THE RELATION BETWEEN THE COMPANION MATRIX AND THE COMPANION POLYNOMIAL IN $\mathbb{R}^4$ ALGEBRAS *

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Dedicated to the memory of Ivo Marek (1933 - 2017)

Abstract. We prove that the quaternionic companion polynomial is identical with the characteristic polynomial of the complex matrix which is obtained by isomorphism from the quaternionic companion matrix. And we show that this is also true for the other three noncommutative algebras in $\mathbb{R}^4$ which include the coquaternions.

Key words. Quaternionic companion matrix, Quaternionic companion polynomial, Characteristic polynomial of quaternionic companion matrix, Coquaternionic companion matrix, Coquaternionic companion polynomial, Characteristic polynomial of coquaternionic companion matrix.

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1. Introduction. We assume that the reader is sufficiently acquainted with the algebra of quaternions and with the problem of finding zeros of unilateral polynomials with quaternionic coefficients as well as with coefficients from nondivision algebras in $\mathbb{R}^4$. For details see [13, 10, 19].

2. Data of problem.

- 1. The field of real numbers, denoted by $\mathbb{R}$, the field of complex numbers, denoted by $\mathbb{C}$, the field of quaternions denoted by $\mathbb{H}$. See [8]. The elements in $\mathbb{H}$ will be denoted by $q = (q_1, q_2, q_3, q_4) \in \mathbb{R}^4$. Numbers of the form $(q_1, 0, 0, 0) \in \mathbb{H}$ will be identified with the real number q_1.
- 2. A given vector

$$a = [a_0, a_1, \ldots, a_n] \in \mathbb{H}^{n+1}, \text{ where } a_0 \text{ is invertible}, a_n = 1. \tag{2.1}$$

- 3. The quaternionic polynomial

$$p(z) = \sum_{j=0}^{n} a_j z^j, \text{ defined by the elements of the given vector } a. \tag{2.2}$$

- 4. The companion polynomial of p defined by

$$c(z) = \sum_{j,k=0}^{n} a_j \overline{a_k} z^{j+k} = \sum_{\ell=0}^{2n} b_\ell z^\ell, \quad b_\ell = \sum_{m=\max(0,\ell-n)}^{\min(\ell,n)} \overline{a_m} a_{\ell-m} \in \mathbb{R}. \tag{2.3}$$

- 5. The companion matrix of p defined by

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & 0 & -a_1 \\ \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & -a_{n-2} \\ 0 & 0 & \ldots & 1 & -a_{n-1} \end{bmatrix} \in \mathbb{H}^{n \times n}. \tag{2.4}$$

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If a, defined in (2.1), is a real or complex vector, we have the following classical theorem.

THEOREM 2.1. *Let the vector a be real or complex. Then the eigenvalues of $\mathbf{C}$ are identical with the zeros of the polynomial p.*

Proof. Horn and Johnson, [9, p. 146/147]. ∎

3. Eigenvalues for quaternionic matrices. The eigenvalue problem for a general quaternionic matrix $\mathbf{A}$ of order n has the form

$$\mathbf{A}\mathbf{x} = \mathbf{x}\lambda, \quad \mathbf{x} \neq 0. \tag{3.1}$$

It has one interesting property which a real or complex eigenvalue problem does not have. Let $h \neq 0$ and multiply equation (3.1) from the right by h. Then,

$$\mathbf{A}\mathbf{x}h = \mathbf{x}\lambda h = \mathbf{x}h(h^{-1}\lambda h).$$

That means if λ is an eigenvalue of $\mathbf{A}$ with respect to the eigenvector $\mathbf{x}$, then the similar value $h^{-1}\lambda h$ is also an eigenvalue of $\mathbf{A}$ with respect to $\mathbf{x}h$. And this is true for all $h \neq 0$. Put $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{H}$. Thus, the similarity class $[\lambda]$ consists only of eigenvalues and in each similarity class there is a unique representative of the complex form

$$\lambda := \lambda_1 + \sqrt{\lambda_2^2 + \lambda_3^2 + \lambda_4^2}\,\mathbf{i}.$$

These eigenvalue are called *standard eigenvalues* of $\mathbf{A}$. A result by Brenner, [1] and Lee, [15] states that all quaternionic matrices of order n have n complex standard eigenvalues. See also Zhang, [23]. Information on similarity and quasi similarity can be gained in [10, Section 2.1]. Already 1936 a paper on similar, quaternionic matrices published by Wolf appeared, [22].

Let $a \in \mathbb{H}$ and $\overline{a}$ be the conjugate of a, also denoted by $\mathrm{conj}(a)$. Then throughout the paper, and also for algebras introduced later, we use the notation

$$\mathrm{abs}_2(a) = a\,\overline{a}.$$

This number is real in $\mathbb{H}$ and also in all other algebras introduced later. In $\mathbb{H}$ we have $\mathrm{abs}_2(a) = ||a||^2$, where $||\cdot||$ is the euclidean $\mathbb{R}^4$ norm. In all algebras a is invertible if and only if $\mathrm{abs}_2(a) \neq 0$. This implies that a and $\overline{a}$ are simultaneously invertible or simultaneously not invertible. At this point we can already prove the following theorem.

