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The Funnel Pre-Compensator

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Abstract

We introduce the funnel pre-compensator as a novel and simple adaptive pre-compensator of "high-gain type". We show that this pre-compensator is feasible for a large class of signal pairs, which satisfy a certain relationship. We show that the funnel pre-compensator guarantees prescribed transient behavior of the compensator error, it is of low complexity and inherently robust since its design is model-free. As an application in adaptive control of nonlinear systems, a cascade of funnel pre-compensators is exploited to obtain an artificial output with explicitly known derivatives which tracks the system output with prescribed transient behavior. In some important cases the interconnection of the system with the pre-compensator cascade is shown to have input-to-state stable internal dynamics. This guarantees feasibility of a novel funnel controller which consists of a funnel pre-compensator cascade in conjunction with a recently developed funnel controller for systems with arbitrary relative degree. We illustrate the application of this interconnection for mechanical systems with relative degree two and three.

KEYWORDS:
funnel pre-compensator; high-gain observer; nonlinear systems; funnel control; adaptive control; internal dynamics.

Nomenclature:

$\mathbb{R}_{\geq 0}$	$= [0, \infty)$
$\mathbb{C}_-$	$= \{ \lambda \in \mathbb{C} \mid \operatorname{Re} \lambda < 0 \}$
$\mathbf{Gl}_n(\mathbb{R})$	the group of invertible matrices in $\mathbb{R}^{n\times n}$
$\sigma(A)$	the spectrum of $A \in \mathbb{R}^{n\times n}$
$\mathcal{L}^\infty_{\text{loc}}(I \to \mathbb{R}^n)$	the set of locally essentially bounded functions $f : I \to \mathbb{R}^n$, $I \subseteq \mathbb{R}$ an interval
$\mathcal{W}^{k,\infty}_{\text{loc}}(I \to \mathbb{R}^n)$	the set of k-times weakly differentiable functions $f : I \to \mathbb{R}^n$ with locally essentially bounded first k weak derivatives $f, \dots, f^{(k)}$
$\mathcal{L}^\infty(I \to \mathbb{R}^n)$	the set of essentially bounded functions $f : I \to \mathbb{R}^n$ with norm
$\|f\|_\infty$	$= \operatorname{ess\,sup}_{t\in I} \|f(t)\|$
$\mathcal{W}^{k,\infty}(I \to \mathbb{R}^n)$	the set of k-times weakly differentiable functions $f : I \to \mathbb{R}^n$ such that $f, \dots, f^{(k)} \in \mathcal{L}^\infty(I \to \mathbb{R}^n)$
$\mathcal{C}^k(I \to \mathbb{R}^n)$	the set of k-times continuously differentiable functions $f : I \to \mathbb{R}^n$
$\mathcal{C}(I \to \mathbb{R}^n)$	$= \mathcal{C}^0(I \to \mathbb{R}^n)$
$f\|_J$	restriction of the function $f : I \to \mathbb{R}^n$ to $J \subseteq I$

1 | INTRODUCTION

In the present paper we propose a novel and simple adaptive pre-compensator of “high-gain type”, the *funnel pre-compensator*. In the case of unknown output derivatives, the funnel pre-compensator may be used to obtain an artificial output, the derivatives of which are known explicitly and which evolves within a prescribed performance funnel around the original output.

In the recent paper Berger et al. (2016) a funnel controller for nonlinear systems with arbitrary known relative degree is developed, which resolves the longstanding open problem of how to handle relative degree higher than one in high-gain adaptive control, cf. Ilchmann (1991); Ilchmann and Ryan (2008); Morse (1996). Earlier works suggested a “backstepping” procedure in conjunction with a filter, see Ilchmann et al. (2006 2007), or a bang-bang funnel controller, see Liberzon and Trenn (2013). Drawbacks are that the backstepping procedure in Ilchmann et al. (2006 2007) is quite complicated and impractical since it involves high powers of a gain function which typically takes large values, cf. (Hackl 2012, Sec. 4.4.3), and the approaches in Berger et al. (2016); Liberzon and Trenn (2013) require availability of the output derivatives which means in practice that measurements have to be differentiated. The latter is an ill-posed problem in particular in the presence of noise, see e.g. (Hackl 2012, Sec. 1.4.4).

We show that these drawbacks may be overcome by incorporating the funnel pre-compensator, so that derivatives of the output are not required anymore. The funnel pre-compensator resembles (and was inspired by) an (adaptive) high-gain observer and was called “funnel observer” in the preprint Berger and Reis (2016b); see the classical works Esfandiari and Khalil (1987); Khalil and Saberi (1987); Saberi and Sannuti (1990); Tornambè (1988) and the recent survey Khalil and Praly (2014) for literature on high-gain observers. However, the funnel pre-compensator does not have the properties of a high-gain observer since the derivatives of the output are not approximated. Rather than that, an alternative “artificial output” is derived which evolves within a prescribed performance funnel around the original output, and derivatives of which are computed exactly.

Nevertheless, since the funnel pre-compensator carries the structure of a high-gain observer, some of its benefits are retained. One advantage of high-gain observers is that they can be used to estimate the system states without knowing the exact parameters (in contrast to observer synthesis, see e.g. Cho and Rajamani (1997); Emel’yanov and Korovin (2004) and the references therein); only some structural assumptions, such as a known relative degree, are necessary. Furthermore, they are robust with respect to input noise. The drawback is that in most cases it is not known a priori how large the high-gain parameter k in the observer must be chosen and appropriate values must be identified by offline simulations. If k is chosen unnecessarily large, the sensitivity to measurement noise increases dramatically.

In order to resolve these problems, the constant high-gain parameter k has been replaced by an adaptation scheme in Bullinger et al. (1998). The gain $k(t)$ is determined by a differential equation depending on the observation error. This leads to a monotonically increasing $k(t)$ as long as the observation error lies outside a predefined λ-strip $[-\lambda, \lambda]$, and it stops increasing as soon as the error enters the strip. The advantage of this observer is that $k(t)$ is adapted online to the actual needed value, which also leads to lower high-gain parameters in general. However, $k(t)$ is monotonically non-decreasing and hence susceptible to unwarranted increase due to perturbations to the system. Furthermore, while convergence of the observation error to the λ-strip is guaranteed, its transient behavior cannot be influenced.

Another high-gain observer with gain adaptation law is proposed in Sanfelice and Praly (2011). Compared to Bullinger et al. (1998) it resolves the drawback of monotonically non-decreasing gain, however a certain condition on the system is necessary which either requires exact knowledge of the high-gain parameter of the system or boundedness of the input $u(t)$. Furthermore, the adaptation law in Sanfelice and Praly (2011) is not able to influence the transient behavior of the observation error, but only its mean value.

Inspired by the adaptive high-gain observer in Bullinger et al. (1998), we introduce the following funnel pre-compensator:

$$
\boxed{\begin{aligned}
\dot z_1(t) &= z_2(t) + \big(q_1 + p_1 k(t)\big)\cdot(y(t) - z_1(t)), \qquad k(t) = \frac{1}{1-\varphi(t)^2\|y(t)-z_1(t)\|^2},\\
\dot z_2(t) &= z_3(t) + \big(q_2 + p_2 k(t)\big)\cdot(y(t) - z_1(t)),\\
&\ \ \vdots\\
\dot z_{r-1}(t) &= z_r(t) + \big(q_{r-1} + p_{r-1} k(t)\big)\cdot(y(t) - z_1(t)),\\
\dot z_r(t) &= \big(q_r + p_r k(t)\big)\cdot(y(t) - z_1(t)) + \tilde\Gamma u(t),
\end{aligned}} \tag{1}
$$

where the design parameters $p_i > 0$, $q_i > 0$, $\tilde\Gamma \in \mathbb{R}^{p\times m}$ and the function $\varphi : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ are explained in detail in Section 2.

We like to emphasize that:

- The proposed adaptation scheme for $k(t)$ is simple, non-dynamic, non-monotone,

- and it guarantees prescribed transient behavior of the compensator error;
- the pre-compensator (1) is of low complexity and inherently robust since its design is model-free.

Another advantage of the funnel pre-compensator (1) is that no higher powers of the gain function k are involved in (1), thus typical challenges in the numerical implementation are avoided without the need for any estimates of the underlying model as discussed for high-gain observers in Astolfi and Marconi (2015); Khalil (2016).

In contrast to other approaches, the signals u and y given to the funnel pre-compensator (1) are not necessarily the input and output corresponding to some system or plant. We only assume that they are some signals belonging to the following, very large set:

$$\mathcal{P}_r := \left\{ (u,y) \in \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times \mathcal{W}^{r,\infty}_{\mathrm{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^p) \;\middle|\; \begin{array}{l} \exists\, \Gamma \in \mathcal{C}^1(\mathbb{R}_{\geq 0} \to \mathbb{R}^{m\times p}) : \\ \Gamma y^{(r-1)} \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m), \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\Gamma y^{(r-1)}\right) - u \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \end{array} \right\},$$

where $r \in \mathbb{N}_0$. The situation is depicted in Figure 1 . We stress that knowledge of the matrix-valued function Γ is *not* assumed, only that of the signals u and y (which can be viewed as the external signals corresponding to some plant) and the number $r \in \mathbb{N}_0$ (which can be viewed as the "relative degree" of the possibly underlying plant). For instance, if $u \in \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ is the input and $y \in \mathcal{C}^r(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ is the output of the system

$$y^{(r)}(t) = f\left(d(t), y(t), \dots, y^{(r-1)}(t)\right) + Bu(t),$$

where f is a suitable continuous function, $B \in \mathbf{Gl}_m(\mathbb{R})$ and $y, \dot{y}, \dots, y^{(r-1)}$ are bounded, then $(u,y) \in \mathcal{P}_r$ with $\Gamma = B^{-1}$. Clearly, the signal set $\mathcal{P}_r$ allows for much larger classes of systems involving functional-differential, partial differential and/or differential-algebraic equations, see e.g. Berger et al. (2016 2014); Ilchmann et al. (2002b) and Section 3. We will show that for signals $(u,y) \in \mathcal{P}_r$ with $r \geq 2$, the funnel pre-compensator (1) has a weakly differentiable and bounded solution $(z_1, \dots, z_r)$ such that k is bounded and

$$\exists\, \varepsilon > 0\ \forall\, t > 0 :\ \|y(t) - z_1(t)\| < \varphi(t)^{-1} - \varepsilon. \tag{2}$$

Furthermore, the derivative $\dot{z}_1$ is known explicitly.

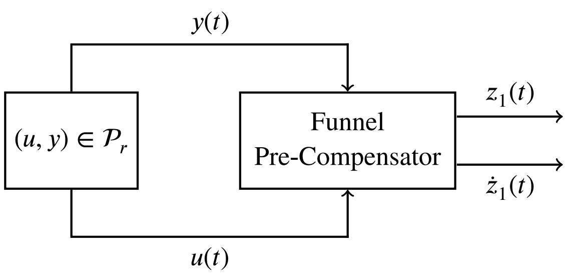

FIGURE 1 Application of the funnel pre-compensator (1) to signals $(u,y) \in \mathcal{P}_r$.

We stress that condition (2) means prescribed transient behavior of the compensator error $e_1(t) := y(t) - z_1(t)$ in the sense that it is pointwise below a given funnel function $1/\varphi$, see Figure 2 . To achieve this, the compensator gain will be increased whenever $\|e_1(t)\|$ approaches the funnel boundary. High values of the gain function lead to a faster decay of the compensator error.

While the funnel pre-compensator yields $\dot{z}_1$ explicitly, higher derivatives remain unknown. To resolve this problem we show that an application of a cascade of funnel pre-compensators yields

- an estimate z for the signal y with prescribed transient behavior and
- the derivatives $\dot{z}, \dots, z^{(r-1)}$ are known explicitly.

As an application of this cascade in adaptive control we investigate its use for output trajectory tracking by funnel control. Given a certain class of systems with input-to-state stable internal dynamics, we show that the interconnection of the system with the pre-compensator cascade has again input-to-state stable internal dynamics. This allows for the application of available funnel

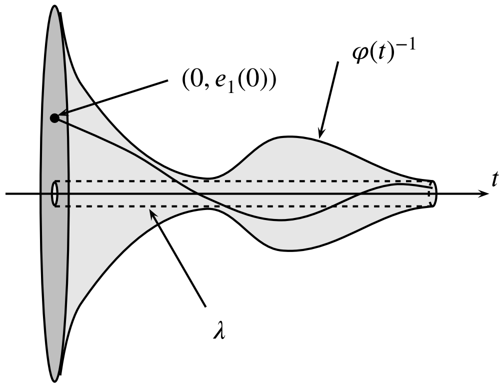

FIGURE 2 Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $\varphi(t)^{-1}$ for $t > 0$.

control techniques to the interconnection in order to achieve tracking with prescribed transient behavior of the tracking error without the requirement to compute derivatives of the system output as in Berger et al. (2016). However, this result is limited to systems with relative degree two or three; for higher relative degree it remains an open problem.

The present paper is organized as follows: The funnel pre-compensator is introduced in Section 2 and feasibility is proved. Furthermore, we show that the funnel pre-compensator cascade achieves the desired properties. The application in output trajectory tracking is discussed in Section 3. The interconnection of the funnel pre-compensator with the funnel controller from Berger et al. (2016) as a funnel controller for systems with higher relative degree which does not require the output derivatives, is presented in Section 4 for relative degree two and three. A simulation of this interconnection for a mass-spring system mounted on a car is provided in Section 5. Some conclusions are given in Section 6.

