On observers for nonlinear differential-algebraic systems

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Abstract—**We extend a recent approach to observer design for linear differential-algebraic systems to impulse observable systems with Lipschitz nonlinearities. The observer design further extends the standard Luenberger type observer design. We show that the design parameters for the observer can be obtained by the solution of a Riccati type inequality. The solutions of the latter can in turn be obtained by solving a set of LMIs and BMIs which provides a computational procedure. A feature of our observer design is the possibility of reformulation as an ordinary differential equation.**

Index Terms—**Differential-algebraic systems, nonlinear systems, observers, Riccati inequality, LMIs.**

Nomenclature:

$\mathbb{N}$, $\mathbb{N}_0$	set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{R}^{n\times m}$	the set of real $n\times m$ matrices
$\mathrm{rk}A$, $\mathrm{im}A$	rank and image of $A \in \mathbb{R}^{n\times m}$
$\mathbf{Gl}_n(\mathbb{R})$	the group of invertible matrices in $\mathbb{R}^{n\times n}$
$M >_{\mathscr{V}} 0$	$:\iff \forall x \in \mathscr{V}\setminus\{0\}: \ x^\top M x > 0$, for a matrix $M \in \mathbb{R}^{n\times n}$ and a subspace $\mathscr{V} \subseteq \mathbb{R}^n$
$\mathscr{C}^k(X \to Y)$	set of k-times continuously differentiable functions $f: X \to Y$, $k \in \mathbb{N}_0 \cup \{\infty\}$; $\mathscr{C}(X\to Y) := \mathscr{C}^0(X \to Y)$; if $k=\infty$ the function f is called *smooth*
$\mathrm{dom}\, f$	the domain of the function f
$f\|_I$	restriction of the function f to the set I

I. Introduction

We study observer design for nonlinear systems governed by differential-algebraic equations (DAEs). We follow the recent approach to observer design developed in [1] for linear DAE systems. In the main result in Theorem III.2 we show that an asymptotic observer can be designed whenever a certain Riccati inequality is solvable. We later show that solvability of certain linear and bilinear matrix inequalities (LMIs and BMIs) is sufficient for solvability of the Riccati inequality.

We consider DAE systems of the form

$$\begin{aligned} \dot x_1(t) &= Ax_1(t) + Bx_2(t) + f_1\big(x_1(t), x_2(t), u(t), y(t)\big),\\ 0 &= Cx_1(t) + Dx_2(t) + f_2\big(u(t), y(t)\big),\\ y(t) &= Fx_1(t) + Gx_2(t) + h\big(u(t)\big), \end{aligned} \tag{1}$$

with $A \in \mathbb{R}^{r\times r}$ and all other matrices of appropriate dimensions so that

$$\begin{bmatrix} A & B\\ C & D\end{bmatrix} \in \mathbb{R}^{l\times n}, \quad [F,G] \in \mathbb{R}^{p\times n};$$

furthermore, $f_1 \in \mathscr{C}^1(X_1 \times X_2 \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^r)$, $f_2 \in \mathscr{C}^1(\mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^{l-r})$, $h \in \mathscr{C}^1(\mathbb{R}^m \to \mathbb{R}^p)$, where $X_1 \subseteq \mathbb{R}^r, X_2 \subseteq \mathbb{R}^{n-r}$ are open, such that the following Lipschitz condition is satisfied:

$$\begin{aligned} &\exists L>0 \ \forall\ (x_1^i, x_2^i) \in X_1 \times X_2 \ \ i=1,2 \ \forall\ (u,y) \in \mathbb{R}^m \times \mathbb{R}^p:\\ &\|f_1(x_1^1,x_2^1,u,y) - f_1(x_1^2,x_2^2,u,y)\| \le L\|(x_1^1-x_1^2, x_2^1-x_2^2)\|. \end{aligned} \tag{2}$$

The functions $u: I \to \mathbb{R}^m$ and $y: I \to \mathbb{R}^p$ are called *input* and *output* of the system, resp. Note that although y from the last equation in (1) may be inserted in the first equation which may hence be written in the form

$$\dot x_1(t) = Ax_1(t) + Bx_2(t) + \tilde f_1\big(x_1(t), x_2(t), u(t)\big),$$

this would be a smaller class of systems as we would need to require $\tilde f_1$ to be Lipschitz continuous w.r.t. x_1 and x_2, while f_1 does not need to be Lipschitz w.r.t. y.

The system class (1) includes any linear DAE system and numerous important classes of nonlinear DAE systems (e.g. chemical process systems [2], mechanical systems [3], [4] and modified nodal analysis models of electrical circuits [5]). Nonlinear DAE systems seem to have been first considered by Luenberger [6]; see also the textbooks [7], [8] and the recent works [9], [10].

The design of estimators for DAE systems similar to (1) has been studied in [11]–[14]. In [11] the regularity of the linear part is assumed, while in [12] it only needs to be square and in this way quite general results are obtained. A unified approach is presented, where existence of the designed estimator is shown to depend on the solvability of certain LMIs. Due to the allowed Lipschitz continuity of the nonlinearities it is clear that the Lipschitz constant must be small enough (compared to the linear part) for an estimator to exist; this can be made precise in terms of the solvability of LMIs. A similar approach has been taken before in [15], [16] for nonlinear systems of ordinary differential equations (ODEs) with unknown inputs, which may be treated as DAE systems (1) as well. Recently, the approach from [12] has been extended in [13], where actuator and sensor faults (similar to [11]) as well as uncertainties are incorporated, and in [14], where the Lipschitz nonlinearities are also allowed in the output equation. Furthermore, the observer design in [13] and [14] additionally requires the solvability of certain BMIs. Different approaches are taken in [17], where the system is completely nonlinear, but semi-explicit and of index 1, and a nonlinear estimator is constructed, and in [18] where a nonlinear generalized PI observer design is used, see also the references therein.

In the present paper, we present a more general observer design than in [12], based on the recent approach in [1], which extends the standard Luenberger type observer design. In particular, our design does not require the linear part of the system to be regular or square. The features of our approach are as follows:

- we do not restrict ourselves to regular or square systems,
- our observer design reduces to Luenberger type observers only in special cases,
- our observers can always be reformulated as ODE systems.