THEOREM 3.1. *Let p be a given polynomial as described in the second section. Then all solutions z of $p(z) = 0$ are invertible.*

Proof. First we obseve that $p(0) = a_0$ is not zero. Let z be a noninvertible solution of $p(z) = 0$. We multiply p from the right by $\overline{z}$. Then $p(z)\overline{z} = a_0\overline{z} + \sum_{j=1}^n a_j z^{j-1} z\overline{z} = a_0\overline{z} = 0$. Since a_0 is invertible, this implies $\overline{z} = z = 0$ which is a contradiction. ∎

We will use the following lemma.

LEMMA 3.2. *Let $a, b \in \mathbb{C}$ with nonnegative imaginary part. The two complex numbers a, b are similar in $\mathbb{H}$, if and only if they are equal.*

Proof. A necessary and sufficient condition for similarity of two elements $a, b \in \mathbb{H}$ is $\Re(a) = \Re(b)$ and $\mathrm{abs}_2(a) = \mathrm{abs}_2(b)$. See [10, Section 2.1]. Let $a = a_1 + a_2\mathbf{i}, b = b_1 + b_2\mathbf{i}$ with $a_2 \geq 0, b_2 \geq 0$. Then the first similarity condition implies $a_1 = b_1$ and the second $a_1^2 + a_2^2 = b_1^2 + b_2^2$ which altogether implies $a = b$. ∎

In order to compute the eigenvalues of the quaternionic matrix $\mathbf{A}$, we use the isomorphic representation of the quaternion $q = (q_1, q_2, q_3, q_4)$ in the complex (2×2) matrix form

$$\imath(q) := \begin{bmatrix} q_1 + q_2\mathbf{i} & q_3 + q_4\mathbf{i} \\ -q_3 + q_4\mathbf{i} & q_1 - q_2\mathbf{i} \end{bmatrix} =: \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix}. \tag{3.2}$$

See v. d. Waerden, [21, p. 55]. We observe that $\det(\imath(q)) = \mathrm{abs}_2(q) = ||q||^2$. Put

$$\begin{aligned} \mathbf{A} &= (\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4) = \mathbf{A}_1 + \mathbf{A}_2\mathbf{i} + \mathbf{A}_3\mathbf{j} + \mathbf{A}_4\mathbf{k},\ \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4 \in \mathbb{R}^{n\times n}, \\ \imath(\mathbf{A}) &:= \begin{bmatrix} \mathbf{A}_1 + \mathbf{A}_2\mathbf{i} & \mathbf{A}_3 + \mathbf{A}_4\mathbf{i} \\ -\mathbf{A}_3 + \mathbf{A}_4\mathbf{i} & \mathbf{A}_1 - \mathbf{A}_2\mathbf{i} \end{bmatrix} =: \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ -\overline{\mathbf{B}_2} & \overline{\mathbf{B}_1} \end{bmatrix} \in \mathbb{C}^{2n\times 2n}, \end{aligned} \tag{3.3}$$

which is in coincidence with (3.2). The matrix $\imath(\mathbf{A})$ is a complex 2×2 block matrix, where the blocks are of order n. The quantities $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ denote the 2nd, 3rd, and 4th standard unit vector in $\mathbb{R}^4$, respectively, regarded as algebra elements.

Though $\imath(\mathbf{A})$ is complex the eigenvalues of $\imath(\mathbf{A})$ are real or appear in pairs of complex conjugate numbers and the eigenvalues with nonnegative imaginary part coincide with the standard eigenvalues of $\mathbf{A}$. This result can be found in Zhang, [23]. The eigenvalues of a quaternionic matrix were used for finding quaternionic polynomial zeros by Serôdio, Pereira, Vitória, [19].

We also use the notation

$$\mathbf{A}_1 = \Re(\mathbf{A}),\ \mathbf{A}_2 = \Im(\mathbf{A}),\ \mathbf{A}_3 = \Im_3(\mathbf{A}),\ \mathbf{A}_4 = \Im_4(\mathbf{A}).$$

