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# The McMillan degree of implicit transfer functions of RLC networks

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**Abstract** The McMillan degree of an implicit network transfer function defines the minimum number of dynamic elements which are necessary to fully describe the network. It is therefore a measure for the complexity of a network. Using modified nodal analysis models, which are linked directly to the natural network topology, we show that the McMillan degree equals the sum of the number of capacitors and inductors minus the number of fundamental loops of capacitors and fundamental cutsets of inductors. Exploiting this representation we derive a minimal realization of a RLC network, that is one where the number of involved (independent) differential equations equals the McMillan degree.

**Keywords** RLC networks · modified nodal analysis · McMillan degree · minimal realization

## 1 Introduction

In the present paper we consider the McMillan degree as a complexity measure for RLC networks. We investigate models of RLC networks (without sources) which arise from modified nodal analysis (MNA), see [10] and the survey [17], and can thus be described by a linear differential-algebraic equation of the form

$$E\dot{x}(t) = Ax(t), \tag{1}$$

where

$$sE - A = \begin{bmatrix} sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top & A_{\mathcal{L}} \\ -A_{\mathcal{L}}^\top & s\mathcal{L} \end{bmatrix} \in \mathbb{R}[s]^{n\times n}, \tag{2}$$

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$$x = \begin{pmatrix} \eta \\ i_{\mathcal{L}} \end{pmatrix}, \tag{3}$$

and

$$\left.\begin{aligned} &\mathcal{C} \in \mathbb{R}^{n_{\mathcal{C}} \times n_{\mathcal{C}}}, \mathcal{G} \in \mathbb{R}^{n_{\mathcal{G}} \times n_{\mathcal{G}}}, \mathcal{L} \in \mathbb{R}^{n_{\mathcal{L}} \times n_{\mathcal{L}}}, \\ &A_{\mathcal{C}} \in \mathbb{R}^{n_e \times n_{\mathcal{C}}}, A_{\mathcal{R}} \in \mathbb{R}^{n_e \times n_{\mathcal{G}}}, A_{\mathcal{L}} \in \mathbb{R}^{n_e \times n_{\mathcal{L}}}, \\ &n = n_e + n_{\mathcal{L}}. \end{aligned}\right\} \tag{4}$$

Here $\mathbb{R}[s]$ denotes the ring of polynomials with coefficients in the set of real numbers $\mathbb{R}$. $\mathcal{C}$, $\mathcal{G}$ and $\mathcal{L}$ are the matrices expressing the constitutive relations of capacitances, resistances and inductances, $\eta(t)$ is the vector of node potentials[1] and $i_{\mathcal{L}}(t)$ is the vector of currents through inductances. By $n_{\mathcal{C}}, n_{\mathcal{G}}, n_{\mathcal{L}}$ we denote the number of capacitances, resistances and inductances in the network, resp., and $n_e+1$ is the number of nodes in the network graph. The matrix pencil $sE - A$ is regular, i.e., $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$, provided that the network is connected and passive, cf. [4,14]. Then the implicit transfer function associated with (1) exists and is given by $G(s) = (sE - A)^{-1}$.

The complexity analysis of RLC networks is related to the problem of network redesign, see [11,12], i.e., the desire to change the natural dynamics of the network by modification of its elements and/or topology. In order to identify appropriate changes it is necessary to have a measure for the complexity of the network. Such a measure is provided by the McMillan degree of the implicit transfer function $G(s)$. Roughly speaking, the McMillan degree defines the minimum number of dynamic elements which are necessary to fully describe the network. A result which is intuitively known but not rigorously proven in the circuit literature [8,19,20] is that the McMillan degree equals the number of independent capacitors and inductors in the network. In order to rigorously define what "independent" means, we use the concepts of fundamental loops and cutsets and show that the McMillan degree equals the sum of the number of capacitors and inductors minus the number of fundamental loops of capacitors and fundamental cutsets of inductors. This significantly improves earlier results obtained in [16].

Network redesign problems are often considered in the context of impedance and admittance models (see [21]) as discussed in [2,13,14,16]. In the present paper we consider models arising from modified nodal analysis, which are linked directly to the natural network topology. This allows to derive the representation of the McMillan degree in terms of the network topology.

As a second main result, we exploit the representation of the McMillan degree to derive a minimal realization of a given RLC network, that is one where the number of involved (independent) differential equations equals the McMillan degree. We illustrate our results by means of two examples.

[1] The node potential $\eta_i$ expresses the voltage between the $i$th node in the network graph and the ground node.

## 2 Graph theoretical preliminaries

In this section we introduce the graph theoretical concepts (cf. for instance [6]) on which the modified nodal analysis is based. We further introduce the notions of fundamental loops and cutsets and characterize their number in terms of the incidence matrix of the network graph.

**Definition 1** A *graph* is a triple $\mathcal{G} = (V, E, \varphi)$ consisting of a *node set* $V$ and a *branch set* $E$ together with an *incidence map*

$$\varphi : E \to V \times V, \quad e \mapsto \varphi(e) = (\varphi_1(e), \varphi_2(e)),$$

where $\varphi_1(e) \neq \varphi_2(e)$ for all $e \in E$, i.e., the graph does not contain self-loops. If $\varphi(e) = (v_1, v_2)$, we call $e$ to be *directed from* $v_1$ *to* $v_2$; $v_1$ is called the *initial node* and $v_2$ the *terminal node* of $e$.