In order to achieve an observer design which can be reformulated as an ODE, while at the same time the system does not need to be square, the assumption of impulse observability is crucial. Applications of our observer design are e.g. error detection and fault diagnosis, disturbance (or unknown input) estimation and feedback control.

The present paper is organized as follows: In Section II we recall some basic definitions and concepts. Our observer design is presented in Section III and we prove in the main result Theorem III.2 that it works provided a certain Riccati type inequality is solvable. In Section IV we show that solvability of certain LMIs and BMIs is sufficient for solvability of the Riccati inequality. The case where

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our observer design reduces to standard Luenberger type observers is discussed in Section V along with consequences thereof. Some illustrative examples are given in Section VI.

II. Preliminaries

In order to define (asymptotic) observers for nonlinear DAE systems we consider the general class of nonlinear systems governed by DAEs of the form

$$\begin{aligned} E\dot{x}(t) &= f\big(x(t),u(t),y(t)\big), \\ y(t) &= h\big(x(t),u(t)\big), \end{aligned} \tag{3}$$

where $X \subseteq \mathbb{R}^n$ is open, $f \in \mathscr{C}^1(X \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^l)$, $h \in \mathscr{C}^1(X \times \mathbb{R}^m \to \mathbb{R}^p)$ and $E \in \mathbb{R}^{l\times n}$. Since solutions not necessarily exist globally (e.g. finite escape times may arise) we consider local solutions of (3). A trajectory $(x,u,y) \in \mathscr{C}(I \to X \times \mathbb{R}^m \times \mathbb{R}^p)$ is called a *solution* of (3), if $I = \operatorname{dom} x \subseteq \mathbb{R}$ is an open interval, $x \in \mathscr{C}^1(I \to \mathbb{R}^l)$ and (x,u,y) solves (3) for all $t \in I$. Note that the interval of definition I of a solution of (3) depends on the choice of the input u and that a solution does not need to be maximal. The *behavior* of (3) is defined as the set of all possible solution trajectories

$$\begin{aligned} \mathfrak{B}_{(3)} := \{\, (x,u,y) \in \mathscr{C}(I \to X \times \mathbb{R}^m \times \mathbb{R}^p) \mid I \subseteq \mathbb{R} \text{ open interval}, \\ (x,u,y) \text{ is a solution of (3)} \}. \end{aligned}$$

We recall that a linear DAE system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

is called *impulse observable* if, and only if, $\ker E \cap A^{-1}(\operatorname{im} E) \cap \ker C = \{0\}$; for a rigorous time domain definition and a detailed discussion we refer to the survey [19]. The aforementioned condition for impulse observability is equivalent to the rank condition

$$\operatorname{rk}\begin{bmatrix} E & A \\ 0 & C \\ 0 & E \end{bmatrix} = n + \operatorname{rk} E$$

as shown e.g. in [20], [21]. This condition can be generalized to nonlinear DAE systems (3) as follows.

Definition II.1. A DAE system (3) is called *impulse observable*, if

$$\forall\, (x,u,y) \in X \times \mathbb{R}^m \times \mathbb{R}^p : \quad \operatorname{rk}\begin{bmatrix} E & \frac{\partial f}{\partial x}(x,u,y) \\ 0 & \frac{\partial h}{\partial x}(x,u) \\ 0 & E \end{bmatrix} = n + \operatorname{rk} E.$$

A novel observer design for linear DAE systems has been introduced in [1]. Here, we extend this approach and the accompanying concepts of (asymptotic) observers to nonlinear DAE systems.

Definition II.2. Consider a system (3). A system

$$\begin{aligned} E_o\dot{x}_o(t) &= f_o\big(x_o(t),u(t),y(t)\big), \\ z(t) &= h_o\big(x_o(t),u(t),y(t)\big), \end{aligned} \tag{4}$$

where $E_o \in \mathbb{R}^{l_o \times n_o}$, $f_o \in \mathscr{C}^1(X_o \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^{l_o})$, $h_o \in \mathscr{C}^1(X_o \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^{p_o})$, $X_o \subseteq \mathbb{R}^{n_o}$ open, is called an *acceptor for* (3), if for all $(x,u,y) \in \mathfrak{B}_{(3)}$ with $I = \operatorname{dom} x$, there exist $x_o \in \mathscr{C}^1(I \to \mathbb{R}^{n_o})$, $z \in \mathscr{C}(I \to \mathbb{R}^{p_o})$ such that

$$\big(x_o, \tbinom{u}{y}, z\big) \in \mathfrak{B}_{(4)}.$$

We stress that there is a directed signal flow from (3) to its acceptor (4) via input and output, see Fig. 1. That is, (3) may influence (4) but not vice-versa.

Definition II.3. Consider a system (3). Then a system (4) with $p_o = n$ is called

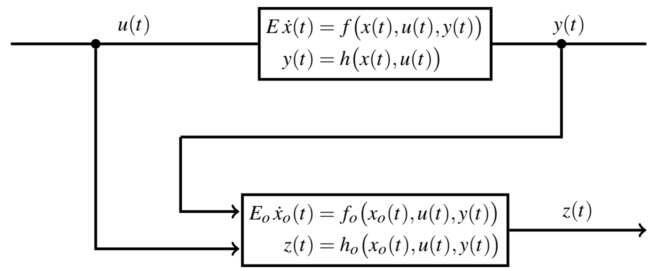

Fig. 1: Interconnection with an acceptor

a) an *observer for* (3), if it is an acceptor for (3), and

$$\begin{aligned} &\forall I \subseteq \mathbb{R} \text{ open intvl. } \forall t_0 \in I \\ &\forall (x,u,y,x_o,z) \in \mathscr{C}(I \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^{n_o} \times \mathbb{R}^n): \\ &\big((x,u,y) \in \mathfrak{B}_{(3)} \,\wedge\, (x_o, \tbinom{u}{y}, z) \in \mathfrak{B}_{(4)} \,\wedge\, Ez(t_0) = Ex(t_0)\big) \\ &\qquad\qquad \implies z = x. \end{aligned}$$

b) an *asymptotic observer for* (3), if it is an observer for (3), and

$$\begin{aligned} &\forall t_0 \in \mathbb{R}\ \forall (x,u,y,x_o,z) \in \mathscr{C}([t_0,\infty) \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^{n_o} \times \mathbb{R}^n): \\ &\big((x,u,y) \in \mathfrak{B}_{(3)} \,\wedge\, (x_o, \tbinom{u}{y}, z) \in \mathfrak{B}_{(4)}\big) \\ &\qquad\qquad \implies \lim_{t\to\infty} z(t) - x(t) = 0. \end{aligned}$$

We like to note that while for *linear* impulse observable DAE systems there always exists an observer as shown in [1] (and it can even be reformulated as an ODE, cf. [22]), this is not true for nonlinear systems in general.

