Fault tolerant funnel control for uncertain linear systems

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Abstract

We study adaptive fault tolerant tracking control for uncertain linear systems. Based on recent results in funnel control and the time-varying Byrnes-Isidori form, we develop a low-complexity model-free controller which achieves prescribed performance of the tracking error for any given sufficiently smooth reference signal. Within the considered system class, we allow for more inputs than outputs as long as a certain redundancy of the actuators is satisfied. An important role in the controller design is played by the controller weight matrix, which is a rectangular input transformation chosen such that in the resulting system the zero dynamics, which are assumed to be uniformly exponentially stable, are independent of the new input. We illustrate the fault tolerant funnel controller by an example of a linearized model for the lateral motion of a Boeing 737 aircraft.

Keywords: linear systems; fault tolerant control; model-free control; funnel control; relative degree; zero dynamics.

1. Introduction

Being able to handle system uncertainties and, at the same time, failures or degrading efficiency of actuators is an important task in the design of control techniques. There are basically four different research directions in fault tolerant control, see the nice literature survey in [38]. These are

- multiple-model, switching, and tuning,
- direct and indirect adaptive designs,
- fault detection and diagnosis,
- robust control design.

We also refer to the exhaustive review paper [47] and the recent surveys [13, 14] for more references.

The uncertainties and actuator faults appearing in the system are usually unknown both in their nature and extent. In this framework, an adaptive control approach seems a suitable choice. The fault tolerant funnel controller that we introduce in the present paper is such a direct adaptive design. The area of adaptive design methods is quite active, see the recent articles [12, 42, 43, 45, 46]. Different approaches have been pursued, such as filter design and backstepping [12], strategies based on solving optimal control problems [31, 42, 43, 44] and (model-free) adaptive control techniques [38, 39, 45, 46].

In the present paper we consider adaptive fault tolerant tracking control for uncertain linear systems with prescribed performance of the tracking error. The uncertainties incorporate modelling errors and process faults as well as bounded noises and disturbances. The actuator faults encompass possible failures and degrading efficiency of the actuators as well as actuator stuck, locked actuator faults, actuator bias and actuator saturation. In the literature, some types of faults are often excluded; in [42, 45, 46] no total faults are allowed, in [38, 39] only actuator stuck is considered, and actuator saturation is considered in none of the aforementioned works.

Most results in fault tolerant control are model-based, cf. [13, 43]. The approaches presented in [45, 46] are completely model-free, however only single-input, single-output systems with trivial internal dynamics are considered and total faults are excluded. The approaches in [38, 39] require only little knowledge about the system parameters.

As the first result in fault tolerant tracking control that the author is aware of, the design in [45] is able to achieve prescribed performance of the tracking error and it is based on the approach of *Prescribed Performance Control* developed in [1], see also the recent works [2, 40]. However, in the present paper we follow the complementary approach of *Funnel Control* which was developed in [24], see also the survey [22] and the references therein. The funnel controller is an adaptive controller of high-gain type and thus inherently robust, cf. [16], which makes it a suitable choice for fault tolerant control tasks. The funnel controller has been successfully applied e.g. in temperature control of chemical reactor models [27], control of industrial servo-systems [17, 18, 26], DC-link power flow control [37], voltage and current control of electrical circuits [7], and control of peak inspiratory pressure [36].

Since it is usually not possible to foresee which actuator

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may fail during the operation of a system, a certain redundancy of the actuators is required, so that the remaining actuators are able to compensate for the (total) fault of others. This leads to systems with more inputs than outputs and thus additionally complicates the control task. For instance, funnel control has only been investigated for systems with the same number of inputs and outputs, see e.g. [19, 22, 24]. The funnel control design that we introduce in the present paper extends the recently developed funnel controller for systems with arbitrary relative degree [5].

We provide extensions of the above mentioned results for uncertain linear systems in the following regard:

- the allowed uncertainties and actuator faults encompass essentially all relevant cases,
- the control design is model-free,[1]
- more inputs than outputs are allowed, which in particular extends available results in funnel control,
- the relative degree of the system may be arbitrary, but known, and the zero dynamics may be nontrivial,
- prescribed performance of the tracking error is achieved,
- the controller is simple in its design and of low complexity.

In order to handle the problem of more inputs than outputs together with possible actuator faults we develop an extension of the time-varying Byrnes-Isidori form from [21]. We derive a characterization for the existence of a class of rectangular input transformations such that in the resulting system the zero dynamics are independent of the new input; in the case where the number of linearly independent actuators without total fault equals the number of outputs, the latter is already true for the original input.

1.1. Nomenclature

$\mathbb{R}_{\geq 0}$	$= [0,\infty)$
$\mathbb{C}_-$, $(\mathbb{C}_+)$	the set of complex numbers with negative (positive) real part
$\mathbf{Gl}_n(\mathbb{R})$	the group of invertible matrices in $\mathbb{R}^{n\times n}$
$\sigma(A)$	the spectrum of $A\in\mathbb{R}^{n\times n}$
$M^\dagger$	the Moore-Penrose pseudoinverse of $M\in\mathbb{R}^{n\times m}$
$\mathcal{L}^\infty(I\to\mathbb{R}^n)$	the set of essentially bounded functions $f: I\to\mathbb{R}^n$ with norm
$\|f\|_\infty$	$= \operatorname{ess\,sup}_{t\in I}\|f(t)\|$
$\mathcal{L}^\infty_{\rm loc}(I\to\mathbb{R}^n)$	the set of locally essentially bounded functions $f: I\to\mathbb{R}^n$
$\mathcal{W}^{k,\infty}(I\to\mathbb{R}^n)$	the set of k-times weakly differentiable functions $f: I\to\mathbb{R}^n$ such that $f,\ldots,f^{(k)}\in\mathcal{L}^\infty(I\to\mathbb{R}^n)$
$\mathcal{C}^k(I\to\mathbb{R}^n)$	the set of k-times continuously differentiable functions $f: I\to\mathbb{R}^n$, $k\in\mathbb{N}_0\cup\{\infty\}$
$\mathcal{C}(I\to\mathbb{R}^n)$	$=\mathcal{C}^0(I\to\mathbb{R}^n)$
$f\vert_J$	restriction of the function $f: I\to\mathbb{R}^n$ to $J\subseteq I$

1.2. System class

In the present paper we consider linear systems with time-varying and nonlinear uncertainties and possible actuator faults of the form

$$\begin{aligned}\dot x(t) &= Ax(t) + BL(t)u(t) + f\big(t, x(t), u(t)\big), \quad x(0)=x^0\\ y(t) &= Cx(t)\end{aligned} \tag{1}$$

where $x^0\in\mathbb{R}$, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{p\times n}$ with $m\geq p$, $f\in\mathcal{C}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n)$ is bounded and the following properties are satisfied:

(P1) $L\in\mathcal{C}^\infty(\mathbb{R}\to\mathbb{R}^{m\times m})$ such that $L,\dot L,\ldots,L^{(n)}$ are bounded and $\operatorname{rk} BL(t) = q\geq p$ for all $t\in\mathbb{R}$, $q\in\mathbb{N}$;

(P2) the system has (strict) relative degree $r\in\mathbb{N}$, i.e.,

- $CA^kBL(\cdot)=0$ and $CA^kf(\cdot)=0$ for all $k=0,\ldots,r-2$ and
- the "high-frequency gain matrix" $\Gamma := CA^{r-1}B\in\mathbb{R}^{p\times m}$ and L satisfy $\operatorname{rk}\Gamma L(t) = p$ for all $t\in\mathbb{R}$.

The functions $u:\mathbb{R}_{\geq 0}\to\mathbb{R}^m$ and $y:\mathbb{R}_{\geq 0}\to\mathbb{R}^p$ are called *input* and *output* of the system (1), resp. Some comments on the system class (1) are warranted.

(i) Since the control objective is fault tolerant control (see Subsection 1.3) a certain redundancy of the actuators is necessary in (1), thus m is usually much larger than p and the matrix B does not have full column rank. A typical situation is that $\operatorname{rk} B = p$, i.e., the number of linearly independent actuators equals the number of outputs of the system; one may think of p groups of actuators, where actuators in the same group perform the same control task. This situation is close to the concept of *uniform actuator redundancy*, see [48]. Clearly, $\operatorname{rk} B\geq p$ is necessary for the application of adaptive control techniques.

(ii) The (unknown) time-varying matrix function L from (P1) describes the *reliability* of the actuators. Typically we have $L(t)=\operatorname{diag}(l_1(t),\ldots,l_m(t))$ with $l_i\in\mathcal{C}^\infty(\mathbb{R}\to[0,1])$ monotonically non-increasing and $l_i(0)=1$; in this way, possible failures and degrading efficiency of the actuators may be described, cf. also [31, 42, 43, 44]. In our framework we allow for a general smooth matrix-valued function such that

[1]Some knowledge of system parameters helps with the construction of the controller weight matrix. For instance, knowledge of the high-frequency gain matrix is sufficient, if the number of linearly independent inputs equals the number of outputs.

the rank of $BL(t)$ is constant, i.e., the number of linearly independent actuators without total fault does not change.

(iii) The (unknown) nonlinearity f describes possible modelling errors or process faults, uncertainties, bounded noises and disturbances, and types of actuator failures not covered by the matrix function L as described above, see e.g. [15]. The latter means for instance locked actuator faults, actuator bias or actuator saturation, i.e., $f(u_i) = \mathrm{sat}(u_i) + b_i$ with $i \in \{1, \dots, m\}$, $b_i \in \mathbb{R}$ and $\mathrm{sat}(u_i) = \mathrm{sgn}(u_i)\,\hat{u}_i$ for $|u_i| \geq \hat{u}_i$ and $\mathrm{sat}(u_i) = u_i$ for $|u_i| < \hat{u}_i$, cf. [11].

(iv) The conditions in (P2) are slightly stronger than the assumption of a *strict and uniform relative degree* $r \in \mathbb{N}$ as introduced for time-varying nonlinear systems with $m = p$ in [21, Def. 2.2]. We use the stronger concept, and call it (strict) relative degree again, since, in view of (iii), we do not want to impose any differentiability assumptions on the nonlinearity f which are required in [21]. For the linear part of (1), i.e., $f(\cdot) = 0$ and $L(\cdot) = I_m$, the notion of strict relative degree as in (P2) is justified (note that $m > p$ is possible) since by [4, Def. B.1] the transfer function $G(s) = C(sI - A)^{-1}B$ has vector relative degree $(r, \dots, r)$, cf. also [29, 35].

(v) We assume that the system parameters A, B, C, $L(\cdot)$, $f(\cdot)$, x^0 are unknown; in particular the state space dimension n does not need to be known. We only require knowledge of the relative degree $r \in \mathbb{N}$. Furthermore, we will derive a class of rectangular input transformations of the form $u(t) = K(t)v(t)$, where $K \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{m\times p})$, such that in the resulting system the zero dynamics are independent of the new input v. As a structural assumption, we will require that the zero dynamics of the time-varying linear system $(A, BL(\cdot)K(\cdot), C)$ are uniformly exponentially stable for one (and hence any) K in this class; it is hence independent of the choice of K. Some additional knowledge of system parameters, such as the high-frequency gain matrix $\Gamma = CA^{r-1}B$ from (P2), may be helpful for the construction of K.

The roles of the reliability matrix function L and the uncertainties f are illustrated in Figure 1. We stress that even in the case $L(\cdot) = I_m$ the results of the present paper are new when $m > p$, since $m = p$ is usually assumed in funnel control.