Let $V' \subseteq V$ and let $E'$ be a set of branches satisfying

$$E' \subseteq E|_{V'} := \{ \; e \in E \;\mid \varphi_1(e) \in V' \text{ and } \varphi_2(e) \in V' \; \}.$$

Further let $\varphi|_{E'}$ be the restriction of $\varphi$ to $E'$. Then the triple $\mathcal{K} := (V', E', \varphi|_{E'})$ is called a *subgraph of* $\mathcal{G}$. If $V' = V$, then $\mathcal{K}$ is called a *spanning subgraph.* A *proper subgraph* is one with $E \neq E'$.

For each branch $e$, define an additional branch $-e$ being directed from the terminal to the initial node of $e$, that is $\varphi(-e) = (\varphi_2(e), \varphi_1(e))$ for $e \in E$. Now define the set $\tilde{E} = \{ \; e \;\mid e \in E \text{ or } -e \in E \; \}$. A tuple $w = (w_1, \ldots, w_r) \in \tilde{E}^r$, where for $i = 1, \ldots, r-1$,

$$v_0 := \varphi_1(v_1), \quad v_i := \varphi_2(w_i) = \varphi_1(w_{i+1})$$

is called *path from* $v_0$ *to* $v_r$; $w$ is called *elementary path*, if $v_1, \ldots, v_r$ are distinct. A *loop* is an elementary path with $v_0 = v_r$. Two nodes $v, v'$ are called *connected*, if there exists a path from $v$ to $v'$. The graph itself is called connected, if any two nodes are connected. A subgraph $\mathcal{K} = (V', E', \varphi|_{E'})$ is called a *component of connectivity*, if it is connected and $\mathcal{K}^c := (V \setminus V', E \setminus E', \varphi|_{E\setminus E'})$ is a subgraph.

A *tree* is a minimally connected graph, i.e., it is connected without having any connected proper spanning subgraph. A spanning subgraph of a connected graph $\mathcal{G}$, which is a tree, is called a *tree in* $\mathcal{G}$. If $\mathcal{G}$ is not connected, with $k$ components of connectivity, and $\mathcal{T}_i$ is a tree in any such component for $i = 1, \ldots, k$, then $\mathcal{T} = \mathcal{T}_1 \cup \ldots \cup \mathcal{T}_k$ is called a *forest in* $\mathcal{G}$.

A spanning subgraph $\mathcal{K} = (V, E', \varphi|_{E'})$ is called a *cutset* of $\mathcal{G} = (V, E, \varphi)$, if its branch set is non-empty, $\mathcal{G} - \mathcal{K} := (V, E \setminus E', \varphi|_{E\setminus E'})$ is a disconnected subgraph and $\mathcal{G} - \mathcal{K}'$ is a connected subgraph for any proper spanning subgraph $\mathcal{K}'$ of $\mathcal{K}$.

In this work we consider only *finite graphs*, i.e., graphs with finite node set and finite branch set.

**Definition 2** Let $\mathcal{G}$ be a graph, $\mathcal{K}, \mathcal{L}$ be spanning subgraphs of $\mathcal{G}$, and $\ell$ be a path of $\mathcal{G}$.

(i) $\mathcal{L}$ is called a *$\mathcal{K}$-cutset*, if $\mathcal{L}$ is a subgraph of $\mathcal{K}$ and a cutset of $\mathcal{G}$.
(ii) $\ell$ is called a *$\mathcal{K}$-loop*, if $\ell$ is a loop of $\mathcal{K}$.

A graph can have many $\mathcal{K}$-loops and $\mathcal{K}$-cutsets, resp., but not all of them are independent. In the following we introduce the crucial notions of fundamental $\mathcal{K}$-loops and $\mathcal{K}$-cutsets, which generalize the notions of fundamental loops and cutsets given e.g. in [1].

**Definition 3** Let $\mathcal{G}$ be a graph and $\mathcal{K}$ be a spanning subgraph of $\mathcal{G}$. Further let $\mathcal{T}_1$ be a forest in $\mathcal{K}$ and $\mathcal{T}_2$ be a forest in $\mathcal{G} - \mathcal{K}$. Then

(i) every branch in $\mathcal{K} - \mathcal{T}_1$ closes a unique loop in $\mathcal{K}$ that consists of that branch and branches from $\mathcal{T}_1$ only. These loops are called *fundamental $\mathcal{K}$-loops of $\mathcal{G}$*.
(ii) $\mathcal{T}_2$ can be completed to a tree $\mathcal{T}_3$ in $\mathcal{G}$ by adding branches from $\mathcal{K}$ (if necessary). Every branch in $\mathcal{T}_3 - \mathcal{T}_2$ defines a unique cutset of $\mathcal{G}$ that consists of that branch and branches which are common to $\mathcal{G} - \mathcal{T}_3$ and $\mathcal{K}$ only. These cutsets are called *fundamental $\mathcal{K}$-cutsets of $\mathcal{G}$*.