Example II.4. Consider the system

$$0 = x(t)^2 - 1, \quad y(t) = 0 \tag{5}$$

in the form (3) with

$$f : X \to \mathbb{R},\ x \mapsto x^2 - 1, \quad X := \mathbb{R}\setminus\{0\}.$$

Since $\frac{\partial f}{\partial x}(x) = 2x \neq 0$ for all $x \in X$, system (5) is impulse observable. However, the system has two different solutions, $x_1(t) \equiv 1$ and $x_2(t) \equiv -1$, and it is impossible to reconstruct the solution from the information of the output $y(t)$ and the initial condition $Ex(t_0)$, as $E = 0$. Therefore, there does not exist an observer for system (5).

As a consequence, we restrict ourselves to nonlinear systems of the form (1).

III. Observer design by Riccati inequality

In this section we propose a design of asymptotic observers for systems (1). We improve upon earlier approaches by allowing a larger class of systems and we use an observer design which extends the Luenberger type observer design. We present a Riccati type inequality whose solutions are used for the observer design. In the subsequent Section IV we show that the solution of certain LMIs and BMIs yields a solution to this Riccati inequality. The LMIs and BMIs then yield a computational procedure for obtaining the observer.

First, we record the following observation.

Lemma III.1. *A system* (1) *is impulse observable if, and only if,*

$$\operatorname{rk}\begin{bmatrix} D \\ G \end{bmatrix} = n - r. \tag{6}$$

Proof. Observe that impulse observability of (1) means

$$\operatorname{rk}\begin{bmatrix} I_r & 0 & A+\frac{\partial f_1}{\partial x_1}(x_1,x_2,u,y) & B+\frac{\partial f_1}{\partial x_2}(x_1,x_2,u,y) \\ 0 & 0 & C & D \\ 0 & 0 & F & G \\ 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = n+r$$

for all $(x_1,x_2,u,y)\in X_1\times X_2\times\mathbb{R}^m\times\mathbb{R}^p$. This is equivalent to (6). □

Note that impulse observability implies $n\le l+p$, since otherwise $n-r>l+p-r$ by which condition (6) could never be true.

Motivated by the observer design in [1] (which is closely related to that in [23] for behavioral systems) we propose the following observer, which consists of an internal model of the system (1) driven by additional "innovations" terms:

$$\begin{aligned} \dot z_1(t) &= Az_1(t)+Bz_2(t)+f_1\big(z_1(t),z_2(t),u(t),y(t)\big)+L_1d(t),\\ 0 &= Cz_1(t)+Dz_2(t)+f_2\big(u(t),y(t)\big)+L_2d(t),\\ 0 &= Fz_1(t)+Gz_2(t)+h\big(u(t)\big)-y(t)+L_3d(t),\\ z(t) &= \left(\begin{smallmatrix} z_1(t)\\ z_2(t)\end{smallmatrix}\right), \end{aligned} \tag{7}$$

where $L_1\in\mathbb{R}^{r\times k}, L_2\in\mathbb{R}^{(l-r)\times k}, L_3\in\mathbb{R}^{p\times k}$, $k\in\mathbb{N}_0$, and the additional observer state $d(t)$ represents the innovations; the complete observer state is

$$x_o(t)=\begin{pmatrix} z_1(t)\\ z_2(t)\\ d(t)\end{pmatrix}.$$

The innovations in the observer design have been first introduced by Polderman and Willems [24, p. 351] in order to "express how far the actual observed output differs from what we would have expected to observe". Set

$$\mathscr{E}=\begin{bmatrix} I_r & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad \mathscr{A}=\begin{bmatrix} A & B & L_1\\ C & D & L_2\\ F & G & L_3\end{bmatrix} \tag{8}$$

and

$$\mathscr{V}=\operatorname{im}\begin{bmatrix} I_r\\ M\end{bmatrix},\quad M=-\begin{bmatrix} D & L_2\\ G & L_3\end{bmatrix}^{-1}\begin{bmatrix} C\\ F\end{bmatrix}, \tag{9}$$

provided that $\left[\begin{smallmatrix} D & L_2\\ G & L_3\end{smallmatrix}\right]$ is invertible. In the following main result of the paper we show that there exists an asymptotic observer of the form (7), if the Riccati type inequality

$$\mathscr{A}^\top P\mathscr{E}+\mathscr{E}^\top P\mathscr{A}+\frac{1}{\delta}\mathscr{E}^\top P^2\mathscr{E}+(\delta L^2)I_{n+k} <_{\mathscr{V}} 0 \tag{10}$$

has a solution L_1,L_2,L_3, $\delta>0$ and $P=P^\top\in\mathbb{R}^{(n+k)\times(n+k)}$ such that

$$[I_r,0]P\begin{bmatrix} I_r\\ 0\end{bmatrix}>0.$$

Riccati inequalities of the form (10), i.e., restricted to certain subspaces, have been studied before, see e.g. [25] and the references therein.

Theorem III.2. *Consider a system* (1) *which satisfies* (2) *and is impulse observable, i.e.,* (6) *holds. Let* $k=l+p-n$ *and assume that* $L_1\in\mathbb{R}^{r\times k}, L_2\in\mathbb{R}^{(l-r)\times k}, L_3\in\mathbb{R}^{p\times k}$, $\delta>0$ *and* $P=P^\top\in\mathbb{R}^{(n+k)\times(n+k)}$ *solve* (10) *such that*

$$\begin{bmatrix} D & L_2\\ G & L_3\end{bmatrix}\in\mathbf{Gl}_{n-r+k}(\mathbb{R})\quad\text{and}\quad [I_r,0]P\begin{bmatrix} I_r\\ 0\end{bmatrix}>0.$$

Then (7) *is an asymptotic observer for* (1).

