmatrix} \implies \hat{z} = z. \end{aligned} \tag{24}$$

Let $(x,\left(\begin{smallmatrix} u\\ d\end{smallmatrix}\right),y,z) \in \mathscr{B}_{[E,A,[B,Q],C_1,C_2,0,0]}$ and $(x_o,\left(\begin{smallmatrix} u\\ y\end{smallmatrix}\right),\hat{z}) \in \mathscr{B}_{[E_o,A_o,B_o,C_o,D_o]}$ be smooth trajectories such that $z(0) = \hat{z}(0)$. Furthermore, by (21) and Lemma 4.3 there exist $\tilde{x} \in \mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^n)$, $\tilde{x}_o \in \mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^{n_o})$ such that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E & 0\\ 0 & E_o\\ 0 & 0\end{bmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{x}_o\end{pmatrix} = \begin{bmatrix} A & 0\\ B_{o,2}C_1 & A_o\\ D_{o,2}C_1 - C_2 & C_o\end{bmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{x}_o\end{pmatrix} + \begin{bmatrix} B & Q\\ B_{o,1} & 0\\ D_{o,1} & 0\end{bmatrix}\begin{pmatrix} u\\ d\end{pmatrix}.$$

The new variables $v := \tilde{x} - x$ and $w := \tilde{x}_o - x_o$ satisfy

$$\begin{aligned} E\dot{v} &= Av, \\ E_o\dot{w} &= A_o w + B_{o,2}C_1 v, \end{aligned}$$

and

$$\hat{z} - z = C_o x_o + D_{o,1}u + D_{o,2}C_1 x - C_2 x \underbrace{- (D_{o,2}C_1 - C_2)\tilde{x} - C_o\tilde{x}_o - D_{o,1}u}_{=0} = (C_2 - D_{o,2}C_1)v - C_o w,$$

thus

$$C_o w + D_{o,2}C_1 v = C_2 v - (\hat{z} - z).$$

Therefore,

$$(v, 0, C_1 v, C_2 v) \in \mathscr{B}_{[E,A,B,C_1,C_2,0,0]} \quad \wedge \quad \left(w, \left(\begin{smallmatrix} 0 \\ C_1 v \end{smallmatrix}\right), C_2 v - (\hat{z} - z)\right) \in \mathscr{B}_{[E_o,A_o,B_o,C_o,D_o]},$$

and since $[E_o, A_o, B_o, C_o, D_o]$ is a partial state observer and $C_2 v(0) = C_2 v(0) - (\hat{z}(0) - z(0))$ we obtain

$$C_2 v = C_2 v - (\hat{z} - z),$$

hence $\hat{z} = z$, which proves (24). It remains to consider $x \in \mathscr{C}^\infty(\mathbb{R} \to \mathbb{R}^n)$ and $d \in \mathscr{C}^\infty(\mathbb{R} \to \mathbb{R}^q)$ such that $\left(x, \left(\begin{smallmatrix} 0 \\ d \end{smallmatrix}\right), 0, C_2 x\right) \in \mathscr{B}_{[E,A,[B,Q],C_1,C_2,0,0]}$ with $C_2 x(0) = 0$. Since $(0, \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), 0) \in \mathscr{B}_{[E_o,A_o,B_o,C_o,D_o]}$ it follows from (24) that $C_2 x = 0$, i.e., we have shown that

$$\forall x \in \mathscr{C}^\infty(\mathbb{R} \to \mathbb{R}^n),\ d \in \mathscr{C}^\infty(\mathbb{R} \to \mathbb{R}^q): \ \left(E\dot{x} = Ax + Qd \ \wedge\ C_1 x = 0 \ \wedge\ C_2 x(0) = 0\right) \implies C_2 x = 0.$$

By Lemma A.2 this implies (ii) and (23). Invoking Lemma 2.1, (23) is equivalent to (iii). □

Remark 5.2.

(i) It may seem strange that condition (i) in Theorem 5.1 involves a condition on B, while in the ODE case the solution of the disturbance decoupled estimation problem does not involve such a condition, see [43]. Condition (i) is specific to the DAE case and always satisfied in the ODE case, since it characterizes the existence of a solution $x \in \mathscr{C}^\infty(\mathbb{R} \to \mathbb{R}^n)$ of

$$E\dot{x} = Ax + Bu + Qd$$

for any given input $u \in \mathscr{C}^\infty(\mathbb{R} \to \mathbb{R}^m)$ and any disturbance $d \in \mathscr{C}^\infty(\mathbb{R} \to \mathbb{R}^q)$. If $sE - A$ is not regular, then this condition may not be met.

(ii) The intuition for conditions (ii) and (iii) in Theorem 5.1 is as follows: Since only inputs and outputs are known to the partial state observer, the dynamics of the system

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} Ex &= Ax + Qd, \\ 0 &= C_1 x \end{aligned} \tag{25}$$

are hidden from the observer. More precisely, for any $(x, \left(\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right), y, z) \in \mathscr{B}_{[E,A,[B,Q],C_1,C_2,0,0]}$ and a solution $(\tilde{x}, d)$ of (25) we have

$$(x + \tilde{x}, \left(\begin{smallmatrix} u \\ d \end{smallmatrix}\right), y, z + C_2\tilde{x}) \in \mathscr{B}_{[E,A,[B,Q],C_1,C_2,0,0]},$$

i.e., another solution with the same input u and the same output y; the disturbance d is not visible for the observer. Now, conditions (ii) and (iii) guarantee that the partial state observer is still able to reconstruct the controlled output from its initial condition, i.e., that if $C_2\tilde{x}(0) = 0$, then we have $C_2\tilde{x} = 0$, see also Lemma A.2.

Utilizing a more geometric interpretation, condition (ii) states that the unobservable space of the ODE part of the DAE (25), where $\binom{x}{d}$ is treated as the state, with respect to the controlled output $z = C_2x$ is equal to the kernel of the output matrix C_2, i.e., if the initial state is not visible at the output, then it is not visible at the output for all other time instants. The unobservable space is given by $\mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right], 0, [C_2, 0]\right]}$, see [12]. Condition (iii) guarantees that the completely controllable part of $\frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + Qd$ (where d is viewed as an input) restricted to $\ker C_1$, i.e., of the system

$$ET\dot{x} = ATx + Qd, \quad \operatorname{im} T = \ker C_1,$$

lies in $\ker C_2$, so that it is not visible in the controlled output $z = C_2x$. Geometrically, the completely controllable part is characterized by the reachable space of the system, which in turn is given by the intersection of the generalized Wong sequences $\mathscr{V}^*_{[E,A,Q,C_1]} \cap \mathscr{W}^*_{[E,A,Q,C_1]}$, see [8, 15].