Similar to [1] we may show that any $\mathcal{K}$-loop/$\mathcal{K}$-cutset can be expressed in terms of fundamental $\mathcal{K}$-loops/$\mathcal{K}$-cutsets, for any fix choice of trees/forests $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ as in Definition 3. Therefore, in particular, the number of fundamental $\mathcal{K}$-loops/$\mathcal{K}$-cutsets in a graph $\mathcal{G}$ is independent of the choice of the trees/forests and we may define, using the notation from Definition 3,

$$\begin{aligned}\mathrm{FL}_{\mathcal{K}} &:= |\{\ \ell \mid \ell \text{ is a fundamental } \mathcal{K}\text{-loop of } \mathcal{G} \text{ corresponding to } \mathcal{T}_1\ \}| \\ &= |\{\ e \mid e \text{ is a branch of } \mathcal{K} - \mathcal{T}_1\ \}|, \\ \mathrm{FC}_{\mathcal{K}} &:= |\{\ c \mid c \text{ is a fundamental } \mathcal{K}\text{-cutset of } \mathcal{G} \text{ corresponding to } \mathcal{T}_2 \text{ and } \mathcal{T}_3\ \}| \\ &= |\{\ e \mid e \text{ is a branch of } \mathcal{T}_3 - \mathcal{T}_2\ \}|.\end{aligned}$$

In the following we introduce the notion of an incidence matrix, which is helpful in describing the topology of RLC networks. In particular, we derive formulas for $\mathrm{FL}_{\mathcal{K}}$ and $\mathrm{FC}_{\mathcal{K}}$ using incidence matrices.

**Definition 4** Let a graph $\mathcal{G} = (V, E, \varphi)$ with $l$ branches $E = \{e_1, \ldots, e_l\}$ and $k$ nodes $V = \{v_1, \ldots, v_k\}$ be given. Then the *all-node incidence matrix* of $\mathcal{G}$ is given by $A_0 = (a_{ij}) \in \mathbb{R}^{k \times l}$, where

$$a_{ij} = \begin{cases} 1, & \text{if } \varphi_1(e_j) = v_i, \\ -1, & \text{if } \varphi_2(e_j) = v_i, \\ 0, & \text{otherwise.} \end{cases}$$

Since the rows of $A_0$ sum up to the zero row vector, one may delete an arbitrary row of $A_0$ to obtain a matrix $A$ having the same rank as $A_0$. We call $A$ an *incidence matrix* of $\mathcal{G}$. Usually, the chosen row corresponds to the ground node from $V$.

A spanning subgraph $\mathcal{K}$ of the graph $\mathcal{G}$ has an incidence matrix $A_\mathcal{K}$ which is constructed by deleting columns of the incidence matrix $A$ of $\mathcal{G}$ corresponding to the branches of the complementary spanning subgraph $\mathcal{G}-\mathcal{K}$. By a suitable reordering of the branches, the incidence matrix reads

$$A = \begin{bmatrix} A_\mathcal{K} \; A_{\mathcal{G}-\mathcal{K}} \end{bmatrix}. \tag{5}$$

In the following result we derive the number of fundamental $\mathcal{K}$-loops/$\mathcal{K}$-cutsets in terms of the incidence matrices $A_\mathcal{K}$, $A_{\mathcal{G}-\mathcal{K}}$; this improves the result in [18, Lem. 2.1 & Lem. 2.3].

**Theorem 1** *Let $\mathcal{G}$ be a connected graph with incidence matrix $A \in \mathbb{R}^{(k-1)\times l}$. Further, let $\mathcal{K}$ be a spanning subgraph and assume that the branches of $\mathcal{G}$ are sorted in a way that* (5) *is satisfied. Then the following holds true:*

(i) $\mathrm{FL}_\mathcal{K} = \dim\ker A_\mathcal{K}$,

(ii) $\mathrm{FC}_\mathcal{K} = \dim\ker A_{\mathcal{G}-\mathcal{K}}^\top$.

*Proof* Let $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ be trees/forests as in Definition 3.
We show (i): Let $m$ denote the number of branches of $\mathcal{K}$, $n$ the number of its node and $p$ the number of its components of connectivity. Note that $A_\mathcal{K} \in \mathbb{R}^{(k-1)\times m}$. Since $\mathcal{T}_1$ is a forest in $\mathcal{K}$, it has $n-p$ branches, hence

$$\mathrm{FL}_\mathcal{K} = m - (n-p).$$

By [17, Thm. 4.3] we have that $\operatorname{rk} A_\mathcal{K} = n-p$, and hence

$$\mathrm{FL}_\mathcal{K} = m - \operatorname{rk} A_\mathcal{K} = \dim\ker A_\mathcal{K}.$$

We show (ii): Since $\mathcal{T}_3$ is a tree in $\mathcal{G}$, it has $k-1$ branches. Therefore, we have

$$\mathrm{FC}_\mathcal{K} = k-1-r,$$

where $r$ is the number of branches in $\mathcal{T}_2$. Since $\mathcal{T}_2$ is a forest in $\mathcal{G}-\mathcal{K}$, it has $k-q$ branches, where $q$ is the number of components of connectivity of $\mathcal{G}-\mathcal{K}$, thus

$$\mathrm{FC}_\mathcal{K} = q-1.$$

By [17, Thm. 4.3] we have that $\operatorname{rk} A_{\mathcal{G}-\mathcal{K}} = k-q$, where $A_{\mathcal{G}-\mathcal{K}} \in \mathbb{R}^{(k-1)\times(l-m)}$, and hence