Proof. System (7) is an acceptor for (1) since for any $(x,u,y)\in\mathfrak{B}_{(1)}$ we have that $\left(\left(\begin{smallmatrix} x_1\\ x_2\\ 0\end{smallmatrix}\right),\left(\begin{smallmatrix} u\\ y\end{smallmatrix}\right),x\right)\in\mathfrak{B}_{(7)}$.

Step 1: We show that (7) is an observer for (1). To this end, let $I\subseteq\mathbb{R}$ be an open interval, $t_0\in I$ and $(\left(\begin{smallmatrix} x_1\\ x_2\end{smallmatrix}\right),u,y)\in\mathfrak{B}_{(1)}$, $\left(\left(\begin{smallmatrix} z_1\\ z_2\\ d\end{smallmatrix}\right),\left(\begin{smallmatrix} u\\ y\end{smallmatrix}\right),z\right)\in\mathfrak{B}_{(7)}$ be defined on I such that $x_1(t_0)=z_1(t_0)$. From (7) we have that

$$\begin{pmatrix} z_2(t)\\ d(t)\end{pmatrix}=-\begin{bmatrix} D & L_2\\ G & L_3\end{bmatrix}^{-1}\left(\begin{bmatrix} C\\ F\end{bmatrix}z_1(t)+\begin{pmatrix} f_2\big(u(t),y(t)\big)\\ h\big(u(t)\big)-y(t)\end{pmatrix}\right)$$

for all $t\in I$, and from (1), in a similar way,

$$\begin{pmatrix} x_2(t)\\ 0\end{pmatrix}=-\begin{bmatrix} D & L_2\\ G & L_3\end{bmatrix}^{-1}\left(\begin{bmatrix} C\\ F\end{bmatrix}x_1(t)+\begin{pmatrix} f_2\big(u(t),y(t)\big)\\ h\big(u(t)\big)-y(t)\end{pmatrix}\right).$$

With

$$\begin{pmatrix} g_1(x_1,u,y)\\ g_2(x_1,u,y)\end{pmatrix}:=-\begin{bmatrix} D & L_2\\ G & L_3\end{bmatrix}^{-1}\left(\begin{bmatrix} C\\ F\end{bmatrix}x_1+\begin{pmatrix} f_2(u,y)\\ h(u)-y\end{pmatrix}\right) \tag{11}$$

for $(x_1,u,y)\in X_1\times\mathbb{R}^m\times\mathbb{R}^p$ we thus have

$$\begin{aligned} \dot x_1(t) &= Ax_1(t)+Bg_1\big(x_1(t),u(t),y(t)\big)\\ &\quad+f_1\big(x_1(t),g_1\big(x_1(t),u(t),y(t)\big),u(t),y(t)\big)+L_1g_2\big(x_1(t),u(t),y(t)\big) \end{aligned}$$

for all $t\in I$, and z_1 solves the same ODE with the same initial value $z_1(t_0)=x_1(t_0)$. Therefore, since g_1 and g_2 are linear in x_1 and

$$X_1\ni x_1\mapsto f_1\big(x_1,g(x_1,u,y),u,y\big)$$

is Lipschitz in x_1 for all $(u,y)\in\mathbb{R}^m\times\mathbb{R}^p$, the uniqueness theorem for ODEs (see [26, Thm. 4.17]) yields that $x_1(t)=z_1(t)$ for all $t\in I$. Moreover,

$$z_2(t)=g_1\big(z_1(t),u(t),y(t)\big)=g_1\big(x_1(t),u(t),y(t)\big)=x_2(t)$$

for all $t\in I$, and this shows that (7) is an observer.

Step 2: We determine the observation error dynamics. Let $e_1:=z_1-x_1$ and $e_2:=z_2-x_2$, then

$$\begin{aligned} \dot e_1(t) &= Ae_1(t)+Be_2(t)+L_1d(t)\\ &\quad+f_1\big(z_1(t),z_2(t),u(t),y(t)\big)-f_1\big(x_1(t),x_2(t),u(t),y(t)\big),\\ 0 &= Ce_1(t)+De_2(t)+L_2d(t),\\ 0 &= Fe_1(t)+Ge_2(t)+L_3d(t). \end{aligned} \tag{12}$$

Let M and $\mathscr{V}$ be as in (9), then

$$\begin{pmatrix} e_2(t)\\ d(t)\end{pmatrix}=Me_1(t),$$

and hence any solution (e_1,e_2,d) of (12) evolves in $\mathscr{V}$.

Step 3: We show that the observer (7) is asymptotic. To this end, let $t_0\in\mathbb{R}$ and $(\left(\begin{smallmatrix} x_1\\ x_2\end{smallmatrix}\right),u,y)\in\mathfrak{B}_{(1)}$, $\left(\left(\begin{smallmatrix} z_1\\ z_2\\ d\end{smallmatrix}\right),\left(\begin{smallmatrix} u\\ y\end{smallmatrix}\right),z\right)\in\mathfrak{B}_{(7)}$ be defined on $[t_0,\infty)$. The corresponding observation errors solve (12) and hence in particular, by Step 2,

$$\forall t\ge t_0:\ w(t):=\begin{pmatrix} e_1(t)\\ e_2(t)\\ d(t)\end{pmatrix}\in\mathscr{V}. \tag{13}$$

Use $\mathscr{E}$ and $\mathscr{A}$ from (8) and let

$$V:\mathbb{R}^{n+k}\to\mathbb{R},\ w\mapsto w^\top\mathscr{E}^\top P\mathscr{E}w.$$

Let $\chi_z(t) := \big(z_1(t),z_2(t),u(t),y(t)\big)$ and $\chi_x(t) :=$

$\big(z_1(t), z_2(t), u(t), y(t)\big)$, then by (12) we have

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t} V\big(w(t)\big)\\
&= w(t)^\top \big(\mathscr{A}^\top P \mathscr{E} + \mathscr{E}^\top P \mathscr{A}\big) w(t)\\
&\quad + w(t)^\top \mathscr{E}^\top P \left[\begin{smallmatrix} I_r \\ 0 \\ 0 \end{smallmatrix}\right] \Big(f_1\big(\chi_z(t)\big) - f_1\big(\chi_x(t)\big)\Big)\\
&\quad + \left(\left[\begin{smallmatrix} I_r \\ 0 \\ 0 \end{smallmatrix}\right] \Big(f_1\big(\chi_z(t)\big) - f_1\big(\chi_x(t)\big)\Big)\right)^\top P \mathscr{E} w(t)\\
&= w(t)^\top \big(\mathscr{A}^\top P \mathscr{E} + \mathscr{E}^\top P \mathscr{A}\big) w(t) + w(t)^\top \mathscr{E}^\top P \tilde{f}(t) + \tilde{f}(t)^\top P \mathscr{E} w(t),
\end{aligned}$$

where

$$\tilde{f}(t) := \left[\begin{smallmatrix} I_r \\ 0 \\ 0 \end{smallmatrix}\right] \Big(f_1\big(\chi_z(t)\big) - f_1\big(\chi_x(t)\big)\Big).$$