(iii) In [43] the following result has been derived: For $[I, A, B, C_1, C_2, 0, 0] \in \Sigma_{n,n,m,p_1,p_2}$ and $Q \in \mathbb{R}^{l \times q}$ there exists $[I, A_o, B_o, C_o, D_o] \in \Sigma_{n_o,n_o,m+p_1,p_2}$ such that the closed-loop system (18) has zero transfer function, i.e.,

$$\begin{bmatrix} D_{o,2}C_1 - C_2 & C_o \end{bmatrix} \begin{bmatrix} sI - A & 0 \\ -B_{o,2}C_1 & sI - A_o \end{bmatrix}^{-1} \begin{bmatrix} B & Q \\ B_{o,1} & 0 \end{bmatrix} + \begin{bmatrix} D_{o,1} & 0 \end{bmatrix} = 0$$

if, and only if,

$$\mathscr{W}^*_{[I,A,Q,C_1]} \subseteq \ker C_2. \tag{26}$$

This may be compared to Theorem 5.1 as follows: Condition (i) is always satisfied since $E = I$; condition (ii) is due to the different definition of the partial state observer compared to [43], see Remark 3.6 (v), hence such a condition does not appear there; condition (iii) is weaker than (26), where $\mathscr{W}^*_{[I,A,Q,C_1]}$ is considered instead of $\mathscr{V}^*_{[I,A,Q,C_1]} \cap \mathscr{W}^*_{[I,A,Q,C_1]}$. The reason for this weaker condition is that we allow for a DAE observer (i.e., E_o in (3) is not invertible) even if the plant is governed by an ODE.

The existence of a disturbance decoupled partial state observer for a general system $[E, A, B, C_1, C_2, D_1, D_2] \in \Sigma_{l,n,m,p_1,p_2}$ and disturbance matrix $Q = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} \in \mathbb{R}^{(l+p_1+p_2)\times q}$, i.e., with nonzero feedthrough matrices, follows from Theorem 5.1 and Lemma 4.6. In the case of an undisturbed controlled output, that is $Q_3 = 0$, the conditions simplify a lot; this is the result of the following theorem.

Theorem 5.3. *There exists a disturbance decoupled partial state observer* $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p_1,p_2}$ *for a system* $[E, A, B, C_1, C_2, D_1, D_2] \in \Sigma_{l,n,m,p_1,p_2}$ *and disturbance matrix* $Q = \begin{bmatrix} Q_1 \\ Q_2 \\ 0 \end{bmatrix} \in \mathbb{R}^{(l+p_1+p_2)\times q}$ *if, and only if, the following statements hold:*

(i) $\operatorname{im}[B, Q_1] \subseteq E\mathscr{V}^*_{[E,A,0,0]} + A\mathscr{W}^*_{[E,A,0,0]}$,

(ii) $\mathscr{V}^*_{\left[\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & Q_1 \\ C_1 & Q_2 \end{bmatrix}, 0, [C_2, 0]\right]} = \mathscr{V}^*_{\left[\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & Q_1 \\ C_1 & Q_2 \end{bmatrix}, 0, 0\right]} \cap \ker[C_2, 0]$,

(iii) $\mathscr{V}^*_{\left[\begin{bmatrix} E \\ 0 \end{bmatrix}, \begin{bmatrix} A \\ C_1 \end{bmatrix}, \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, 0\right]} \cap \mathscr{W}^*_{\left[\begin{bmatrix} E \\ 0 \end{bmatrix}, \begin{bmatrix} A \\ C_1 \end{bmatrix}, \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, 0\right]} \subseteq \ker C_2$.

Proof. Using the notation in (20), Lemma 4.6 and Theorem 5.1 give that there exists a disturbance decoupled partial state observer for $[E, A, B, C_1, C_2, D_1, D_2]$ and disturbance matrix $Q = \begin{bmatrix} Q_1 \\ Q_2 \\ 0 \end{bmatrix}$ if, and only if,

(a) $\operatorname{im}[\hat{B}, \hat{Q}] \subseteq \hat{E}\mathscr{V}^*_{[\hat{E},\hat{A},0,0]} + \hat{A}\mathscr{W}^*_{[\hat{E},\hat{A},0,0]}$,

(b) $\mathscr{V}^*_{\left[\begin{bmatrix} \hat{E} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \hat{A} & \hat{Q} \\ \hat{C}_1 & 0 \end{bmatrix}, 0, [\hat{C}_2, 0]\right]} = \mathscr{V}^*_{\left[\begin{bmatrix} \hat{E} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \hat{A} & Q \\ \hat{C}_1 & 0 \end{bmatrix}, 0, 0\right]} \cap \ker[\hat{C}_2, 0]$,

(c) $\mathscr{V}^*_{[\hat{E},\hat{A},\hat{Q},\hat{C}_1]} \cap \mathscr{W}^*_{[\hat{E},\hat{A},\hat{Q},\hat{C}_1]} \subseteq \ker \hat{C}_2$.

We proof that (a)–(b) are equivalent to (i)–(iii) by proceeding in several steps.

Step 1: We show that (a) is equivalent to (i). A simple inductive argument gives that

$$\begin{aligned} \mathscr{V}^i_{[\hat{E},\hat{A},0,0]} &= \mathscr{V}^i_{[E,A,0,0]} \times \{0\} \times \{0\} \times \{0\} \times \{0\}, \\ \mathscr{W}^i_{[\hat{E},\hat{A},0,0]} &= \mathscr{W}^i_{[E,A,0,0]} \times \mathbb{R}^{p_1} \times \mathbb{R}^{p_1} \times \mathbb{R}^{p_2} \times \mathbb{R}^{p_2}, \end{aligned}$$

hence

$$\hat{E}\mathscr{V}^*_{[\hat{E},\hat{A},0,0]} + \hat{A}\mathscr{W}^*_{[\hat{E},\hat{A},0,0]} = (E\mathscr{V}^*_{[E,A,0,0]} + A\mathscr{W}^*_{[E,A,0,0]}) \times \mathbb{R}^{p_1} \times \mathbb{R}^{p_1} \times \mathbb{R}^{p_2} \times \mathbb{R}^{p_2}.$$

The statement then follows from observing that

$$\operatorname{im}[\hat{B}, \hat{Q}] = \operatorname{im} \begin{bmatrix} B & Q_1 \\ D_1 & 0 \\ 0 & Q_2 \\ D_2 & 0 \\ 0 & Q_3 \end{bmatrix}.$$

Step 2: We show that (b) is equivalent to (ii). We first prove by induction that

$$\forall i \in \mathbb{N}: \mathscr{V}^i_{\left[\begin{bmatrix} \hat{E} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \hat{A} & \hat{Q} \\ \hat{C}_1 & 0 \end{bmatrix}, 0, [\hat{C}_2, 0]\right]} = \begin{bmatrix} I_n & 0 \\ 0 & 0 \\ -C_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I_q \end{bmatrix} \mathscr{V}^i_{\left[\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & Q_1 \\ C_1 & Q_2 \end{bmatrix}, 0, [C_2, 0]\right]}.$$