$$q = k - \operatorname{rk} A_{\mathcal{G}-\mathcal{K}}^\top = k - \left(k-1-\dim\ker A_{\mathcal{G}-\mathcal{K}}^\top\right) = \dim\ker A_{\mathcal{G}-\mathcal{K}}^\top + 1,$$

which completes the proof of the theorem. □

## 3 Network equations

It is well-known [5,10] that the graph underlying an electrical network can be described by an incidence matrix $\mathbf{A} \in \mathbb{R}^{(k-1)\times l}$, which can be decomposed into submatrices

$$\mathbf{A} = \begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{L}} \end{bmatrix}$$

for the quantities in (4), where $n_e = k-1$ and $l = n_{\mathcal{C}} + n_{\mathcal{G}} + n_{\mathcal{L}}$. Each submatrix is the incidence matrix of a specific subgraph of the network graph. $A_{\mathcal{C}}$ is the incidence matrix of the subgraph consisting of all network nodes and all branches corresponding to capacitors. Similarly, $A_{\mathcal{R}}$ and $A_{\mathcal{L}}$ are the incidence matrices corresponding to the resistor and inductor subgraphs, resp. Then, using the standard MNA modeling procedure [10], see also the survey [17], which is just a clever arrangement of Kirchhoff's laws together with the characteristic equations of the devices, results in a differential-algebraic equation (1) with (2)–(4). $\mathcal{C}$, $\mathcal{G}$ and $\mathcal{L}$ are the matrices expressing the constitutive relations of capacitances, resistances and inductances, $\eta(t)$ is the vector of node potentials and $i_{\mathcal{L}}(t)$ is the vector of currents through inductances.

**Definition 5** For a given RLC network, any differential-algebraic equation (1) satisfying (2)–(4), which arises from the MNA modeling procedure [10], is said to be an MNA model of the network.

It is a reasonable assumption that an electrical network is connected (see also [14]); otherwise, since the components of connectivity do not physically interact, one might consider them separately. Furthermore, in the present paper we consider networks with *passive* devices. These assumptions lead to the following assumptions on an MNA model (2)–(4) of the network (cf. [17]).

**(A1)** $\operatorname{rk}\begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{L}} \end{bmatrix} = n_e$,

**(A2)** $\mathcal{C} = \mathcal{C}^\top > 0, \mathcal{L} = \mathcal{L}^\top > 0, \mathcal{G} + \mathcal{G}^\top > 0$.

It is shown in [4, Cor. 4.5] that under the conditions (A1) and (A2), the pencil $sE - A$ in (2) is regular.

## 4 The McMillan degree

In this section we investigate the McMillan degree of implicit network transfer functions and derive a formula as well as a topological interpretation.

The McMillan degree of a rational matrix $G(s) \in \mathbb{R}(s)^{n\times n}$, where $\mathbb{R}(s)$ is the quotient field of $\mathbb{R}[s]$, is the total number of its poles, and can be defined via its *Smith-McMillan form*

$$U^{-1}(s)G(s)V^{-1}(s) = \operatorname{diag}\left(\frac{\varepsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\varepsilon_r(s)}{\psi_r(s)}, 0, \ldots, 0\right) \in \mathbb{R}(s)^{n\times n}, \quad (6)$$

where $U(s)$, $V(s) \in \mathbb{R}[s]^{n\times n}$ are unimodular (i.e. invertible over $\mathbb{R}[s]^{n\times n}$), $\operatorname{rk} G(s) = r$, $\varepsilon_i(s), \psi_i(s) \in \mathbb{R}[s]$ are monic, coprime and satisfy $\varepsilon_i(s) \mid \varepsilon_{i+1}(s)$, $\psi_{i+1}(s) \mid \psi_i(s)$ for all $i = 1, \ldots, r-1$.

**Definition 6** Consider $G(s) \in \mathbb{R}(s)^{n\times n}$ with Smith-McMillan form (6). Then we call

$$\delta_M G(s) := \deg \prod_{i=1}^{r} \psi_i(s)$$

the *McMillan degree* of $G(s)$.

For the implicit transfer function $G(s) = (sE - A)^{-1}$ of a system (1) with regular matrix pencil $sE - A$, it is a consequence of the Weierstraß canonical form (see [7]) that

$$\delta_M G(s) = \deg \det(sE - A). \tag{7}$$

We are now in a position to derive the first main result of the present paper.

**Theorem 2** *Consider a MNA model* (1) *with* (2)–(4) *of a RLC network. Then, for* $G(s) = (sE - A)^{-1}$*, we have*

$$\delta_M G(s) = n_{\mathcal{L}} + \operatorname{rk} A_{\mathcal{C}} - \dim\ker\,[A_{\mathcal{R}}, A_{\mathcal{C}}]^\top.$$