2 | THE FUNNEL PRE-COMPENSATOR

In this section we consider the funnel pre-compensator (1) as a new adaptive pre-compensator of high-gain type. Following the methodology of funnel control, see Ilchmann and Ryan (2008); Ilchmann et al. (2002b) and the references therein, it is our aim to construct a dynamical system which receives signals $(u, y) \in \mathcal{P}_r$ and has output z such that the derivatives of z up to order $r-1$, where $r \in \mathbb{N}$, are known explicitly and the error $e = y - z$ evolves within a prescribed performance funnel

$$\mathcal{F}_\varphi := \left\{ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^p \;\middle|\; \varphi(t)\|e\| < 1 \right\}. \tag{3}$$

This performance funnel is determined by a function φ belonging to

$$\Phi := \left\{ \varphi \in C^1(\mathbb{R}_{\geq 0} \to \mathbb{R}) \;\middle|\; \varphi, \dot{\varphi} \text{ are bounded, } \varphi(s) > 0 \text{ for all } s > 0, \text{ and } \liminf_{s\to\infty} \varphi(s) > 0 \right\}.$$

Note that the funnel boundary is given by the reciprocal of φ, see Figure 2 . The case $\varphi(0) = 0$ is explicitly allowed and puts no restriction on the initial value since $\varphi(0)\|e(0)\| < 1$; in this case the funnel boundary $1/\varphi$ has a pole at $t = 0$.

An important property of the funnel class Φ is that each performance funnel $\mathcal{F}_\varphi$ with $\varphi \in \Phi$ is bounded away from zero, i.e., due to boundedness of φ there exists $\lambda > 0$ such that $1/\varphi(t) \geq \lambda$ for all $t > 0$. The funnel boundary is not necessarily monotonically decreasing, while in most situations it is convenient to choose a monotone funnel. However, there are situations where widening the funnel over some later time interval might be beneficial, e.g., in the presence of periodic disturbances or when the signal y changes strongly.

Our first objective is robust estimation of the signal y so that the derivative of the compensator state z_1 in (1) is known explicitly, the compensator error $e_1 = y - z_1$ evolves within the funnel $\mathcal{F}_\varphi$ and all variables are bounded. To achieve this objective we consider the funnel pre-compensator (1) with initial conditions

$$z_i(0) = z_i^0 \in \mathbb{R}^p, \quad i = 1, \dots, r, \tag{4}$$

where $\varphi \in \Phi$, $\tilde{\Gamma} \in \mathbb{R}^{p\times m}$ and $q_i > 0$, $p_i > 0$ for all $i = 1, \dots, r$. The functions $z_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$, $i = 1, \dots, r$, are the compensator states and $k : \mathbb{R}_{\geq 0} \to [1, \infty)$ is the compensator gain. The constants $q_i > 0$ are such that the matrix

$$A = \begin{bmatrix} -q_1 & 1 & & \\ \vdots & & \ddots & \\ -q_{r-1} & & & 1 \\ -q_r & & & 0 \end{bmatrix} \in \mathbb{R}^{r\times r}$$

is Hurwitz, i.e., $\sigma(A) \subseteq \mathbb{C}_-$. The constants p_i depend on the choice of the q_i in the following way: Let $Q = Q^\top > 0$ and

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^\top & P_4 \end{bmatrix}, \quad P_1 \in \mathbb{R},\ P_2 \in \mathbb{R}^{1\times(r-1)},\ P_4 \in \mathbb{R}^{(r-1)\times(r-1)}$$

be such that

$$A^\top P + PA + Q = 0, \quad P = P^\top > 0.$$

The matrix P depends only on the choice of the constants q_i and the matrix Q. The constants p_i must then satisfy

$$\begin{pmatrix} p_1 \\ \vdots \\ p_r \end{pmatrix} = P^{-1} \begin{pmatrix} P_1 - P_2 P_4^{-1} P_2^\top \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -P_4^{-1} P_2^\top \end{pmatrix}. \tag{5}$$

In passing, we note for later use that any such P satisfies

$$(1, -P_2 P_4^{-1})\, P \begin{pmatrix} 1 \\ (-P_2 P_4^{-1})^\top \end{pmatrix} = P_1 - P_2 P_4^{-1} P_2^\top > 0. \tag{6}$$

We will see later that the above condition guarantees that P defines a quadratic Lyapunov function for the error dynamics of the funnel pre-compensator.

While the funnel pre-compensator (1) resembles a high-gain observer, it is different in its structure when compared to the high-gain observers in Bullinger et al. (1998); Tornambè (1992), where the gain enters with power k^i into the equation for $\dot{z}_i$. Furthermore, the constants q_i are not present in Bullinger et al. (1998); Tornambè (1992), but we show that they are important to ensure boundedness of the error dynamics even when $k(t)$ is small.

Although the pre-compensator (1) is a nonlinear and time-varying system, it is simple in its structure and its dimension depends only on the "relative degree" r given by $\mathcal{P}_r$. The set $\mathcal{P}_r$ of signals u and y ensures error evolution within the funnel: by the design of the pre-compensator (1), the gain $k(t)$ increases if the norm of the error $\|y(t) - z_1(t)\|$ approaches the funnel boundary $1/\varphi(t)$, and decreases if a high gain is not necessary.

For a sketch of the construction of the funnel pre-compensator (1) see also Figure 3 .

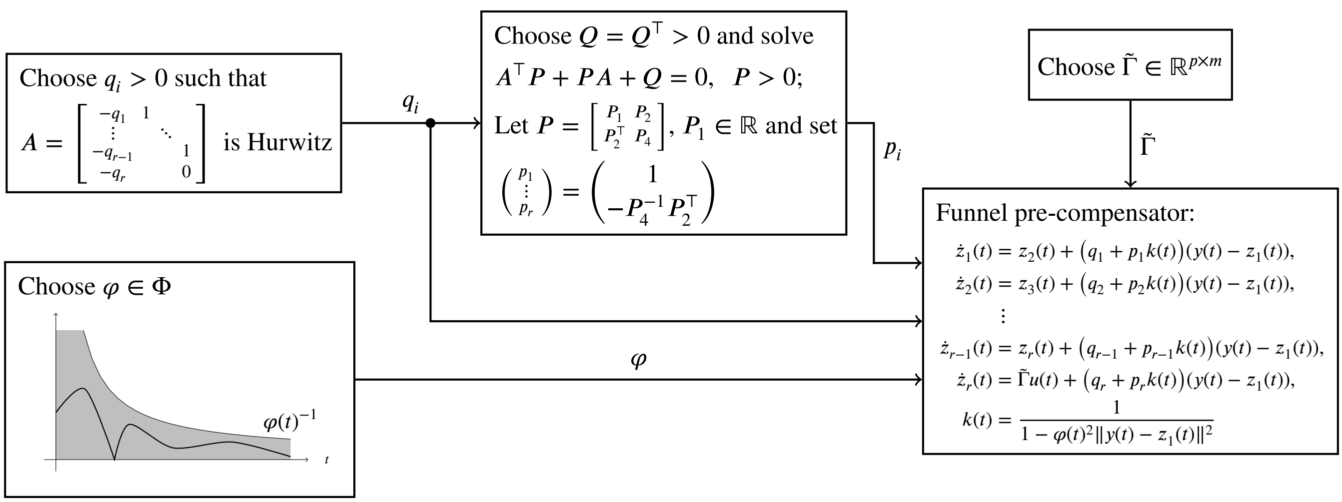

FIGURE 3 Construction of the funnel pre-compensator (1) depending on its design parameters.

We now show that the funnel pre-compensator achieves its objective; note that we only consider the relevant case $r \geq 2$.

Proposition 2.1
Consider $(u, y) \in \mathcal{P}_r$ so that $r \geq 2$, and the funnel pre-compensator (1), (4) with $\varphi \in \Phi$ such that

$$\varphi(0)\|y(0) - z_1^0\| < 1,$$

$\tilde{\Gamma} \in \mathbb{R}^{p\times m}$ and $q_i > 0$, $p_i > 0$ such that (5) is satisfied for corresponding matrices A, P, Q.
Then (1), (4) has a weakly differentiable solution $z = (z_1, \dots, z_r) \in \mathcal{L}^\infty\left(\mathbb{R}_{\geq 0} \to (\mathbb{R}^p)^r\right)$ with $k \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to [1, \infty))$ and

$$\exists\, \varepsilon > 0\ \forall\, t > 0:\ \|y(t) - z_1(t)\| < \varphi(t)^{-1} - \varepsilon. \tag{7}$$

Furthermore, using the errors

$$\begin{aligned} e_i &:= y^{(i-1)} - z_i, \quad i = 1, \dots, r-1 \\ e_r &:= \tilde{\Gamma}\Gamma y^{(r-1)} - z_r, \end{aligned} \tag{8}$$

and the constants

$$M_1 := \|(I - \tilde{\Gamma}\Gamma)y^{(r-1)}\|_\infty, \qquad M_2 := \|\tilde{\Gamma}(\dot{\Gamma}y^{(r-1)} + \Gamma y^{(r)} - u)\|_\infty, \tag{9}$$

which are well-defined by $(u, y) \in \mathcal{P}_r$, with $M = (M_1^2 + M_2^2)^{\frac{1}{2}}$ we have

$$\limsup_{t\to\infty} \|e(t)\| \leq \frac{2\, M\, \lambda_{\max}(P)^2}{\lambda_{\min}(Q)\, \lambda_{\min}(P)}. \tag{10}$$

Here $\lambda_{\max}(P)$ denotes the largest eigenvalue of the positive definite matrix P, and $\lambda_{\min}(P)$ denotes its smallest eigenvalue.

Proof. We proceed in several steps.
Step 1: We show existence of a local solution of (1), (4). *Set*

$$\mathcal{D} := \left\{\ (t, e_1, \dots, e_r) \in \mathbb{R}_{\geq 0} \times (\mathbb{R}^p)^r \ \middle|\ \varphi(t)\|e_1\| < 1\ \right\}$$

and

$$\begin{aligned} f(t) &:= \tilde{\Gamma}\big(\dot{\Gamma}(t)y^{(r-1)}(t) + \Gamma(t)y^{(r)}(t) - u(t)\big) \\ g(t) &:= \big(I - \tilde{\Gamma}\Gamma(t)\big)y^{(r-1)}(t), \qquad t \geq 0. \end{aligned}$$

Invoking $r \geq 2$ we find that the errors (8) *satisfy*

$$\begin{aligned} \dot{e}_1(t) &= e_2(t) - \big(q_1 + p_1 k(t)\big)\cdot e_1(t), \qquad k(t) = \frac{1}{1 - \varphi(t)^2\|e_1(t)\|^2}. \\ &\ \vdots \\ \dot{e}_{r-2}(t) &= e_{r-1}(t) - \big(q_{r-2} + p_{r-2}k(t)\big)\cdot e_1(t), \\ \dot{e}_{r-1}(t) &= e_r(t) - \big(q_{r-1} + p_{r-1}k(t)\big)\cdot e_1(t) + g(t), \\ \dot{e}_r(t) &= \quad - \big(q_r + p_r k(t)\big)\cdot e_1(t) + f(t), \end{aligned} \tag{11}$$

By the existence theorem for ordinary differential equations (see e.g. (Walter 1998, § 10, Thm. XX)), there exists a maximal weakly differentiable solution $e = (e_1, \dots, e_r) : [0, \omega) \to (\mathbb{R}^p)^r$, $\omega \in (0, \infty]$, of (11) *satisfying the initial conditions*

$$\begin{aligned} e_i(0) &= y^{(i-1)}(0) - z_i^0, \quad i = 1, \dots, r, \\ e_r(0) &= \tilde{\Gamma}\Gamma(0)y^{(r-1)}(0) - z_r^0, \end{aligned}$$

and $(t, e(t)) \in \mathcal{D}$ for all $t \in [0, \omega)$. Furthermore, the closure of the graph of e, i.e., the set

$$\overline{\text{graph}\ e} := \overline{\{\ (t, e(t))\ \mid t \in [0, \omega)\ \}},$$

is not a compact subset of $\mathcal{D}$. Thus, a maximal solution $(z_1, \dots, z_r)$ of (1), (4) *can be reconstructed.*
Step 2: We show that $e \in \mathcal{L}^\infty([0, \omega) \to (\mathbb{R}^p)^r)$. Recalling that the Kronecker product of two matrices $V \in \mathbb{R}^{l\times n}$ and $W \in \mathbb{R}^{j\times q}$ is given by

$$V \otimes W = \begin{bmatrix} v_{11}W & \cdots & v_{1n}W \\ \vdots & & \vdots \\ v_{l1}W & \cdots & v_{ln}W \end{bmatrix} \in \mathbb{R}^{lj\times nq}, \tag{12}$$

let

$$\hat{A} := A \otimes I_p = \begin{bmatrix} -q_1 I_p & I_p & & \\ \vdots & & \ddots & \\ -q_{r-1} I_p & & & I_p \\ -q_r I_p & & & 0 \end{bmatrix} \in \mathbb{R}^{rp\times rp}, \quad \hat{P} := P \otimes I_p \in \mathbb{R}^{rp\times rp}, \quad \text{and} \quad \hat{Q} = Q \otimes I_p \in \mathbb{R}^{rp\times rp}.$$