Let $\mathbf{C}$ now be the companion matrix defined in (2.4). If we apply (3.3) to $\mathbf{C}$ we obtain the complex matrix

$$\imath(\mathbf{C}) := \left[\begin{array}{ccccc|ccccc} 0 & 0 & \dots & 0 & \alpha_0 & 0 & 0 & \dots & 0 & \beta_0 \\ 1 & 0 & \dots & 0 & \alpha_1 & 0 & 0 & \dots & 0 & \beta_1 \\ & \ddots & \ddots & & & & \ddots & \ddots & & \\ 0 & 0 & \ddots & 0 & \alpha_{n-2} & 0 & 0 & \ddots & 0 & \beta_{n-2} \\ 0 & 0 & \dots & 1 & \alpha_{n-1} & 0 & 0 & \dots & 0 & \beta_{n-1} \\ \hline 0 & 0 & \dots & 0 & -\overline{\beta_0} & 0 & 0 & \dots & 0 & \overline{\alpha_0} \\ 0 & 0 & \dots & 0 & -\overline{\beta_1} & 1 & 0 & \dots & 0 & \overline{\alpha_1} \\ & \ddots & \ddots & & & & \ddots & \ddots & & \\ 0 & 0 & \ddots & 0 & -\overline{\beta_{n-2}} & 0 & 0 & \ddots & 0 & \overline{\alpha_{n-2}} \\ 0 & 0 & \dots & 0 & -\overline{\beta_{n-1}} & 0 & 0 & \dots & 1 & \overline{\alpha_{n-1}} \end{array}\right]. \tag{3.4}$$

where $\alpha_\ell, \beta_\ell \in \mathbb{C},\ \ell = 0, 1, \dots, n-1$ are defined in (3.2) and also in (3.5), (3.6).

The quaternionic polynomial p will have quaternionic zeros which in general are not comparable with the complex eigenvalues of $\imath(\mathbf{C})$. Therefore, it is in general not possible, that the zeros of p coincide with the eigenvalues of $\mathbf{C}$ and Theorem 2.1 is not valid in the quaternionic case.

LEMMA 3.3. *The coefficients $a_0, a_1, \dots, a_{n-1}$ (which are defined in (2.1) and occur in (2.2), (2.3), and in (2.4)) can be recovered from $\imath(\mathbf{C})$ which is defined in (3.4).*

Proof. By definition (see (3.2) and (3.3)) we have

$$\alpha_\ell = -\big(\Re(a_\ell) + \Im(a_\ell)\mathbf{i}\big), \tag{3.5}$$

$$\beta_\ell = -\big(\Im_3(a_\ell) + \Im_4(a_\ell)\mathbf{i}\big),\ \ell = 0, 1, \dots, n-1. \tag{3.6}$$

It follows that

$$\alpha_\ell + \beta_\ell\mathbf{j} = -a_\ell, \quad \overline{\alpha_\ell + \beta_\ell\mathbf{j}} = \overline{\alpha_\ell} - \mathbf{j}\,\overline{\beta_\ell} = -\overline{a_\ell},\ \ell = 0, 1, \dots, n-1. \tag{3.7}$$

□

We will use the following notation for the characteristic polynomial of $\imath(\mathbf{C})$.

$$\chi(\lambda) := \det(\mathbf{I}\lambda - \imath(\mathbf{C})) = \sum_{j=0}^{2n} \chi_j \lambda^j, \quad \chi_{2n} = 1,\ \chi_0 \text{ invertible.} \tag{3.8}$$

The theorem we want to prove is the following.

THEOREM 3.4. *The eigenvalues of the matrix $\imath(\mathbf{C})$ are identical with the roots of the companion polynomial c. Or in other words: The characteristic polynomial χ of $\imath(\mathbf{C})$ coincides with the companion polynomial c of p. Or: The standard eigenvalues of $\mathbf{C}$ coincide with the roots of c which have nonnegative imaginary parts.*

This Theorem was already conjectured in [10, Conjecture 8.1].

We use a polynomial p of degree 6 as an example in order to show that there is high evidence for Theorem 3.4. Cf. Example 3.8 from [13].

EXAMPLE 3.5. Let the coefficients of p (in the order of (2.1)) be

$$a = [-\mathbf{i}, -\mathbf{j}, -1, 0, \mathbf{i}, \mathbf{j}, 1] \in \mathbb{H}^7. \tag{3.9}$$

The companion polynomial c of p is

$$c(z) = z^{12} + z^{10} - z^8 - 2z^6 - z^4 + z^2 + 1 \tag{3.10}$$

and the 12 roots of c are

$$\pm 1,\ \pm 1,\ \pm\mathbf{i},\ \pm\mathbf{i},\ (\pm 1 \pm \sqrt{3}\,\mathbf{i})/2. \tag{3.11}$$

Because of the symmetry of the coefficients of c the set of the inverse roots of c is the same as the set of the roots. However, this does not play a role here. If we compute the complex matrix $\imath(\mathbf{C})$ for these data, we find that the 12 eigenvalues of $\imath(\mathbf{C})$ are the same as the 12 roots of c. The matrix $\imath(\mathbf{C})$ is in this case

$$\imath(\mathbf{C}) = \left[\begin{array}{cccccr|cccccr}
0 & 0 & 0 & 0 & 0 & \mathbf{i} & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -\mathbf{i} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{i} \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \mathbf{i} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]. \tag{3.12}$$

We have tried out many examples numerically and in all cases Theorem 3.4 was true. At least to computer precision. For zeros of p see [10, 13, 16, 17, 19].