1.3. Control objective

The objective is fault tolerant tracking of a reference trajectory $y_{\mathrm{ref}} \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^p)$ with prescribed performance, i.e., we seek an output error derivative feedback such that in the closed-loop system the tracking error $e(t) = y(t) - y_{\mathrm{ref}}(t)$ evolves within a prescribed performance funnel

$$\mathcal{F}_\varphi := \{\ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^p \ \mid \varphi(t)\|e\| < 1\ \}, \qquad (2)$$

which is determined by a function φ belonging to

$$\Phi_r := \left\{ \varphi \in \mathcal{C}^r(\mathbb{R}_{\geq 0} \to \mathbb{R}) \;\middle|\; \begin{array}{l} \varphi, \dot{\varphi}, \dots, \varphi^{(r)} \text{ are bounded,} \\ \varphi(\tau) > 0 \text{ for all } \tau > 0, \\ \text{and } \liminf_{\tau\to\infty} \varphi(\tau) > 0 \end{array} \right\}.$$

Furthermore, the state x and the input u in (1) should remain bounded.

The funnel boundary is given by the reciprocal of φ, see Fig. 2. The case $\varphi(0) = 0$ means that there is no restriction on the initial error $e(0)$, i.e., the funnel boundary $1/\varphi$ has a pole at $t = 0$.

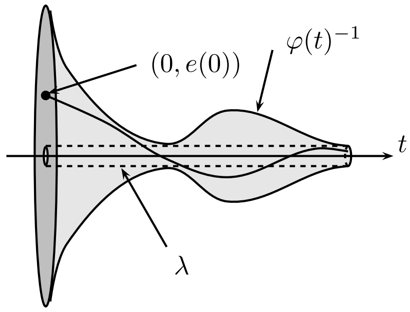

Figure 2: Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $\varphi(t)^{-1}$ for $t > 0$.

We stress that each performance funnel $\mathcal{F}_\varphi$ with $\varphi \in \Phi_r$ is bounded away from zero, i.e., because of boundedness of φ there exists $\lambda > 0$ such that $1/\varphi(t) \geq \lambda$ for all $t > 0$. While it is often convenient to choose a monotonically decreasing funnel boundary, it might be advantageous to widen the funnel over some later time interval, for instance in the presence of periodic disturbances or strongly varying reference signals. For typical choices of funnel boundaries see e.g. [19, Sec. 3.2].

1.4. Organization of the present paper

The paper is structured as follows. In Section 2 we derive a normal form for system (1) which extends the Byrnes-Isidori form for time-varying linear systems from [21]. We derive a class of rectangular input transformations such that in the resulting system the zero dynamics are independent of the new input. Uniform exponential stability of these zero dynamics, which is an important assumption for funnel control, is characterized in Section 3. The rectangular input transformation is exploited as controller weight matrix in the design of a fault tolerant funnel controller in Section 4. Possible choices for the weight matrix are discussed in the cases $q = p$ and $q > p$. The performance of the proposed funnel controller is illustrated by means of a linearized model for the lateral motion of a Boeing 737 aircraft in Section 5. Some conclusions are given in Section 6.

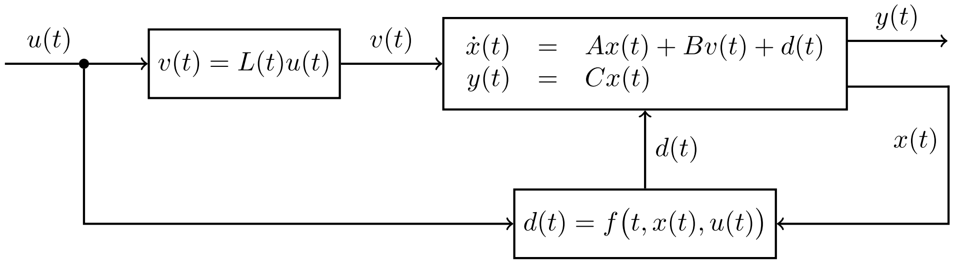

Figure 1: Interplay of system (1), reliability matrix function and uncertainties.

2. A time-varying normal form

We derive a normal form for systems (1) which is an extension of the Byrnes-Isidori form for time-varying linear systems from [21]. In this paper, with "normal form" we do not mean a "canonical form" which would be a unique representative of its equivalence class with respect to a certain set of transformations (or the mapping to this representative, resp.), but rather a weaker notion. We will see that the freedom left within the non-zero entries of the derived decomposition of (1) can be specified and is not significant which justifies to call it "normal form".

We introduce the following matrix-valued functions for all $t \in \mathbb{R}$:

$$\begin{aligned}\mathcal{B}(t) := \Big[BL(t), \big(\tfrac{\mathrm{d}}{\mathrm{d}t} - A\big)\big(BL(t)\big),\\ \dots, \big(\tfrac{\mathrm{d}}{\mathrm{d}t} - A\big)^{r-1}\big(BL(t)\big)\Big] \in \mathbb{R}^{n\times rm},\end{aligned}$$

$$\mathcal{C} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-1}\end{bmatrix} \in \mathbb{R}^{rm\times n}.$$

Let $\rho := \operatorname{rk}\mathcal{C}$, choose $V \in \mathbb{R}^{n\times(n-\rho)}$ such that $\operatorname{im} V = \ker \mathcal{C}$ and define

$$\mathcal{N}(t) := V^\dagger \Big[I_n - \mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger \mathcal{C}\Big] \in \mathbb{R}^{(n-\rho)\times n}, \quad t \in \mathbb{R},$$

where $M^\dagger$ denotes the *Moore-Penrose pseudoinverse* of a matrix M. Set

$$U(t) := \begin{bmatrix}\mathcal{C}\\ \mathcal{N}(t)\end{bmatrix} \in \mathbb{R}^{(n-\rho+pr)\times n}, \quad t \in \mathbb{R}. \tag{3}$$

Lemma 2.1. *Consider a system* (1) *with (P1) and (P2). Then we have for all $t \in \mathbb{R}$ that*

$$\rho = \operatorname{rk}\mathcal{C} = \operatorname{rk}\mathcal{C}\mathcal{B}(t) = pr$$

and

$$\mathcal{C}\mathcal{B}(t) = \begin{bmatrix} 0 & & (-1)^{r-1}\Gamma L(t) \\ & \cdot^{\cdot^{\cdot}} & \\ \Gamma L(t) & & * \end{bmatrix}. \tag{4}$$

Furthermore, $U(\cdot)$ as in (3) *is invertible with*

$$U(t)^{-1} = \Big[\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger, V\Big], \quad t \in \mathbb{R}. \tag{5}$$

Proof. Similar to the proof of [21, Prop. 3.1] and using (P2) it is straightforward to show that (4) holds. Hence $\operatorname{rk}\mathcal{C}\mathcal{B}(t) = pr$ for all $t \in \mathbb{R}$ since $\operatorname{rk}\Gamma L(t) = p$. Therefore, for all $t \in \mathbb{R}$,

$$\begin{aligned} &\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger = \big(\mathcal{C}\mathcal{B}(t)\big)^\top \Big(\mathcal{C}\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\top\Big)^{-1} \\ \Longrightarrow\ &\mathcal{C}\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger = I_{pr} \\ \Longrightarrow\ &\begin{bmatrix}\mathcal{C}\\ \mathcal{N}(t)\end{bmatrix}\Big[\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger, V\Big] = \begin{bmatrix} I_{pr} & 0 \\ 0 & I_{n-\rho}\end{bmatrix} \\ \Longrightarrow\ &n \ge \operatorname{rk} U(t) = \operatorname{rk}\begin{bmatrix}\mathcal{C}\\ \mathcal{N}(t)\end{bmatrix} = n - \rho + pr \\ \Longrightarrow\ &pr \le \rho = \operatorname{rk}\mathcal{C} \le pr. \end{aligned}$$

This shows $\rho = \operatorname{rk}\mathcal{C} = pr$ and (5). ☐

The matrix function U as in (3) will serve as a time-varying state space transformation in the following. Therefore, it will be important that U is a Lyapunov transformation.

Definition 2.2. We call a matrix function $M \in \mathcal{C}^1(\mathbb{R} \to \mathbf{Gl}_n(\mathbb{R}))$ a *Lyapunov transformation*, if M, M^{-1} and $\dot{M}$ are bounded.

By (3) and (5) it is straightforward to see that

$$\begin{aligned} &U(\cdot) \text{ and } U(\cdot)^{-1} \text{ are bounded} \\ \Longleftrightarrow\ &\mathcal{B}(\cdot)\big(\mathcal{C}\mathcal{B}(\cdot)\big)^\dagger \text{ is bounded.}\end{aligned} \tag{6}$$

Furthermore, $\dot{U}$ is bounded if, and only if, $\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{B}(\cdot)\big(\mathcal{C}\mathcal{B}(\cdot)\big)^\dagger\Big)$ is bounded; together with (6) we thus obtain a characterization for U being a Lyapunov transformation. A simpler condition, which however is only sufficient, is given in the next result.

Lemma 2.3. *Consider a system* (1) *with (P1) and (P2). Then U as in* (3) *is a Lyapunov transformation, if*

$$\exists\, \alpha > 0\ \forall\, t \in \mathbb{R}:\ \det\Big(\mathcal{C}\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\top\Big) \ge \alpha. \tag{7}$$

The converse implication is false in general.

Proof. First we show that a Lyapunov transformation U does not necessarily satisfy (7). To this end, consider $A = 0$, $B = C = 1$, $f = 0$ and $L(t) = \frac{1}{t^2+1}$. Then (P1) and (P2) are satisfied with $r = 1$ and $U = 1$ is a Lyapunov transformation, but (7) is not satisfied.

Now we show that (7) implies that U is a Lyapunov transformation. Since it follows from (P1) that $\mathcal{B}$ is bounded, we may infer from

$$\Big(\mathcal{C}\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\top\Big)^{-1} = \frac{\operatorname{adj}\Big(\mathcal{C}\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\top\Big)}{\det\Big(\mathcal{C}\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\top\Big)}$$

and (7) that $\Big(\mathcal{C}\mathcal{B}(\cdot)\big(\mathcal{C}\mathcal{B}(\cdot)\big)^\top\Big)^{-1}$ is bounded. Therefore, $\mathcal{B}(\cdot)\big(\mathcal{C}\mathcal{B}(\cdot)\big)^\dagger$ is bounded and hence it follows from (6) that U and U^{-1} are bounded. To show that $\dot{U}$ is bounded we compute

$$\begin{aligned}\dot{U}(t) &= \begin{bmatrix} 0 \\ \dot{\mathcal{N}}(t)\end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -V^\dagger \left[\dot{\mathcal{B}}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger\mathcal{C} + \mathcal{B}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger\right)\mathcal{C}\right]\end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger &= \big(\mathcal{C}\dot{\mathcal{B}}(t)\big)^\top\Big(\mathcal{C}\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\top\Big)^{-1} \\ &\quad + \big(\mathcal{C}\mathcal{B}(t)\big)^\top\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{C}\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\top\Big)^{-1}.\end{aligned}$$

Since $\mathcal{B}$ and $\dot{\mathcal{B}}$ are bounded by (P1) it follows that $\dot{U}$ is bounded, if $\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{C}\mathcal{B}(\cdot)\big(\mathcal{C}\mathcal{B}(\cdot)\big)^\top\Big)^{-1}$ is bounded. For any pointwise invertible $M \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{k\times k})$ we have

$$\begin{aligned}&\tfrac{\mathrm{d}}{\mathrm{d}t}M(t)^{-1} \\ &\quad = \frac{\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\operatorname{adj} M(t)\right)\det M(t) - \operatorname{adj} M(t)\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\det M(t)\right)}{\big(\det M(t)\big)^2}.\end{aligned}$$

For $M(t) = \mathcal{C}\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\top$ we observe that $\operatorname{adj} M$, $\det M$, $\tfrac{\mathrm{d}}{\mathrm{d}t}\operatorname{adj} M$ and $\tfrac{\mathrm{d}}{\mathrm{d}t}\det M$ are bounded by boundedness of $\mathcal{B}$ and $\dot{\mathcal{B}}$. Finally, (7) yields that $\tfrac{\mathrm{d}}{\mathrm{d}t}M(\cdot)^{-1}$ is bounded. □

Note that, in view of Lemma 2.1, $t \mapsto \det\Big(\mathcal{C}\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\top\Big)$ is a positive and smooth function and condition (7) only requires that, roughly speaking, it does not decay to zero for $t \to \infty$ (or $t \to -\infty$, if this is of relevance).