From (Bernstein 2009, Fact 7.4.34) it follows that

$$\sigma(\hat{A}) = \sigma(A), \quad \sigma(\hat{Q}) = \sigma(Q), \quad \sigma(\hat{P}) = \sigma(P) \tag{13}$$

and so $A^\top P + PA + Q = 0$ *gives that* $\hat{P} = \hat{P}^\top > 0$, $\hat{Q} = \hat{Q}^\top > 0$ *and*

$$\hat{A}^\top \hat{P} + \hat{P}\hat{A} + \hat{Q} = 0.$$

Since $P_2^\top + P_4 \begin{pmatrix} p_2 \\ \vdots \\ p_r \end{pmatrix} = 0$ *by* (5) *we find*

$$\hat{P} \begin{bmatrix} p_1 I_p \\ \vdots \\ p_r I_p \end{bmatrix} = \begin{bmatrix} (P_1 - P_2 P_4^{-1} P_2^\top) I_p \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $P_1 - P_2 P_4^{-1} P_2^\top \overset{(6)}{>} 0$. *Observe that we may write* (11) *in the form*

$$\dot{e}(t) = \hat{A} e(t) - k(t) \begin{bmatrix} p_1 I_p \\ \vdots \\ p_r I_p \end{bmatrix} e_1(t) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g(t) \\ f(t) \end{pmatrix}.$$

Since $(u, y) \in \mathcal{P}_r$, *the constants* M_1, M_2 *in* (9) *are well-defined and we have* $\|g(t)\| \le M_1$ *and* $\|f(t)\| \le M_2$ *for almost all* $t \in [0, \omega)$. *With* $M = (M_1^2 + M_2^2)^{\frac{1}{2}}$ *we may now calculate that, for almost all* $t \in [0, \omega)$,

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} e(t)^\top \hat{P} e(t) &= e(t)^\top \hat{A}^\top \hat{P} e(t) + e(t)^\top \hat{P}\hat{A} e(t) - 2k(t) e(t)^\top \hat{P} \begin{bmatrix} p_1 I_p \\ \vdots \\ p_r I_p \end{bmatrix} e_1(t) + 2e(t)^\top \hat{P} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g(t) \\ f(t) \end{pmatrix} \\ &\le -e(t)^\top \hat{Q} e(t) - 2k(t)(P_1 - P_2 P_4^{-1} P_2^\top) \|e_1(t)\|^2 + 2M \|\hat{P}\| \|e(t)\| \\ &\le -\mu e(t)^\top \hat{P} e(t) + 2M \|\hat{P}\| \|e(t)\|, \end{aligned}$$

where $\mu = \lambda_{\min}(\hat{Q}) / \lambda_{\max}(\hat{P})$. *With* $\delta := \frac{1}{2}\mu \lambda_{\min}(\hat{P})$ *and using that* $ab \le \frac{1}{2}(a^2 + b^2)$ *for all* $a, b \ge 0$, *it follows that*

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} e(t)^\top \hat{P} e(t) &\le -\mu e(t)^\top \hat{P} e(t) + \left(\sqrt{2\delta} \|e(t)\|\right) \left(\frac{2M\|\hat{P}\|}{\sqrt{2\delta}} \right) \\ &\le -\mu e(t)^\top \hat{P} e(t) + \delta \|e(t)\|^2 + \frac{M^2 \|\hat{P}\|^2}{\delta} \\ &\le -\tfrac{\mu}{2} e(t)^\top \hat{P} e(t) + \frac{2M^2 \|\hat{P}\|^2}{\mu \lambda_{\min}(\hat{P})} \end{aligned}$$

for almost all $t \in [0, \omega)$. *Gronwall's lemma now implies that*

$$e(t)^\top \hat{P} e(t) \le e(0)^\top \hat{P} e(0) e^{-\frac{\mu}{2} t} + \frac{2M^2 \|\hat{P}\|^2}{\mu^2 \lambda_{\min}(\hat{P})},$$

and hence

$$\|e(t)\|^2 \le \frac{\lambda_{\max}(\hat{P})}{\lambda_{\min}(\hat{P})} e^{-\frac{\mu}{2} t} \|e(0)\|^2 + \frac{2M^2 \|\hat{P}\|^2}{\mu^2 \lambda_{\min}(\hat{P})^2} \tag{14}$$

for all $t \in [0, \omega)$. *Equation* (14) *in particular implies that* $e \in \mathcal{L}^\infty([0, \omega) \to (\mathbb{R}^p)^r)$.

Step 3: We show that $k \in \mathcal{L}^\infty([0,\omega) \to \mathbb{R})$. Let $\kappa \in (0,\omega)$ be arbitrary but fixed and $\lambda := \inf_{t\in(0,\omega)} \varphi(t)^{-1} > 0$. Since $\dot\varphi$ is bounded and $\liminf_{t\to\infty} \varphi(t) > 0$ we find that $\frac{\mathrm{d}}{\mathrm{d}t} \varphi|_{[\kappa,\infty)}(\cdot)^{-1}$ is bounded and hence there exists a Lipschitz bound $L > 0$ of $\varphi|_{[\kappa,\infty)}(\cdot)^{-1}$. By Step 2, e_2 is bounded and we may choose $\varepsilon > 0$ small enough so that

$$\varepsilon \le \min\left\{ \frac{\lambda}{2}, \inf_{t\in(0,\kappa]} (\varphi(t)^{-1} - \|e_1(t)\|) \right\}$$

and

$$L \le - \sup_{t\in[0,\omega)} \|e_2(t)\| - M_1 + \frac{q_1\lambda}{2} + \frac{\lambda^2}{4\varepsilon}; \tag{15}$$

feasibility of this choice is guaranteed by $r \ge 2$. Using a standard argument in funnel control, see e.g. (Ilchmann 2013, pp. 241–242), it is then straightforward to show that

$$\forall\, t \in (0,\omega) :\ \varphi(t)^{-1} - \|e_1(t)\| \ge \varepsilon. \tag{16}$$

The estimate (16) clearly implies boundedness of k.

Step 4: We show $\omega = \infty$. Assume that $\omega < \infty$. Then, since e and k are bounded by Steps 2 and 3, it follows that graph e is a compact subset of $\mathcal{D}$, a contradiction. Therefore, $\omega = \infty$. Together with Step 3, this in particular implies (7). Inequality (10) is an immediate consequence of (14) together with the observation that by (13) we have $\lambda_{\min}(\hat P) = \lambda_{\min}(P)$, $\lambda_{\max}(\hat P) = \lambda_{\max}(P)$, $\lambda_{\min}(\hat Q) = \lambda_{\min}(Q)$ and, since $\hat P$ is positive definite, $\|\hat P\| = \lambda_{\max}(\hat P) = \lambda_{\max}(P)$. □

In (Sanfelice and Praly 2011, Thm. 2.2), using the adaptive high-gain observer proposed therein, bounds for the mean value of the observation error e_i (defined similar to (8)) are given; we stress that both the bounds in (Sanfelice and Praly 2011, (14)) and in (10) cannot be made arbitrarily small in general, they depend on the given signals.

Remark 2.2. We consider two special cases for signals $(u,y) \in \mathcal{P}_r$, the funnel pre-compensator (1) and the resulting estimate (10).

(i) $\tilde\Gamma = 0$. It is immediate from (9) that in this case $M_1 = \|y^{(r-1)}\|_\infty$ and $M_2 = 0$, thus $M = \|y^{(r-1)}\|_\infty$ in (10). Note that the choice of $\tilde\Gamma$ is independent of the signals u and y.

(ii) $p = m$, $\Gamma \in \mathbf{Gl}_m(\mathbb{R})$, $\tilde\Gamma = \Gamma^{-1}$ and we have $\Gamma y^{(r)} = u$. This means the signals satisfy a very simple relation and we have exact knowledge of the invertible matrix Γ. Then $M_1 = M_2 = 0$ in (9) and hence $M = 0$ in (10). In particular, this implies that $e(t) \to 0$ and $k(t) \to 1$ for $t \to \infty$.

Remark 2.3. If the signal y is subject to noise, i.e., the funnel pre-compensator (1) receives $y + n$ instead of y, where $n \in C^r([-h,\infty) \to \mathbb{R}^p)$ is such that, for Γ as in $\mathcal{P}_r$, $n, \dot n, \dots, n^{(r-2)}, \Gamma n^{(r-1)}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\left(\Gamma n^{(r-1)}\right)$ are bounded, then $(u, y+n) \in \mathcal{P}_r$ with Γ. Therefore, Proposition 2.1 yields that the funnel pre-compensator may also be applied to u and $y+n$ and achieves that

$$\forall\, t > 0 :\ \varphi(t)\|y(t) + n(t) - z_1(t)\| < 1,$$

which implies

$$\forall\, t > 0 :\ \frac{\varphi(t)}{1 + \varphi(t)\|n(t)\|}\|y(t) - z_1(t)\| < 1,$$

i.e., $y - z_1$ evolves in the funnel $\mathcal{F}_\psi$, where $\psi = \frac{\varphi(t)}{1+\varphi(t)\|n(t)\|}$. If an upper bound for n is known, say $\|n(t)\| \le \nu$ for all $t \ge 0$, then

$$\forall\, t > 0 :\ \|y(t) - z_1(t)\| < \varphi(t)^{-1} + \nu.$$

Hence, the actual error remains in the wider funnel obtained by adding the corresponding bound of the noise to the funnel bounds used for the pre-compensator. The bound in (10) changes as follows: Modify M_1 and M_2 from (9) to

$$\tilde M_1 := \|g + (I - \tilde\Gamma\Gamma)n^{(r-1)}\|_\infty, \qquad \tilde M_2 := \left\| f + \tilde\Gamma \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\Gamma n^{(r-1)}\right)\right\|_\infty.$$

Then, with $\tilde M := (\tilde M_1^2 + \tilde M_2^2)^{\frac{1}{2}}$, we have that

$$\limsup_{t\to\infty} \|e(t)\| \le \frac{2\,\tilde M\,\lambda_{\max}(P)^2}{\lambda_{\min}(Q)\,\lambda_{\min}(P)} + \left\|\left(n, \dot n, \dots, n^{(r-2)}, \tilde\Gamma\Gamma n^{(r-1)}\right)\right\|_\infty.$$

If the signal u is subject to noise before the funnel pre-compensator receives it, i.e., $u + v$ enters the pre-compensator (1), where $v \in \mathcal{L}^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$, then obviously $(u+v, y) \in \mathcal{P}_r$ and hence Proposition 2.1 yields that the funnel pre-compensator may also be applied to $u + v$ and y; in particular, (7) is achieved.

While Proposition 2.1 shows that the funnel pre-compensator is able to achieve prescribed transient behavior of the compensator error $e_1 = y - z_1$ and that the errors $e_2, \dots, e_r$ as in (8) converge to a certain strip, we like to stress that no transient behavior can be prescribed for $e_2, \dots, e_r$ since $\dot{y}, \dots, y^{(r-1)}$ are not known. Therefore, $z_2, \dots, z_r$ from the funnel pre-compensator cannot be viewed as estimates for $\dot{y}, \dots, y^{(r-1)}$. In the following we show that a successive application of the funnel pre-compensator to the signals u and z_1 results in a cascade of pre-compensators which yields, as desired,

- an estimate z for the signal y with prescribed transient behavior (i.e., $(t, y(t) - z(t)) \in \mathcal{F}_\varphi$) and
- the derivatives $\dot{z}, \dots, z^{(r-1)}$ are known explicitly.

We introduce a *cascade of funnel pre-compensators*

$$FP_{r-1} \circ \dots \circ FP_1$$

where the funnel pre-compensators

$$FP_i(p_i, q_i, \tilde{\Gamma}_i, \varphi_i) : \ (u, z_{i-1,1}) \mapsto z_{i,1},$$

are specified, for $i = 1, \dots, r-1$, as follows:

$$\begin{aligned}
\dot{z}_{i,1}(t) &= z_{i,2}(t) + \big(q_{i,1} + p_{i,1} k_i(t)\big) \cdot (z_{i-1,1}(t) - z_{i,1}(t)), \\
\dot{z}_{i,2}(t) &= z_{i,3}(t) + \big(q_{i,2} + p_{i,2} k_i(t)\big) \cdot (z_{i-1,1}(t) - z_{i,1}(t)), \\
&\ \vdots \\
\dot{z}_{i,r-1}(t) &= z_{i,r}(t) + \big(q_{i,r-1} + p_{i,r-1} k_i(t)\big) \cdot (z_{i-1,1}(t) - z_{i,1}(t)), \\
\dot{z}_{i,r}(t) &= \phantom{z_{i,r}(t)} + \big(q_{i,r} + p_{i,r} k_i(t)\big) \cdot (z_{i-1,1}(t) - z_{i,1}(t)) + \tilde{\Gamma}_i u(t),
\end{aligned} \qquad k_i(t) = \frac{1}{1 - \varphi_i(t)^2 \|z_{i-1,1}(t) - z_{i,1}(t)\|^2}. \tag{17}$$

where $z_{0,1} := y$, $\tilde{\Gamma}_i \in \mathbb{R}^{p \times m}$,

$$\varphi_i \in \Phi_{r-1} := \Phi \cap \left\{ \varphi \in C^{r-1}(\mathbb{R}_{\geq 0} \to \mathbb{R}) \ \middle| \ \dot{\varphi}, \dots, \varphi^{(r-1)} \text{ bounded} \right\}$$

and $q_{i,j} > 0$, $p_{i,j} > 0$ are such that (5) is satisfied for corresponding matrices A_i, P_i, Q_i for $i = 1, \dots, r-1$. The initial values are

$$z_{i,j}(0) = z_{i,j}^0 \in \mathbb{R}^p, \quad i = 1, \dots, r-1, \ j = 1, \dots, r. \tag{18}$$

The situation is illustrated in Figure 4 .