*Proof* Choose matrices $V, W$ with full column rank such that

$$\operatorname{im} V = \operatorname{im}[A_{\mathcal{R}}, A_{\mathcal{C}}], \qquad \operatorname{im} W = \ker\,[A_{\mathcal{R}}, A_{\mathcal{C}}]^\top = \big(\operatorname{im} V\big)^\perp,$$

and let $m := \operatorname{rk} W$. With

$$T := \begin{bmatrix} V & W & 0 \\ 0 & 0 & I_{n_{\mathcal{L}}} \end{bmatrix} \in \mathbf{GL}_{n_e+n_{\mathcal{L}}},$$

where $\mathbf{GL}_n$ denotes the set of invertible matrices from $\mathbb{R}^{n\times n}$, we obtain

$$T^\top (sE - A) T = \begin{bmatrix} V^\top\big(sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top\big)V & 0 & V^\top A_{\mathcal{L}} \\ 0 & 0 & W^\top A_{\mathcal{L}} \\ -A_{\mathcal{L}}^\top V & -A_{\mathcal{L}}^\top W & s\mathcal{L} \end{bmatrix}. \tag{8}$$

Then, the Schur complement formula (see e.g. [9, Lem. A.1.17]), yields

$$\begin{aligned}\det(sE - A) &= \det\big(V^\top\big(sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top\big)V\big)\\ &\quad\cdot \det\begin{bmatrix} 0 & W^\top A_{\mathcal{L}} \\ -A_{\mathcal{L}}^\top W & s\mathcal{L} + A_{\mathcal{L}}^\top V\left(V^\top\big(sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top\big)V\right)^{-1} V^\top A_{\mathcal{L}} \end{bmatrix}.\end{aligned}$$

Let

$$P(s) := s\mathcal{L} + A_{\mathcal{L}}^\top V\left(V^\top\big(sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top\big)V\right)^{-1} V^\top A_{\mathcal{L}},$$

then, again using the Schur complement formula,

$$\begin{aligned}&\det(sE - A)\\ &\quad= \det\big(V^\top\big(sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top\big)V\big)\, \det P(s)\, \det W^\top A_{\mathcal{L}} P(s)^{-1} A_{\mathcal{L}}^\top W.\end{aligned}$$

We show that $A_{\mathcal{L}}^\top W$ has full column rank: Let $x \in \mathbb{R}^m$ be such that $A_{\mathcal{L}}^\top W x = 0$, then

$$Wx \in \ker A_{\mathcal{L}}^\top \cap \operatorname{im} W = \ker A_{\mathcal{L}}^\top \cap \ker [A_{\mathcal{R}}, A_{\mathcal{C}}]^\top = \ker [A_{\mathcal{R}}, A_{\mathcal{C}}, A_{\mathcal{L}}]^\top \overset{(A1)}{=} \{0\},$$

and the full column rank of $W$ implies $x = 0$. Write $P(s) = s\mathcal{L} + G_p(s)$, where $G_p(s)$ is proper. Then

$$P(s)^{-1} = (s\mathcal{L})^{-1}\big(I + (s\mathcal{L})^{-1} G_p(s)\big)^{-1} = \sum_{k=0}^{\infty} (-1)^k s^{-k-1} \mathcal{L}^{-k-1} G_p(s)^k,$$

and

$$W^\top A_{\mathcal{L}} P(s)^{-1} A_{\mathcal{L}}^\top W = s^{-1} W^\top A_{\mathcal{L}} \mathcal{L}^{-1} A_{\mathcal{L}}^\top W + s^{-1} G_{sp}(s),$$

where $G_{sp}(s)$ is strictly proper. Since $W^\top A_{\mathcal{L}} \mathcal{L}^{-1} A_{\mathcal{L}}^\top W \in \mathbf{GL}_m$, the highest power of $s$ appearing in $\det W^\top A_{\mathcal{L}} P(s)^{-1} A_{\mathcal{L}}^\top W$ is $s^{-m}$. Furthermore, the highest power of $s$ appearing in $\det P(s)$ is $s^{n_{\mathcal{L}}}$. By (7) and the above observations we obtain

$$\delta_M G(s) = \deg\det\big(V^\top\big(sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top\big)V\big) + n_{\mathcal{L}} - m.$$

We consider the matrix pencil $s\tilde{E} - \tilde{A} := V^\top\big(sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top\big)V$. We show that $\ker \tilde{E} \cap \ker(\tilde{A} + \tilde{A}^\top) = \{0\}$: Let $x \in \ker \tilde{E} \cap \ker(\tilde{A} + \tilde{A}^\top)$, then

$$x^\top V^\top A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top V x = 0 \quad \text{and} \quad x^\top V^\top A_{\mathcal{R}}(\mathcal{G} + \mathcal{G}^\top)A_{\mathcal{R}}^\top V x = 0,$$

which implies, using (A2), that $A_{\mathcal{C}}^\top V x = 0$ and $A_{\mathcal{R}}^\top V x = 0$. Therefore,

$$Vx \in \ker [A_{\mathcal{R}}, A_{\mathcal{C}}]^\top \cap \operatorname{im} V = \ker [A_{\mathcal{R}}, A_{\mathcal{C}}]^\top \cap \operatorname{im}[A_{\mathcal{R}}, A_{\mathcal{C}}] = \{0\},$$

and full column rank of $V$ implies $x = 0$. Invoking that $\ker \tilde{A} \subseteq \ker(\tilde{A} + \tilde{A}^\top)$ by (A2), it now follows from [4, Cor 2.6 & Lem. 2.6] that $s\tilde{E} - \tilde{A}$ is regular. We show that its index (see e.g. [15, Def. 2.9] for a definition) is at most one. Seeking a contradiction, assume that it is larger than one. Then [3, Prop. 2.10] implies that there exist $x, y \in \mathbb{R}^q \setminus \{0\}$, where $q = \operatorname{rk} V$, such that $\tilde{E}y = \tilde{A}x$ and $\tilde{E}x = 0$. Therefore,