A crucial tool for the proof of the main result of this section is the following relation.

Lemma 2.4. *Consider a system* (1) *with (P1) and (P2) such that $q = p$ in (P1). Then*

$$\begin{aligned}\forall\, t \in \mathbb{R}:\quad &\mathcal{B}(t) - \mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger\mathcal{C}\mathcal{B}(t) \\ &= \Big(I_n - \mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger\mathcal{C}\Big)\mathcal{B}(t) = 0\end{aligned}\tag{8}$$

and, as a consequence, $\mathcal{N}(t)\mathcal{B}(t) = 0$ for all $t \in \mathbb{R}$.

Proof. First observe that by property (P1) and [30, Thm. 3.9] there exists $W \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n\times q})$ with $\operatorname{rk} W(t) = q$ for all $t \in \mathbb{R}$ and pointwise orthogonal $[R_1, R_2] \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{m\times(q+(m-q))})$ such that

$$\forall\, t \in \mathbb{R}:\ BL(t)[R_1(t), R_2(t)] = [W(t), 0].\tag{9}$$

Define, for all $t \in \mathbb{R}$,

$$\begin{aligned}\mathcal{W}(t) &:= \left[W(t), \left(\tfrac{\mathrm{d}}{\mathrm{d}t} - A\right)W(t), \ldots, \left(\tfrac{\mathrm{d}}{\mathrm{d}t} - A\right)^{r-1}W(t)\right], \\ \mathcal{R}(t) &:= \left(\mathcal{R}_{i,j}\right)_{i,j=1,\ldots,r}, \\ \mathcal{R}_{i,j}(t) &:= \begin{cases}\binom{j-1}{i-1}R_1^{(j-i)}(t)^\top, & j \ge i, \\ 0, & j < i.\end{cases}\end{aligned}$$

Step 1: Fix $t \in \mathbb{R}$. We show that $\mathcal{W}(t)\mathcal{R}(t) = \mathcal{B}(t)$ by proving that

$$\begin{aligned}\mathcal{W}_j(t) := \mathcal{W}(t)\mathcal{R}(t)\begin{bmatrix}0_{(j-1)m\times m} \\ I_m \\ 0_{(r-j)m\times m}\end{bmatrix} \\ = \mathcal{B}(t)\begin{bmatrix}0_{(j-1)m\times m} \\ I_m \\ 0_{(r-j)m\times m}\end{bmatrix} =: \mathcal{B}_j(t)\end{aligned}$$

for all $j = 1, \ldots, r$. For any $j \in \{1, \ldots, r\}$ we have

$$\mathcal{W}_j(t) = \sum_{i=1}^{j}\left[\left(\tfrac{\mathrm{d}}{\mathrm{d}t} - A\right)^{i-1}W(t)\right]\binom{j-1}{i-1}R_1^{(j-i)}(t)^\top$$

and invoking formula [21, (3.2)] with $C = I$ and $i = 0$ it follows that

$$\mathcal{W}_j(t) = \sum_{i=0}^{j-1}\sum_{k=0}^{i}(-1)^k\binom{i}{k}\binom{j-1}{i}A^kW^{(i-k)}(t)R_1^{(j-i-1)}(t)^\top.$$

Changing the summation over the "triangle" in the double sum we obtain

$$\begin{aligned}&\mathcal{W}_j(t) \\ &= \sum_{k=0}^{j-1}\sum_{i=k}^{j-1}(-1)^k\binom{i}{k}\binom{j-1}{i}A^kW^{(i-k)}(t)R_1^{(j-i-1)}(t)^\top \\ &= \sum_{k=0}^{j-1}\sum_{l=0}^{j-k-1}(-1)^k\binom{j-1}{l+k}\binom{l+k}{k}A^kW^{(l)}(t)R_1^{(j-l-k-1)}(t)^\top \\ &= \sum_{k=0}^{j-1}(-1)^k\binom{j-1}{k}\sum_{l=0}^{j-k-1}\binom{j-k-1}{l}A^kW^{(l)}(t)R_1^{(j-l-k-1)}(t)^\top \\ &= \sum_{k=0}^{j-1}(-1)^k\binom{j-1}{k}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{j-k-1}\left[A^kW(t)R_1(t)^\top\right],\end{aligned}$$

where we used

$$\binom{j-1}{l+k}\binom{l+k}{k} = \binom{j-1}{k}\binom{j-k-1}{l}.$$

Again using formula [21, (3.2)] we finally obtain

$$\begin{aligned}\mathcal{W}_j(t) &= (\tfrac{\mathrm{d}}{\mathrm{d}t} - A)^{j-1}\big[W(t)R_1(t)^\top\big] \\ &= (\tfrac{\mathrm{d}}{\mathrm{d}t} - A)^{j-1}\big[BL(t)\big] = \mathcal{B}_j(t).\end{aligned}$$

Step 2: We show that $\operatorname{rk}\mathcal{B}(t) = pr$ for all $t \in \mathbb{R}$. It follows from (P2) and (9) that $CA^kW(t) = 0$ for $k = 0,\ldots,r-2$ and $\operatorname{rk} CA^{r-1}W(t) = p$ for all $t \in \mathbb{R}$, by which, using $q = p$, $CA^{r-1}W(t) \in \mathbf{Gl}_p(\mathbb{R})$. Therefore, [21, Cor. 3.3] implies that $\operatorname{rk}\mathcal{W}(t) = pr$ for all $t \in \mathbb{R}$. Furthermore, it is clear that $\operatorname{rk}\mathcal{R}(t) = pr$ for all $t \in \mathbb{R}$ and hence we may infer from Sylvester's rank inequality that

$$\begin{aligned}pr = \operatorname{rk}\mathcal{W}(t) + \operatorname{rk}\mathcal{R}(t) - pr &\le \operatorname{rk}\mathcal{W}(t)\mathcal{R}(t) \\ &\le \min\{\operatorname{rk}\mathcal{W}(t), \operatorname{rk}\mathcal{R}(t)\} = pr.\end{aligned}$$

This shows $\operatorname{rk}\mathcal{B}(t) = \operatorname{rk}\mathcal{W}(t)\mathcal{R}(t) = pr$ for all $t \in \mathbb{R}$.

Step 3: We show the assertion of the lemma. Since $\mathcal{B}$ has constant rank we may again apply [30, Thm. 3.9] to find $Y \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n\times pr})$ with $\operatorname{rk} Y(t) = pr$ for all $t \in \mathbb{R}$ and pointwise orthogonal $[V_1, V_2] \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{rm\times(pr+(m-p)r)})$ such that

$$\forall\, t \in \mathbb{R}:\ \mathcal{B}(t)[V_1(t), V_2(t)] = [Y(t), 0].$$

Then using that $\mathcal{B}(t) = Y(t)V_1(t)^\top$ and that $[V_1, V_2]$ is pointwise orthogonal, by which $V_1(t)^\top V_1(t) = I_{pr}$ for all $t \in \mathbb{R}$, we obtain

$$\begin{aligned}&\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger \\ &= \mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\top\Big(\mathcal{C}\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\top\Big)^{-1} \\ &= Y(t)V_1(t)^\top V_1(t)\big(\mathcal{C}Y(t)\big)^\top\Big(\mathcal{C}Y(t)V_1(t)^\top V_1(t)\big(\mathcal{C}Y(t)\big)^\top\Big)^{-1} \\ &= Y(t)\big(\mathcal{C}Y(t)\big)^\dagger.\end{aligned}$$

Clearly, $\operatorname{rk}\mathcal{C}Y(t) = \operatorname{rk}\mathcal{C}\mathcal{B}(t)[V_1(t), V_2(t)] = \operatorname{rk}\mathcal{C}\mathcal{B}(t) = pr$ and hence $\mathcal{C}Y(t) \in \mathbf{Gl}_{pr}(\mathbb{R})$ for all $t \in \mathbb{R}$. Therefore, it finally follows that

$$\begin{aligned}\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger\mathcal{C}\mathcal{B}(t) &= Y(t)\big(\mathcal{C}Y(t)\big)^{-1}\mathcal{C}Y(t)V_1(t)^\top \\ &= Y(t)V_1(t)^\top = \mathcal{B}(t).\end{aligned}$$

□

We are now in the position to state the main result on the time-varying normal form.

Theorem 2.5. *Consider a system* (1) *with (P1) and (P2) such that U as in* (3) *is a Lyapunov transformation. Then*

$$(\hat{A}, \hat{B}, \hat{C}) := \Big((UA + \dot{U})U^{-1}, UBL, CU^{-1}\Big) \tag{10}$$

and

$$\hat{f}(t,z,u) := U(t)f\big(t, U(t)^{-1}z, u\big), \quad (t,z,u) \in \mathbb{R}^{1+n+m}$$

satisfy

$$\begin{aligned}\hat{A}(t) &= \begin{bmatrix} 0 & I_p & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_p & & & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & I_p & 0 \\ R_1(t) & R_2(t) & \cdots & R_{r-1}(t) & R_r(t) & S(t) \\ P_1(t) & P_2(t) & \cdots & P_{r-1}(t) & P_r(t) & Q(t) \end{bmatrix}, \\ \hat{B}(t) &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Gamma L(t) \\ N(t) \end{bmatrix}, \qquad \hat{f}(t,z,u) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f_r(t,z,u) \\ f_\eta(t,z,u) \end{pmatrix}, \\ \hat{C} &= \begin{bmatrix} I_p & 0 & \cdots & 0 \end{bmatrix},\end{aligned} \tag{11}$$

where $R_i \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{p\times p})$, $P_i, S^\top \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{(n-pr)\times p})$, $Q \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{(n-pr)\times(n-pr)})$, $N \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{(n-pr)\times m})$, $f_r \in \mathcal{C}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^p)$, $f_\eta \in \mathcal{C}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^{n-pr})$ are all bounded. Furthermore, the following holds true:

(i) *There exists $K \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{m\times p})$ such that $\Gamma L(t)K(t) = I_p$ and $N(t)K(t) = 0$ for all $t \in \mathbb{R}$ if, and only if,*

$$\forall\, t \in \mathbb{R}:\ \operatorname{im}\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger\begin{bmatrix}0\\ I_p\end{bmatrix} \subseteq \operatorname{im} BL(t); \tag{12}$$

in this case we may choose

$$K(t) := \big(BL(t)\big)^\dagger\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger\begin{bmatrix}0\\ I_p\end{bmatrix}. \tag{13}$$

(ii) *If $q = p$ for q in (P1), then we have*

$$P_2 = P_3 = \ldots = P_r = 0 \quad \text{and} \quad N = 0.$$

Proof. By the choice of U as in (3) and the representation of its inverse as in Lemma 2.1 it follows immediately that $\hat{A}, \hat{B}, \hat{C}, \hat{f}$ have the structure as in the statement of the theorem, cf. also [21, Thm. 3.5]. Boundedness of all entries follows from the fact that U is a Lyapunov transformation. It remains to show (i) and (ii).