Since for any vectors $u, v \in \mathbb{R}^q$ we have

$$u^\top v + v^\top u \le \delta u^\top u + \frac{1}{\delta} v^\top v$$

we obtain the inequality

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} V\big(w(t)\big) &\le w(t)^\top \big(\mathscr{A}^\top P \mathscr{E} + \mathscr{E}^\top P \mathscr{A}\big) w(t) + \frac{1}{\delta} w(t)^\top \mathscr{E}^\top P^2 \mathscr{E} w(t)\\
&\quad + \delta \tilde{f}(t)^\top \tilde{f}(t)\\
&\overset{(2)}{\le} w(t)^\top \big(\mathscr{A}^\top P \mathscr{E} + \mathscr{E}^\top P \mathscr{A} + \frac{1}{\delta} \mathscr{E}^\top P^2 \mathscr{E} + \delta L^2 I_{n+k}\big) w(t)\\
&= -w(t)^\top Q w(t)
\end{aligned}$$

for all $t \ge t_0$, where

$$Q := -\big(\mathscr{A}^\top P \mathscr{E} + \mathscr{E}^\top P \mathscr{A} + \frac{1}{\delta} \mathscr{E}^\top P^2 \mathscr{E} + \delta L^2 I_{n+k}\big).$$

By (10) we have $Q >_{\mathscr{V}} 0$ and thus, invoking (13), it follows that

$$\tfrac{\mathrm{d}}{\mathrm{d}t} V\big(w(t)\big) \le -c V\big(w(t)\big), \quad t \ge t_0,$$

for some $c > 0$, hence an application of Gronwall's lemma yields $\lim_{t\to\infty} V\big(w(t)\big) = 0$. Invoking that $[I_r, 0] P \left[\begin{smallmatrix} I_r \\ 0 \end{smallmatrix}\right] > 0$ we obtain that e_1 converges to zero, and hence $\left(\begin{smallmatrix} e_2 \\ d \end{smallmatrix}\right) = M e_1$ converges to zero, too. Therefore,

$$\lim_{t\to\infty} z(t) - x(t) = 0,$$

and this completes the proof of the theorem. □

We like to stress that since $\left[\begin{smallmatrix} D & L_2 \\ G & L_3 \end{smallmatrix}\right]$ is invertible the asymptotic observer (7) can be reformulated as an ODE system as follows:

$$\begin{aligned}
\dot{z}_1(t) &= \left(A - [B, L_1] \begin{bmatrix} D & L_2 \\ G & L_3 \end{bmatrix}^{-1} \begin{bmatrix} C \\ F \end{bmatrix}\right) z_1(t)\\
&\quad + f_1\big(z_1(t), g_1\big(z_1(t), u(t), y(t)\big), u(t), y(t)\big)\\
&\quad - [B, L_1] \begin{bmatrix} D & L_2 \\ G & L_3 \end{bmatrix}^{-1} \begin{pmatrix} f_2\big(u(t), y(t)\big) \\ h\big(u(t)\big) - y(t) \end{pmatrix},\\
z(t) &= \begin{pmatrix} z_1(t) \\ g_1\big(z_1(t), u(t), y(t)\big) \end{pmatrix},
\end{aligned} \tag{14}$$

where g_1 is as in (11). However, since this structure is quite complicated, (7) may be preferred for implementation and numerical computations. We also stress that (14) is not of Luenberger type in general.

IV. Observer design by solution of LMIs and BMIs

In this section we derive a set of LMIs and BMIs and show how their solution yields a solution of the Riccati inequality (10). LMIs impose convex problems and can be solved efficiently with standard algorithms, however the drawback of our general approach is that the Riccati inequality (10) cannot be completely reformulated as an LMI, but we show that it is possible to obtain its solution from a set of LMIs with an additional BMI constraint. This is the basis for a computational procedure in order to construct the asymptotic observer (7).

Lemma IV.1. *Let $A \in \mathbb{R}^{r\times r}, B \in \mathbb{R}^{r\times(n-r)}, C \in \mathbb{R}^{(l-r)\times r}, D \in \mathbb{R}^{(l-r)\times(n-r)}, F \in \mathbb{R}^{p\times r}, G \in \mathbb{R}^{p\times(n-r)}$ and $L > 0$ be such that* (6) *holds. Assume that $V \in \mathbf{Gl}_{l+p-r}(\mathbb{R}), W \in \mathbb{R}^{r\times r}, P = P^\top \in \mathbb{R}^{(n+k)\times(n+k)}$ and $\delta > 0$, where $k = l + p - n$, solve the following set of LMIs*

$$\begin{bmatrix} W^\top + W + (\delta L^2) I_r & [C^\top, F^\top] V^\top & [I_r, 0] P^\top \\ V \left[\begin{smallmatrix} C \\ F \end{smallmatrix}\right] & -\frac{1}{\delta L^2} I_{l+p-r} & 0 \\ P \left[\begin{smallmatrix} I_r \\ 0 \end{smallmatrix}\right] & 0 & -\delta I_{n+k} \end{bmatrix} < 0, \tag{15a}$$

$$V \begin{bmatrix} D \\ G \end{bmatrix} = \begin{bmatrix} I_{n-r} \\ 0 \end{bmatrix}, \tag{15b}$$

$$P_{11} := [I_r, 0] P \begin{bmatrix} I_r \\ 0 \end{bmatrix} > 0, \tag{15c}$$

under the additional BMI constraint that

$$A - [B, L_1] V \begin{bmatrix} C \\ F \end{bmatrix} = P_{11}^{-\top} W \tag{16}$$

for some $L_1 \in \mathbb{R}^{r\times k}$. Then, with

$$\begin{bmatrix} L_2 \\ L_3 \end{bmatrix} := V^{-1} \begin{bmatrix} 0 \\ I_k \end{bmatrix},$$

we have that the Riccati inequality (10) *is satisfied, where we use the notation from* (8) *and* (9).