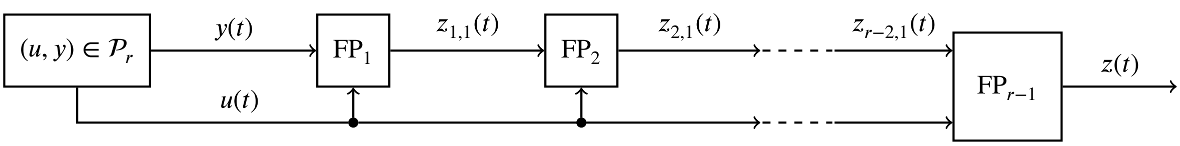

FIGURE 4 Cascade of funnel pre-compensators (1) applied to signals $(u, y) \in \mathcal{P}_r$.

We show that the cascade (17) with $\operatorname{rk} \tilde{\Gamma}_i = m$ applied to signals $(u, y) \in \mathcal{P}_r$, where additionally $y, \dot{y}, \dots, y^{(r-1)}$ are bounded, yields an interconnection with output $z = z_{r-1,1}$ such that $y - z$ has prescribed transient behavior and $\dot{z}, \dots, z^{(r-1)}$ are known explicitly.

Theorem 2.4
Consider $(u, y) \in \mathcal{P}_r$ so that $r \geq 2$, and assume that $y, \dot{y}, \dots, y^{(r-1)}$ are bounded. Consider the cascade of funnel pre-compensators $FP_{r-1} \circ \dots \circ FP_1$ defined in (17) for $\varphi_i \in \Phi_{r-1}$, $\tilde{\Gamma}_i \in \mathbb{R}^{p \times m}$ with $\operatorname{rk} \tilde{\Gamma}_i = m$ and $q_{i,j} > 0$, $p_{i,j} > 0$ are such that (5) is satisfied for corresponding matrices A_i, P_i, Q_i, and initial data (18) such that

$$\varphi_i(0) \| z_{i-1,1}(0) - z_{i,1}^0 \| < 1, \quad i = 1, \dots, r-1,$$

where $z_{0,1} := y$. Then (17), (18) has weakly differentiable solutions $z_{i,j} \in \mathcal{L}^\infty \left(\mathbb{R}_{\geq 0} \to \mathbb{R}^p\right)$ with $k_i \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to [1, \infty))$ for $i = 1, \dots, r-1$, $j = 1, \dots, r$ and

$$\forall\, i \in \{1, \dots, r-1\}\ \exists\, \varepsilon_i > 0\ \forall\, t > 0 : \quad \| z_{i-1,1}(t) - z_{i,1}(t) \| < \varphi_i(t)^{-1} - \varepsilon_i. \tag{19}$$

Furthermore, for $z := z_{r-1,1}$ we have that

$$\forall\, t > 0: \ \|y(t) - z(t)\| < \sum_{i=1}^{r-1} \varphi_i(t)^{-1} - \varepsilon_i. \tag{20}$$

Proof. *We show existence of bounded weakly differentiable solutions for each pre-compensator in* (17) *and the property* (19) *by induction. Note that* (20) *is a consequence of* (19).
For $i = 1$ we have $z_{0,1} = y$ and hence the existence of bounded global solutions follows from Proposition 2.1. We may calculate that

$$z_{i,1}^{(j)}(t) = z_{i,j+1}(t) + \sum_{l=0}^{j-1} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^l \big(q_{i,j-l} + p_{i,j-l}k_i(t)\big)\big(z_{i-1,1}(t) - z_{i,1}(t)\big) \tag{21}$$

for $i = 1, \ldots, r-1$ and $j = 0, \ldots, r$, where $z_{i,r+1} := \tilde{\Gamma}_i u$. With $w_i(t) := z_{i-1,1}(t) - z_{i,1}(t)$ we calculate

$$\dot{k}_i(t) = 2k_i(t)^2 \left(\varphi_i(t)\dot{\varphi}_i(t) w_i(t)^\top w_i(t) + \varphi_i(t)^2 w_i(t)^\top \dot{w}_i(t)\right) \tag{22}$$

for all $i = 1, \ldots, r-1$. In particular, for $i = 1$ we obtain that $\dot{z}_{1,1}, \ldots, z_{1,1}^{(r-1)}$ are bounded since $y, \ldots, y^{(r-1)}, \varphi_1, \ldots, \varphi_1^{(r-1)}$ are bounded and $z_{1,1}, \ldots, z_{1,r}$, and k_1 are bounded by Proposition 2.1. Now assume that the statement is true for $i \in \{1, \ldots, r-2\}$ such that $\dot{z}_{i,1}, \ldots, z_{i,1}^{(r-1)}$ are bounded. Choosing $\Gamma_i \in \mathbb{R}^{m\times p}$ such that $\Gamma_i\tilde{\Gamma}_i = I_m$ it follows from (21) *with $j = r$ that*

$$\Gamma_i z_{i,1}^{(r)} - u = \sum_{l=0}^{r-1} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^l \left(q_{i,r-l-1} + p_{i,r-l-1}k_i\right)\left(z_{i-1,1} - z_{i,1}\right) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m),$$

and hence $(u, z_{i,1}) \in \mathcal{P}_r$, by which an application of Proposition 2.1 is feasible and yields existence of bounded global solutions such that k_{i+1} is bounded. Again invoking (21) *yields boundedness of $\dot{z}_{i+1,1}, \ldots, z_{i+1,1}^{(r-1)}$.* □

Remark 2.5. Use the notation and assumptions from Theorem 2.4. Then the derivatives $\dot{z}, \ldots, z^{(r-1)}$ are known explicitly as

$$z^{(j)}(t) = z_{r-1,j+1}(t) + \tilde{P}_j^{r-1}(t), \qquad j = 0, \ldots, r-1,$$

where the functions $\tilde{P}_j^i$ are defined in a recursive way:

$$\begin{aligned} P_0^{a,b}(k, \varphi_0, e_0) :=& (q_{a,b} + p_{a,b}k)e_0, \\ P_{i+i}^{a,b}(k, \varphi_0, \ldots, \varphi_{i+1}, e_0, \ldots, e_{i+1}) :=& \frac{\partial P_i^{a,b}}{\partial k}\left(2k^2(\varphi_0\varphi_1 e_0^\top e_0 + \varphi_0^2 e_0^\top e_1)\right) + \frac{\partial P_i^{a,b}}{\partial \varphi_0}\varphi_1 + \ldots + \frac{\partial P_i^{a,b}}{\partial \varphi_i}\varphi_{i+1} \\ &+ \frac{\partial P_i^{a,b}}{\partial e_0}e_1 + \ldots + \frac{\partial P_i^{a,b}}{\partial e_i}e_{i+1} \end{aligned}$$

for $a, b \in \{1, \ldots, r-1\}$ and $i \geq 0$, where $k, \varphi_i \in \mathbb{R}$ and $e_i \in \mathbb{R}^p$ for each $i \geq 0$. Further define, using (17),

$$\tilde{P}_j^i(t) := \sum_{l=0}^{j-1} P_l^{i,j-l}\Big(k_i(t), \varphi_i(t), \ldots, \varphi_i^{(l)}(t), z_{i-1,1}(t) - z_{i,1}(t), \ldots, z_{i-1,1}^{(l)}(t) - z_{i,1}^{(l)}(t)\Big)$$

for $i = 1, \ldots, r-1$ and $j = 0, \ldots, r-1$. We will show that

$$z_{i,1}^{(j)}(t) = z_{i,j+1}(t) + \tilde{P}_j^i(t), \qquad i = 1, \ldots, r-1, \ \ j = 0, \ldots, r-1. \tag{23}$$

To this end, observe that it follows from (22) and a simple induction that

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^l \left(q_{i,j-l} + p_{i,j-l}k_i(t)\right)w_i(t) = P_l^{i,j-l}\left(k_i(t), \varphi_i(t), \dot{\varphi}_i(t), \ldots, \varphi_i^{(l)}(t), w_i(t), \dot{w}_i(t), \ldots, w_i^{(l)}(t)\right)$$

for $i = 1, \ldots, r-1$, $j = 0, \ldots, r-1$ and $l = 0, \ldots, j-1$. Then (21) immediately implies (23).
By definition, $\tilde{P}_j^{r-1}(t)$ depends on the derivatives of $z_{r-2,1}$ and $z_{r-1,1} = z$ up to order $j-1$. The dependencies on $\dot{z}, \ldots, z^{(j-1)}$ may be immediately resolved by applying the same formula again, thus $z^{(j)}$ depends on $z_{r-1,1}, \ldots, z_{r-1,j+1}$ and on $z_{r-2,1}, \dot{z}_{r-2,1}, \ldots, z_{r-2,1}^{(j-1)}$. Applying (23) in a recursive way to $\dot{z}_{r-2,1}, \ldots, z_{r-2,1}^{(j-1)}$ we obtain dependencies as depicted in Figure 5 .

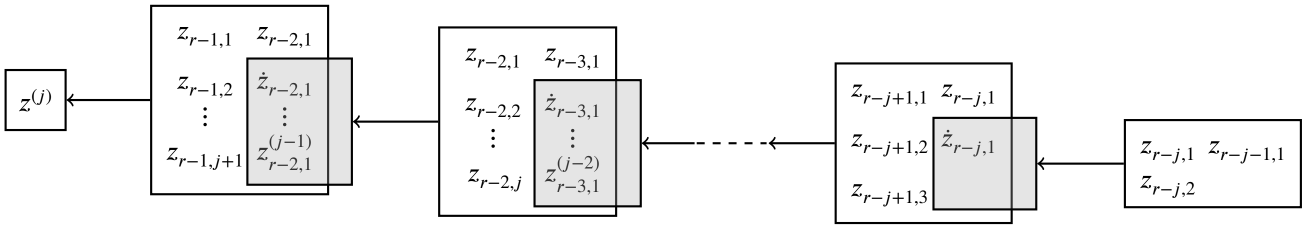

FIGURE 5 Dependency of $z^{(j)}$ on the compensator states. Note that $z_{r-j-1,1} = z_{0,1} = y$ for $j = r - 1$.

3 | APPLICATION TO MINIMUM PHASE SYSTEMS

A possible application of the funnel pre-compensator cascade developed in Section 2 is in high-gain adaptive control in order to solve the longstanding open question of how systems with relative degree larger than one may be appropriately treated, see Ilchmann (1991); Ilchmann and Ryan (2008); Morse (1996). Recently, a funnel controller has been designed in Berger et al. (2016) which is able to achieve tracking with prescribed transient performance for nonlinear systems of arbitrary relative degree. However, the derivatives of the output must be available for the controller. In practice this means that measurements must be differentiated, which is an ill-posed problem, in particular in the presence of noise, see e.g. (Hackl 2012, Sec. 1.4.4). In order to resolve this problem, the funnel pre-compensator cascade may be applied to the system which results in an interconnection with new output z satisfying (20), and the derivatives of which are known. Then the funnel controller from Berger et al. (2016) may be applied to the interconnection in order to achieve tracking with prescribed transient behavior without the need to calculate output derivatives; for linear minimum phase systems with relative degree two this configuration was successfully implemented in Berger and Reis (2016a). The situation is depicted in Figure 6 .

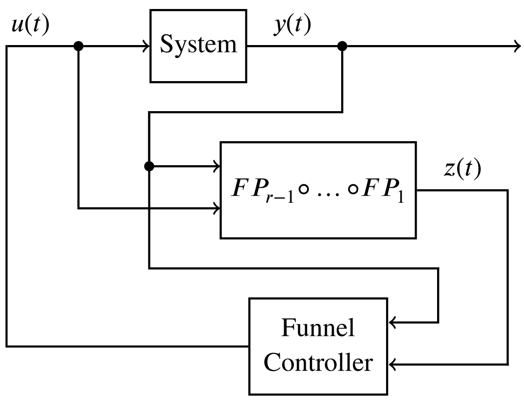

FIGURE 6 Interconnection of a system with funnel pre-compensator cascade and funnel controller.

For the solution of tracking problems, a crucial condition is the input-to-state stability of the internal dynamics (the minimum phase property in case of linear systems), cf. Byrnes and Isidori (1984); Ilchmann and Ryan (2008); Sastry and Isidori (1989). The funnel controller in Berger et al. (2016) requires this as well, and hence we need to ensure that for a minimum phase system, the interconnection with the funnel pre-compensator cascade is again minimum phase. In the following we show that this can be achieved for a special class of systems which are linear up to the influence of an operator T and have relative degree two or three. For relative degree larger than three this remains an open problem; we show explicitly where our proof does not work in this case and conjecture that some kind of small gain condition is needed then.