(i): The existence of K with the mentioned properties is equivalent to

$$\begin{aligned}&\hat{B}(t)K(t) = \begin{bmatrix}0_{p(r-1)\times p} \\ I_p \\ 0_{(n-pr)\times p}\end{bmatrix} \\ \overset{(10)}{\iff}\quad &BL(t)K(t) = U(t)^{-1}\begin{bmatrix}0_{p(r-1)\times p} \\ I_p \\ 0_{(n-pr)\times p}\end{bmatrix} \\ &\overset{(5)}{=} \mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger\begin{bmatrix}0_{p(r-1)\times p} \\ I_p\end{bmatrix}\end{aligned} \tag{14}$$

for some $K \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{m\times p})$. Clearly, (14) implies (12). Conversely, if (12) holds, then

$$\exists\, X : \mathbb{R} \to \mathbb{R}^{m\times pr} :\ \mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger \begin{bmatrix} 0 \\ I_p \end{bmatrix} = BL(t)X(t) \quad (15)$$

and hence K as in (13) satisfies

$$\begin{aligned} BL(t)K(t) &\overset{(13)}{=} BL(t)\big(BL(t)\big)^\dagger \mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger \begin{bmatrix} 0 \\ I_p \end{bmatrix} \\ &\overset{(15)}{=} BL(t)\big(BL(t)\big)^\dagger BL(t)X(t) \\ &= BL(t)X(t) \overset{(15)}{=} \mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger \begin{bmatrix} 0 \\ I_p \end{bmatrix}. \end{aligned}$$

In view of (14), this finishes the proof of (i).

(ii): If $q = p$, then Lemma 2.4 yields that $\mathcal{N}(t)\mathcal{B}(t) = 0$ for all $t \in \mathbb{R}$. This implies

$$\begin{aligned} N(t) &\overset{(10)}{=} [0, I_{n-pr}]U(t)BL(t) \\ &\overset{(3)}{=} \mathcal{N}(t)BL(t) = \mathcal{N}(t)\mathcal{B}(t)\begin{bmatrix} I_m \\ 0 \end{bmatrix} = 0 \end{aligned}$$

for all $t \in \mathbb{R}$. It remains to show that $P_2 = \ldots = P_r = 0$. Fix $t \in \mathbb{R}$ and note that

$$\begin{aligned} &[P_1(t), \ldots, P_r(t), Q(t)] \\ &\overset{(11)}{=} [0, I_{n-pr}]\hat{A}(t) \\ &\overset{(10)}{=} [0, I_{n-pr}](U(t)A + \dot{U}(t))U(t)^{-1} \\ &\overset{(3),(5)}{=} \big(\mathcal{N}(t)A + \dot{\mathcal{N}}(t)\big)[\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger, V], \end{aligned}$$

hence

$$\begin{aligned} &[P_1(t), \ldots, P_r(t)]\mathcal{C}\mathcal{B}(t) \\ &= \big(\mathcal{N}(t)A + \dot{\mathcal{N}}(t)\big)\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger \mathcal{C}\mathcal{B}(t) \\ &\overset{(8)}{=} \big(\mathcal{N}(t)A + \dot{\mathcal{N}}(t)\big)\mathcal{B}(t). \end{aligned}$$

Since $\mathcal{N}(\cdot)\mathcal{B}(\cdot) = 0$ we find that $\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{N}(\cdot)\mathcal{B}(\cdot)\big) = 0$, hence

$$\dot{\mathcal{N}}(t)\mathcal{B}(t) = -\mathcal{N}(t)\dot{\mathcal{B}}(t).$$

Therefore, we have

$$\begin{aligned} &[P_1(t), \ldots, P_r(t)]\mathcal{C}\mathcal{B}(t) \\ &= \big(\mathcal{N}(t)A + \dot{\mathcal{N}}(t)\big)\mathcal{B}(t) \\ &= -\mathcal{N}(t)\big(\tfrac{\mathrm{d}}{\mathrm{d}t} - A\big)\big(\mathcal{B}(t)\big) \\ &= -\mathcal{N}(t)\left[\big(\tfrac{\mathrm{d}}{\mathrm{d}t} - A\big)\big(BL(t)\big), \ldots, \big(\tfrac{\mathrm{d}}{\mathrm{d}t} - A\big)^r\big(BL(t)\big)\right] \\ &\overset{\mathcal{N}(t)\mathcal{B}(t)=0}{=} \left[0, \ldots, 0, -\mathcal{N}(t)\big(\tfrac{\mathrm{d}}{\mathrm{d}t} - A\big)^r\big(BL(t)\big)\right]. \end{aligned}$$

We may infer, using (4), that $P_r(t)\Gamma L(t) = 0$, hence $P_r(t) = 0$ since $\Gamma L(t)$ has full row rank p by (P2). Successively, we thus obtain that $P_{r-1}(t) = \ldots = P_2(t) = 0$ and this finishes the proof of the theorem. □

Using that the time-varying Byrnes-Isidori form from [21] has a certain uniqueness property as derived in [6, Thm. B.7], with the same proof we obtain the following result.

Corollary 2.6. *Consider a system* (1) *with (P1) and (P2) such that $q = p$ for q in (P1). Then uniqueness of the entries in the normal form* (11) *holds as follows:*

(i) *the entries $[R_1, \ldots, R_r] = CA^r\mathcal{B}\big(\mathcal{C}\mathcal{B}\big)^\dagger$ are uniquely defined;*

(ii) *the time-varying linear (sub-)system (Q, P_1, S) is unique up to $\big((WQ + \dot{W})W^{-1}, WP_1, SW^{-1}\big)$ for any Lyapunov transformation $W \in \mathcal{C}^\infty(\mathbb{R} \to$ $\mathbf{Gl}_{n-rm}(\mathbb{R}))$.*

We stress that the uniqueness property from Corollary 2.6 is not true for $q > p$ in general. That Q in $\hat{A}$ is *not* unique up to a Lyapunov transformation in general is shown in the following Example 2.7. In the case $q > p$ it is thus important to follow exactly the construction procedure which leads to the transformation U in (3).

Example 2.7. Consider a system (1) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}, \quad C = [1, 0]$$

and $f = 0$, $L = I_2$. Then we have $q = 2 > 1 = p$, $r = 1$ and (A, B, C) is already in the normal form (11) with $Q = 1$. However, if we compute U as in (3), then we obtain

$$U = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad U^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix},$$

and hence

$$\hat{A} = \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \hat{C} = C.$$

In particular, the new Q-block is given by $\hat{Q} = -1$, which cannot be obtained by a Lyapunov transformation of Q: If there would exists $W \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R} \setminus \{0\})$ with $\hat{Q} = \big(W(t)Q + \dot{W}(t)\big)W(t)^{-1}$, then $-1 = 1 + \frac{\dot{W}(t)}{W(t)}$ which means $\dot{W}(t) = -2W(t)$. Since $W(0) \neq 0$ it follows that $W(t) = e^{-2t}W(0)$, but then W^{-1} is not bounded and hence no Lyapunov transformation.

We also stress that condition (12) is not always satisfied. A counterexample is given in the following.

Example 2.8. Consider a system (1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = [1, 0, 0, 0]$$

and $f = 0$, $L = I_2$. Then we have $q = 2 > 1 = p$ and $r = 2$ with $\Gamma = CAB = [1, 0]$. We compute

$$(\mathcal{CB})^\dagger = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \\ -1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

and hence

$$\operatorname{im} \mathcal{B}(\mathcal{CB})^\dagger \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \operatorname{im} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{1}{2} \end{bmatrix} \not\subseteq \operatorname{im} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \operatorname{im} B,$$

thus condition (12) is not satisfied for this system. And indeed, computing

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

we obtain

$$\hat{B} = UB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ -\frac{1}{2} & 0 \end{bmatrix}$$

with $N = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{bmatrix}$, where obviously the conditions $\Gamma K(t) = 1$ and $NK(t) = 0$ cannot be satisfied for some K.

Remark 2.9. For the construction of the fault tolerant funnel controller in Section 4, the first step is to apply a rectangular input transformation $u(t) = K(t)v(t)$, where $K \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{m \times p})$ is such that $\Gamma L(t)K(t)$ is invertible and $N(t)K(t) = 0$; the latter is important since in the resulting system the so called zero dynamics (see Section 3) need to be independent of the new input v. Existence of such a class of matrices K is actually characterized by condition (12) since an additional invertible transformation does not change this condition: Assuming existence of $K \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{m \times p})$ and $G \in \mathcal{C}^\infty(\mathbb{R} \to \mathbf{Gl}_p(\mathbb{R}))$ such that $\Gamma L(t)K(t) = G(t)$ and $N(t)K(t) = 0$ for all $t \in \mathbb{R}$ leads to $\Gamma L(t)F(t) = I_p$ and $N(t)F(t) = 0$ for $F(t) = K(t)G(t)^{-1}$, $t \in \mathbb{R}$. We stress that by Theorem 2.5, clearly (12) is always satisfied in the case $q = p$.

Example 2.10. As a running example we consider a linearized model for the lateral motion of a Boeing 737 aircraft, which is taken from [38, Sec. 5.4]. The model is of the form (1) with $n = 5$, $m = 4$, $p = 2$,

$$A = \begin{bmatrix} -0.13858 & 14.326 & -219.04 & 32.167 & 0 \\ -0.02073 & -2.1692 & 0.91315 & 0.000256 & 0 \\ 0.00289 & -0.16444 & -0.15768 & -0.00489 & 0 \\ 0 & 1 & 0.00618 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.15935 & 0.15935 & 0.00211 & 0.00211 \\ 0.01264 & 0.01264 & 0.21326 & 0.21326 \\ -0.12879 & -0.12879 & 0.00171 & 0.00171 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$x = (v_b, p_b, r_b, \phi, \psi)^\top, \quad u = (d_{r1}, d_{r2}, d_{a1}, d_{a2})^\top.$$

Here v_b denotes the lateral velocity, p_b the roll rate, r_b the yaw rate, ϕ the roll angle and ψ the yaw angle. The roll angle ϕ and the yaw angle ψ are chosen as outputs of the system. The inputs consist of the rudder position $d_{r1} + d_{r2}$ and the aileron position $d_{a1} + d_{a2}$, so we see that we have two groups of actuators which are both double redundant.

For this example we assume that no faults occur in the actuators d_{r1} and d_{a1}, while d_{r2} and d_{a2} experience some faults, i.e.,

$$L(t) = \operatorname{diag}\big(1, l_2(t), 1, l_4(t)\big), \quad f(t, u) = B \begin{pmatrix} 0 \\ f_2(t, u_2) \\ 0 \\ f_4(t, u_4) \end{pmatrix}$$

where $u = (u_1, u_2, u_3, u_4)^\top$, for some bounded $l_2, l_4 \in \mathcal{C}^\infty(\mathbb{R} \to [0, 1])$ (with bounded derivatives) and bounded $f_2, f_4 \in \mathcal{C}(\mathbb{R}^2 \to \mathbb{R})$ to be specified later.