Proof. With $K := P_{11}^{-\top} W$ and $Z := P \left[\begin{smallmatrix} I_r \\ 0 \end{smallmatrix}\right]$ we find that, by (15a),

$$\begin{bmatrix} [K^\top, 0] Z + Z^\top \left[\begin{smallmatrix} K \\ 0 \end{smallmatrix}\right] + (\delta L^2) I_r & [C^\top, F^\top] V^\top & Z^\top \\ V \left[\begin{smallmatrix} C \\ F \end{smallmatrix}\right] & -\frac{1}{\delta L^2} I_{l+p-r} & 0 \\ Z & 0 & -\delta I_{n+k} \end{bmatrix} < 0.$$

Using the Schur complement lemma this is equivalent to

$$\begin{bmatrix} [K^\top, 0] Z + Z^\top \left[\begin{smallmatrix} K \\ 0 \end{smallmatrix}\right] + (\delta L^2) \big(I_r + [C^\top, F^\top] V^\top V \left[\begin{smallmatrix} C \\ F \end{smallmatrix}\right]\big) & Z^\top \\ Z & -\delta I_{n+k} \end{bmatrix} < 0. \tag{17}$$

Now, invoking (15b), we have

$$V = \begin{bmatrix} D & L_2 \\ G & L_3 \end{bmatrix}^{-1}$$

and hence we may calculate

$$\begin{aligned}
\begin{bmatrix} K \\ 0_{(l+p-r)\times r} \end{bmatrix} &\overset{(16)}{=} \begin{bmatrix} A - [B, L_1] \left[\begin{smallmatrix} D & L_2 \\ G & L_3 \end{smallmatrix}\right]^{-1} \left[\begin{smallmatrix} C \\ F \end{smallmatrix}\right] \\ 0_{(l+p-r)\times r} \end{bmatrix}\\
&= \begin{bmatrix} A \\ C \\ F \end{bmatrix} - \begin{bmatrix} B & L_1 \\ D & L_2 \\ G & L_3 \end{bmatrix} \begin{bmatrix} D & L_2 \\ G & L_3 \end{bmatrix}^{-1} \begin{bmatrix} C \\ F \end{bmatrix} = \begin{bmatrix} A & B & L_1 \\ C & D & L_2 \\ F & G & L_3 \end{bmatrix} \begin{bmatrix} I_r \\ M \end{bmatrix},
\end{aligned}$$

where M is defined in (9). Furthermore,

$$[C^\top, F^\top] V^\top V \left[\begin{smallmatrix} C \\ F \end{smallmatrix}\right] = M^\top M,$$

and hence (17) becomes

$$\begin{bmatrix} [I_r, M^\top] \mathscr{A}^\top Z + Z^\top \mathscr{A} \left[\begin{smallmatrix} I_r \\ M \end{smallmatrix}\right] + (\delta L^2) \big(I_r + M^\top M\big) & Z^\top \\ Z & -\delta I_{n+k} \end{bmatrix} < 0. \tag{18}$$

Applying the Schur complement lemma to (18) yields

$$[I_r, M^\top] \mathscr{A}^\top Z + Z^\top \mathscr{A} \left[\begin{smallmatrix} I_r \\ M \end{smallmatrix}\right] + \frac{1}{\delta} Z^\top Z + (\delta L^2) \Big(I_r + M^\top M\Big) < 0$$

and this is equivalent to

$$[I_r, M^\top]\left(\mathscr{A}^\top[Z,0] + \begin{bmatrix} Z^\top \\ 0 \end{bmatrix}\mathscr{A} + \frac{1}{\delta}\begin{bmatrix} Z^\top \\ 0 \end{bmatrix}[Z,0] + (\delta L^2) I_{l+p}\right)\begin{bmatrix} I_r \\ M \end{bmatrix} < 0.$$

Invoking (9) this is in turn the same as

$$\mathscr{A}^\top[Z,0] + \begin{bmatrix} Z^\top \\ 0 \end{bmatrix}\mathscr{A} + \frac{1}{\delta}\begin{bmatrix} Z^\top \\ 0 \end{bmatrix}[Z,0] + (\delta L^2) I_{l+p} <_{\mathscr{V}} 0.$$

Observing that

$$[Z,0] = P\mathscr{E}$$

yields (10) and this finishes the proof. □

Remark IV.2.

(i) Note that we refer to (16) as a BMI since the equality can be equivalently written as two inequalities.

(ii) While the equality constraint in (15b) is not a LMI at first sight, it can be incorporated into the other LMIs and BMIs as follows. Let $V_1, \ldots, V_q$ be a basis of the linear subspace

$$\operatorname{span}\left\{ V \in \mathbb{R}^{(l+p-r)\times(l+p-r)} \;\middle|\; V\begin{bmatrix} D \\ G \end{bmatrix} = 0 \right\}$$

and let $V_0 \in \mathbb{R}^{(l+p-r)\times(l+p-r)}$ be such that

$$V_0 \begin{bmatrix} D \\ G \end{bmatrix} = \begin{bmatrix} I_{n-r} \\ 0 \end{bmatrix}.$$

Then solve (15a), (15c) and (16) with $V = V_0 + \sum_{i=1}^q \alpha_i V_i$ for W, P, L_1, δ and $\alpha_1, \ldots, \alpha_q \in \mathbb{R}$. Therefore, it is common to refer to an equality constraint as in (15b) as a LMI as well.

(iii) We like to stress that the solutions of (15a)–(15c) and (16) have to satisfy the additional constraint that V must be invertible. This condition is equivalent to the non-convex quadratic inequality $V^\top V > 0$. By introducing a new variable $J = V$, this inequality can also be equivalently written as a BMI as follows

$$V^\top J > 0, \quad V - J \geq 0, \quad J - V \geq 0.$$

Therefore, the problems (15a)–(15c), (16) consist only of LMIs and BMIs which may be solved by standard MATLAB toolboxes like YALMIP [27] and PENLAB [28]. For other algorithmic approaches see e.g. the tutorial paper [29].