In the following we consider systems described by functional differential equations of the form

$$\begin{aligned} y^{(r)}(t) &= \sum_{i=1}^{r} R_i y^{(i-1)}(t) + f\big(d(t), T(y, \dot y, \ldots, y^{(r-1)})(t)\big) + \Gamma u(t), \\ y|_{[-h,0]} &= y^0 \in \mathcal{W}^{r-1,\infty}([-h,0] \to \mathbb{R}^m), \end{aligned} \tag{24}$$

where $h > 0$ is the "memory" of the system, $r \in \mathbb{N}$ is the strict relative degree, and

- $d \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^p)$, $p \in \mathbb{N}$, is a disturbance;
- $f \in \mathcal{C}(\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^m)$, $q \in \mathbb{N}$;
- $\Gamma \in \mathbf{Gl}_m(\mathbb{R})$ is the high-frequency gain matrix;
- $T : \mathcal{C}([-h,\infty) \to \mathbb{R}^m)^r \to \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$ is an operator with the following properties:

  a) T maps bounded trajectories to bounded trajectories, i.e., for all $c_1 > 0$ there exists $c_2 > 0$ such that for all $\zeta \in \mathcal{C}([-h,\infty) \to \mathbb{R}^m)^r$:

  $$\sup_{t\in[-h,\infty)} \|\zeta(t)\| \leq c_1 \;\Rightarrow\; \sup_{t\in[0,\infty)} \|T(\zeta)(t)\| \leq c_2;$$

  b) T is causal, i.e., for all $t \geq 0$ and all $\zeta, \xi \in \mathcal{C}([-h,\infty) \to \mathbb{R}^m)^r$:

  $$\zeta|_{[-h,t)} = \xi|_{[-h,t)} \;\Rightarrow\; T(\zeta)|_{[0,t]} \overset{\text{a.e.}}{=} T(\xi)|_{[0,t]}\,;$$

  c) T is "locally Lipschitz" continuous in the following sense: for all $t \geq 0$ there exist $\tau, \delta, c > 0$ such that for all $\zeta, \Delta\zeta \in \mathcal{C}([-h,\infty) \to \mathbb{R}^m)^r$ with $\Delta\zeta|_{[-h,t]} = 0$ and $\|\, \Delta\zeta|_{[t,t+\tau]}\, \|_\infty < \delta$ we have

  $$\left\| \left(T(\zeta+\Delta\zeta) - T(\zeta)\right)\Big|_{[t,t+\tau]} \right\|_\infty \leq c \|\, \Delta\zeta|_{[t,t+\tau]}\, \|_\infty .$$

The functions $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ and $y : [-h,\infty) \to \mathbb{R}^m$ are called *input* and *output* of the system (24), respectively. For fixed $u \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ we call $y \in \mathcal{C}^{r-1}([-h,\omega) \to \mathbb{R}^m)$ a solution of (24) on $[-h,\omega)$, $\omega \in (0,\infty]$, if $y|_{[-h,0]} = y^0$ and $y^{(r-1)}|_{[0,\omega)}$ is weakly differentiable and satisfies the differential equation in (24) for almost all $t \in [0,\omega)$; y is called *maximal*, if it has no right extension that is also a solution. Existence of maximal solutions of (24) for every $y^0 \in \mathcal{W}^{r-1,\infty}([-h,0] \to \mathbb{R}^m)$ and every $u \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ is guaranteed by (Ilchmann et al. 2002b, Thm. 5); if $y, \dot{y}, \dots, y^{(r-1)}$ are bounded, then $\omega = \infty$. In this case we clearly have $(u,\, y|_{[0,\infty)}) \in \mathcal{P}_r$.

The input-to-state stability of the internal dynamics of (24), i.e., the minimum phase property, is modelled by the property a) of the operator T in (24). It is shown in Berger et al. (2016) that funnel control is feasible for systems of the class (24), provided that Γ is positive or negative definite. In the case of relative degree one, i.e., $r = 1$, systems similar to (24) are well studied, see Ilchmann et al. (2002ab 2005); Ryan and Sangwin (2001). For relative degree two systems see Hackl et al. (2013), and for higher relative degree see Ilchmann et al. (2007). In the aforementioned references it is shown that the class of systems (24) encompasses linear and nonlinear systems with existing strict relative degree and input-to-state stable internal dynamics and the operator T allows for infinite-dimensional linear systems, systems with hysteretic effects or nonlinear delay elements and combinations thereof. In particular, the class (24) contains the system classes discussed in Hackl (2011); Ilchmann et al. (2006 2007) and the nonlinear systems in Jiang et al. (2004) provided that the internal dynamics are input-to-state stable.

In order to show that the minimum phase property of systems (24) is preserved by the cascade of funnel pre-compensators, we additionally need that the operator T is bounded whenever the output y is bounded, i.e., we replace property a) with the stronger condition

a') for all $c_1 > 0$ there exists $c_2 > 0$ such that for all $\zeta_1, \dots, \zeta_r \in \mathcal{C}([-h,\infty) \to \mathbb{R}^m)$:

$$\sup_{t\in[-h,\infty)} \|\zeta_1(t)\| \leq c_1 \;\Rightarrow\; \sup_{t\in[0,\infty)} \|T(\zeta_1,\dots,\zeta_r)(t)\| \leq c_2.$$

The class of systems (24) where T satisfies a') in particular contains the class of nonlinear systems in input-normalized Byrnes-Isidori form with exponentially stable zero dynamics as considered in Bullinger and Allgöwer (2005), provided the high-frequency gain matrix is constant. We show that, if $r = 2$ or $r = 3$, the interconnection of (24) with the cascade of funnel pre-compensators, where $\tilde{\Gamma}_i = \tilde{\Gamma}$ is invertible, has again relative degree r and input-to-state stable internal dynamics in the sense that it can be rewritten as

$$z^{(r)}(t) = F\left(\tilde{d}(t), \tilde{T}(z, \dot{z}, \dots, z^{(r-1)})(t)\right) + \tilde{\Gamma} u(t),$$

where $\tilde{T}$ is an operator with the properties a)–c).

Theorem 3.1
Consider a system (24) with $r \in \{2,3\}$, $y^0 \in \mathcal{W}^{r-1,\infty}([-h,0] \to \mathbb{R}^m)$ and assume that $\Gamma = \Gamma^\top > 0$ and the operator T satisfies a'). Further consider the cascade of funnel pre-compensators $FP_{r-1} \circ \ldots \circ FP_1$ defined by (17), (18) with $\varphi_i \in \Phi_{r-1}$ such that

$$\varphi_i(0)\|z_{i-1,1}(0) - z^0_{i,1}\| < 1,$$

where $z_{0,1} := y$ and $q_{i,j} = q_j > 0$, $p_{i,j} = p_j > 0$ are such that (5) is satisfied for corresponding matrices A, P, Q for all $i = 1, \ldots, r-1$, $j = 1, \ldots, r$. Moreover, assume that $\tilde{\Gamma}_i = \tilde{\Gamma} \in \mathbb{R}^{m\times m}$, $i = 1, \ldots, r-1$, such that $\tilde{\Gamma} = \tilde{\Gamma}^\top > 0$ and,

$$\text{if } r = 3, \text{ then } \; I - \Gamma\tilde{\Gamma}^{-1} = \left(I - \Gamma\tilde{\Gamma}^{-1}\right)^\top > 0. \tag{25}$$

Then the conjunction of (24) and (17) with input u and output $z := z_{r-1,1}$ can be equivalently written as

$$z^{(r)}(t) = F\left(\tilde{d}(t), \tilde{T}(z, \dot{z}, \ldots, z^{(r-1)})(t)\right) + \tilde{\Gamma}u(t), \quad z(0) = z^0_{r-1,1}, \tag{26}$$

for $\tilde{d}(t) := \left(\varphi_{r-1}(t), \dot{\varphi}_{r-1}(t), \ldots, \varphi^{(r-1)}_{r-1}(t)\right)^\top \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^r)$, some $F \in \mathcal{C}(\mathbb{R}^r \times \mathbb{R}^{\tilde{q}} \to \mathbb{R}^m)$ and an operator $\tilde{T} : \mathcal{C}([-h,\infty) \to \mathbb{R}^m)^r \to \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^{\tilde{q}})$ which satisfies the properties a)–c). Furthermore, for any $u \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ and any solution of (17), (24) we have (20) and the derivatives of the compensator states satisfy (23).

Proof. Step 1: We start with several transformations of the error dynamics between two successive systems.
Step 1a: Define $v_{i,j} := z_{i-1,j} - z_{i,j}$ *for* $i = 2, \ldots, r-1$ *and* $j = 1, \ldots, r$. *Then*

$$\begin{aligned}
\dot{v}_{i,1}(t) &= v_{i,2}(t) - \left(q_1 + p_1 k_i(t)\right) \cdot v_{i,1}(t) + \left(q_1 + p_1 k_{i-1}(t)\right) \cdot v_{i-1,1}(t),\\
&\;\;\vdots\\
\dot{v}_{i,r-1}(t) &= v_{i,r}(t) - \left(q_{r-1} + p_{r-1} k_i(t)\right) \cdot v_{i,1}(t) + \left(q_{r-1} + p_{r-1} k_{i-1}(t)\right) \cdot v_{i-1,1}(t),\\
\dot{v}_{i,r}(t) &= - \left(q_r + p_r k_i(t)\right) \cdot v_{i,1}(t) + \left(q_r + p_r k_{i-1}(t)\right) \cdot v_{i-1,1}(t).
\end{aligned}$$

Step 1b: Defining $e_{1,j}(t) := y^{(j-1)}(t) - z_{1,j}(t)$ *for* $j = 1, \ldots, r-1$ *and* $e_{1,r}(t) := y^{(r-1)}(t) - \Gamma\tilde{\Gamma}^{-1} z_{1,r}(t)$ *we obtain*

$$\begin{aligned}
\dot{e}_{1,1}(t) &= e_{1,2}(t) - \left(q_1 + p_1 k_1(t)\right) \cdot e_{1,1}(t),\\
&\;\;\vdots\\
\dot{e}_{1,r-2}(t) &= e_{1,r-1}(t) - \left(q_{r-2} + p_{r-2} k_1(t)\right) \cdot e_{1,1}(t),\\
\dot{e}_{1,r-1}(t) &= e_{1,r}(t) - \left(q_{r-1} + p_{r-1} k_1(t)\right) \cdot e_{1,1}(t) + (\Gamma\tilde{\Gamma}^{-1} - I) \cdot z_{1,r}(t),\\
\dot{e}_{1,r}(t) &= - \Gamma\tilde{\Gamma}^{-1}\left(q_r + p_r k_1(t)\right) \cdot e_{1,1}(t) + \sum_{i=1}^{r} R_i y^{(i-1)}(t) + f\left(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t)\right).
\end{aligned}$$

Set $v_{1,1}(t) := e_{1,1}(t)$ *and* $\tilde{v}(t) := \sum_{i=1}^{r-1} v_{i,1}(t)$*, then we may define* $v_{1,j}(t) := e_{1,j}(t) - \sum_{k=1}^{j-1} R_{r-j+k+1}\tilde{v}^{(k-1)}(t)$ *and obtain*

$$\begin{aligned}
\dot{v}_{1,1}(t) &= v_{1,2}(t) - \left(q_1 + p_1 k_1(t)\right) \cdot v_{1,1}(t) + R_r \tilde{v}(t),\\
\dot{v}_{1,2}(t) &= v_{1,3}(t) - \left(q_2 + p_2 k_1(t)\right) \cdot v_{1,1}(t) + R_{r-1} \tilde{v}(t),\\
&\;\;\vdots\\
\dot{v}_{1,r-2}(t) &= v_{1,r-1}(t) - \left(q_{r-2} + p_{r-2} k_1(t)\right) \cdot v_{1,1}(t) + R_3 \tilde{v}(t),\\
\dot{v}_{1,r-1}(t) &= v_{1,r}(t) - \left(q_{r-1} + p_{r-1} k_1(t)\right) \cdot v_{1,1}(t) + R_2 \tilde{v}(t) + (\Gamma\tilde{\Gamma}^{-1} - I) z_{1,r}(t),\\
\dot{v}_{1,r}(t) &= - \Gamma\tilde{\Gamma}^{-1}\left(q_r + p_r k_1(t)\right) \cdot v_{1,1}(t) + R_1 \tilde{v}(t) + \sum_{i=1}^{r} R_i \left(y^{(i-1)}(t) - \tilde{v}^{(i-1)}(t)\right)\\
&\quad + f\left(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t)\right).
\end{aligned}$$

Now we observe that

$$\begin{aligned}
y(t) - \tilde{v}(t) &= y(t) - v_{1,1}(t) - v_{2,1}(t) - \ldots - v_{r-1,1}(t)\\
&= y(t) - \left(y(t) - z_{1,1}(t)\right) - \left(z_{1,1}(t) - z_{2,1}(t)\right) - \ldots - \left(z_{r-2,1}(t) - z_{r-1,1}(t)\right) = z_{r-1,1}(t) = z(t).
\end{aligned}$$