We compute the transformation matrix U as in (3) and the normal form (11). Since $CB = 0$ and

$$\Gamma = CAB = \begin{bmatrix} 0.01184 & 0.01184 & 0.21327 & 0.21327 \\ -0.12879 & -0.12879 & 0.00171 & 0.00171 \end{bmatrix}$$

has full row rank, we find that the relative degree of the system is $r = 2$. Using MATLAB we calculate that, approximately,

$$U = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0.00618 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -0.0198 & 1.23534 & -14.16314 & 219.17412 \end{bmatrix},$$

$$\hat{A} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -0.29312 & 4.53957 & -2.17063 & 0.95118 & -0.02071 \\ 0.03604 & -0.63341 & -0.16438 & -0.16023 & 0.00289 \\ 30.2546 & 29.50071 & 0 & 0 & -0.1346 \end{bmatrix},$$

$$\hat{B}(t) = \begin{bmatrix} 0 \\ \Gamma L(t) \\ 0 \end{bmatrix}, \quad \hat{C} = [I_2, 0], \quad \hat{f}(t, u) = \begin{bmatrix} 0 \\ \Gamma \\ 0 \end{bmatrix} \begin{pmatrix} 0 \\ f_2(t, u_2) \\ 0 \\ f_4(t, u_4) \end{pmatrix}.$$

In particular, U and $\hat{A}$ are time-invariant and independent of l_2 and l_4, and U is a Lyapunov transformation.

3. The zero dynamics

An important system property in high-gain based adaptive control is a bounded-input, bounded-output property of the internal dynamics of the system, see e.g. [5, 22, 24]. In the case of linear time-invariant systems, this is implied by (but not equivalent to) asymptotic stability of the zero dynamics of the system; the latter property is extensively studied in the literature, see [10, 33, 34], and commonly known as the *minimum phase* property, although this is not completely correct, see [28] and the references therein.

For time-varying linear systems

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t),\\ y(t) &= C(t)x(t),\end{aligned} \tag{16}$$

denoted by $\big(A(\cdot), B(\cdot), C(\cdot)\big)$, where $A \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n\times n})$, $B, C^\top \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{n\times p})$ the zero dynamics are the set of those solution trajectories (x,u) of (16) where the output vanishes, i.e.,

$$\mathcal{ZD}_{(A,B,C)} := \left\{ (x,u) \in \mathcal{C}^1(\mathbb{R} \to \mathbb{R}^{n+p}) \;\middle|\; \begin{array}{l}(x,u) \text{ satisfies (16)}\\ \text{and } y(t)=0\\ \text{for all } t \in \mathbb{R}\end{array} \right\}$$

In particular, the zero dynamics are a *behavior* of a time-varying linear system, see e.g. [6, 20]. Similarly, the behavior of the homogeneous time-varying linear differential equation $\dot{x}(t) = A(t)x(t)$ is given by

$$\mathfrak{B}_A := \left\{ x \in \mathcal{C}^1(\mathbb{R} \to \mathbb{R}^n) \;\middle|\; \dot{x}(t) = A(t)x(t) \text{ for all } t \in \mathbb{R} \right\}.$$

We use the following definition from [6, Def. 5.1].

Definition 3.1. Let $\mathfrak{B} \subseteq \mathcal{C}^1(\mathbb{R} \to \mathbb{R}^q)$ be a linear behavior, i.e., for any $w_1, w_2 \in \mathfrak{B}$ and $\alpha \in \mathbb{R}$ we have $\alpha w_1 + w_2 \in \mathfrak{B}$. Then $\mathfrak{B}$ is called *uniformly exponentially stable*, if there exist $M, \mu > 0$ such that

$$\forall\, w \in \mathfrak{B}\ \forall\, t \ge t_0 \ge 0 : \ \|w(t)\| \le M e^{-\mu(t-t_0)} \|w(t_0)\|.$$

The usual assumption on system (1) would be that the zero dynamics of the linear part (ignoring the bounded nonlinearity) are uniformly exponentially stable. However, for system (1) with $f = 0$, a fixed output $y \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^p)$ does not uniquely define (up to initial values) a corresponding state x and input u since the actuator redundancy leads to $BL(t)$ not having full column rank in general. In other words, the zero dynamics are not necessarily *autonomous*, cf. [6]. To circumvent this problem we apply a rectangular input transformation $u(t) = K(t)G(t)^{-1}v(t)$ to system (1) for K and G such that $\Gamma L(t)K(t) = G(t)$ and $N(t)K(t) = 0$, cf. Remark 2.9. Since

$$\begin{aligned} BL(t)u(t) &= BL(t)K(t)G(t)^{-1}v(t)\\ &= \mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger \begin{bmatrix}0\\ I_p\end{bmatrix} v(t)\end{aligned}$$

by Theorem 2.5, this leads to the time-varying linear system $\Big(A, \mathcal{B}(\cdot)\big(\mathcal{C}\mathcal{B}(\cdot)\big)^\dagger \left[\begin{smallmatrix}0\\ I_p\end{smallmatrix}\right], C\Big)$ and hence we may assume that the zero dynamics $\mathcal{ZD}_{\left(A,\mathcal{B}(\cdot)\left(\mathcal{C}\mathcal{B}(\cdot)\right)^\dagger \left[\begin{smallmatrix}0\\ I_p\end{smallmatrix}\right],C\right)}$ are a uniformly exponentially stable behavior; this assumption is independent of the existence of K and G.

Proposition 3.2. *Consider a system* (1) *with (P1) and (P2) such that U as in* (3) *is a Lyapunov transformation and $(\hat{A}, \hat{B}, \hat{C}, \hat{f})$ are as in Theorem 2.5. Then*

$$\begin{aligned}&\mathcal{ZD}_{\left(A,\mathcal{B}(\cdot)\left(\mathcal{C}\mathcal{B}(\cdot)\right)^\dagger \left[\begin{smallmatrix}0\\ I_p\end{smallmatrix}\right],C\right)}\\ &= \left\{ (V\eta, -S\eta) \;\middle|\; \begin{array}{l}\eta \in \mathcal{C}^1(\mathbb{R} \to \mathbb{R}^{n-rp}) \text{ solves}\\ \dot{\eta}(t) = Q(t)\eta(t)\end{array}\right\}\end{aligned}$$

and, furthermore, $\mathcal{ZD}_{\left(A,\mathcal{B}(\cdot)\left(\mathcal{C}\mathcal{B}(\cdot)\right)^\dagger \left[\begin{smallmatrix}0\\ I_p\end{smallmatrix}\right],C\right)}$ *are uniformly exponentially stable if, and only if, $\mathfrak{B}_Q$ is uniformly exponentially stable.*

Proof. Since

$$U(t)\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger \begin{bmatrix}0\\ I_p\end{bmatrix} = \begin{bmatrix}0_{(r-1)p\times p}\\ I_p\\ 0_{(n-rp)\times p}\end{bmatrix}$$

it follows from Theorem 2.5 that

$$\begin{aligned} &(x,u) \in \mathcal{ZD}_{\left(A,\mathcal{B}(\cdot)\left(\mathcal{C}\mathcal{B}(\cdot)\right)^\dagger \left[\begin{smallmatrix}0\\ I_p\end{smallmatrix}\right],C\right)}\\ \iff\ & (Ux,u) \in \mathcal{ZD}_{\left(\hat{A}, \left[\begin{smallmatrix}0_{(r-1)p\times p}\\ I_p\\ 0_{(n-rp)\times p}\end{smallmatrix}\right], \hat{C}\right)}\\ \iff\ & x = V\eta \ \wedge\ u = -S\eta \ \wedge\ \dot{\eta} = Q\eta\end{aligned}$$

from which the representation of the zero dynamics follows. Since U is a Lyapunov transformation, and hence in particular S is bounded, we may infer that $\mathcal{ZD}_{\left(A,\mathcal{B}(\cdot)\left(\mathcal{C}\mathcal{B}(\cdot)\right)^\dagger \left[\begin{smallmatrix}0\\ I_p\end{smallmatrix}\right],C\right)}$ are uniformly exponentially stable if, and only if, $\mathfrak{B}_Q$ is uniformly exponentially stable. ☐

Remark 3.3. If $L(\cdot)$ is constant, then $U \in \mathbf{Gl}_n(\mathbb{R})$ and $\hat{A}, \hat{B}, \hat{C}$ in Theorem 2.5 are constant matrices. In this case, we obtain the following criterion for the (uniform) exponential stability of the zero dynamics from [3, Lem. 4.3.9]: $\mathcal{ZD}_{\left(A,\mathcal{B}\left(\mathcal{C}\mathcal{B}\right)^\dagger \left[\begin{smallmatrix}0\\ I_p\end{smallmatrix}\right],C\right)}$ are exponentially stable (equivalently, $\sigma(Q) \subseteq \mathbb{C}_-$) if, and only if,

$$\forall\, \lambda \in \overline{\mathbb{C}_+} : \ \det \begin{bmatrix} A - \lambda I_n & \mathcal{B}(\mathcal{C}\mathcal{B})^\dagger \left[\begin{smallmatrix}0\\ I_p\end{smallmatrix}\right]\\ C & 0\end{bmatrix} \ne 0. \tag{17}$$

Example 3.4. We revisit the model of a Boeing 737 aircraft from Example 2.10. We calculate that

$$\mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger \begin{bmatrix} 0 \\ I_p \end{bmatrix} = \begin{bmatrix} 0.0198 & -1.23546 \\ 1 & -0.00618 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is time-invariant and in particular independent of l_2 and l_4. We show that $\mathcal{ZD}_{\left(A,\mathcal{B}(\cdot)\left(\mathcal{C}\mathcal{B}(\cdot)\right)^\dagger \left[\begin{smallmatrix} 0 \\ I_p \end{smallmatrix}\right], C\right)}$ are uniformly exponentially stable. Since all involved matrices are time-invariant we may use the criterion from Remark 3.3 and calculate that, approximately,

$$\det \begin{bmatrix} A - \lambda I_n & \mathcal{B}(\mathcal{C}\mathcal{B})^\dagger \left[\begin{smallmatrix} 0 \\ I_p \end{smallmatrix}\right] \\ C & 0 \end{bmatrix} = -(\lambda + 0.1346) =: p(\lambda).$$

Therefore, condition (17) is satisfied and hence $\mathcal{ZD}_{\left(A,\mathcal{B}(\cdot)\left(\mathcal{C}\mathcal{B}(\cdot)\right)^\dagger \left[\begin{smallmatrix} 0 \\ I_p \end{smallmatrix}\right], C\right)}$ are uniformly exponentially stable. This is also a validation of Proposition 3.2: From Example 2.10 we find that Q as in Theorem 2.5 is given by

$$Q = -0.1346 \in \mathbb{C}_-$$

for Q as in Theorem 2.5, hence also $p(\lambda) = -(\lambda - Q)$.