$$x^\top(\tilde{A} + \tilde{A}^\top)x = x^\top \tilde{E} y + y^\top \tilde{E} x = 0,$$

hence $(\tilde{A} + \tilde{A}^\top)x = 0$ which gives $x \in \ker \tilde{E} \cap \ker(\tilde{A} + \tilde{A}^\top) = \{0\}$, a contradiction. Since the index of $s\tilde{E} - \tilde{A}$ is at most one we find that (see e.g. [15])

$$\deg\det(s\tilde{E} - \tilde{A}) = \operatorname{rk} \tilde{E}.$$

Furthermore,

$$\operatorname{rk} \tilde{E} = \operatorname{rk} V^\top A_{\mathcal{C}} = \operatorname{rk} \begin{bmatrix} V^\top A_{\mathcal{C}} \\ 0 \end{bmatrix} = \operatorname{rk} \begin{bmatrix} V^\top A_{\mathcal{C}} \\ W^\top A_{\mathcal{C}} \end{bmatrix} = \operatorname{rk} \begin{bmatrix} V^\top \\ W^\top \end{bmatrix} A_{\mathcal{C}} = \operatorname{rk} A_{\mathcal{C}},$$

and so we finally obtain

$$\delta_M G(s) = \operatorname{rk} A_{\mathcal{C}} + n_{\mathcal{L}} - m = \operatorname{rk} A_{\mathcal{C}} + n_{\mathcal{L}} - \dim\ker [A_{\mathcal{R}}, A_{\mathcal{C}}]^\top.$$

□

In the following we will use expressions like $\mathcal{C}$-loop for a loop in the circuit graph whose branch set consists only of branches corresponding to capacitors. Likewise, a $\mathcal{L}$-cutset is a cutset in the circuit graph whose branch set consists only of branches corresponding to inductors. We present an interpretation of Theorem 2 in terms of the network topology. By Theorem 1, $\dim\ker A_{\mathcal{C}}$ equals the number of fundamental $\mathcal{C}$-loops and $\dim\ker\,[A_{\mathcal{R}}, A_{\mathcal{C}}]^\top$ equals the number of fundamental $\mathcal{L}$-cutsets in the network, thus the following is an immediate consequence of Theorems 1 and 2.

**Corollary 1** *Using the notation from Theorem 2 we have that*

$$\delta_M G(s) = n_{\mathcal{C}} + n_{\mathcal{L}} - \mathrm{FL}_{\mathcal{C}} - \mathrm{FC}_{\mathcal{L}}.$$

## 5 Minimal realization

In this section we derive a minimal realization of a given RLC network in the following sense.

**Definition 7** A system of the form (1) is called a *minimal realization* of a RLC network, if its number of (independent) differential equations equals the McMillan degree of the implicit transfer function $G(s)$ of the network, i.e., $\operatorname{rk} E = \delta_M G(s)$, and there is a one-to-one correspondence to the solutions of an MNA model of the network.

In order to obtain a minimal realization we start with an MNA model (1) satisfying (2)–(4) of the RLC network and its transformation in (8), using the notation from the proof of Theorem 2. Now let $Y$ be a matrix with full column rank such that

$$\operatorname{im} Y = \ker W^\top A_{\mathcal{L}} = \left(\operatorname{im} A_{\mathcal{L}}^\top W\right)^\perp,$$

and, recalling that $A_{\mathcal{L}}^\top W$ has full column rank,

$$S := \begin{bmatrix} I_{n_e} & 0 & 0 \\ 0 & A_{\mathcal{L}}^\top W & Y \end{bmatrix} \in \mathbf{GL}_{n_e+n_{\mathcal{L}}}.$$

Then

$$\begin{aligned} &S^\top T^\top (sE-A)TS \\ &= \begin{bmatrix} V^\top\left(sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top\right)V & 0 & V^\top A_{\mathcal{L}}A_{\mathcal{L}}^\top W & V^\top A_{\mathcal{L}} Y \\ 0 & 0 & W^\top A_{\mathcal{L}}A_{\mathcal{L}}^\top W & 0 \\ -W^\top A_{\mathcal{L}}A_{\mathcal{L}}^\top V & -W^\top A_{\mathcal{L}}A_{\mathcal{L}}^\top W & sW^\top A_{\mathcal{L}}\mathcal{L}A_{\mathcal{L}}^\top W & sW^\top A_{\mathcal{L}}\mathcal{L}Y \\ -Y^\top A_{\mathcal{L}}^\top V & 0 & sY^\top \mathcal{L}A_{\mathcal{L}}^\top W & sY^\top \mathcal{L} Y \end{bmatrix}. \end{aligned}$$