Furthermore,

$$z_{1,r}(t) = z^{(r-1)}_{1,1}(t) - \sum_{i=0}^{r-2} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^i \left[\left(q_{r-i-1} + p_{r-i-1} k_1(t)\right) v_{1,1}(t)\right]$$

and

$$z_{1,1}(t) = y(t) - v_{1,1}(t) = z(t) + \tilde{v}(t) - v_{1,1}(t) = z(t) + \sum_{i=2}^{r-1} v_{i,1}(t),$$

hence

$$z_{1,r}(t) = z^{(r-1)}(t) + \sum_{i=2}^{r-1} v_{i,1}(t) - \sum_{i=0}^{r-2} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{i} \left[\left(q_{r-i-1} + p_{r-i-1}k_1(t)\right)v_{1,1}(t)\right].$$

Step 1c: Define $w_{i,j}(t) := v_{i,j}(t)$ for $i = 2, \dots, r-1$ and $j = 1, \dots, r$ and $w_{1,r}(t) := v_{1,r}(t)$,

$$w_{1,r-j}(t) := v_{1,r-j}(t) + G\left[\sum_{i=2}^{r-1} v_{i,1}^{(r-j-1)}(t) - \sum_{i=j}^{r-2}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{i-j}\left[\left(q_{r-i-1} + p_{r-i-1}k_1(t)\right)v_{1,1}(t)\right]\right]$$

for $j = 1, \dots, r-1$, where $G := (I - \Gamma\tilde{\Gamma}^{-1})$; in particular we have

$$w_{1,1}(t) = v_{1,1}(t) + G\sum_{i=2}^{r-1} v_{i,1}(t).$$

With $\tilde{w}(t) := \sum_{i=2}^{r-1} w_{i,1}(t)$ we find

$$\begin{aligned}
\dot{w}_{1,1}(t) &= w_{1,2}(t) - \Gamma\tilde{\Gamma}^{-1}\left(q_1 + p_1k_1(t)\right)\cdot\left(w_{1,1}(t) - G\tilde{w}(t)\right) + R_r w_{1,1}(t) + R_r\Gamma\tilde{\Gamma}^{-1}\tilde{w}(t),\\
\dot{w}_{1,2}(t) &= w_{1,3}(t) - \Gamma\tilde{\Gamma}^{-1}\left(q_2 + p_2k_1(t)\right)\cdot\left(w_{1,1}(t) - G\tilde{w}(t)\right) + R_{r-1} w_{1,1}(t) + R_{r-1}\Gamma\tilde{\Gamma}^{-1}\tilde{w}(t),\\
&\ \ \vdots\\
\dot{w}_{1,r-2}(t) &= w_{1,r-1}(t) - \Gamma\tilde{\Gamma}^{-1}\left(q_{r-2} + p_{r-2}k_1(t)\right)\cdot\left(w_{1,1}(t) - G\tilde{w}(t)\right) + R_3 w_{1,1}(t) + R_3\Gamma\tilde{\Gamma}^{-1}\tilde{w}(t),\\
\dot{w}_{1,r-1}(t) &= w_{1,r}(t) - \Gamma\tilde{\Gamma}^{-1}\left(q_{r-1} + p_{r-1}k_1(t)\right)\cdot\left(w_{1,1}(t) - G\tilde{w}(t)\right) + R_2 w_{1,1}(t) + R_2\Gamma\tilde{\Gamma}^{-1}\tilde{w}(t) - Gz^{(r-1)}(t),\\
\dot{w}_{1,r}(t) &= - \Gamma\tilde{\Gamma}^{-1}\left(q_r + p_rk_1(t)\right)\cdot\left(w_{1,1}(t) - G\tilde{w}(t)\right) + R_1 w_{1,1}(t) + R_1\Gamma\tilde{\Gamma}^{-1}\tilde{w}(t)\\
&\quad + \sum_{i=1}^{r} R_i z^{(i-1)}(t) + f\left(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t)\right),\\
k_1(t) &= \frac{1}{1 - \varphi_1(t)^2\|w_{1,1}(t) - G\tilde{w}(t)\|^2}.
\end{aligned} \tag{27a}$$

and, for $i = 2, \dots, r-1$,

$$\begin{aligned}
\dot{w}_{i,1}(t) &= w_{i,2}(t) - \left(q_1 + p_1k_i(t)\right)\cdot w_{i,1}(t) + \left(q_1 + p_1k_{i-1}(t)\right)\cdot\underbrace{w_{i-1,1}(t)}_{=\left(w_{1,1}(t) - G\tilde{w}(t)\right)\ \text{if } i=2},\\
&\ \ \vdots\\
\dot{w}_{i,r-1}(t) &= w_{i,r}(t) - \left(q_{r-1} + p_{r-1}k_i(t)\right)\cdot w_{i,1}(t) + \left(q_{r-1} + p_{r-1}k_{i-1}(t)\right)\cdot\underbrace{w_{i-1,1}(t)}_{=\left(w_{1,1}(t) - G\tilde{w}(t)\right)\ \text{if } i=2},\\
\dot{w}_{i,r}(t) &= - \left(q_r + p_rk_i(t)\right)\cdot w_{i,1}(t) + \left(q_r + p_rk_{i-1}(t)\right)\cdot\underbrace{w_{i-1,1}(t)}_{=\left(w_{1,1}(t) - G\tilde{w}(t)\right)\ \text{if } i=2},\\
k_i(t) &= \frac{1}{1 - \varphi_i(t)^2\|w_{i,1}(t)\|^2}.
\end{aligned} \tag{27b}$$

Step 2: We define the operator $\tilde{T} : C([-h,\infty) \to \mathbb{R}^m)^r \to \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^{\tilde{q}})$, where $\tilde{q} = (r-1)rm + r$, (essentially) as the solution operator of (27)*, i.e., for $\zeta_1, \dots, \zeta_r \in C([-h,\infty) \to \mathbb{R}^m)$ let $w_{ij} : [0,\beta) \to \mathbb{R}^m$, $\beta \in (0,\infty]$, be the unique maximal solution of* (27) *for $z = \zeta_1, \dot{z} = \zeta_2, \dots, z^{(r-1)} = \zeta_r$ with appropriate initial values according to the transformation which leads to* (27)*, and define*

$$\tilde{T}(\zeta_1, \dots, \zeta_r)(t) := \left(w_{1,1}(t), \dots, w_{1,r}(t), w_{2,1}(t), \dots, w_{r-1,r}(t), k_1(t), \dots, k_r(t)\right)^\top, \quad t \in [0,\beta).$$

We stress that $y, \dot{y}, \dots, y^{(r-1)}$ in (27a) *can be replaced by $w_{i,j}$ and $z, \dot{z}, \dots, z^{(r-1)}$ using $y^{(i)} = z^{(i)} + w_{1,1}^{(i)} + \Gamma\tilde{\Gamma}^{-1}\tilde{w}^{(i)}$ and the differential equations* (27)*. Furthermore, the operator $\tilde{T}$ depends on the disturbance d and several initial values. In the* *following we show that $\tilde{T}$ is well-defined, i.e., $\beta = \infty$, and has the properties a)–c). Note that for*

$$\mathcal{D} := \left\{ \left(t, w_{1,1}, \ldots, w_{1,r}, w_{2,1}, \ldots, w_{r-1,r}\right) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{rm} \;\middle|\; \varphi_1(t) \left\| w_{1,1} - G \sum_{i=2}^{r-1} w_{i,1} \right\| < 1,\ \varphi_i(t)\|w_{i,1}\| < 1,\ i = 2, \ldots, r-1 \right\}$$

we have $\left(t, w_{1,1}(t), \ldots, w_{1,r}(t), w_{2,1}(t), \ldots, w_{r-1,r}(t)\right) \in \mathcal{D}$ for all $t \in [0, \beta)$, and the closure of the graph of the solution $\left(w_{1,1}, \ldots, w_{1,r}, w_{2,1}, \ldots, w_{r-1,r}\right)$ is not a compact subset of $\mathcal{D}$.

Step 2a: First assume that $\zeta_1, \ldots, \zeta_r$ are bounded on $[0, \beta)$. We show that $w_{i,j}$ and k_i are bounded as well. As the solution evolves in $\mathcal{D}$, it is clear that $w_{1,1} - G\tilde{w}, w_{2,1}, \ldots, w_{r-1,1}$ are bounded, and thus also $w_{1,1}$ is bounded. Since $y = z + w_{1,1} + \Gamma\tilde{\Gamma}^{-1}\tilde{w}$, it follows that y is bounded and hence $T(y, \dot{y}, \ldots, y^{(r-1)})$ is bounded by property a'). Boundedness of d and continuity of f then imply that $f\left(d(\cdot), T(y, \dot{y}, \ldots, y^{(r-1)})(\cdot)\right)$ is bounded.

Now let $w_i := (w_{i,1}^\top, \ldots, w_{i,r}^\top)^\top$, then it follows from (27) that

$$\begin{aligned}
\dot{w}_1(t) &= \hat{A}w_1(t) - k_1(t)\bar{P}\Gamma\tilde{\Gamma}^{-1}\left(w_{1,1}(t) - G\tilde{w}(t)\right) + B_1(t),\\
\dot{w}_2(t) &= \hat{A}w_2(t) - k_2(t)\bar{P}w_{2,1}(t) + k_1(t)\bar{P}\left(w_{1,1}(t) - G\tilde{w}(t)\right) + B_2(t),\\
\dot{w}_i(t) &= \hat{A}w_i(t) - k_i(t)\bar{P}w_{i,1}(t) + k_{i-1}(t)\bar{P}w_{i-1,1}(t) + B_i(t)
\end{aligned} \tag{28}$$

for $i = 3, \ldots, r-1$, where $\hat{A}$ is as in the proof of Proposition 2.1, B_i is some suitable bounded function and

$$\bar{P} := \begin{bmatrix} p_1 I_m \\ \vdots \\ p_r I_m \end{bmatrix}.$$

Recall that $\hat{A}^\top \hat{P} + \hat{P}\hat{A} + \hat{Q} = 0$, where $\hat{P} > 0$ and $\hat{Q} > 0$, and that

$$\bar{P}^\top \hat{P} = [\tilde{p}I_m, 0, \ldots, 0], \quad \tilde{p} := (P_1 - P_2 P_4^{-1} P_2^\top) > 0.$$

We consider the cases $r = 2$ and $r = 3$ separately.

Step 2b: Assume that $r = 2$. Then (28) reads

$$\dot{w}_1(t) = \hat{A}w_1(t) - k_1(t)\bar{P}\Gamma\tilde{\Gamma}^{-1}w_{1,1}(t) + B_1(t).$$

Using the Lyapunov function $V(w_1) = w_1^\top \hat{P} w_1$ one can then show, as in the proof of Proposition 2.1, that w_1 and k_1 are bounded on $[0, \beta)$.

Step 2c: Assume that $r = 3$. Then (28) reads

$$\begin{aligned}
\dot{w}_1(t) &= \hat{A}w_1(t) - k_1(t)\bar{P}\Gamma\tilde{\Gamma}^{-1}\left(w_{1,1}(t) - Gw_{2,1}(t)\right) + B_1(t),\\
\dot{w}_2(t) &= \hat{A}w_2(t) - k_2(t)\bar{P}w_{2,1}(t) + k_1(t)\bar{P}\left(w_{1,1}(t) - Gw_{2,1}(t)\right) + B_2(t).
\end{aligned}$$

From condition (25) we obtain that $G = G^\top > 0$, hence $G\Gamma\tilde{\Gamma}^{-1} = \left(G\Gamma\tilde{\Gamma}^{-1}\right)^\top > 0$ has a unique matrix square root. Let $K := I_m \otimes \left(G\Gamma\tilde{\Gamma}^{-1}\right)^{\frac{1}{2}} > 0$ (recall the Kronecker product $\otimes$ from the proof of Proposition 2.1) and define the Lyapunov function $V(w_1, w_2) := w_1^\top \hat{P} w_1 + w_2^\top K^\top \hat{P} K w_2$ for $w_1, w_2 \in \mathbb{R}^{3m}$. Then, for all $t \in [0, \beta)$,

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} V\left(w_1(t), w_2(t)\right) = {} & w_1(t)^\top\left(\hat{A}^\top \hat{P} + \hat{P}\hat{A}\right)w_1(t) - 2k_1(t)w_1(t)^\top \hat{P}\bar{P}\Gamma\tilde{\Gamma}^{-1}\left(w_{1,1}(t) - G\tilde{w}(t)\right)\\
& + 2w_1(t)^\top B_1(t) + w_2(t)^\top\left(\hat{A}^\top K^\top \hat{P} K + K^\top \hat{P} K \hat{A}\right)w_2(t) - 2k_2(t)w_2(t)^\top K^\top \hat{P} K \bar{P} w_{2,1}(t)\\
& + 2w_2(t)^\top K^\top \hat{P} K B_2(t) + 2k_1(t)w_2(t)^\top K^\top \hat{P} K \bar{P}\left(w_{1,1}(t) - Gw_{2,1}(t)\right),
\end{aligned}$$

and since it is easy to see that $\hat{A}$ and K commute and $K^\top \hat{P} K \bar{P} = \tilde{p}[I_m, 0, \ldots, 0]^\top G\Gamma\tilde{\Gamma}^{-1}$, it follows that, for some positive $\alpha_1, \alpha_2, M_1, M_2$,

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} V\left(w_1(t), w_2(t)\right) &\leq -\alpha_1\|w_1(t)\|^2 - \alpha_2\|w_2(t)\|^2 - 2k_1(t)\left(\tilde{p}w_{1,1}^\top \Gamma\tilde{\Gamma}^{-1} - \tilde{p}w_{2,1}^\top G\Gamma\tilde{\Gamma}^{-1}\right)\left(w_{1,1}(t) - Gw_{2,1}(t)\right)\\
&\quad + M_1\|w_1(t)\| + M_2\|w_2(t)\|\\
&= -\alpha_1\|w_1(t)\|^2 - \alpha_2\|w_2(t)\|^2 + M_1\|w_1(t)\| + M_2\|w_2(t)\|\\
&\quad - 2\tilde{p}k_1(t)\left(w_{1,1} - Gw_{2,1}\right)^\top \Gamma\tilde{\Gamma}^{-1}\left(w_{1,1}(t) - Gw_{2,1}(t)\right)\\
&\leq -\alpha_1\|w_1(t)\|^2 - \alpha_2\|w_2(t)\|^2 + M_1\|w_1(t)\| + M_2\|w_2(t)\|.
\end{aligned}$$

As in the proof of Proposition 2.1 we may now show that w_1 and w_2 are bounded and that k_1 and k_2 are bounded as well on $[0,\beta)$.