In the next two sections we show directly that Theorem 3.4 is true for $n = 1$, $n = 2$, and $n = 3$. In a subsequent section we try to use the block structure of the matrix $\imath(\mathbf{C})$ for a proof, however, this approach is in the end not successful.

4. The cases $n \le 2$. Let $n = 1$ and $p(z) = z + a$, where a is invertible. The companion polynomial of p is $c(z) = z^2 + 2\Re(a)z + \mathrm{abs}_2(a)$. The companion matrix is $\mathbf{C} = [-a]$ and, using (3.2),

$$\mathbf{I}\lambda - \imath(\mathbf{C}) = \begin{bmatrix} \lambda + \Re(a) + \Im(a)\mathbf{i} & \Im_3(a) + \Im_4(a)\mathbf{i} \\ -\Im_3(a) + \Im_4(a)\mathbf{i} & \lambda + \Re(a) - \Im(a)\mathbf{i} \end{bmatrix}.$$

The determinant of this matrix, $\chi(\lambda)$, coincides with the companion polynomial $c(\lambda)$.

Let $n = 2$. The companion matrix has in this case the form

$$\mathbf{C} = \begin{bmatrix} 0 & -a_0 \\ 1 & -a_1 \end{bmatrix} \in \mathbb{H}^{2\times 2}.$$

The corresponding complex eigenvalue problem is defined by

$$\mathbf{I}\lambda - \imath(\mathbf{C}) = \begin{bmatrix} \lambda & -\alpha_0 & 0 & -\beta_0 \\ -1 & \lambda - \alpha_1 & 0 & -\beta_1 \\ 0 & \overline{\beta_0} & \lambda & -\overline{\alpha_0} \\ 0 & \overline{\beta_1} & -1 & \lambda - \overline{\alpha_1} \end{bmatrix} \in \mathbb{C}^{4\times 4}. \tag{4.1}$$

For the definition of $\alpha_\ell, \beta_\ell \in \mathbb{C},\ \ell = 0, 1$ see (3.5), (3.6). We compute the eigenvalues by using the expansion formula:

$$\det(\mathbf{I}\lambda - \imath(\mathbf{C})) = \lambda \det \begin{bmatrix} \lambda - \alpha_1 & 0 & -\beta_1 \\ \overline{\beta_0} & \lambda & -\overline{\alpha_0} \\ \overline{\beta_1} & -1 & \lambda - \overline{\alpha_1} \end{bmatrix} + \det \begin{bmatrix} -\alpha_0 & 0 & -\beta_0 \\ \overline{\beta_0} & \lambda & -\overline{\alpha_0} \\ \overline{\beta_1} & -1 & \lambda - \overline{\alpha_1} \end{bmatrix},$$

$$\det \begin{bmatrix} \lambda - \alpha_1 & 0 & -\beta_1 \\ \overline{\beta_0} & \lambda & -\overline{\alpha_0} \\ \overline{\beta_1} & -1 & \lambda - \overline{\alpha_1} \end{bmatrix} = \lambda \det \begin{bmatrix} \lambda - \alpha_1 & -\beta_1 \\ \overline{\beta_1} & \lambda - \overline{\alpha_1} \end{bmatrix} + \det \begin{bmatrix} \lambda - \alpha_1 & -\beta_1 \\ \overline{\beta_0} & -\overline{\alpha_0} \end{bmatrix} =$$

$$\lambda\big((\lambda - \alpha_1)(\lambda - \overline{\alpha_1}) + ||\beta_1||^2\big) - (\lambda - \alpha_1)\overline{\alpha_0} + \overline{\beta_0}\beta_1,$$

$$\det \begin{bmatrix} -\alpha_0 & 0 & -\beta_0 \\ \overline{\beta_0} & \lambda & -\overline{\alpha_0} \\ \overline{\beta_1} & -1 & \lambda - \overline{\alpha_1} \end{bmatrix} = \lambda \det \begin{bmatrix} -\alpha_0 & -\beta_0 \\ \overline{\beta_1} & \lambda - \overline{\alpha_1} \end{bmatrix} + \det \begin{bmatrix} -\alpha_0 & -\beta_0 \\ \overline{\beta_0} & -\overline{\alpha_0} \end{bmatrix} =$$

$$\lambda\big(- \alpha_0(\lambda - \overline{\alpha_1}) + \beta_0\overline{\beta_1}\big) + ||\alpha_0||^2 + ||\beta_0||^2.$$

If we put all results together we obtain the characteristic polynomial of (4.1):

$$\begin{aligned}\chi(\lambda) = \det(\mathbf{I}\lambda - \imath(\mathbf{C})) = \lambda^4 - 2\Re(\alpha_1)\lambda^3 + \big(||\alpha_1||^2 + ||\beta_1||^2 - 2\Re(\alpha_0)\big)\lambda^2 + \\ \big(2\Re(\alpha_0\overline{\alpha_1} + \beta_0\overline{\beta_1})\big)\lambda + ||\alpha_0||^2 + ||\beta_0||^2.\end{aligned} \tag{4.2}$$

A comparison with the corresponding companion polynomial (see [17, Section 3]) reveals that (4.2) coincides with the companion polynomial. Thus, Theorem 3.4 is valid for $n = 1$ and $n = 2$ by using a technical proof.