For $i = 1$ the statement follows from

$$\begin{aligned} &\mathscr{V}^1_{\left[\begin{bmatrix} \hat{E} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \hat{A} & \hat{Q} \\ \hat{C}_1 & 0 \end{bmatrix}, 0, [\hat{C}_2, 0]\right]} \\ &= \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} \in \mathbb{R}^{n+2p_1+2p_2+q} \;\middle|\; \begin{array}{rclrcl} \exists y_1 \in \mathbb{R}^n: \ Ax_1 + Q_1x_6 &=& Ey_1, & x_5 &=& 0, \\ x_2 &=& 0, & C_1x_1 + x_2 + x_3 &=& 0, \\ -x_3 + Q_2x_6 &=& 0, & C_2x_1 + x_4 + x_5 &=& 0, \\ x_4 &=& 0, & & & \end{array} \right\} \\ &= \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} \in \mathbb{R}^{n+2p_1+2p_2+q} \;\middle|\; \begin{array}{l} \begin{bmatrix} A \\ C_1 \end{bmatrix} x_1 + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} x_6 \in \operatorname{im} \begin{bmatrix} E \\ 0 \end{bmatrix}, \ C_2x_1 = 0, \\ x_2 = 0, \ x_3 = -C_1x_1, \ x_4 = 0, \ x_5 = 0 \end{array} \right\}. \end{aligned}$$

Assuming that the statement is true for some $i \in \mathbb{N}$ we obtain, with a similar calculation as above,

$$\begin{aligned}
&\mathscr{V}^{i+1}_{\left[\left[\begin{smallmatrix} \hat{E} & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} \hat{A} & \hat{Q} \\ \hat{C}_1 & 0 \end{smallmatrix}\right], 0, [\hat{C}_2, 0]\right]} \\
&= \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} \in \mathbb{R}^{n+2p_1+2p_2+q} \;\middle|\; \begin{array}{l} \begin{bmatrix} A \\ C_1 \end{bmatrix} x_1 + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} x_6 \in \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \mathscr{V}^i_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q_1 \\ C_1 & Q_2 \end{smallmatrix}\right], 0, [C_2, 0]\right]}, \\ C_2 x_1 = 0,\ x_2 = 0,\ x_3 = -C_1 x_1,\ x_4 = 0,\ x_5 = 0 \end{array} \right\}
\end{aligned}$$

and this proves the statement. Analogously, it can be shown that

$$\forall\, i \in \mathbb{N}:\ \mathscr{V}^i_{\left[\left[\begin{smallmatrix} \hat{E} & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} \hat{A} & \hat{Q} \\ \hat{C}_1 & 0 \end{smallmatrix}\right], 0, 0\right]} = \begin{bmatrix} I_n & 0 \\ 0 & 0 \\ -C_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I_q \end{bmatrix} \mathscr{V}^i_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q_1 \\ C_1 & Q_2 \end{smallmatrix}\right], 0, 0\right]}.$$

Now we have

$$\begin{aligned}
\text{(b)} \iff & \begin{bmatrix} I_n & 0 \\ 0 & 0 \\ -C_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I_q \end{bmatrix} \mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q_1 \\ C_1 & Q_2 \end{smallmatrix}\right], 0, [C_2, 0]\right]} = \begin{bmatrix} I_n & 0 \\ 0 & 0 \\ -C_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I_q \end{bmatrix} \mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q_1 \\ C_1 & Q_2 \end{smallmatrix}\right], 0, 0\right]} \cap \ker[C_2, 0, 0, I_{p_2}, I_{p_2}, 0] \\
\iff & \begin{bmatrix} I_n & 0 \\ 0 & 0 \\ -C_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I_q \end{bmatrix} \mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q_1 \\ C_1 & Q_2 \end{smallmatrix}\right], 0, [C_2, 0]\right]} = \begin{bmatrix} I_n & 0 \\ 0 & 0 \\ -C_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I_q \end{bmatrix} \mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q_1 \\ C_1 & Q_2 \end{smallmatrix}\right], 0, 0\right]} \cap \ker[C_2, 0, \ldots, 0] \iff \text{(ii)}.
\end{aligned}$$

Step 3: We show that (c) is equivalent to (iii). For brevity we denote, for $i \in \mathbb{N}_0$,

$$\hat{\mathscr{V}}_i := \mathscr{V}^i_{[\hat{E}, \hat{A}, \hat{Q}, \hat{C}_1]}, \quad \mathscr{V}_i := \mathscr{V}^i_{\left[\left[\begin{smallmatrix} E \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A \\ C_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} Q_1 \\ Q_2 \end{smallmatrix}\right], 0\right]}, \quad \hat{\mathscr{W}}_i := \mathscr{W}^i_{[\hat{E}, \hat{A}, \hat{Q}, \hat{C}_1]}, \quad \mathscr{W}_i := \mathscr{W}^i_{\left[\left[\begin{smallmatrix} E \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A \\ C_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} Q_1 \\ Q_2 \end{smallmatrix}\right], 0\right]}.$$

We show by induction that

$$\forall\, i \in \mathbb{N}:\ \hat{\mathscr{V}}_i = \begin{bmatrix} I_n \\ 0 \\ -C_1 \\ 0 \\ 0 \end{bmatrix} \mathscr{V}_i.$$

For $i = 1$ this follows from

$$\begin{aligned}
\hat{\mathscr{V}}_1 &= \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} \in \mathbb{R}^{n+2p_1+2p_2} \;\middle|\; \begin{array}{l} \exists\, y \in \mathbb{R}^n, z \in \mathbb{R}^q:\ Ax_1 = Ey + Q_1 z, \\ x_2 = 0,\ -x_3 = Q_2 z,\ x_4 = 0, \\ x_5 = 0,\ C_1 x_1 + x_2 + x_3 = 0 \end{array} \right\} \\
&= \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} \in \mathbb{R}^{n+2p_1+2p_2} \;\middle|\; \begin{array}{l} x_1 \in \mathscr{V}_1,\ x_2 = 0,\ x_3 = -C_1 x_1, \\ x_4 = 0,\ x_5 = 0 \end{array} \right\}.
\end{aligned}$$