(iv) A careful inspection of the proof of Lemma IV.1 reveals that the opposite implication is true as well, that is if $L_1 \in \mathbb{R}^{r\times k}, L_2 \in \mathbb{R}^{(l-r)\times k}, L_3 \in \mathbb{R}^{p\times k}$, $P = P^\top \in \mathbb{R}^{(n+k)\times(n+k)}$ and $\delta > 0$ solve (10) such that (15c) holds and $\left[\begin{smallmatrix} D & L_2 \\ G & L_3 \end{smallmatrix}\right]$ is invertible, then there exists $W \in \mathbb{R}^{r\times r}$ such that $V = \left[\begin{smallmatrix} D & L_2 \\ G & L_3 \end{smallmatrix}\right]^{-1}, W, P, L_1$ and δ solve (15a)–(15c) and (16). Therefore, solvability of the LMIs and BMIs is necessary and sufficient for solvability of the Riccati inequality (10).

(v) In the case $C = 0$ and $F = 0$ the LMIs (15a)–(15c) always have a solution. We may choose $P = I_{n+k}$ and V such that (15b) is satisfied. With $W = W^\top$ the LMI (15a) leads to the condition $W < -\frac{1}{2}\left(\delta L^2 + \frac{1}{\delta}\right) I_r$. The BMI (16) yields $A = P_{11}^{-\top} W = W$ and choosing $\delta = \frac{1}{L}$ we obtain the condition

$$A < -L I_r$$

on the system data in this case. This is a reasonable condition from the point of view that then the error dynamics

$$\begin{aligned} \dot{e}_1(t) &= A e_1(t) + f_1\big(z_1(t), z_2(t), u(t), y(t)\big) \\ &\quad - f_1\big(x_1(t), x_2(t), u(t), y(t)\big), \\ \begin{pmatrix} e_2(t) \\ d(t) \end{pmatrix} &= 0, \end{aligned}$$

are asymptotically stable for all f_1 which satisfy (2).

V. Observer of Luenberger type

In order to illustrate the observer (7) and Lemma IV.1 we consider the question as to when the observer (7) is of Luenberger type as considered e.g. in [12]. If we would have $L_3 = I_k$, then we can eliminate the variable d in (7) and reformulate it as

$$\begin{aligned} \dot{z}_1(t) &= ([A,B] - L_1[F,G])\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} + L_1\big(y(t) - h\big(u(t)\big)\big) \\ &\quad + f_1\big(z_1(t), z_2(t), u(t), y(t)\big), \\ 0 &= ([C,D] - L_2[F,G])\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} + L_2\big(y(t) - h\big(u(t)\big)\big) \\ &\quad + f_2\big(u(t), y(t)\big), \\ z(t) &= \left(\begin{smallmatrix} z_1(t) \\ z_2(t) \end{smallmatrix}\right), \end{aligned}$$

and the first two equations are equivalent to

$$\begin{aligned} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} &= \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}[F,G]\right)\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \\ &\quad + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}\big(y(t) - h\big(u(t)\big)\big) + \begin{pmatrix} f_1\big(z_1(t), z_2(t), u(t), y(t)\big) \\ f_2\big(u(t), y(t)\big) \end{pmatrix}. \end{aligned}$$

This is a Luenberger type observer for system (1) with gain $\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]$. With $e_1 := z_1 - x_1$ and $e_2 := z_2 - x_2$ the error dynamics (12) become

$$\begin{aligned} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \end{pmatrix} &= \begin{bmatrix} A - L_1 F & B - L_1 G \\ C - L_2 F & D - L_2 G \end{bmatrix}\begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} \\ &+ \begin{pmatrix} f_1\big(z_1(t), z_2(t), u(t), y(t)\big) - f_1\big(x_1(t), x_2(t), u(t), y(t)\big) \\ 0 \end{pmatrix}. \end{aligned} \tag{19}$$

Having a look at dimension, $L_3 = I_k$ can only be true if $k = p$ or, equivalently, $l = n$. If the latter is the case, then the matrix D is square, i.e., $D \in \mathbb{R}^{(n-r)\times(n-r)}$ and hence condition (6) implies existence of $L_2 \in \mathbb{R}^{(n-r)\times k}$ such that $D - L_2 G$ is invertible. Therefore, the matrix $\left[\begin{smallmatrix} D & L_2 \\ G & I_k \end{smallmatrix}\right]$ is invertible, i.e., it is always possible to choose $L_3 = I_k$. In this case, the second equation in (19) can be solved for e_2 and with $Y := (D - L_2 G)^{-1}(C - L_2 F)$ we find

$$\begin{aligned} \dot{e}_1(t) &= \Big((A - L_1 F) - (B - L_1 G)Y\Big) e_1(t) \\ &\quad + f_1\big(z_1(t), z_2(t), u(t), y(t)\big) - f_1\big(x_1(t), x_2(t), u(t), y(t)\big), \\ e_2(t) &= -Y e_1(t). \end{aligned} \tag{20}$$

Summarizing, we find that if $l = n$, then the observer (7) can be chosen to be of Luenberger type by $k = p$, $L_3 = I_k$ and L_2 such that $D - L_2 G$ is invertible.

Finally, we have a look at the solvability of the LMIs (15a)–(15c) together with the BMI (16) in this case. Use the notation from Lemma IV.1, then

$$V = \begin{bmatrix} D & L_2 \\ G & I_k \end{bmatrix}^{-1} = \begin{bmatrix} (D - L_2 G)^{-1} & -(D - L_2 G)^{-1} L_2 \\ -G(D - L_2 G)^{-1} & I_k + G(D - L_2 G)^{-1} L_2 \end{bmatrix}.$$

We restrict ourselves to $P = I_{n+k}$ and $W = W^\top$. Then (15a) reads

$$\begin{bmatrix} 2W + \delta L^2 I_r & Y^\top & (F - GY)^\top & [I_r, 0] \\ Y & -\frac{1}{\delta L^2} I_{n-r} & 0 & 0 \\ F - GY & 0 & -\frac{1}{\delta L^2} I_k & 0 \\ \left[\begin{smallmatrix} I_r \\ 0 \end{smallmatrix}\right] & 0 & 0 & -\delta I_{n+k} \end{bmatrix} < 0.$$