5. The case $n = 3$. Let $n = 3$. We suppose that there is a given vector $a = [a_0, a_1, a_2, a_3] \in \mathbb{H}^4$, where a_0 is invertible and $a_3 = 1$. The quaternionic polynomial p defined by the elements of the given vector a has the form

$$p(z) = \sum_{j=0}^{3} a_j z^j \,. \tag{5.1}$$

The companion polynomial c of p reads as

$$c(z) = \sum_{j,k=0}^{3} a_j \overline{a_k} z^{j+k} = \sum_{\ell=0}^{6} b_\ell z^\ell, \quad b_\ell = \sum_{m=\max(0,\ell-3)}^{\min(\ell,3)} \overline{a_m} a_{\ell-m} \in \mathbb{R}\,. \tag{5.2}$$

The coefficients $b_\ell,\ \ell = 0, 1, \ldots, 6$ read explicitly:

$$\begin{array}{rcl}
b_0 & = & \overline{a_0}a_0, \\
b_1 & = & \overline{a_0}a_1 + \overline{a_1}a_0, \\
b_2 & = & \overline{a_0}a_2 + \overline{a_1}a_1 + \overline{a_2}a_0, \\
b_3 & = & \overline{a_0}a_3 + \overline{a_1}a_2 + \overline{a_2}a_1 + \overline{a_3}a_0, \\
b_4 & = & \overline{a_1}a_3 + \overline{a_2}a_2 + \overline{a_3}a_1, \\
b_5 & = & \overline{a_2}a_3 + \overline{a_3}a_2, \\
b_6 & = & \overline{a_3}a_3.
\end{array}$$

Because $a_3 = 1$, the coefficients of the companion polynomial c are:

$$\begin{aligned}
&b_0 = \operatorname{abs}_2(a_0), \\
&b_1 = 2\Re(\overline{a_0}a_1), \\
&b_2 = \operatorname{abs}_2(a_1) + 2\Re(\overline{a_0}a_2), \\
&b_3 = 2\Re(a_0 + \overline{a_1}a_2), \\
&b_4 = \operatorname{abs}_2(a_2) + 2\Re(a_1), \\
&b_5 = 2\Re(a_2), \\
&b_6 = 1.
\end{aligned} \tag{5.3}$$

The companion matrix has the form

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} \in \mathbb{H}^{3\times 3}\,.$$

The corresponding eigenvalue problem is defined by

$$\mathbf{I}\lambda - \imath(\mathbf{C}) = \left[\begin{array}{ccc|ccc}
\lambda & 0 & -\alpha_0 & 0 & 0 & -\beta_0 \\
-1 & \lambda & -\alpha_1 & 0 & 0 & -\beta_1 \\
0 & -1 & \lambda - \alpha_2 & 0 & 0 & -\beta_2 \\
\hline
0 & 0 & \overline{\beta_0} & \lambda & 0 & -\overline{\alpha_0} \\
0 & 0 & \overline{\beta_1} & -1 & \lambda & -\overline{\alpha_1} \\
0 & 0 & \overline{\beta_2} & 0 & -1 & \lambda - \overline{\alpha_2}
\end{array}\right] \in \mathbb{C}^{6\times 6}\,. \tag{5.4}$$

The expansion formula gives

$$\chi(\lambda) = \det(\mathbf{I}\lambda - \imath(\mathbf{C})) =$$

$$= \lambda \det \begin{bmatrix} \lambda & -\alpha_1 & 0 & 0 & -\beta_1 \\ -1 & \lambda-\alpha_2 & 0 & 0 & -\beta_2 \\ 0 & \overline{\beta_0} & \lambda & 0 & -\overline{\alpha_0} \\ 0 & \overline{\beta_1} & -1 & \lambda & -\overline{\alpha_1} \\ 0 & \overline{\beta_2} & 0 & -1 & \lambda-\overline{\alpha_2} \end{bmatrix} + \det \begin{bmatrix} 0 & -\alpha_0 & 0 & 0 & -\beta_0 \\ -1 & \lambda-\alpha_2 & 0 & 0 & -\beta_2 \\ 0 & \overline{\beta_0} & \lambda & 0 & -\overline{\alpha_0} \\ 0 & \overline{\beta_1} & -1 & \lambda & -\overline{\alpha_1} \\ 0 & \overline{\beta_2} & 0 & -1 & \lambda-\overline{\alpha_2} \end{bmatrix} =$$