Obviously, $W^\top A_{\mathcal{L}}A_{\mathcal{L}}^\top W \in \mathbf{GL}_m$ and hence there is a one-to-one correspondence between the solutions of the MNA model (1) and the solutions of the system

$$\begin{aligned} V^\top A_\mathcal{C}\mathcal{C}A_\mathcal{C}^\top V\,\dot{x}_1(t) &= -V^\top A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top V\,x_1(t) - V^\top A_\mathcal{L} Y\,x_4(t),\\ W^\top A_\mathcal{L}\mathcal{L}Y\,\dot{x}_4(t) &= W^\top A_\mathcal{L}A_\mathcal{L}^\top V\,x_1(t) + W^\top A_\mathcal{L}A_\mathcal{L}^\top W\,x_2(t),\\ Y^\top\mathcal{L}Y\,\dot{x}_4(t) &= Y^\top A_\mathcal{L}^\top V\,x_1(t). \end{aligned}$$

Again using that $W^\top A_\mathcal{L}A_\mathcal{L}^\top W \in \mathbf{GL}_m$, the second equation can be solved for $x_2$ and we obtain a one-to-one correspondence to the solutions of the system

$$\tilde{E}\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t), \qquad \tilde{x}(t) = \begin{pmatrix} x_1(t) \\ x_4(t) \end{pmatrix}, \tag{9}$$

with

$$s\tilde{E} - \tilde{A} = \begin{bmatrix} V^\top\left(sA_\mathcal{C}\mathcal{C}A_\mathcal{C}^\top + A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top\right)V & V^\top A_\mathcal{L}Y \\ -Y^\top A_\mathcal{L}^\top V & sY^\top\mathcal{L}Y \end{bmatrix}.$$

The variables $x_1$ in (9) may be interpreted as those corresponding to independent capacitors in the network and $x_4$ as those corresponding to independent inductors.

**Theorem 3** *Consider a MNA model* (1) *with* (2)–(4) *of a RLC network. Then the system* (9) *is a minimal realization of that network. In particular, for the matrices in* (9) *we find*

$$\operatorname{rk} A_\mathcal{C}^\top V = n_\mathcal{C} - \mathrm{FL}_\mathcal{C} \quad \textit{and} \quad \operatorname{rk} Y = n_\mathcal{L} - \mathrm{FC}_\mathcal{L}.$$

*Proof* It is obvious that there is a one-to-one correspondence between the solutions of (1) and (9). Furthermore,

$$\operatorname{rk}\tilde{E} = \operatorname{rk} V^\top A_\mathcal{C} + \operatorname{rk} Y^\top\mathcal{L}Y = \operatorname{rk} A_\mathcal{C} + \operatorname{rk} Y,$$

where we have used that $\operatorname{rk} V^\top A_\mathcal{C} = \operatorname{rk} A_\mathcal{C}$ as shown in proof of Theorem 2. We may further calculate that

$$\operatorname{rk} Y = \dim\ker W^\top A_\mathcal{L} = n_\mathcal{L} - \operatorname{rk} W^\top A_\mathcal{L} = n_\mathcal{L} - \operatorname{rk} A_\mathcal{L}^\top W = n_\mathcal{L} - m,$$

since $A_\mathcal{L}^\top W$ has full column rank $m = \operatorname{rk} W$. Therefore,

$$\operatorname{rk} Y = n_\mathcal{L} - \operatorname{rk} W = n_\mathcal{L} - \dim\ker\,[A_\mathcal{R}, A_\mathcal{C}]^\top$$

and hence

$$\operatorname{rk}\tilde{E} = \operatorname{rk} A_\mathcal{C} + n_\mathcal{L} - \dim\ker\,[A_\mathcal{R}, A_\mathcal{C}]^\top = \delta_M G(s)$$

for $G(s) = (sE - A)^{-1}$ by Theorem 2. The last statement is a consequence of Corollary 1. □

## 6 Examples

We illustrate our obtained results by means of two examples.

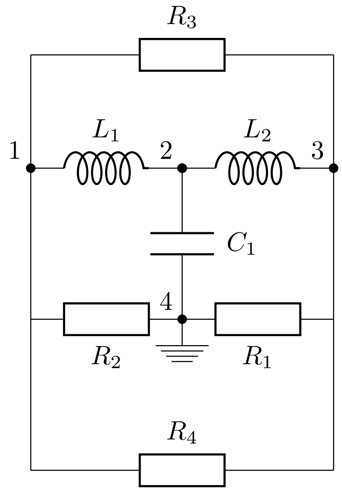


Fig. 1: RLC network

6.1 Example 1

Consider the RLC network depicted in Figure 1.

According to the numbering of the nodes, the element-related incidence matrices are as follows:

$$A_{\mathcal{R}} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 0 & 0 \end{bmatrix}, \quad A_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad A_{\mathcal{L}} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

and

$$\mathcal{G} = \mathrm{diag}\,(R_1^{-1}, R_2^{-1}, R_3^{-1}, R_4^{-1}), \quad \mathcal{C} = [C_1], \quad \mathcal{L} = \mathrm{diag}\,(L_1, L_2).$$