Successively applying the Schur complement lemma yields the equivalent inequality

$$2W + \delta L^2\left(I_r + (F - GY)^\top (F - GY) + Y^\top Y\right) + \frac{1}{\delta} I_r < 0. \tag{21}$$

The equation (16) becomes

$$A - L_1F + L_1GY - BY = W,$$

which needs to be solved for L_1. Together with (21) we obtain the condition

$$\begin{aligned} &A - L_1F + L_1GY - BY \\ &\quad < -\frac{\delta L^2}{2}\left(I_r + (F-GY)^\top(F-GY) + Y^\top Y\right) - \frac{1}{2\delta}I_r. \quad (22) \end{aligned}$$

on L_1, where Y depends on the choice of L_2. Choosing $\delta = \frac{1}{L}$ we obtain

$$A - L_1F + L_1GY - BY < -LI_r - \frac{L}{2}\left((F-GY)^\top(F-GY) + Y^\top Y\right). \quad (23)$$

This condition immediately implies the asymptotic stability of the error dynamics (20) for any f_1 satisfying (2) since

$$\begin{aligned} &\|f_1\big(z_1(t), z_2(t), u(t), y(t)\big) - f_1\big(x_1(t), x_2(t), u(t), y(t)\big)\| \\ &\le L\|(e_1(t), e_2(t))\| \le L\left\|\begin{bmatrix} I_r \\ -Y \end{bmatrix}\right\| \|e_1(t)\| \end{aligned}$$

which explains the term $Y^\top Y$ on the right hand side of (23). The term $(F-GY)^\top(F-GY)$ is due to the fact that the original Riccati inequality (10) is formulated on the space $\mathscr{V}$ for the variables (e_1, e_2, d), so it is actually possible to allow f_1 to depend on the "innovations" $d = (F-GY)e_1$, which is expected by the inequality (23) in some sense.

Different special cases may be discussed in terms of (22). For instance, if $F = 0$ and $C = 0$, then $Y = 0$ and (22) reduces to the inequality discussed in Remark IV.2 (v).

VI. Examples

We consider two illustrative examples with $l \neq n$, i.e., the corresponding observer cannot be reformulated as an observer of Luenberger type as discussed in Section V. In particular, this shows that our observer design is applicable to a larger class of systems than those presented in [11], [12] for instance.

A. Example

Consider (1) with

$$\begin{aligned} &A = [-1], \quad B = [1], \quad C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ &F = [1], \quad G = [0] \end{aligned}$$

and some functions f_1, f_2, h such that (2) is satisfied. Then $k = l + p - n = 2$ and in order to find a solution to (15a)–(15c) and (16) we set $P = I_4$ and

$$L_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad L_3 = [0, 1],$$

thus

$$\begin{bmatrix} D & L_2 \\ G & L_3 \end{bmatrix} = I_3 = V^{-1}.$$

The LMIs (15b) and (15c) are already satisfied and (15a) reads

$$\begin{bmatrix} 2W + \delta L^2 + \frac{1}{\delta} & [0,1,1] \\ \left[\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix}\right] & -\frac{1}{\delta L^2}I_3 \end{bmatrix} < 0$$

which is equivalent to

$$W < -\frac{\delta L^2 + \frac{1}{\delta}}{2} - \delta L^2 = -2L$$

where in the latter equality we chose $\delta = \frac{1}{L}$. Now we choose $W = -2L - 1$. It remains to find L_1 so that (16) is satisfied, i.e.,

$$L_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -1 - W = 2L,$$

which is fulfilled by

$$L_1 = [L, L].$$

Now, the observer (7) reads

$$\begin{aligned} \dot{z}_1(t) &= -z_1(t) + z_2(t) + Ld_1(t) + Ld_2(t) + f_1\big(z_1(t), z_2(t), u(t), y(t)\big), \\ 0 &= z_2(t) + f_{2,1}\big(u(t), y(t)\big), \\ 0 &= z_1(t) + d_1(t) + f_{2,2}\big(u(t), y(t)\big), \\ 0 &= z_1(t) + d_2(t) + h\big(u(t)\big) - y(t), \\ z(t) &= \left(\begin{smallmatrix} z_1(t) \\ z_2(t) \end{smallmatrix}\right). \end{aligned}$$

This can be simplified to the ODE observer

$$\begin{aligned} \dot{z}_1(t) &= -(2L+1)z_1(t) + f_1\Big(z_1(t), -f_{2,1}\big(u(t), y(t)\big), u(t), y(t)\Big) \\ &\quad - f_{2,1}\big(u(t), y(t)\big) - L\Big(f_{2,2}\big(u(t), y(t)\big) + h\big(u(t)\big) - y(t)\Big), \\ z(t) &= \begin{pmatrix} z_1(t) \\ -f_{2,1}\big(u(t), y(t)\big) \end{pmatrix}. \end{aligned}$$

B. Example

In Example A the LMIs and BMIs were solvable for any Lipschitz constant $L > 0$, but usually the solvability depends on the magnitude of L. Consider Example A with the modification

$$C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad F = [0]$$

and choose P, L_2, L_3 and V as before. Then (16) reads

$$-1 - [1, L_1]\left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right] = W,$$

i.e., $W = -2$. From (15a) we obtain $W < -\delta L^2 - \frac{1}{2\delta}$ which gives

$$L^2 < -\frac{1}{2\delta^2} + \frac{2}{\delta}$$

and the right hand side is maximal for $\delta = \frac{1}{2}$. Therefore, we obtain the constraint

$$L < \sqrt{2}$$

on the Lipschitz constant and the LMIs and BMIs are solvable only in this case in general.

VII. Conclusion

In the present paper we developed a novel observer design for DAE systems with Lipschitz nonlinearities. The design parameters of the asymptotic observer are constructed from the solutions of a Riccati type inequality. We have further shown that the solution of certain LMIs and BMIs yields a solution to this Riccati inequality. The solvability of the LMIs and BMIs depends on the magnitude of the Lipschitz constant in general.

The present work is the basis for extensions in several directions such as systems which are not impulse observable. Incorporating the presence of actuator and sensor faults as well as nonlinearities in the output equation (as discussed in [13], [14] for Luenberger type observers) is another interesting extension for future work.

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