$$= \lambda^6 + \big(-\overline{\alpha_2}-\alpha_2\big)\lambda^5 + \big(\|\beta_2\|^2+\|\alpha_2\|^2-\overline{\alpha_1}-\alpha_1\big)\lambda^4 + \big(\overline{\alpha_1}\alpha_2+\overline{\beta_1}\beta_2+\overline{\beta_2}\beta_1+\overline{\alpha_2}\alpha_1-\overline{\alpha_0}-\alpha_0\big)\lambda^3 +$$
$$+ \big(\|\alpha_1\|^2+\|\beta_1\|^2+\overline{\alpha_0}\alpha_2+\overline{\beta_0}\beta_2+\overline{\beta_2}\beta_0+\overline{\alpha_2}\alpha_0\big)\lambda^2 + \big(\overline{\alpha_1}\alpha_0+\overline{\alpha_0}\alpha_1+\overline{\beta_0}\beta_1+\overline{\beta_1}\beta_0\big)\lambda + \|\alpha_0\|^2+\|\beta_0\|^2,$$

which represents the characteristic polynomial χ of $\imath(\mathbf{C})$. We have to show that it coincides with the companion polynomial c of p. For this purpose we use the identities (3.5), (3.6) to replace α_j, β_j by $a_j,\ j = 0, 1, 2$. We compare coefficient by coefficient.

$$\begin{aligned}
\chi_0 &= ||\alpha_0||^2 + ||\beta_0||^2 = \mathrm{abs}_2(a_0) = b_0, \\
\chi_1 &= 2\Re(\overline{\alpha_0}\alpha_1 + \overline{\beta_0}\beta_1) = 2\Re(\overline{a_0}a_1) = b_1, \\
\chi_2 &= \|\alpha_1\|^2 + \|\beta_1\|^2 + \overline{\alpha_0}\alpha_2 + \overline{\beta_0}\beta_2 + \overline{\beta_2}\beta_0 + \overline{\alpha_2}\alpha_0 = \mathrm{abs}_2(a_1) + 2\Re(\overline{\alpha_0}\alpha_2 + \overline{\beta_0}\beta_2) = \\
&= \mathrm{abs}_2(a_1) + 2\Re(\overline{a_0}a_2) = b_2, \\
\chi_3 &= \overline{\alpha_1}\alpha_2 + \overline{\beta_1}\beta_2 + \overline{\beta_2}\beta_1 + \overline{\alpha_2}\alpha_1 - \overline{\alpha_0} - \alpha_0 = \\
&= 2\Re\big(\overline{\alpha_1}\alpha_2 + \overline{\beta_1}\beta_2 - \alpha_0\big) = 2\Re(\overline{a_1}a_2 + a_0) = b_3, \\
\chi_4 &= \|\beta_2\|^2 + \|\alpha_2\|^2 - \overline{\alpha_1} - \alpha_1 = \mathrm{abs}_2(a_2) - 2\Re(\alpha_1) = \mathrm{abs}_2(a_2) + 2\Re(a_1) = b_4, \\
\chi_5 &= -\overline{\alpha_2} - \alpha_2 = -2\Re(\alpha_2) = 2\Re(a_2) = b_5, \\
\chi_6 &= 1 = b_6.
\end{aligned}$$

Thus, for $n = 3$ the companion polynomial c of p and the characteristic polynomial χ of $\imath(\mathbf{C})$ coincide and Theorem 3.4 is valid in this case.

6. The use of the block structure. The definition of $\imath(\mathbf{C})$ has the form of a 2×2 block matrix. See (3.3). Let $\mathbf{I}_n, \mathbf{I}_{2n}$ be two identity matrices with order $n, 2n$, respectively. We can therefore write

$$\lambda \mathbf{I}_{2n} - \imath(\mathbf{C}) =: \begin{bmatrix} \lambda\mathbf{I}_n - \mathbf{B}_1 & \mathbf{B}_2 \\ -\overline{\mathbf{B}_2} & \lambda\mathbf{I}_n - \overline{\mathbf{B}_1} \end{bmatrix} \in \mathbb{C}^{2n\times 2n}, \tag{6.1}$$

and use the fact that $\mathbf{B}_1$ is again a companion matrix and that the rank of $\mathbf{B}_2$ is one and use a determinant formula for a block matrix. For a block matrix there is the following determinant formula: Assume that $\lambda\mathbf{I}_n - \overline{\mathbf{B}_1}$ is invertible. Then

$$\det\big(\lambda\mathbf{I}_{2n} - \imath(\mathbf{C})\big) = \det\Big((\lambda\mathbf{I}_n - \mathbf{B}_1)(\lambda\mathbf{I}_n - \overline{\mathbf{B}_1}) - \mathbf{B}_2(\lambda\mathbf{I}_n - \overline{\mathbf{B}_1})^{-1}(-\overline{\mathbf{B}}_2)(\lambda\mathbf{I}_n - \overline{\mathbf{B}_1})\Big).$$

See [20]. The condition that $\lambda\mathbf{I}_n - \overline{\mathbf{B}_1}$ is invertible means, that λ is not an eigenvalue of $\overline{\mathbf{B}_1}$. If we use Example 3.5 again, we find that the eigenvalues of $\overline{\mathbf{B}_1}$ are

$$\pm\mathbf{i},\ \pm1,\ \pm\sqrt{0.5}(1-\mathbf{i}).$$

However, the eigenvalues of $\imath(\mathbf{C})$ contain the values $\pm1, \pm\mathbf{i}$ (see (3.11)) such that an exclusion of these values will not allow to find the full spectrum of $\imath(\mathbf{C})$ by application of the block matrix formula. Thus, the idea of using the block structure is not leading to success.