Now assume that the statement is true for some $i \in \mathbb{N}$, then

$$\begin{aligned}
\hat{\mathscr{V}}_{i+1} &= \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} \in \mathbb{R}^{n+2p_1+2p_2} \;\middle|\; \begin{array}{l} \exists\, y \in \mathscr{V}_i, z \in \mathbb{R}^q:\ Ax_1 = Ey + Q_1 z, \\ x_2 = 0,\ -x_3 = Q_2 z,\ x_4 = 0, \\ x_5 = 0,\ C_1 x_1 + x_2 + x_3 = 0 \end{array} \right\} \\
&= \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} \in \mathbb{R}^{n+2p_1+2p_2} \;\middle|\; \begin{array}{l} x_1 \in \mathscr{V}_{i+1},\ x_2 = 0,\ x_3 = -C_1 x_1, \\ x_4 = 0,\ x_5 = 0 \end{array} \right\}.
\end{aligned}$$

Next, we show by induction that

$$\forall\, i \in \mathbb{N}: \ \hat{\mathscr{W}}_i = (\mathscr{W}_i \times \mathbb{R}^{2p_1+2p_2}) \cap \ker \hat{C}_1.$$

For $i = 1$ we have

$$\begin{aligned}\hat{\mathscr{W}}_1 &= \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} \in \mathbb{R}^{n+2p_1+2p_2} \ \middle| \ \begin{array}{l} \exists\, z \in \mathbb{R}^q: \ Ex_1 = Q_1 z, \\ 0 = Q_2 z, \ C_1 x_1 + x_2 + x_3 = 0 \end{array} \right\} \\ &= (\mathscr{W}_1 \times \mathbb{R}^{2p_1+2p_2}) \cap \ker \hat{C}_1.\end{aligned}$$

Assume that the statement is true for some $i \in \mathbb{N}$, then we have

$$\hat{\mathscr{W}}_{i+1} = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} \in \mathbb{R}^{n+2p_1+2p_2} \ \middle| \ \exists \begin{pmatrix} y_1 \\ \vdots \\ y_5 \end{pmatrix} \in \hat{\mathscr{W}}_i, \ z \in \mathbb{R}^q: \ \begin{array}{l} Ex_1 = Ay_1 + Q_1 z, \ y_2 = 0, \\ y_3 = Q_2 z, \ y_4 = 0, \ y_5 = 0, \\ C_1 x_1 + x_2 + x_3 = 0 \end{array} \right\}$$

and since $\hat{\mathscr{W}}_i \subseteq \ker \hat{C}_1$ it follows $0 = C_1 y_1 + y_2 + y_3 = C_1 y_1 + Q_2 z$ and hence

$$\begin{aligned}\hat{\mathscr{W}}_{i+1} &= \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} \in \mathbb{R}^{n+2p_1+2p_2} \ \middle| \ \exists\, y_1 \in \mathscr{W}_i, \ z \in \mathbb{R}^q: \ \begin{array}{l} Ex_1 = Ay_1 + Q_1 z, \\ 0 = C_1 y_1 + Q_2 z, \\ C_1 x_1 + x_2 + x_3 = 0 \end{array} \right\} \\ &= (\mathscr{W}_{i+1} \times \mathbb{R}^{2p_1+2p_2}) \cap \ker \hat{C}_1.\end{aligned}$$

Invoking that $\mathscr{V}^*_{[\hat{E},\hat{A},\hat{Q},\hat{C}_1]} \subseteq \ker \hat{C}_1$ we may infer that

$$\begin{aligned}\text{(c)} \iff & \begin{bmatrix} I_n \\ 0 \\ -C_1 \\ 0 \\ 0 \end{bmatrix} \mathscr{V}^*_{\left[\left[\begin{smallmatrix} E \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A \\ C_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} Q_1 \\ Q_2 \end{smallmatrix}\right], 0\right]} \cap \left(\mathscr{W}^*_{\left[\left[\begin{smallmatrix} E \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A \\ C_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} Q_1 \\ Q_2 \end{smallmatrix}\right], 0\right]} \times \mathbb{R}^{2p_1+2p_2} \right) \subseteq \ker[C_2, 0, 0, I_{p_2}, I_{p_2}] \\ \iff & \mathscr{V}^*_{\left[\left[\begin{smallmatrix} E \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A \\ C_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} Q_1 \\ Q_2 \end{smallmatrix}\right], 0\right]} \cap \mathscr{W}^*_{\left[\left[\begin{smallmatrix} E \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A \\ C_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} Q_1 \\ Q_2 \end{smallmatrix}\right], 0\right]} \subseteq \ker C_2 \iff \text{(iii)}. \qquad \square\end{aligned}$$

In the remainder of this section we consider the special case of full state estimation (i.e., $C_2 = I$) and show that the conditions (i)–(iii) in Theorem 5.1 simplify in this case.

Corollary 5.4. *Let $[E, A, B, C_1, I_n, 0, 0] \in \Sigma_{l,n,m,p_1,p_2}$ and $Q \in \mathbb{R}^{l \times q}$ with $\operatorname{rk} Q = q$. There exists a partial state observer $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p_1,p_2}$ such that* (21) *is satisfied if, and only if, the following statements hold:*

(a) $\operatorname{im}[B, Q] \subseteq E\mathscr{V}^*_{[E,A,0,0]} + A\mathscr{W}^*_{[E,A,0,0]}$,

(b) $\operatorname{rk}_{\mathbb{R}[s]} \begin{bmatrix} sE - A & Q \\ C_1 & 0 \end{bmatrix} = n + q.$

Proof. By Theorem 5.1 it suffices to prove that conditions (ii) and (iii) in Theorem 5.1 are equivalent to condition (b).

$\Rightarrow$: Assume that conditions (ii) and (iii) in Theorem 5.1 are true. By Lemma 2.1, (iii) is equivalent to

$$\mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right], 0, 0\right]} \cap \mathscr{W}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right], 0, 0\right]} \subseteq \ker[I_n, 0]. \tag{27}$$

Now let

$$\begin{pmatrix} 0 \\ x_2 \end{pmatrix} \in \mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right], 0, 0\right]} \cap \mathscr{W}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right], 0, 0\right]},$$

then

$$\begin{aligned} \begin{pmatrix} Qx_2 \\ 0 \end{pmatrix} = \begin{bmatrix} A & Q \\ C_1 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ x_2 \end{pmatrix} &\in \begin{bmatrix} A & Q \\ C_1 & 0 \end{bmatrix} \left(\mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right], 0, 0\right]} \cap \mathscr{W}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right], 0, 0\right]} \right) \\ &= \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \left(\mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right], 0, 0\right]} \cap \mathscr{W}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right], 0, 0\right]} \right), \end{aligned}$$

where the last equality follows from [13, Lem. 4.4]. Then (27) implies that $Qx_2 = 0$ and hence $x_2 = 0$ as $\operatorname{rk} Q = q$, thus we have shown

$$\mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right], 0, 0\right]} \cap \mathscr{W}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right], 0, 0\right]} = \{0\}. \tag{28}$$

By a combination of [8, Cors. 5.1 & 5.2] and [13, Thm. 2.6] it follows that condition (28) is equivalent to (b).