Step 2d: We show $\beta = \infty$ (not assuming boundedness of $\zeta_1,\dots,\zeta_r$). Assume that $\beta < \infty$. Then $\zeta_1,\dots,\zeta_r$ are bounded on $[0,\beta)$ and hence $w_{i,j}$ and k_i are bounded by Steps 2a–2c. Therefore, it follows that the closure of the graph of the solution $\big(w_{1,1},\dots,w_{1,r},w_{2,1},\dots,w_{r-1,r}\big)$ is a compact subset of $\mathcal{D}$, a contradiction, thus $\beta = \infty$.

Step 2e: It remains to show that $\tilde{T}$ has the properties a)–c). Properties b) and c) are clear and property a) is an immediate consequence of Steps 2a–2c.

Step 3: By Step 2 we may write the conjunction of (24) and (17) with input u and output $z = z_{r-1,1}$ in the form

$$z^{(r)}(t) = \tilde{\Gamma}u(t) + \sum_{j=0}^{r-1}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^j\left[\big(q_{r-j}+p_{r-j}k_{r-1}(t)\big)w_{r-1,1}(t)\right]$$

and hence

$$z^{(r)}(t) = F\big(\tilde{d}(t),\tilde{T}(z,\dot{z},\dots,z^{(r-1)})(t)\big) + \tilde{\Gamma}u(t)$$

for $\tilde{d}(t) := \big(\varphi_{r-1}(t),\dot{\varphi}_{r-1}(t),\dots,\varphi_{r-1}^{(r-1)}(t)\big)^\top \in \mathcal{L}^\infty(\mathbb{R}_{\ge 0}\to\mathbb{R}^r)$, some $F \in \mathcal{C}(\mathbb{R}^r\times\mathbb{R}^{\tilde{q}}\to\mathbb{R}^m)$ and the operator $\tilde{T} : \mathcal{C}([-h,\infty)\to\mathbb{R}^m)^r \to \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\ge 0}\to\mathbb{R}^{\tilde{q}})$ which satisfies the properties a)–c). It is clear that any solution of (17), (24) satisfies the properties (23) and (20). □

Remark 3.2. A careful inspection of the proof of Theorem 3.1 reveals that in order for Theorem 3.1 to hold true for $r \ge 4$ we would need to show that (28) has bounded solutions. However, we were only able to find suitable Lyapunov functions in the cases $r = 2$ and $r = 3$, thus the proof for $r \ge 4$ remains an open problem; in particular, a recursive Lyapunov function of the form $V_i(w_1,\dots,w_i) = V_{i-1}(w_1,\dots,w_{i-1}) + w_i^\top K_i^\top \hat{P} K_i w_i$ does not exist in the latter case. It is worth noting that in the case $r = 2$ no condition on $\tilde{\Gamma}$ is present and for $r = 3$ condition (25) means, roughly speaking, that we need to choose $\tilde{\Gamma}$ "larger than" Γ, which resembles a small gain condition, cf. Dashkovskiy et al. (2007). We conjecture that some kind of small gain condition is needed in the case $r \ge 4$.

4 | FUNNEL CONTROL VIA FUNNEL PRE-COMPENSATOR

As discussed in Section 3, in virtue of Theorem 3.1 we may apply the funnel controller from Berger et al. (2016) to the interconnection of system (24) with the funnel pre-compensator cascade in the cases $r = 2$ and $r = 3$, cf. Figure 6 . For completeness we state the resulting controller structure and the corresponding feasibility result. The funnel controller as in Berger et al. (2016) is given by

$$\boxed{\begin{aligned} e_0(t) &= e(t) = y(t) - y_{\mathrm{ref}}(t), & k_i(t) &= \frac{1}{1-\varphi_i(t)^2\|e_i(t)\|^2}, \quad i = 0,\dots,r-1,\\ e_1(t) &= \dot{e}_0(t) + k_0(t)\cdot e_0(t), & u(t) &= -k_{r-1}(t)e_{r-1}(t),\\ e_2(t) &= \dot{e}_1(t) + k_1(t)\cdot e_1(t), & &\\ &\ \,\vdots & &\\ e_{r-1}(t) &= \dot{e}_{r-2}(t) + k_{r-2}(t)\cdot e_{r-2}(t), & & \end{aligned}} \tag{29}$$

where $r \in \mathbb{N}$ is the relative degree and the reference signal and funnel functions satisfy

$$y_{\mathrm{ref}} \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\ge 0}\to\mathbb{R}^m), \qquad \varphi_0 \in \Phi_r,\ \ \varphi_1 \in \Phi_{r-1},\ \dots,\ \varphi_{r-1} \in \Phi_1. \tag{30}$$

In Berger et al. (2016), the existence of solutions of the initial value problem resulting from the application of the funnel controller (29) to a system (24) is investigated (actually, a much larger class of nonlinear systems is allowed in Berger et al. (2016)). The combination of the funnel controller (29) with the cascade of funnel pre-compensators $FP_{r-1}\circ\dots\circ FP_1$ defined by (17), (18) reads as follows, where we only consider the two cases $r = 2$ and $r = 3$:

$$\boxed{\begin{aligned} &\textbf{Case } r = 2\textbf{:} & &\\ &\dot{z}_1(t) = z_2(t) + \big(q_1+p_1k_2(t)\big)\cdot(y(t)-z_1(t)), & k_0(t) &= \tfrac{1}{1-\varphi_0(t)^2\|e_0(t)\|^2},\\ &\dot{z}_2(t) = \big(q_2+p_2k_2(t)\big)\cdot(y(t)-z_1(t)) + \tilde{\Gamma}u(t), & k_1(t) &= \tfrac{1}{1-\varphi_1(t)^2\|e_1(t)\|^2},\\ &e_0(t) = z_1(t) - y_{\mathrm{ref}}(t), & k_2(t) &= \tfrac{1}{1-\varphi_2(t)^2\|y(t)-z_1(t)\|^2},\\ &e_1(t) = \dot{e}_0(t) + k_0(t)e_0(t), & u(t) &= -k_1(t)\cdot e_1(t), \end{aligned}} \tag{31}$$

Case $r = 3$:

$$\begin{aligned}
\dot z_{1,1}(t) &= z_{1,2}(t) + \big(q_1 + p_1 k_3(t)\big)\cdot(y(t) - z_{1,1}(t)), \\
\dot z_{1,2}(t) &= z_{1,3}(t) + \big(q_2 + p_2 k_3(t)\big)\cdot(y(t) - z_{1,1}(t)), \\
\dot z_{1,3}(t) &= \big(q_3 + p_3 k_3(t)\big)\cdot(y(t) - z_{1,1}(t)) + \tilde\Gamma\, u(t), \\
\dot z_{2,1}(t) &= z_{2,2}(t) + \big(q_1 + p_1 k_4(t)\big)\cdot(z_{1,1}(t) - z_{2,1}(t)), \\
\dot z_{2,2}(t) &= z_{2,3}(t) + \big(q_2 + p_2 k_4(t)\big)\cdot(z_{1,1}(t) - z_{2,1}(t)), \\
\dot z_{2,3}(t) &= \big(q_3 + p_3 k_4(t)\big)\cdot(z_{1,1}(t) - z_{2,1}(t)) + \tilde\Gamma\, u(t), \\
e_0(t) &= z_{2,1}(t) - y_{\rm ref}(t), \\
e_1(t) &= \dot e_0(t) + k_0(t)\cdot e_0(t), \\
e_2(t) &= \dot e_1(t) + k_1(t)\cdot e_1(t),
\end{aligned}
\qquad
\begin{aligned}
k_0(t) &= \tfrac{1}{1-\varphi_0(t)^2\|e_0(t)\|^2}, \\
k_1(t) &= \tfrac{1}{1-\varphi_1(t)^2\|e_1(t)\|^2}, \\
k_2(t) &= \tfrac{1}{1-\varphi_2(t)^2\|e_2(t)\|^2}, \\
k_3(t) &= \tfrac{1}{1-\varphi_3(t)^2\|y(t)-z_{1,1}(t)\|^2}, \\
k_4(t) &= \tfrac{1}{1-\varphi_4(t)^2\|z_{1,1}(t)-z_{2,1}(t)\|^2}, \\
u(t) &= -k_2(t)\cdot e_2(t),
\end{aligned}
\tag{32}$$

where $y_{\rm ref}$ and $\varphi_0, \dots, \varphi_{r-1}$ satisfy (30), $\tilde\Gamma = \tilde\Gamma^\top > 0$, $\varphi_r, \dots, \varphi_{2r-2} \in \Phi_{r-1}$ and $q_1, \dots, q_r, p_1, \dots, p_r > 0$ are such that (5) is satisfied for corresponding matrices P and Q. In a slightly different structure, the controller (31) for the case $r = 2$ was already successfully implemented in Berger and Reis (2016a), see also the discussion therein.

Feasibility of (31) and (32) in the respective cases may now be inferred.

Corollary 4.1
Consider a system (24) with $r \in \{2, 3\}$, $y^0 \in \mathcal{W}^{r-1,\infty}([-h, 0] \to \mathbb{R}^m)$ and assume that $\Gamma = \Gamma^\top > 0$ and the operator T satisfies a'). Let $y_{\rm ref}$ and $\varphi_0, \dots, \varphi_{r-1}$ be such that (30) holds and $\varphi_r, \dots, \varphi_{2r-2} \in \Phi_{r-1}$ be such that z_1, e_0, e_1 as defined in (31) or $z_{1,1}, z_{2,1}, e_0, e_1, e_2$ as defined in (32), resp., with initial data (18) satisfy

$$\varphi_i(0)\|e_i(0)\| < 1, \quad \text{for all } i = 0, \dots, r-1,$$

and

$$\begin{aligned}
\varphi_2(0)\|y(0) - z_1(0)\| &< 1, \quad \text{if } r = 2, \\
\varphi_3(0)\|y(0) - z_{1,1}(0)\| < 1 \quad \text{and} \quad \varphi_4(0)\|z_{1,1}(0) - z_{2,1}(0)\| &< 1, \quad \text{if } r = 3.
\end{aligned}$$

Further let $q_1, \dots, q_r, p_1, \dots, p_r > 0$ be such that (5) is satisfied for corresponding matrices A, P, Q, and let $\tilde\Gamma = \tilde\Gamma^\top > 0$ be such that (25) is satisfied.

Then the application of the funnel controller (31) (if $r = 2$) or (32) (if $r = 3$), resp., to (24) yields an initial-value problem, which has a solution, and every solution can be extended to a maximal solution $y : [-h, \omega) \to \mathbb{R}^m$, $\omega \in (0, \infty]$, which has the following properties:

(i) The solution is global (i.e., $\omega = \infty$).

(ii) The input $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, the compensator states z_1, z_2 or $z_{1,1}, \dots, z_{2,3}$, resp., the gain functions $k_0, \dots, k_{2r-2} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $y, \dots, y^{(r-1)} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ are bounded.