7. The proof. We will furnish a nontechnical proof of Theorem 3.4 which goes as follows.

Proof. The authors Serôdio, Pereira, Vitória, [19], have shown, that Niven's algorithm will find all zeros of a given quaternionic polynomial p if the standard eigenvalues (complex with nonnegative imaginary part) of the quaternionic companion matrix (see (2.4)) are used to solve Niven's algorithm. The standard eigenvalues serve as a means in Niven's algorithm to find a similar but in general quaternionic value which will be a zero of p. Recently, it was shown by Opfer, [17], that the same can be accomplished by using the roots of the companion polynomial (see (2.3)). In principle it would be sufficient that the standard eigenvalues and the roots of the companion polynomial would be similar. However, Lemma 3.2 says that similarity of complex numbers implies equality. $\square$

8. An extension to the other noncommutative $\mathbb{R}^4$ algebras. We will see that Theorem 3.4 carries over to other $\mathbb{R}^4$ algebras, namely to $\mathbb{H}_{\rm coq}$, the algebra of coquaternions, to $\mathbb{H}_{\rm nec}$ the algebra of nectarines and to $\mathbb{H}_{\rm con}$ the algebra of conectarines. The algebra $\mathbb{H}_{\rm coq}$ was introduced by Cockle, [2, 1849], [3] and the other two algebras by Schmeikal, [18, 2014]. All three algebras are noncommutative, nondivision algebras. For completeness we present the multiplication rules for $\mathbb{H}$, $\mathbb{H}_{\rm coq}$, $\mathbb{H}_{\rm nec}$, $\mathbb{H}_{\rm con}$ in Table 8.1.

TABLE 8.1. The multiplication tables for $\mathbb{H}$, $\mathbb{H}_{\rm coq}$, $\mathbb{H}_{\rm nec}$, $\mathbb{H}_{\rm con}$.

(8.1)

$\mathbb{H}$	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	-1	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	-1	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	-1

$\mathbb{H}_{\rm coq}$	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	-1	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	1	$-\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$	1

(8.2)

$\mathbb{H}_{\rm nec}$	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	1	$\mathbf{k}$	$\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	-1	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$-\mathbf{j}$	$-\mathbf{i}$	1

$\mathbb{H}_{\rm con}$	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	1	$\mathbf{k}$	$\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	1	$-\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$-\mathbf{j}$	$\mathbf{i}$	-1

Detailed information on these algebras is given in [12, 17]. If p is a polynomial as given in (2.2), the finding of zeros of p in one of these three algebras is very similar, but not the same as finding them in $\mathbb{H}$. There is one principle difference. The number of zeros for polynomials over $\mathbb{H}$ is limited to the degree n, see [6]. In the other three algebras there may be up to $2n(n-1)$ zeros. See [10]. It was shown by Opfer [17] that Niven's algorithm could also be used in the three nondivision algebras $\mathbb{H}_{\rm coq}, \mathbb{H}_{\rm nec}, \mathbb{H}_{\rm con}$ by applying the roots of the companion polynomial. The algebras mentioned belong to the class of Clifford algebras. Introductions to Clifford algebras can be found in [4], [7]. Algebras over $\mathbb{R}^N$ are also called geometric algebras, [5].

In order to find the eigenvalues of the companion matrix, defined over $\mathbb{H}_{\rm coq}, \mathbb{H}_{\rm nec}, \mathbb{H}_{\rm con}$, proceed analogously to (3.2), (3.3), but use the isomorphisms

$$\imath_2 : \mathbb{H}_{\rm coq} \to \mathbb{R}^{2\times 2}, \quad \imath_3 : \mathbb{H}_{\rm nec} \to \mathbb{R}^{2\times 2}, \quad \imath_4 : \mathbb{H}_{\rm con} \to \mathbb{R}^{2\times 2}$$

defined by

$$\imath_2(q) := \begin{bmatrix} q_1+q_4 & q_2+q_3 \\ -q_2+q_3 & q_1-q_4 \end{bmatrix}, \tag{8.3}$$

$$\imath_3(q) := \begin{bmatrix} q_1-q_4 & q_2+q_3 \\ q_2-q_3 & q_1+q_4 \end{bmatrix}, \tag{8.4}$$

$$\imath_4(q) := \begin{bmatrix} q_1-q_3 & q_2+q_4 \\ q_2-q_4 & q_1+q_3 \end{bmatrix}, \tag{8.5}$$

respectively, where $q = (q_1, q_2, q_3, q_4)$ in all algebras.