$\Leftarrow$: As shown above it follows from (b) that (28) is true and this implies (iii) in Theorem 5.1. Since

$$\ker \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \subseteq \mathscr{W}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right], 0, 0\right]}$$

it follows that

$$\begin{aligned} \ker[I_n, 0] \cap \mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right], 0, 0\right]} &\subseteq \ker \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \cap \mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right], 0, 0\right]} \\ &\subseteq \mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right], 0, 0\right]} \cap \mathscr{W}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right], 0, 0\right]} = \{0\}. \end{aligned}$$

Furthermore, since $\operatorname{rk} Q = q$ and $C_2 = I_n$ we have that

$$\mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right], 0, [I_n, 0]\right]} \subseteq \mathscr{V}^1_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right], 0, [I_n, 0]\right]} = \ker \begin{bmatrix} A & Q \\ C_1 & 0 \end{bmatrix} \cap \ker[I_n, 0] = \{0\},$$

and hence we may conclude that (ii) in Theorem 5.1 is true. □

Note that, if $sE - A$ is regular, then, using the notation from Corollary 5.4, condition (i) in Corollary 5.4 is always satisfied by Remark 5.2, hence in this case the DDEP with $C_2 = I_n$ and $\operatorname{rk} Q = q$ is solvable if, and only if, condition (ii) in Corollary 5.4 is true.

6 Disturbance decoupled asymptotic estimation

In this section we consider the problem of disturbance decoupled asymptotic estimation and derive a characterization for the case of zero feedthrough matrices. Similar to Section 5, the general case follows from Lemma 4.6. We omit the analog of Theorem 5.3 here.

Theorem 6.1. *Let* $[E, A, B, C_1, C_2, 0, 0] \in \Sigma_{l,n,m,p_1,p_2}$ *and* $Q \in \mathbb{R}^{l \times q}$. *There exists an asymptotic partial state observer* $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p_1,p_2}$ *such that* (21) *is satisfied if, and only if, the following statements hold:*

(i) $\operatorname{im}[B,Q] \subseteq E\mathscr{V}^*_{[E,A,0,0]} + A\mathscr{W}^*_{[E,A,0,0]}$,

(ii) $\mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right],\left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right],0,[C_2,0]\right]} = \mathscr{V}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right],\left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right],0,0\right]} \cap \ker[C_2,0]$,

(iii) $\forall \lambda \in \overline{\mathbb{C}_+}: \mathscr{W}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right],\left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right]\right],\lambda} \subseteq \ker_{\mathbb{C}}[C_2,0]$.

Proof. $\Leftarrow$: As in the proof of Theorem 5.1 we choose $[E_o,A_o,B_o,C_o,D_o]$ to be of the form (6) with $k=q$, $L_x=Q$, $L_y=0$, $L_z=0$, and we already know that $[E_o,A_o,B_o,C_o,D_o]$ is a disturbance decoupled partial state observer for $[E,A,B,C_1,C_2,0,0]$. To show that $[E_o,A_o,B_o,C_o,D_o]$ is asymptotic we need to show that (11) is true for $L_x=Q$, $L_y=0$ and $L_z=0$, but this is immediate from (iii).

$\Rightarrow$: Let an asymptotic partial state observer $[E_o,A_o,B_o,C_o,D_o] \in \Sigma_{l_o,n_o,m+p_1,p_2}$ such that (21) is satisfied be given. Then Theorem 5.1 implies that (i) and (ii) are satisfied. For (iii) we show that

$$\begin{aligned} &\forall (x,u,d,y,z,x_o,\hat{z}) \in \mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^q\times\mathbb{R}^{p_1}\times\mathbb{R}^{p_2}\times\mathbb{R}^{n_o}\times\mathbb{R}^{p_2}):\\ &\Big((x,\left(\begin{smallmatrix} u \\ d \end{smallmatrix}\right),y,z) \in \mathscr{B}_{[E,A,[B,Q],C_1,C_2,0,0]} \ \wedge\ (x_o,\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right),\hat{z}) \in \mathscr{B}_{[E_o,A_o,B_o,C_o,D_o]}\Big) \implies \lim_{t\to\infty}\big(\hat{z}(t)-z(t)\big)=0. \end{aligned} \tag{29}$$

Let $(x,\left(\begin{smallmatrix} u \\ d \end{smallmatrix}\right),y,z) \in \mathscr{B}_{[E,A,[B,Q],C_1,C_2,0,0]}$ and $(x_o,\left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right),\hat{z}) \in \mathscr{B}_{[E_o,A_o,B_o,C_o,D_o]}$ be smooth trajectories. By (21) there exist $\tilde{x} \in \mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^n)$, $\tilde{x}_o \in \mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^{n_o})$ such that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E & 0 \\ 0 & E_o \\ 0 & 0 \end{bmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{x}_o \end{pmatrix} = \begin{bmatrix} A & 0 \\ B_{o,2}C_1 & A_o \\ D_{o,2}C_1 - C_2 & C_o \end{bmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{x}_o \end{pmatrix} + \begin{bmatrix} B & Q \\ B_{o,1} & 0 \\ D_{o,1} & 0 \end{bmatrix}\begin{pmatrix} u \\ d \end{pmatrix}.$$

As in the proof of Theorem 5.1 we may show that the new variables $v := \tilde{x} - x$ and $w := \tilde{x}_o - x_o$ satisfy

$$(v,0,C_1v,C_2v) \in \mathscr{B}_{[E,A,B,C_1,C_2,0,0]} \quad \wedge \quad (w,\left(\begin{smallmatrix} 0 \\ C_1v \end{smallmatrix}\right),C_2v-(\hat{z}-z)) \in \mathscr{B}_{[E_o,A_o,B_o,C_o,D_o]}.$$

Since $[E_o,A_o,B_o,C_o,D_o]$ is an asymptotic partial state observer we obtain

$$0 = \lim_{t\to\infty}\Big(\big(C_2v(t)-(\hat{z}(t)-z(t))\big)-C_2v(t)\Big) = \lim_{t\to\infty}\big(\hat{z}(t)-z(t)\big),$$

which proves (29). It remains to consider $x \in \mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^n)$ and $d \in \mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^q)$ such that $\big(x,\left(\begin{smallmatrix} 0 \\ d \end{smallmatrix}\right),0,C_2x\big) \in \mathscr{B}_{[E,A,[B,Q],C_1,C_2,0,0]}$. Since $(0,\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right),0) \in \mathscr{B}_{[E_o,A_o,B_o,C_o,D_o]}$ it follows from (29) that $\lim_{t\to\infty} C_2x(t) = 0$, i.e., we have shown that

$$\forall x \in \mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^n),\ d \in \mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^q): \ \big(E\dot{x} = Ax + Qd \ \wedge\ C_1x = 0\big) \implies \lim_{t\to\infty} C_2x(t) = 0.$$

By Lemma A.4 this implies (iii). □

The characterization of existence of disturbance decoupled asymptotic partial state observers for ODE systems ($E=I$) can be compared to the result in [43] similar as in Remark 5.2.