An essential step is now to observe that one of the four node potentials can be chosen freely. Therefore, we may, for instance, choose the potential at node 4 to be zero, which is equivalent to choosing this node as the ground node as in Figure 1. As a result, the corresponding node potential is not relevant in the modified nodal model and we may delete the corresponding row (here it is the last row) in the incidence matrices, that is

$$A_{\mathcal{R}} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix}, \quad A_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad A_{\mathcal{L}} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Therefore, the matrix pencil (2) corresponding to the MNA model is

$$sE - A = \begin{bmatrix} sA_\mathcal{C}\mathcal{C}A_\mathcal{C}^\top + A_\mathcal{R}\mathcal{R}A_\mathcal{R}^\top & A_\mathcal{L} \\ -A_\mathcal{L}^\top & s\mathcal{L} \end{bmatrix}$$
$$= \begin{bmatrix} R_1^{-1} + R_3^{-1} + R_4^{-1} & 0 & -R_3^{-1} - R_4^{-1} & 1 & 0 \\ 0 & sC_1 & 0 & -1 & 1 \\ -R_3^{-1} - R_4^{-1} & 0 & R_2^{-1} + R_3^{-1} + R_4^{-1} & 0 & -1 \\ -1 & 1 & 0 & sL_1 & 0 \\ 0 & -1 & 1 & 0 & sL_2 \end{bmatrix}.$$

We calculate

$$\det(sE-A) = s^3\, C_1 L_1 L_2 (R_1^{-1}R_2^{-1} + R_1^{-1}R_3^{-1} + R_1^{-1}R_4^{-1} + R_2^{-1}R_3^{-1} + R_2^{-1}R_4^{-1}) + s^2(\ldots) + s(\ldots) + R_1^{-1} + R_2^{-1},$$

and hence the McMillan degree of $G(s) = (sE - A)^{-1}$ is $\delta_M G(s) = 3$. This is the same value as we obtain from Theorem 2:

$$n_\mathcal{L} + \operatorname{rk} A_\mathcal{C} - \dim\ker\,[A_\mathcal{R}, A_\mathcal{C}]^\top = 2 + 1 - 0 = 3,$$

and we observe that the network neither contains $\mathcal{C}$-loops nor $\mathcal{L}$-cutsets. Since $\operatorname{rk} E = 3$, the MNA model itself is already a minimal realization.

## 6.2 Example 2

Consider the RLC network depicted in Figure 2.

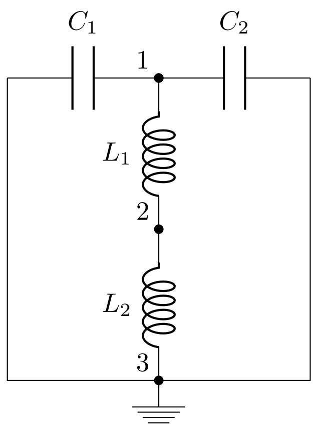


Fig. 2: RLC network

After deleting the row corresponding to the ground node the incidence matrices read

$$A_\mathcal{C} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_\mathcal{L} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},$$

$$\mathcal{C} = \operatorname{diag}(C_1, C_2), \quad \mathcal{L} = \operatorname{diag}(L_1, L_2).$$

Then

$$sE - A = \begin{bmatrix} s(C_1+C_2) & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & sL_1 & 0 \\ 0 & -1 & 0 & sL_2 \end{bmatrix}$$

and

$$\det(sE-A) = s^2(C_1+C_2)(L_1+L_2)+1,$$

hence the McMillan degree of $G(s) = (sE-A)^{-1}$ is $\delta_M G(s) = 2$. From Theorem 2 we obtain the same value:

$$n_{\mathcal{L}} + \operatorname{rk} A_{\mathcal{C}} - \dim\ker\,[A_{\mathcal{R}}, A_{\mathcal{C}}]^\top = 2+1-1 = 2,$$

and we observe that the circuit contains one fundamental $\mathcal{C}$-loop and one fundamental $\mathcal{L}$-cutset. Since $\operatorname{rk} E = 3 \neq 2 = \delta_M G(s)$, the MNA model is not a minimal realization. We see that $sE-A$ is already in the form (8), so it remains to choose

$$Y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \operatorname{im} Y = \ker W^\top A_{\mathcal{L}} = \ker\,[-1, 1].$$

Then, with

$$S := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \in \mathbf{GL}_4$$

we obtain

$$S^\top(sE-A)S = \begin{bmatrix} s(C_1+C_2) & 0 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ -1 & 2 & sL_1+sL_2 & -sL_1+sL_2 \\ 1 & 0 & -sL_1+sL_2 & sL_1+sL_2 \end{bmatrix},$$

and hence a minimal realization is given by (9) with

$$s\tilde{E} - \tilde{A} = \begin{bmatrix} s(C_1+C_2) & 1 \\ 1 & s(L_1+L_2) \end{bmatrix}.$$

## 7 Conclusion

In the present paper we have shown that the McMillan degree of implicit network transfer functions equals the sum of the number of capacitors and inductors minus the number of fundamental loops of capacitors and fundamental cutsets of inductors; this defines the number of independent dynamic elements which are necessary to fully describe the network. A minimal realization of the RLC network is then derived, where the number of involved (independent) differential equations equals the McMillan degree.

The starting point for our analysis has been the modified nodal analysis model, which preserves the natural graph topology of the network, but in general leads to an implicit non-minimal representation. The results presented here provide an extension to the results derived in [16] based on the impedance-admittance network description, which provides an appropriate framework for network re-engineering. The corresponding integral-differential rational description also leads to a state space description that is in general non-minimal [14], but that preserves the natural nodal/loop topologies of the network. Extending the results on the McMillan degree obtained in the present paper to this alternative description is a topic of future research.

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