4. Fault tolerant control

4.1. Preliminaries

Before we introduce a fault tolerant funnel controller we need some preliminary results for nonlinear systems with arbitrary known relative degree and equal number of inputs and outputs without any faults. A funnel controller for such systems has been developed in [5]. This controller is of the form

$$\boxed{\begin{aligned} e_0(t) &= e(t) = y(t) - y_{\mathrm{ref}}(t), \\ e_1(t) &= \dot e_0(t) + k_0(t)\, e_0(t), \\ e_2(t) &= \dot e_1(t) + k_1(t)\, e_1(t), \\ &\ \ \vdots \\ e_{r-1}(t) &= \dot e_{r-2}(t) + k_{r-2}(t)\, e_{r-2}(t), \\ k_i(t) &= \frac{1}{1 - \varphi_i(t)^2 \|e_i(t)\|^2}, \quad i = 0, \ldots, r-1, \end{aligned}} \tag{18}$$

with feedback law

$$u(t) = -k_{r-1}(t)\, e_{r-1}(t), \tag{19}$$

where the reference signal and funnel functions satisfy:

$$\begin{aligned} &y_{\mathrm{ref}} \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m), \\ &\varphi_0 \in \Phi_r, \ \ \varphi_1 \in \Phi_{r-1}, \ \ldots, \ \ \varphi_{r-1} \in \Phi_1. \end{aligned} \tag{20}$$

The controller (18), (19) is shown to be feasible for a large class of nonlinear systems of the form

$$\begin{aligned} y^{(r)}(t) &= f\big(d(t), T(y, \dot y, \ldots, y^{(r-1)})(t)\big) \\ &\quad + \Gamma\big(d(t), T(y, \dot y, \ldots, y^{(r-1)})(t)\big)\, u(t) \\ y|_{[-h,0]} &= y^0 \in \mathcal{W}^{r-1,\infty}([-h,0] \to \mathbb{R}^m), \end{aligned} \tag{21}$$

where $h > 0$ is the "memory" of the system, $r \in \mathbb{N}$ is the strict relative degree, and

(N1) the disturbance satisfies $d \in \mathcal{L}^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^p)$, $p \in \mathbb{N}$;

(N2) $f \in \mathcal{C}(\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^m)$, $q \in \mathbb{N}$,

(N3) the high-frequency gain matrix function $\Gamma \in \mathcal{C}(\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^{m \times m})$ satisfies $\Gamma(d,\eta) + \Gamma(d,\eta)^\top > 0$ for all $(d,\eta) \in \mathbb{R}^p \times \mathbb{R}^q$;[2]

(N4) $T : \mathcal{C}([-h,\infty) \to \mathbb{R}^{rm}) \to \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\ge 0} \to \mathbb{R}^q)$ is an operator with the following properties:

a) T maps bounded trajectories to bounded trajectories, i.e, for all $c_1 > 0$, there exists $c_2 > 0$ such that for all $\zeta \in \mathcal{C}([-h,\infty) \to \mathbb{R}^{rm})$,

$$\sup_{t \in [-h,\infty)} \|\zeta(t)\| \le c_1 \Rightarrow \sup_{t \in [0,\infty)} \|T(\zeta)(t)\| \le c_2,$$

b) T is causal, i.e, for all $t \ge 0$ and all $\zeta, \xi \in \mathcal{C}([-h,\infty) \to \mathbb{R}^{rm})$,

$$\zeta|_{[-h,t)} = \xi|_{[-h,t)} \Rightarrow T(\zeta)|_{[0,t)} \overset{\text{a.a.}}{=} T(\xi)|_{[0,t)},$$

where "a.a." stands for "almost all".

c) T is locally Lipschitz continuous in the following sense: for all $t \ge 0$ there exist $\tau, \delta, c > 0$ such that for all $\zeta, \Delta\zeta \in \mathcal{C}([-h,\infty) \to \mathbb{R}^{rm})$ with $\Delta\zeta|_{[-h,t)} = 0$ and $\|\Delta\zeta|_{[t,t+\tau]}\|_\infty < \delta$ we have

$$\big\|(T(\zeta + \Delta\zeta) - T(\zeta))\,|_{[t,t+\tau]}\big\|_\infty \le c\|\Delta\zeta|_{[t,t+\tau]}\|_\infty.$$

In [18, 23, 24, 25] it is shown that the class of systems (21) encompasses linear and nonlinear systems with strict relative degree and input-to-state stable internal dynamics and that the operator T allows for infinite-dimensional linear systems, systems with hysteretic effects or nonlinear delay elements, and combinations thereof.

In [5], the existence of solutions of the initial value problem resulting from the application of the funnel controller (18), (19) to a system (21) is investigated. By a *solution* of (18)–(21) on $[-h,\omega)$ we mean a function $y \in \mathcal{C}^{r-1}([-h,\omega) \to \mathbb{R}^m)$, $\omega \in (0,\infty]$, with $y|_{[-h,0]} = y^0$ such that $y^{(r-1)}|_{[0,\omega)}$ is weakly differentiable and satisfies the differential equation in (21) with u defined in (19) for almost all $t \in [0,\omega)$; y is called *maximal*, if it has no right extension that is also a solution. Existence of solutions of functional differential equations has been investigated in [24] for instance.

The following result is from [5].

[2] This is a slightly less restrictive assumption than in [5], but the proofs do not change.

Theorem 4.1. *Consider a system* (21) *with strict relative degree* $r \in \mathbb{N}$ *and properties (N1)–(N4). Let* $y_{\rm ref}$ *and* $\varphi_0, \dots, \varphi_{r-1}$ *be as in* (20) *and* $y|_{[-h,0]} = y^0 \in \mathcal{W}^{r-1,\infty}([-h,0] \to \mathbb{R}^m)$ *be an initial trajectory such that* $e_0, \dots, e_{r-1}$ *as defined in* (18) *satisfy*

$$\varphi_i(0)\|e_i(0)\| < 1 \quad \text{for } i = 0, \dots, r-1.$$

Then the funnel controller (18), (19) *applied to* (21) *yields an initial-value problem which has a solution, and every solution can be extended to a maximal solution* $y : [-h,\omega) \to \mathbb{R}^m$, $\omega \in (0,\infty]$, *which has the following properties:*

(i) *The solution is global (i.e.,* $\omega = \infty$*).*

(ii) *The input* $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, *the gain functions* $k_0, \dots, k_{r-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ *and* $y, \dots, y^{(r-1)} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ *are bounded.*

(iii) *The functions* $e_0, \dots, e_{r-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ *are uniformly bounded away from the funnel boundaries in the following sense:*

$$\forall\, i = 0, \dots, r-1 \;\exists\, \varepsilon_i > 0 \;\forall\, t > 0 : \quad \|e_i(t)\| \leq \varphi_i(t)^{-1} - \varepsilon_i. \tag{22}$$

4.2. Controller structure

We introduce the fault tolerant funnel controller for systems of type (1) as an extension of the controller (18), (19) where we only change the feedback law (19). That is, the fault tolerant funnel controller consists of (18) together with the new feedback law

$$\boxed{u(t) = -k_{r-1}(t)\, K(t)\, e_{r-1}(t),} \tag{23}$$

where the reference signal and funnel functions have the following properties:

$$\boxed{\begin{aligned} &y_{\rm ref} \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^p),\\ &\varphi_0 \in \Phi_r,\;\; \varphi_1 \in \Phi_{r-1},\; \dots,\;\; \varphi_{r-1} \in \Phi_1. \end{aligned}} \tag{24}$$

We choose the bounded controller weight matrix function $K \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{m \times p})$, if possible, such that

$$\boxed{\begin{aligned} &\exists\, \alpha > 0 :\; \Gamma L(t)K(t) + \big(\Gamma L(t)K(t)\big)^\top \geq \alpha I_p\\ &\text{and}\quad N(t)K(t) = 0, \end{aligned}} \tag{25}$$

where we use the notation from Theorem 2.5. Note that condition (25) is not always satisfied under the assumptions (P1) and (P2). Existence and possible choices for K are discussed in Subsection 4.4. The first condition in (25) is required to meet assumption (N3) after a reformulation of the closed-loop system; the second condition is important to make the zero dynamics of (1) with the input transformation $u(t) = K(t)v(t)$ independent of the action of the new input v, cf. Remark 2.9 and Section 3.

We stress that (23) can be interpreted as (19) multiplied with the controller weight $K(t)$. The application of the controller (18), (23) to a system (1) is illustrated in Figure 3.

In the sequel we investigate existence of solutions of the initial value problem resulting from the application of the funnel controller (18), (23) to a system (1). Even if (1) is a linear system with $f = 0$ and $L = I_m$, some care must be exercised with the existence of a solution of (1), (18), (23) since this closed-loop differential equation is time-varying, nonlinear and only defined on an open subset of $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$. By a *solution* of (1), (18), (23) on $[0,\omega)$ we mean a weakly differentiable function $x : [0,\omega) \to \mathbb{R}^n$, $\omega \in (0,\infty]$, which satisfies $x(0) = x^0$ and the differential equation in (1) with u defined in (18), (23) for almost all $t \in [0,\omega)$; x is called *maximal*, if it has no right extension that is also a solution.

4.3. Feasibility of the controller

We show feasibility of the controller (18), (23) for every system (1) which satisfies the assumptions (P1), (P2) and

(P3) U as in (3) is a Lyapunov transformation (this is satisfied if (7) holds),

(P4) $\mathcal{ZD}_{\left(A,\mathcal{B}(\cdot)\left(\mathcal{CB}(\cdot)\right)^\dagger \left[\begin{smallmatrix} 0\\ I_p\end{smallmatrix}\right], C\right)}$ are uniformly exponentially stable.

We stress that assumptions (P1)–(P4) are only of structural nature and hold for a large class of systems.

Under assumptions (P1)–(P3) it follows from Theorem 2.5 that the transformation matrix U from (3) can be used for a state space transformation as follows. Setting $z(t) := U(t)x(t)$ we obtain from (1) that

$$\begin{aligned} \dot z(t) &= \big(U(t)A(t) + \dot U(t)\big)U(t)^{-1}z(t) + U(t)BL(t)u(t)\\ &\quad + U(t)f\big(t, U(t)^{-1}z(t), u(t)\big) \end{aligned}$$

and $y(t) = CU(t)^{-1}z(t)$. By Theorem 2.5 this implies that

$$\begin{aligned} \dot z(t) &= \hat A(t)z(t) + \hat B(t)u(t) + \hat f\big(t, z(t), u(t)\big),\\ y(t) &= \hat C z(t) \end{aligned}$$

and this is equivalent to

$$\begin{aligned} y^{(r)}(t) &= \sum_{i=1}^r R_i(t)y^{(i-1)}(t) + S(t)\eta(t) + \Gamma L(t)u(t)\\ &\quad + f_r\big(t, y(t), \dots, y^{(r-1)}(t), \eta(t), u(t)\big),\\ \dot\eta(t) &= \sum_{i=1}^r P_i(t)y^{(i-1)}(t) + Q(t)\eta(t) + N(t)u(t)\\ &\quad + f_\eta\big(t, y(t), \dots, y^{(r-1)}(t), \eta(t), u(t)\big), \end{aligned} \tag{26}$$

where $z(t) = \big(y(t), \dots, y^{(r-1)}(t), \eta(t)\big)$. By Proposition 3.2 and assumption (P4) it further follows that the behavior $\mathfrak{B}_Q$ is uniformly exponentially stable. Together with (25) these are the main ingredients for the proof of the following result.

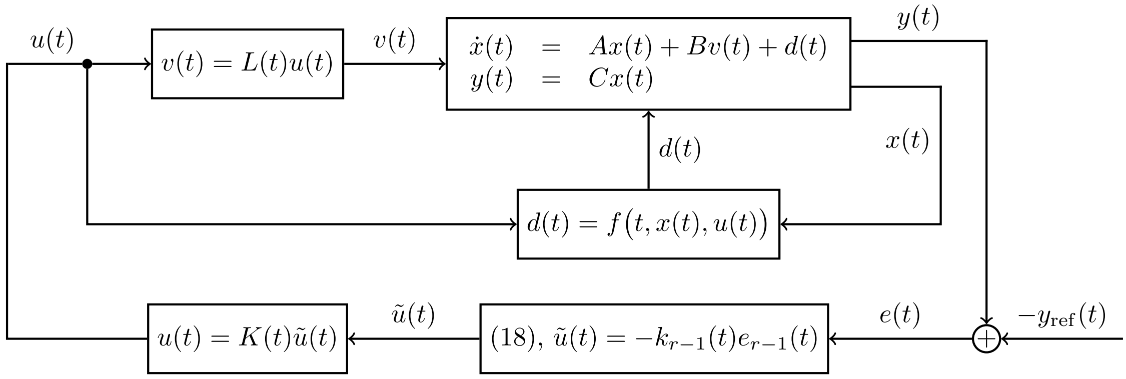

Figure 3: Fault tolerant funnel controller consisting of (18) and (23) applied to system (1).