Concluding this section we consider the special case of full state asymptotic estimation (i.e., $C_2 = I$) and show that the conditions (i)–(iii) in Theorem 6.1 simplify in this case.

Corollary 6.2. *Let* $[E,A,B,C_1,I_n,0,0] \in \Sigma_{l,n,m,p_1,p_2}$ *and* $Q \in \mathbb{R}^{l\times q}$ *with* $\operatorname{rk} Q = q$. *There exists an asymptotic partial state observer* $[E_o,A_o,B_o,C_o,D_o] \in \Sigma_{l_o,n_o,m+p_1,p_2}$ *such that* (21) *is satisfied if, and only if, the following statements hold:*

(a) $\operatorname{im}[B,Q] \subseteq E\mathscr{V}^*_{[E,A,0,0]} + A\mathscr{W}^*_{[E,A,0,0]}$,

(b) $\forall \lambda \in \overline{\mathbb{C}_+}: \ \mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A & Q \\ C_1 & 0 \end{bmatrix} = n+q.$

Proof. By Theorem 6.1 it suffices to prove that conditions (ii) and (iii) in Theorem 6.1 are equivalent to condition (b).

$\Rightarrow$: Assume that conditions (ii) and (iii) in Theorem 6.1 hold true. Let $\lambda \in \overline{\mathbb{C}_+}$ and

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathscr{W}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right]\right], \lambda}.$$

By (iii) we find that $x_1 = 0$ and furthermore

$$\mathscr{W}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right]\right], \lambda} = \begin{bmatrix} A - \lambda E & Q \\ C_1 & 0 \end{bmatrix}^{-1} \left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \mathscr{W}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right]\right], \lambda} \right) = \ker_{\mathbb{C}} \begin{bmatrix} A - \lambda E & Q \\ C_1 & 0 \end{bmatrix}.$$

This implies $Qx_2 = 0$ and hence $x_2 = 0$, thus we obtain

$$\mathscr{W}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right]\right], \lambda} = \{0\}.$$

Since

$$\ker_{\mathbb{C}} \begin{bmatrix} A - \lambda E & Q \\ C_1 & 0 \end{bmatrix} = \mathscr{W}^1_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right]\right], \lambda} \subseteq \mathscr{W}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right]\right], \lambda} = \{0\}$$

we may infer (b).

$\Leftarrow$: Condition (b) in particular implies condition (b) in Corollary 5.4 and hence it follows that condition (ii) in Theorem 6.1 is true. Condition (iii) in Theorem 6.1 follows from the fact that by (b) we have, for all $\lambda \in \overline{\mathbb{C}_+}$,

$$\ker_{\mathbb{C}} \begin{bmatrix} A - \lambda E & Q \\ C_1 & 0 \end{bmatrix} = \{0\},$$

which implies

$$\mathscr{W}^*_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right]\right], \lambda} = \mathscr{W}^1_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & Q \\ C_1 & 0 \end{smallmatrix}\right]\right], \lambda} = \{0\}. \qquad \square$$

For the case of regular $sE - A$, Corollary 6.2 was already proved in [23, Thm. 4-4.3].

7 Conclusion

In the present paper we derived a geometric characterization for solvability of the DDEP. This required a rigorous definition and the characterization of existence of (asymptotic) partial state observers. It turned out that the respective observer design is new even if the plant is governed by an ODE, since partial state observers are DAE systems in general. In the case of full state estimation also algebraic characterizations for the DDEP were derived.

A thorough investigation of the proof of Theorem 5.1 reveals that the freedom of choice in the (partial state) observer design (6) is not fully exploited yet, since we set $L_y = 0$ and $L_z = 0$. Hence, there is still room for additional tasks in the observer design apart from disturbance decoupling. Additional requirements on the observer, such as regularity of $sE_o - A_o$ (and its index being at most one) or estimation of certain components of the disturbance vector, may use the full freedom.

Appendix A

We provide some preliminary results for the characterization of (asymptotic) partial state observers. To this end, we consider the set of homogeneous DAEs

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t), \tag{30}$$

where $E,A \in \mathbb{R}^{l\times n}$, which is denoted by $\Sigma_{l,n}$ and we write $[E,A] \in \Sigma_{l,n}$. The behavior of $[E,A] \in \Sigma_{l,n}$ is given by

$$\mathscr{B}_{[E,A]} := \left\{ x \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}\to\mathbb{R}^n) \;\middle|\; Ex \in \mathscr{AC}(\mathbb{R}\to\mathbb{R}^l) \text{ and } x \text{ satisfies (30) for almost all } t\in\mathbb{R} \right\}.$$

Lemma A.1. *Let $A \in \mathbb{R}^{n\times n}, C \in \mathbb{R}^{p\times n}$ and set $\mathscr{O}(A,C) := [C^\top, A^\top C^\top, \ldots, (A^{n-1})^\top C^\top]^\top$. Then the following two statements are equivalent:*

(i) $\forall x \in \mathscr{B}_{[I,A]}:\ Cx(0) = 0 \ \Rightarrow\ Cx = 0,$

(ii) $\ker \mathscr{O}(A,C) = \ker C.$

Proof. (i)⇒(ii): By definition we have $\ker\mathscr{O}(A,C) \subseteq \ker C$. Let $x^0 \in \ker C$ and set $x(t) := e^{At}x^0$ for all $t\in\mathbb{R}$. Then $x \in \mathscr{B}_{[I,A]}$ and $Cx(0) = Cx^0 = 0$, thus (i) implies that $Ce^{At}x^0 = 0$ for all $t \in \mathbb{R}$. By a classical argument, see e.g. [40, Sec. 3.3], we obtain $x^0 \in \ker\mathscr{O}(A,C)$.

(ii)⇒(i): Let $x \in \mathscr{B}_{[I,A]}$ with $Cx(0) = 0$. Then $x(t) = e^{At}x(0)$ for all $t\in\mathbb{R}$ and $x(0) \in \ker C = \ker\mathscr{O}(A,C)$. According to [40, Sec. 3.3] we find $Cx(t) = 0$ for all $t \in \mathbb{R}$. □

Note that $\ker\mathscr{O}(A,C) = \mathscr{V}^*_{[I,A,0,C]}$ which motivates the following result.