(iii) The functions $e_0, \dots, e_{r-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ and the compensator errors $y - z_1$ or $y - z_{1,1}, z_{1,1} - z_{2,1}$, resp., evolve in their respective performance funnels in the sense

$$\begin{aligned}
\exists\, \varepsilon_0, \dots, \varepsilon_{2r-2} > 0\ \forall\, t > 0 : \qquad \|e_i(t)\| &\leq \varphi_i(t)^{-1} - \varepsilon_i, \quad i = 0, \dots, r-1, \\
\|y(t) - z_1(t)\| &\leq \varphi_2(t)^{-1} - \varepsilon_2, \quad \text{if } r = 2, \\
\|y(t) - z_{1,1}(t)\| \leq \varphi_3(t)^{-1} - \varepsilon_3 \quad \text{and} \quad \|z_{1,1}(t) - z_{2,1}(t)\| &\leq \varphi_4(t)^{-1} - \varepsilon_4, \quad \text{if } r = 3.
\end{aligned}$$

In particular, the tracking error $e(t) = y(t) - y_{\rm ref}(t)$ satisfies, for all $t > 0$,

$$\begin{aligned}
\|e(t)\| &\leq \varphi_0(t)^{-1} + \varphi_2(t)^{-1} - \varepsilon_0 - \varepsilon_2, \quad \text{if } r = 2, \\
\|e(t)\| &\leq \varphi_0(t)^{-1} + \varphi_3(t)^{-1} + \varphi_4(t)^{-1} - \varepsilon_0 - \varepsilon_3 - \varepsilon_4, \quad \text{if } r = 3.
\end{aligned}$$

5 | SIMULATIONS

We illustrate the combined funnel controller and funnel pre-compensator in (31) and (32) by a simulation for a mass-spring system mounted on a car from Seifried and Blajer (2013), see Fig. 7 , and compare it to the simulation of the funnel controller (29) for this system as performed in Berger et al. (2016). As depicted in Fig. 7 , the mass $m_2[kg]$ moves on a ramp which is inclined by the angle $\alpha[rad]$ and mounted on a car with mass $m_1[kg]$. We assume that we may control the force $u = F[N]$ acting on it. The equations of motion are given by

$$\begin{bmatrix} m_1 + m_2 & m_2 \cos\alpha \\ m_2 \cos\alpha & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}(t) \\ \ddot{s}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ ks(t) + d\dot{s}(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ 0 \end{pmatrix}, \tag{33}$$

where $x[m]$ is the horizontal car position and $s[m]$ the relative position of the mass on the ramp. The constants $k[N/m]$, $d[Ns/m]$ are the coefficients of the spring and damper, resp. The output is the horizontal position of the mass on the ramp,

$$y(t) = x(t) + s(t)\cos\alpha.$$

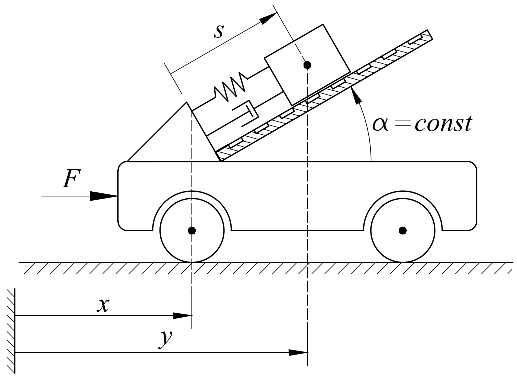

FIGURE 7 Mass on car system.

The system (33) can be reformulated such that it belongs to the class (24), see Seifried and Blajer (2013), with a relative degree r depending on the angle $\alpha[rad]$ and the damping $d[Ns/m]$. We consider the same experimental setup as in Berger et al. (2016) and distinguish two cases.

Case 1: If $0 < \alpha < \frac{\pi}{2}$, see Fig. 7 , then system (33) has relative degree $r = 2$ and the high-frequency gain matrix reads $\Gamma = \frac{\sin^2\alpha}{m_1 + m_2 \sin^2\alpha} > 0$. For the simulation, we choose the reference trajectory $y_{\text{ref}}(t) = \cos t[m]$, the parameters $m_1 = 4[kg]$, $m_2 = 1[kg]$, $k = 2[N/m]$, $d = 1[Ns/m]$, the initial values $x(0) = 0$, $\dot{x}(0) = 0$, $s(0) = 0$, $\dot{s}(0) = 0$ and $\alpha = \frac{\pi}{4}$. For the controller (31) we choose the initial values $z_1(0) = z_2(0) = 0$, the funnel functions

$$\varphi_0(t) = \varphi_2(t) = \tfrac{1}{2}(5e^{-2t} + 0.1)^{-1}, \qquad \varphi_1(t) = (1.3e^{-4t} + 0.01)^{-1},$$

and $\tilde{\Gamma} = \frac{1}{4} > \frac{1}{9} = \Gamma$. The parameters q_i, p_i are determined by the coefficients of the Hurwitz polynomial

$$(s+5)^2 = s^2 + 10s + 25,$$

by which $q_1 = 10$ and $q_2 = 25$. Therefore, $A = \left[\begin{smallmatrix} -10 & 1 \\ -25 & 0 \end{smallmatrix}\right]$ and the Lyapunov equation $A^\top P + PA = -I_2$ has the solution

$$P = \begin{bmatrix} \frac{13}{10} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{63}{250} \end{bmatrix},$$

by which $p_1 = 1$ and $p_2 = \frac{125}{63}$. Obviously the initial errors lie within the respective funnel boundaries and all assumptions of Corollary 4.1 are satisfied, thus it yields that funnel control is feasible. The sum $\varphi_0^{-1} + \varphi_2^{-1}$ equals the funnel boundary as chosen for the simulation in Berger et al. (2016), hence the results may be compared.

The simulation of the controller (31) applied to (33) over the time interval [0, 10] has been performed in MATLAB (solver: `ode15s`, rel. tol.: 10^{-14}, abs. tol.: 10^{-10}) and is depicted in Fig. 8 . Fig. 8 a shows the tracking error and the funnel boundary, while Fig. 8 b shows the corresponding input function generated by the controller. It can be seen that the proposed funnel controller (31) guarantees that the tracking error evolves within the prescribed performance funnel and it yields a similar performance of the input as the controller (29) when we compare it to the simulation results in Berger et al. (2016).

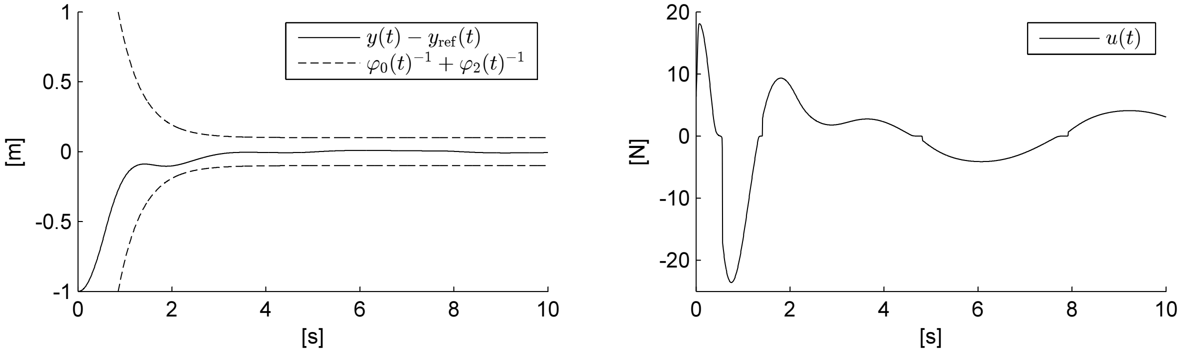

Fig. 8 a: Funnel and tracking error

Fig. 8 b: Input function

FIGURE 8 Simulation of the controller (31) for the mass on car system (33) with $\alpha = \frac{\pi}{4}$.

Case 2: If $\alpha = 0$ and $d \neq 0$, see Fig. 9 , then system (33) has relative degree $r = 3$ and high-frequency gain matrix $\Gamma = \frac{d}{m_1 m_2} > 0$. For the simulation, we choose the reference trajectory $y_{\mathrm{ref}}(t) = \cos t[m]$, the parameters $m_1 = 4[kg], m_2 = 1[kg], k = 2[N/m], d = 1[Ns/m]$ and the initial values $x(0) = 0$, $\dot{x}(0) = 0$, $s(0) = 0$, $\dot{s}(0) = 0$.

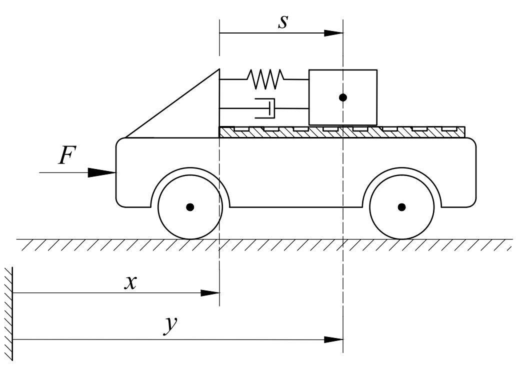

FIGURE 9 Mass on car system with $\alpha = 0$.

For the controller (32) we choose the initial values $z_{i,j}(0) = 0$, $i = 1, 2$, $j = 1, 2, 3$, the funnel functions

$$\varphi_0(t) = \varphi_3(t) = \varphi_4(t) = \tfrac{1}{3}(10e^{-2t} + 0.1)^{-1}, \quad \varphi_1(t) = (2.5e^{-3t} + 0.01)^{-1}, \quad \varphi_2(t) = (1.8e^{-20t} + 0.01)^{-1}$$

and $\tilde{\Gamma} = 0.8 > \frac{1}{4} = \Gamma$ such that (25) is satisfied. The parameters q_i, p_i are determined by the coefficients of the Hurwitz polynomial

$$(s+5)^3 = s^3 + 15s^2 + 75s + 125,$$

by which $q_1 = 15$, $q_2 = 75$ and $q_3 = 125$. Therefore, $A = \begin{bmatrix} -15 & 1 & 0 \\ -75 & 0 & 1 \\ -125 & 0 & 0 \end{bmatrix}$ and the Lyapunov equation $A^\top P + PA = -I_3$ has the solution

$$P = \begin{bmatrix} \frac{58}{5} & -\frac{1}{2} & -\frac{136}{125} \\ -\frac{1}{2} & \frac{136}{125} & -\frac{1}{2} \\ -\frac{136}{125} & -\frac{1}{2} & \frac{1333}{3125} \end{bmatrix},$$

by which $p_1 = 1$, $p_2 = \frac{1383}{391}$ and $p_3 = \frac{2230}{333}$. The initial errors lie within the respective funnel boundaries and all assumptions of Corollary 4.1 are satisfied, thus it yields that funnel control is feasible.

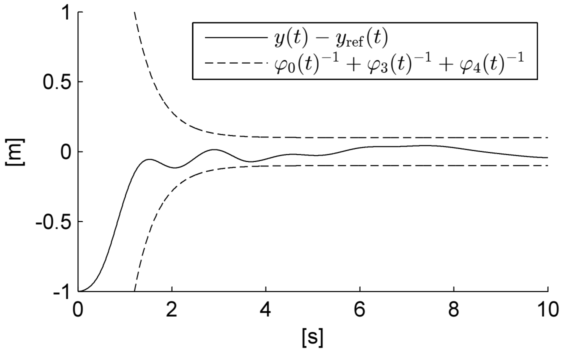

Fig. 10 a: Funnel and tracking error

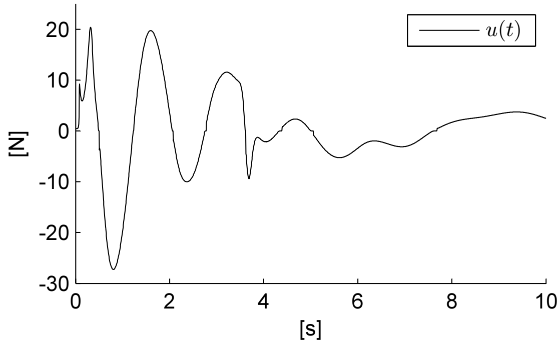

Fig. 10 b: Input function

FIGURE 10 Simulation of the controller (32) for the mass on car system (33) with $\alpha = 0$.

The simulation of the controller (32) applied to (33) over the time interval $[0, 10]$ has been performed in MATLAB (solver: `ode15s`, rel. tol.: 10^{-14}, abs. tol.: 10^{-10}) and is depicted in Fig. 10 , where the tracking error is shown in Fig. 10 a and the input in Fig. 10 b. We see that the funnel controller (32) is able to guarantee that the tracking error evolves within the prescribed performance funnel. The performance of the control input generated by (32) is comparable to that generated by the controller (29) in the simulation results in Berger et al. (2016); we stress that the controller (32) does not require availability of $\dot{y}$ and $\ddot{y}$.

6 | CONCLUSION

In the present paper we have introduced the funnel pre-compensator as a novel and simple adaptive pre-compensator, which resembles the structure of high-gain observers. We showed that the funnel pre-compensator is feasible for the large class of signal pairs $\mathcal{P}_r$. The proposed adaptation scheme for the pre-compensator gain is of low complexity and inherently robust since its design is model-free, and we showed that it guarantees prescribed transient behavior of the compensator error. Using a cascade of funnel pre-compensators, we proved that it is possible to obtain an artificial output with explicitly known derivatives which tracks the system output with prescribed transient behavior. As an application in adaptive control, we show that for a certain class of nonlinear systems, the interconnection with the funnel pre-compensator cascade has input-to-state stable internal dynamics provided the relative degree does not exceed three. This guarantees feasibility of a novel funnel controller which consists of a funnel pre-compensator cascade in conjunction with the recently developed funnel controller from Berger et al. (2016); this new controller does not require the derivatives of the output.

The results that we obtained in Sections 3 and 4 suggest that the funnel pre-compensator is a suitable tool for resolving the problem of higher relative degree in stabilization and tracking problems. If a system has a higher relative degree and derivatives of the output are not available, then a filter or observer is frequently used to obtain approximations of the output derivatives, see the survey Ilchmann and Ryan (2008) and the references therein. As explained there, the concept of funnel control is usually combined with a back-stepping procedure to overcome the higher relative degree, which however is quite complicated and impractical, cf. (Hackl 2012, Sec. 4.4.3). Nevertheless, in the last sentence of (Ilchmann and Ryan 2008, Sec. 6) it is conjectured that the combination of a high-gain observer with a funnel-type controller might be beneficial for tracking of higher relative degree systems. In Section 4 we have shown that the funnel pre-compensator, which resembles a high-gain observer, may be used to achieve this for systems with relative degree two or three. Systems of higher relative degree are the topic of future research.

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