Theorem 4.2. *Consider a system* (1) *which satisfies assumptions (P1)–(P4). Let* $y_{\mathrm{ref}}, \varphi_0, \ldots, \varphi_{r-1}$ *be as in* (24) *and* $x^0 \in \mathbb{R}^n$ *be an initial value such that* $e_0, \ldots, e_{r-1}$ *as defined in* (18) *satisfy*

$$\varphi_i(0)\|e_i(0)\| < 1 \quad \text{for } i = 0, \ldots, r-1. \tag{27}$$

Assume that there exists a bounded $K \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{m\times p})$ *such that* (25) *is satisfied. Then the funnel controller* (18), (23) *applied to* (1) *yields an initial-value problem which has a solution, and every solution can be extended to a maximal solution* $x : [0, \omega) \to \mathbb{R}^n$, $\omega \in (0, \infty]$, *which satisfies:*

(i) *The solution is global (i.e.,* $\omega = \infty$*).*

(ii) *The input* $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$*, the gain functions* $k_0, \ldots, k_{r-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ *and* $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ *are bounded.*

(iii) *The functions* $e_0, \ldots, e_{r-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ *evolve in their respective performance funnels and are uniformly bounded away from the funnel boundaries in the sense:*

$$\forall\, i = 0, \ldots, r-1 \;\exists\, \varepsilon_i > 0 \;\forall\, t > 0 : \quad \|e_i(t)\| \leq \varphi_i(t)^{-1} - \varepsilon_i. \tag{28}$$

In particular, the error $e(t) = y(t) - y_{\mathrm{ref}}(t)$ *evolves in the funnel* $\mathcal{F}_{\varphi_0}$ *as in* (2) *and stays uniformly away from its boundary.*

Proof. We proceed in several steps.

Step 1: We show existence of a solution of (1), (18), (23) and that it can be extended to a maximal solution. By assumptions (P1)–(P3) it follows from Theorem 2.5 that the state space transformation $\big(y(t)^\top, \dot{y}(t)^\top, \ldots, y^{(r-1)}(t)^\top, \eta(t)^\top\big)^\top := U(t)x(t)$ puts system (1) into the form (26). Set $v(t) := -k_{r-1}(t)e_{r-1}(t)$, then $u(t) = K(t)v(t)$. Using the same technique as in Step 1 of the proof of [5, Thm. 3.1] we find that there exist a relatively open set $\mathcal{D} \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}^{rp}$ and $G : \mathcal{D} \to \mathbb{R}^p$ such that

$$v(t) = -\frac{G\big(t, y(t), \dot{y}(t), \ldots, y^{(r-1)}(t)\big)}{1 - \varphi_{r-1}^2(t)\|G\big(t, y(t), \dot{y}(t), \ldots, y^{(r-1)}(t)\big)\|^2}$$

and

$$\big(0, y(0), \dot{y}(0), \ldots, y^{(r-1)}(0)\big) \in \mathcal{D}.$$

Using the notation

$$Y(t) = \big(y(t), \dot{y}(t), \ldots, y^{(r-1)}(t)\big),$$

the closed-loop system (1), (18), (23) can be reformulated as, invoking that $N(t)K(t) = 0$ by (25),

$$\begin{aligned} y^{(r)}(t) &= \sum_{i=1}^{r} R_i(t) y^{(i-1)}(t) + S(t)\eta(t) - \tfrac{\Gamma L(t)K(t)G(t,Y(t))}{1-\varphi_{r-1}^2(t)\|G(t,Y(t))\|^2} \\ &\quad + f_r\left(t, Y(t), \eta(t), \tfrac{-K(t)G\left(t,Y(t)\right)}{1-\varphi_{r-1}^2(t)\|G(t,Y(t))\|^2}\right), \\ \dot{\eta}(t) &= \sum_{i=1}^{r} P_i(t) y^{(i-1)}(t) + Q(t)\eta(t) \\ &\quad + f_\eta\left(t, Y(t), \eta(t), \tfrac{-K(t)G(t,Y(t))}{1-\varphi_{r-1}^2(t)\|G(t,Y(t))\|^2}\right). \end{aligned} \tag{29}$$

It is clear that (29) can be reformulated as a first-order system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(r-1)}(t) \\ \eta(t) \end{pmatrix} = F\left(t, \begin{pmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(r-1)}(t) \\ \eta(t) \end{pmatrix}\right),$$

$$\begin{pmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(r-1)}(0) \\ \eta(0) \end{pmatrix} = U(0)x^0,$$

with a suitable continuous function $F : \mathcal{D} \times \mathbb{R}^{n-rp} \to \mathbb{R}^n$. Furthermore, $(0, U(0)x^0) \in \mathcal{D} \times \mathbb{R}^{n-rp}$ and $\mathcal{D} \times \mathbb{R}^{n-rp}$ is relatively open in $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$. Hence, by [41, § 10, Thm. XX] there exists a weakly differentiable solution of (29) satisfying the initial conditions and every solution can be extended to a maximal solution; let $(y, \ldots, y^{(r-1)}, \eta) : [0, \omega) \to \mathbb{R}^n$, $\omega \in (0, \infty]$, be such a maximal solution.

Step 2: We show that $(y, \ldots, y^{(r-1)}, \eta)$ also solves a closed-loop system which is of the form (21), (18) and $u(t) = v(t)$ in (21). Set

$$d_1(t) := f_\eta\left(t, Y(t), \eta(t), \tfrac{-K(t)G(t,Y(t))}{1-\varphi_{r-1}^2(t)\|G(t,Y(t))\|^2}\right)$$

for $t \in [0,\omega)$. Let $\Phi_Q(\cdot,\cdot)$ be the transition matrix of the linear time-varying system $\dot\eta(t) = Q(t)\eta(t)$. Then, using the variation of constants formula (see e.g. [32, Thm. 2.15]), we find that

$$\begin{aligned}\eta(t) &= \Phi_Q(t,0)\eta(0)\\ &\quad + \int_0^t \Phi_Q(t,s)\left(\sum_{i=1}^r P_i(s)y^{(i-1)}(s) + d_1(s)\right)\mathrm{d}s\end{aligned}$$

for all $t \in [0,\omega)$. Set

$$\begin{aligned}d(t) &:= S(t)\Phi_Q(t,0)\eta(0) + \int_0^t S(t)\Phi_Q(t,s)d_1(s)\,\mathrm{d}s\\ &\quad + f_r\left(t, Y(t), \eta(t), \tfrac{-K(t)G(t,Y(t))}{1-\varphi_{r-1}^2(t)\|G(t,Y(t))\|^2}\right)\end{aligned}$$

for $t \in [0,\omega)$. Since S, f_η and f_r are bounded and the behavior $\mathfrak{B}_Q$ is uniformly exponentially stable by Proposition 3.2 and assumptions (P1)–(P3), it follows that d is bounded on $[0,\omega)$. If $\omega < \infty$, we define $d(t) := 0$ for $t \ge \omega$ and obtain $d \in \mathcal{L}^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^p)$. Define the operator $T : \mathcal{C}([0,\infty) \to \mathbb{R}^{rp}) \to \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\ge 0} \to \mathbb{R}^p)$ by

$$\begin{aligned}T(\zeta_1,\dots,\zeta_r)(t) &= \sum_{i=1}^r R_i(t)\zeta_i(t)\\ &\quad + \sum_{i=1}^r \int_0^t S(t)\Phi_Q(t,s)P_i(s)\zeta_i(s)\,\mathrm{d}s\end{aligned}$$

for $t \ge 0$. Then we have

$$y^{(r)}(t) = T(y,\dots,y^{(r-1)})(t) + d(t) + \Gamma L(t)K(t)v(t)$$

for almost all $t \in [0,\omega)$. Finally, we seek a function $g \in \mathcal{C}(\mathbb{R}^\ell \to \mathbb{R}^{p\times p})$, $\ell \in \mathbb{N}$, and a bounded function $\bar d \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^\ell)$ such that

$$g\big(\bar d(t)\big) = \Gamma L(t)K(t)$$

for all $t \ge 0$ and $g(x) + g(x)^\top > 0$ for all $x \in \mathbb{R}^q$. The construction is as follows: By assumption (25) we have that

$$A(t) := \Gamma L(t)K(t) + \big(\Gamma L(t)K(t)\big)^\top - \tfrac{\alpha}{2}I_p > 0$$

for all $t \ge 0$, hence there exists a pointwise Cholesky decomposition

$$A(t) = H(t)H(t)^\top.$$

For $\alpha > 0$ as above and $x_{ij} \in \mathbb{R}$, $i,j = 1,\dots,p$, define

$$\begin{aligned}g_1 : \mathbb{R}^{p^2} &\to \mathbb{R}^{p\times p},\ (x_{11},\dots,x_{1p},x_{21},\dots,x_{pp})\\ &\mapsto \tfrac12 \begin{bmatrix} x_{11} & \cdots & x_{1p}\\ \vdots & & \vdots\\ x_{p1} & \cdots & x_{pp}\end{bmatrix}\begin{bmatrix} x_{11} & \cdots & x_{1p}\\ \vdots & & \vdots\\ x_{p1} & \cdots & x_{pp}\end{bmatrix}^\top + \tfrac{\alpha}{4}I_p.\end{aligned}$$

Let $H(t) = \big(h_{ij}(t)\big)_{i,j=1,\dots,p}$, then

$$\begin{aligned}g_1\big(h_{11}(t),\dots,h_{pp}(t)\big) &= \tfrac12 H(t)H(t)^\top + \tfrac{\alpha}{4}I_p\\ &= \tfrac12\left(\Gamma L(t)K(t) + \big(\Gamma L(t)K(t)\big)^\top\right).\end{aligned}$$

Further define, for $z_{ij} \in \mathbb{R}$, $i,j = 1,\dots,p$,

$$\begin{aligned}g_2 : \mathbb{R}^{p^2} &\to \mathbb{R}^{p\times p},\ (z_{11},\dots,z_{1p},z_{21},\dots,z_{pp})\\ &\mapsto \tfrac12\left(\begin{bmatrix} z_{11} & \cdots & z_{1p}\\ \vdots & & \vdots\\ z_{p1} & \cdots & z_{pp}\end{bmatrix} - \begin{bmatrix} z_{11} & \cdots & z_{1p}\\ \vdots & & \vdots\\ z_{p1} & \cdots & z_{pp}\end{bmatrix}^\top\right).\end{aligned}$$

Let $\Gamma L(t)K(t) = \big(k_{ij}(t)\big)_{i,j=1,\dots,p}$, then

$$g_2\big(k_{11}(t),\dots,k_{pp}(t)\big) = \tfrac12\left(\Gamma L(t)K(t) - \big(\Gamma L(t)K(t)\big)^\top\right),$$

thus

$$g_1\big(h_{11}(t),\dots,h_{pp}(t)\big) + g_2\big(k_{11}(t),\dots,k_{pp}(t)\big) = \Gamma L(t)K(t).$$

Define

$$g : \mathbb{R}^{p^2} \times \mathbb{R}^{p^2} \to \mathbb{R}^{p\times p}, (x,z) \mapsto g_1(x) + g_2(z),$$

then we find that, for all $x, z \in \mathbb{R}^{p^2}$,

$$\begin{aligned}g(x,z) + g(x,z)^\top &= g_1(x) + g_1(x)^\top + g_2(z) + g_2(z)^\top\\ &= g_1(x) + g_1(x)^\top \ge \tfrac{\alpha}{2}I_p > 0.\end{aligned}$$

With the bounded function

$$\begin{aligned}\bar d(t) &:= \big(h_{11}(t),\dots,h_{pp}(t),k_{11}(t),\dots,k_{pp}(t)\big)\\ &\in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{2p^2})\end{aligned}$$

we finally obtain that the solution $(y,\dots,y^{(r-1)},\eta)$ from Step 1 satisfies

$$y^{(r)}(t) = T(y,\dots,y^{(r-1)})(t) + d(t) + g\big(\bar d(t)\big)v(t) \qquad (30)$$

for almost all $t \in [0,\omega)$, where the input $v(t) = -k_{r-1}(t)e_{r-1}(t)$ is obtained from the controller (18). Invoking boundedness of d and $\bar d$, system (30) satisfies assumptions (N1) and (N2). Assumption (N3) is a consequence of the construction of g. The operator T is clearly causal and locally Lipschitz. Since $\mathfrak{B}_Q$ is uniformly exponentially stable, T maps bounded trajectories to bounded trajectories and therefore (30) satisfies condition (N4).