Lemma A.2. *Let $E,A \in \mathbb{R}^{l\times n}$ and $C \in \mathbb{R}^{p\times n}$. Then the following statements are equivalent:*

(i) $\forall x \in \mathscr{B}_{[E,A]}:\ Cx(0) = 0 \ \Rightarrow\ Cx \overset{\text{a.e.}}{=} 0,$

(ii) $\forall x \in \mathscr{B}_{[E,A]} \cap \mathscr{C}^\infty(\mathbb{R}\to\mathbb{R}^n):\ Cx(0) = 0 \ \Rightarrow\ Cx = 0,$

(iii) *a)* $\mathscr{V}^*_{[E,A,0,C]} = \mathscr{V}^*_{[E,A,0,0]} \cap \ker C,$

b) $\mathscr{V}^*_{[E,A,0,0]} \cap \mathscr{W}^*_{[E,A,0,0]} \subseteq \ker C.$

Proof. Utilizing [14, Cor. 2.3] we may, without loss of generality, assume that the pencil $sE-A$ is in quasi-Kronecker form, i.e.,

$$sE - A = \begin{bmatrix} sE_P - A_P & 0 & 0 & 0 \\ 0 & sI_{n_J} - J & 0 & 0 \\ 0 & 0 & sN - I_{n_N} & 0 \\ 0 & 0 & 0 & sE_Q - A_Q \end{bmatrix}, \tag{31}$$

where

1. $E_P, A_P \in \mathbb{R}^{l_P\times n_P}, l_P > n_P$, are such that $\mathrm{rk}(\lambda E_P - A_P) = n_P$ and $\mathrm{rk}\, E_P = n_P$;

2. $J \in \mathbb{R}^{n_J\times n_J}$;

3. $N \in \mathbb{R}^{n_N\times n_N}$ is nilpotent;

4. $E_Q, A_Q \in \mathbb{R}^{l_Q \times n_Q}, l_Q < n_Q$, are such that $\operatorname{rk}(\lambda E_Q - A_Q) = l_Q$ and $\operatorname{rk} E_Q = l_Q$.

Let $C = [C_1, C_2, C_3, C_4]$ according to the partitioning of $sE - A$ in (31). Then we have that, see [14],

$$\begin{aligned} \mathcal{V}^*_{[E,A,0,0]} &= \mathbb{R}^{n_P} \times \mathbb{R}^{n_J} \times \{0\} \times \{0\}, \\ \mathcal{W}^*_{[E,A,0,0]} &= \mathbb{R}^{n_P} \times \{0\} \times \mathbb{R}^{n_N} \times \{0\}, \\ \mathcal{V}^*_{[E,A,0,C]} &= \mathcal{V}^*_{\left[\left[\begin{smallmatrix} E_P & 0 \\ 0 & I \end{smallmatrix}\right], \left[\begin{smallmatrix} A_P & 0 \\ 0 & J \end{smallmatrix}\right], 0, [C_1, C_2]\right]} \times \{0\} \times \{0\}. \end{aligned} \tag{32}$$

(i)⇒(iii): Assume that $C_1 \neq 0$. Choose $x_1 \in \mathscr{C}^\infty(\mathbb{R} \to \mathbb{R}^{n_P})$ such that $x_1(0) = 0$, $E_P \dot{x}_1 = A_P x_1$ and $C_1 x_1 \neq 0$. Then $x = (x_1^\top, 0, 0, 0)^\top \in \mathscr{B}_{[E,A]}$ with $Cx(0) = 0$ and $Cx \neq 0$, a contradiction. Therefore, we have $C_1 = 0$ and it follows from (32) that

$$\mathcal{V}^*_{[E,A,0,0]} \cap \mathcal{W}^*_{[E,A,0,0]} = \mathbb{R}^{n_P} \times \{0\} \times \{0\} \times \{0\} \subseteq \ker[0, C_2, C_3, C_4] = \ker C,$$

which proves b). Furthermore, it follows from (32) that

$$\mathcal{V}^*_{[E,A,0,C]} = \mathbb{R}^{n_P} \times \mathcal{V}^*_{[I,J,0,C_2]} \times \{0\} \times \{0\}, \quad \mathcal{V}^*_{[E,A,0,0]} \cap \ker C = \mathbb{R}^{n_P} \times \ker C_2 \times \{0\} \times \{0\}.$$

Now, let $x_2 \in \mathscr{B}_{[I,J]}$ with $C_2 x_2(0) = 0$. Then $x = (0, x_2^\top, 0, 0)^\top \in \mathscr{B}_{[E,A]}$ and $Cx(0) = 0$, thus we have $C_2 x_2 = Cx = 0$. Then it follows from Lemma A.1 that $\ker \mathscr{O}(J, C_2) = \mathcal{V}^*_{[I,J,0,C_2]} = \ker C_2$. This proves a).

(iii)⇒(i): It follows from (32) and b) that

$$\mathcal{V}^*_{[E,A,0,0]} \cap \mathcal{W}^*_{[E,A,0,0]} = \mathbb{R}^{n_P} \times \{0\} \times \{0\} \times \{0\} \subseteq \ker[C_1, C_2, C_3, C_4],$$

which implies $C_1 = 0$. Furthermore, as in "(i)⇒(iii)" we have

$$\mathbb{R}^{n_P} \times \mathcal{V}^*_{[I,J,0,C_2]} \times \{0\} \times \{0\} = \mathcal{V}^*_{[E,A,0,C]} \overset{a)}{=} \mathcal{V}^*_{[E,A,0,0]} \cap \ker C = \mathbb{R}^{n_P} \times \ker C_2 \times \{0\} \times \{0\},$$

which gives $\mathcal{V}^*_{[I,J,0,C_2]} = \ker C_2$. Now, let $x = (x_1^\top, x_2^\top, x_3^\top, x_4^\top)^\top \in \mathscr{B}_{[E,A]}$ with $Cx(0) = 0$. First, it follows from [13, Thm. 3.2] that $x_3 \overset{\text{a.e.}}{=} 0$ and $x_4 \overset{\text{a.e.}}{=} 0$. By $C_1 = 0$ it follows $C_2 x_2(0) = 0$ and as $x_2 \in \mathscr{B}_{[I,J]}$ and $\mathcal{V}^*_{[I,J,0,C_2]} = \ker C_2$, we may infer from Lemma A.1 that $C_2 x_2 = 0$, thus $Cx \overset{\text{a.e.}}{=} 0$.

The proof for "(ii)⇔(iii)" is analogous and omitted. □

The next two results are the basis for the characterization of asymptotic partial state observers.

Lemma A.3. *Let $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{p \times n}$. Then the following two statements are equivalent:*

(i) $\forall x \in \mathscr{B}_{[I,A]}: \ \lim_{t \to \infty} Cx(t) = 0$,

(ii) $\forall \lambda \in \overline{\mathbb{C}_+}: \ \ker_{\mathbb{C}}(\lambda I - A)^n \subseteq \ker_{\mathbb{C}} C$.