Step 3: By Steps 1 and 2, the maximal solution $(y,\dots,y^{(r-1)},\eta)$ is also a solution of (30) with (18) and $v(t) = -k_{r-1}(t)e_{r-1}(t)$, hence Theorem 4.1 yields that it can be extended to a global solution, i.e., $\omega = \infty$. Statements (ii) and (iii) are consequences of Theorem 4.1 as well. □

4.4. Discussion of controller weight matrix

We discuss possible choices for the bounded controller weight matrix $K \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{m\times p})$ satisfying (25). We distinguish the two cases $\operatorname{rk} BL(t) = p$ and $\operatorname{rk} BL(t) = q > p$. In practical applications, it is frequently the case that some actuators are used to perform similar control tasks or they can be divided into p groups of actuators with the same physical characteristics, where p is the number of outputs, see e.g. [38]. Due to this redundancy it may be assumed (and actually is quite probable) that in each group at least one actuator remains (partially) functional, i.e., does not experience a total fault. This means that we are in the case $\operatorname{rk} BL(t) = p$. An interesting and relevant example is mentioned in [38, p. 103]:

> "A typical example is modern transport aircraft, which have two or more engines. In longitudinal motion control, the engines are used for forward speed control, and the elevator and stabilizer are used for pitch rate control (in normal flight or emergency situations). In some special designs, the elevator and stabilizer may consist of multiple independently operated segments in order to provide redundancy. We can consider the engines as one group of inputs and the elevator and stabilizer (possibly segmented) as another group."

If there are q groups of actuators and p outputs with $q > p$, then the system typically has an unnecessary high redundancy. When it is still possible to guarantee that at least one actuator without total fault remains in each group, then complete groups of actuators may be switched off so that $q = p$ is achieved. Otherwise, some knowledge of the failures, i.e., of the actuator reliability matrix L from (P1) is inevitable as discussed below.

4.4.1. The case $\operatorname{rk} BL(t) = p$

Under the assumptions (P1)–(P3) it follows from Theorem 2.5 that in the case $q = p$ we have $N = 0$, so the second condition in (25) is satisfied for any choice of K. In order to satisfy the first condition in (25), a natural choice is $K(t) = \Gamma^\top$ and the requirement that

$$\exists\, \alpha > 0\ \forall\, t \in \mathbb{R}:\ \Gamma\big(L(t) + L(t)^\top\big)\Gamma^\top \geq \alpha I_p.$$

This condition means that we have at least p linearly independent actuators, the reliability of which does not converge to zero. In other words, in each group of actuators at least one remains functional, see the discussion above. Clearly, for this choice of K we have to assume that the high frequency gain matrix Γ of (1) is known; apart from that, no knowledge of the system parameters is required.

4.4.2. The case $\operatorname{rk} BL(t) = q > p$

Under the assumptions (P1)–(P3) it follows from Theorem 2.5 and Remark 2.9 that in the case $q > p$ there exists K such that $\Gamma L(t)K(t)$ is invertible and $N(t)K(t) = 0$ for all $t \in \mathbb{R}$ if, and only if, condition (12) is satisfied. In this case $K(t)$ as in (13) is a feasible choice which satisfies $\Gamma L(t)K(t) = I_p$ and hence (25) holds true. However, this requires knowledge of the system parameters and of the reliability matrix function L from (P1).

5. Simulation

We illustrate the fault tolerant funnel controller (18), (23) by applying it to the model of the Boeing 737 aircraft from Example 2.10. As reference trajectories we choose

$$y_{\mathrm{ref},1}(t) = 2\sin t, \quad y_{\mathrm{ref},2}(t) = \cos t,$$

the initial value is $x(0) = 0$, and the funnel functions are chosen as

$$\varphi_0(t) = (5e^{-t} + 0.1)^{-1}, \qquad \varphi_1(t) = \left(\tfrac{5}{2}e^{-\frac{1}{2}t} + 0.1\right)^{-1},$$

hence (24) is satisfied. Obviously, the initial errors lie within the respective funnel boundaries, i.e., (27) is satisfied. The controller weight matrix is chosen as

$$K(t) = \Gamma^\top = \begin{bmatrix} 0.01184 & -0.12879 \\ 0.01184 & -0.12879 \\ 0.21327 & 0.00171 \\ 0.21327 & 0.00171 \end{bmatrix}.$$

For the simulation, we assume that the actuator d_{r2} has a slowly decreasing efficiency to 50% of the original capability on the time interval $[0, 6]$ and at $t = 6$ another fault occurs so that we have an actuator saturation by 1 (which means an effective saturation by 0.5 due to the 50% reduction of efficiency). Using the smooth error function *erf* and the complementary error function *erfc*, which are both implemented in MATLAB, this behavior can be modelled by

$$\begin{aligned} l_2(t) &= \tfrac{1}{4}\operatorname{erfc}(t-3) + \tfrac{1}{4}\operatorname{erfc}\big(100(t-6)\big), \\ f_2(t, u_2) &= \tfrac{1}{4}\big(1 + \operatorname{erf}\big(100(t-6)\big)\big) \cdot \operatorname{sat}_1(u_2), \end{aligned}$$

where $\operatorname{sat}_1(v) = \operatorname{sgn}(v)$ for $|v| \geq 1$ and $\operatorname{sat}_1(v) = v$ for $|v| < 1$; see Fig. 4 for a plot of $l_2(t)$ and $f_2(t, 1)$. We further assume that the actuator d_{a2} has a sudden total fault at $t = 7$, which can be modelled by

$$\begin{aligned} l_4(t) &= \tfrac{1}{2}\operatorname{erfc}\big(20(t-7)\big), \\ f_4(t, u_4) &= 0, \end{aligned}$$

see Fig. 4 for a plot of $l_4(t)$.

After the faults, the effective input actions are

$$\begin{aligned} u_1(t) &= d_{r1}(t), \\ u_2(t) &= l_2(t)d_{r2}(t) + f_2\big(t, d_{r2}(t)\big), \\ u_3(t) &= d_{a1}(t), \\ u_4(t) &= l_4(t)d_{a2}(t). \end{aligned}$$

Since the actuators d_{r1} and d_{a1} are assumed to experience no faults, condition (25) is clearly satisfied. It further follows from Examples 2.10 and 3.4 that (P1)–(P4) are satisfied. Therefore, fault tolerant funnel control is feasible by Theorem 4.2.

The simulation of the controller (18), (23) applied to the model of the Boeing 737 aircraft from Example 2.10 over the time interval $[0, 10]$ has been performed in MATLAB (solver: `ode45`, rel. tol.: 10^{-14}, abs. tol.: 10^{-10}) and is depicted in Fig. 5. Fig. 5a shows the tracking errors for

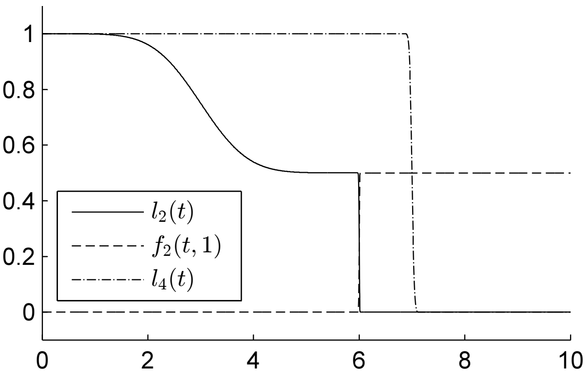

Figure 4: Reliability functions corresponding to actuators d_{r2} and d_{a2}.

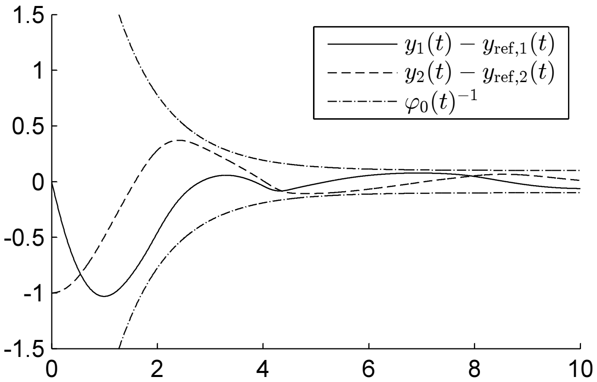

Fig. 5a: Funnel and tracking errors

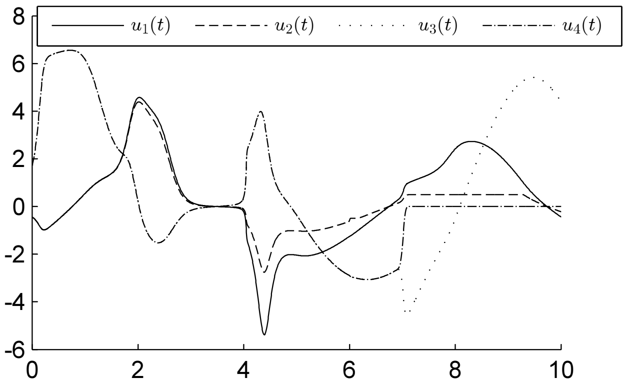

Fig. 5b: Input functions

Figure 5: Simulation of the controller (18), (23) for the Boeing 737 aircraft.

the two outputs and reference trajectories as well as the funnel boundary, while Fig. 5b shows the effective input functions $u_1, \ldots, u_4$. The decreasing efficiency of u_2 can clearly be seen as well as that it is saturated by 0.5 on the interval $[6, 10]$; the saturation is active on the interval $[7, 9]$. Furthermore, it can be seen that at $t = 7$, u_4 has a total fault and hence u_3 needs to increase in order to compensate for this. The tracking performance is not affected at all by these faults; the tracking errors evolve within the prescribed performance funnels.

6. Conclusion

In the present paper we proposed a novel fault-tolerant funnel controller for uncertain linear systems. We allowed for a large class of uncertainties and actuator faults which encompasses essentially all relevant cases. The funnel control design is simple, of low complexity and model-free up to some knowledge of system parameters which may help to construct the controller weight matrix as discussed in Subsection 4.4. The controller achieves prescribed performance of the tracking error for any given sufficiently smooth reference signal. In particular, more inputs than outputs are allowed in the system, as long as a certain actuator redundancy is satisfied, which in particular extends available results in funnel control. There are no restrictions on the relative degree (as long as it is known) and on the dimension of the zero dynamics (as long as after the transformation with the controller weight matrix they are uniformly exponentially stable).

Finally, we like to point out that a drawback of our approach, which still needs to be resolved, is that the derivatives of the output must be available for the controller. However, there are several applications where this condition is not satisfied, and it may even be hard to obtain suitable estimates of the output derivatives. A first approach to treat these problems using a "funnel pre-compensator" has been developed in [8, 9] for systems with relative degree $r = 2$ or $r = 3$.

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