Proof. (i)⇒(ii): Seeking a contradiction we assume that there exist $\lambda \in \overline{\mathbb{C}_+}$ and $x^0 \in \ker_{\mathbb{C}}(\lambda I - A)^n$ with $Cx^0 \neq 0$. Obviously, λ is an eigenvalue of A. By [33, Thm. 2.11] we find that

$$e^{At} x^0 = e^{\lambda t} \sum_{i=0}^{m-1} \frac{t^i}{i!} (A - \lambda I)^i x^0, \quad t \in \mathbb{R},$$

where m is the algebraic multiplicity of λ. Let $k \in \{0,\dots,p\}$ such that $e_k^\top C x^0 \neq 0$, where e_k denotes the kth unit vector. Invoking that for any polynomial $p(t) = a_0 + a_1 t + \ldots + a_q t^q$ with $a_0 \neq 0$ it holds $|p(t)| \geq |a_0|$ for sufficiently large t, we may infer that there exists $T > 0$ such that for all $t \geq T$ we have

$$\left|e_k^\top C e^{At} x^0\right| = \underbrace{|e^{\lambda t}|}_{\geq 1} \cdot \left| e_k^\top C x^0 + \sum_{i=1}^{m-1} \frac{t^i}{i!} e_k^\top C (A - \lambda I)^i x^0 \right| \geq \left|e_k^\top C x^0\right| > 0.$$

This shows that $x(t) := e^{At} x^0$, $t \in \mathbb{R}$, satisfies $x \in \mathscr{B}_{[I,A]}$ and $Cx(t) \not\to 0$, a contradiction.

(ii)$\Rightarrow$(i): Let $x \in \mathscr{B}_{[I,A]}$ and $\lambda_1,\dots,\lambda_r \in \mathbb{C}$ be the pairwise distinct eigenvalues of A. Then [33, Thm. 2.11] gives that

$$x(t) = e^{At} x(0) = \sum_{j=1}^{r} e^{\lambda_j t} \sum_{i=0}^{m_j - 1} \frac{t^i}{i!} (A - \lambda_j I)^i z_j, \quad t \in \mathbb{R},$$

where m_j is the algebraic multiplicity of λ_j and $z_j \in \ker_{\mathbb{C}} (A - \lambda_j I)^{m_j}$ for $j = 1,\dots,r$. We may assume that $\lambda_1,\dots,\lambda_q \in \overline{\mathbb{C}_+}$ and $\lambda_{q+1},\dots,\lambda_r \in \mathbb{C}_-$ for some $q \in \{0,\dots,r\}$. Since $(A - \lambda_j I)^i z_j \in \ker_{\mathbb{C}} (A - \lambda_j I)^n \subseteq \ker_{\mathbb{C}} C$ for all $j = 1,\dots,q$ it follows that

$$Cx(t) = \sum_{j=q+1}^{r} e^{\lambda_j t} \sum_{i=0}^{m_j - 1} \frac{t^i}{i!} C (A - \lambda_j I)^i z_j, \quad t \in \mathbb{R},$$

and hence, obviously, $Cx(t) \to 0$ which completes the proof. □

In the following DAE version we use the space $\mathscr{W}^*_{[E,A],\lambda}$ introduced in Section 2.

Lemma A.4. *Let $E, A \in \mathbb{R}^{l \times n}$ and $C \in \mathbb{R}^{p \times n}$. Then the following statements are equivalent:*

(i) $\forall x \in \mathscr{B}_{[E,A]} : \ \lim_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} \|Cx(t)\| = 0$,

(ii) $\forall x \in \mathscr{B}_{[E,A]} \cap \mathscr{C}^\infty(\mathbb{R} \to \mathbb{R}^n) : \ \lim_{t\to\infty} Cx(t) = 0$,

(iii) $\forall \lambda \in \overline{\mathbb{C}_+} : \ \mathscr{W}^*_{[E,A],\lambda} \subseteq \ker_{\mathbb{C}} C$.

Proof. Without loss of generality we assume that $sE - A$ is in quasi-Kronecker form (31) with the properties stated in the proof of Lemma A.2, and $C = [C_1, C_2, C_3, C_4]$ according to the partitioning of $sE - A$. It is a straightforward calculation, see also [14], that

$$\mathscr{W}^*_{[E,A],\lambda} = \mathbb{C}^{n_P} \times \ker_{\mathbb{C}} (\lambda I_{n_J} - J)^{n_J} \times \{0\} \times \{0\}. \tag{33}$$

(i)$\Rightarrow$(iii): We show that $C_1 = 0$. Assume that $C_1 \neq 0$ and choose $x_1 \in \mathscr{C}^\infty(\mathbb{R} \to \mathbb{R}^{n_P})$ such that $E_P \dot{x}_1 = A_P x_1$ and $\|C_1 x_1(t)\| \to \infty$. Then $x = (x_1^\top, 0, 0, 0)^\top \in \mathscr{B}_{[E,A]}$ with $Cx \not\to 0$, a contradiction. Therefore, $C_1 = 0$ and, in view of (33), it remains to show that for all $\lambda \in \overline{\mathbb{C}_+}$ we have $\ker_{\mathbb{C}}(\lambda I - J)^{n_J} \subseteq \ker_{\mathbb{C}} C_2$. This follows from Lemma A.3 and the fact that for any $x_2 \in \mathscr{B}_{[I,J]}$ we have that $x = (0, x_2^\top, 0, 0)^\top \in \mathscr{B}_{[E,A]}$ and hence $C_2 x_2(t) = Cx(t) \to 0$.

(iii)$\Rightarrow$(i): By (33) we obtain $C_1 = 0$ and hence $\ker_{\mathbb{C}}(\lambda I - J)^{n_J} \subseteq \ker_{\mathbb{C}} C_2$ for all $\lambda \in \overline{\mathbb{C}_+}$. Therefore, Lemma A.3 implies that for all $x_2 \in \mathscr{B}_{[I,J]}$ we have $C_2 x_2(t) \to 0$. If now $x = (x_1^\top, x_2^\top, x_3^\top, x_4^\top)^\top \in \mathscr{B}_{[E,A]}$, then it follows from [13, Thm. 3.2] that $x_3 \overset{\text{a.e.}}{=} 0$ and $x_4 \overset{\text{a.e.}}{=} 0$. Furthermore, $x_2 \in \mathscr{B}_{[I,J]}$ and therefore we obtain

$$\lim_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} \|Cx(t)\| = \lim_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} \|C_2 x_2(t)\| = 0.$$

The proof for "(ii)$\Leftrightarrow$(iii)" is analogous and omitted. □

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