See [12] for more details and also Lam, [14, p. 52]. Observe that in all three algebras $\det(\imath_k(q)) = \mathrm{abs}_2(q)$ and that $\imath_k(q), k = 2, 3, 4$ is a real matrix. If $\mathbf{C}$ is the companion matrix we define $\imath_k(\mathbf{C}), k = 2, 3, 4$ in the same way as (3.3). Since $\imath_k(\mathbf{C}), k = 2, 3, 4$ is a real matrix, the eigenvalues appear in pairs of complex conjugate values or real values. Since the order of $\imath_k(\mathbf{C}), k = 2, 3, 4$ is $2n$, real eigenvalues, if any, appear also in an even number. That there are matrices over nondivision algebras which have no eigenvalues was shown in [11]. In order to show that one has to use the fact that eigenvectors by definition must contain an invertible component. For more details see the already quoted paper [11].

EXAMPLE 8.2. We use the same coefficients a defined in (3.5) but as coefficients of a polynomial in $\mathbb{H}_{\rm coq}$. In this case the companion polynomial is

$$c(z) = z^{12} - z^{10} - z^8 + 2z^6 - z^4 - z^2 + 1.$$

It has the 12 roots

$$\pm 1,\ \pm 1,\ \pm \mathbf{i},\ \pm \mathbf{i},\ (\pm\sqrt{3} \pm \mathbf{i})/2. \tag{8.6}$$

Let $\mathbf{C}$ be the 6×6 companion matrix over $\mathbb{H}_{\rm coq}$ defined by the coefficients a. See (2.4). Then the real 12×12 matrix $\imath_2(\mathbf{C})$ is

$$\imath_2(\mathbf{C}) = \left[\begin{array}{cccccc|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0
\end{array}\right].$$

It has exactly the eigenvalues given as roots of c in (8.6).

In order to show that Theorem 3.4 is also valid for algebras other than quaternions, we show that the algorithm given by Serôdio, Pereira, and Vitória [19] can also be applied to the other algebras. Let $\mathcal{A}$ be one of the three algebras $\mathbb{H}_{\rm coq}, \mathbb{H}_{\rm nec}, \mathbb{H}_{\rm con}$. We have already shown in [17] that the Niven algorithm works in $\mathcal{A}$, where the roots of the companion polynomials were used.

An eigenvalue of a matrix in one of these algebras is an algebra element with a corresponding eigenvector where at least one component is invertible. See [11]. In what follows (up to Theorem 8.5) we use essentially the ideas of the authors of [19].

LEMMA 8.3. *Let $\mathbf{C}$ be the companion matrix as defied in (2.4) but over $\mathcal{A}$. Let λ be a left eigenvalue of the transposed matrix $\mathbf{C}^{\rm T}$, which means that*

$$\mathbf{C}^{\mathbf{T}}\mathbf{x} = \lambda\mathbf{x}, \quad \mathbf{x} \textit{ has at least one invertible component}. \tag{8.7}$$

Then λ is a zero of the polynomial p.

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\rm T}$. The equation (8.7) reads explicitly:

$$\begin{aligned}
x_2 &= \lambda x_1, \\
x_3 &= \lambda x_2, \\
&\vdots \\
x_j &= \lambda x_{j-1}\,, j = 2, 3, \ldots, n, \\
&\vdots \\
-(a_0 x_1 + a_1 x_2 + \cdots + a_{n-1} x_n) &= \lambda x_n.
\end{aligned} \tag{8.8}$$

The first $n-1$ equations can be written as

$$x_j = \lambda^{j-1} x_1, j = 2, 3, \ldots, n. \tag{8.9}$$

If we insert $x_j, j = 2, 3, \ldots, n$ into the last equation we obtain

$$-\,(a_0 x_1 + a_1 \lambda x_1 + \cdots + a_{n-1}\lambda^{n-1} x_1) = \lambda^n x_1. \tag{8.10}$$

Let x_1 be not invertible. Then, from (8.9) it follows that all $x_j, j = 2, 3, \ldots, x_n$ are noninvertible. This contradicts the definition of an eigenvector. Thus, x_1 must be invertible. Multiplying equation (8.10) from the right by x_1^{-1} yields the desired result. ∎

If we apply the previous Theorem 3.1, we conclude that the left eigenvalues λ of $\mathbf{C}^{\rm T}$ are invertible.

LEMMA 8.4. *The left eigenvalues λ of $\mathbf{C}^{\rm T}$ are also the right eigenvalue of $\mathbf{C}^{\rm T}$.*

Proof. We find that $\mathbf{x} = (1, \lambda, \lambda^2, \dots, \lambda^n)^{\mathrm{T}}$ is an eigenvector of the left eigenvalue λ, since

$$\mathbf{C}^{\mathrm{T}}\mathbf{x} = \lambda\mathbf{x}.$$

Because of $\lambda\lambda^j = \lambda^j\lambda$ we have

$$\lambda\mathbf{x} = \mathbf{x}\lambda$$

and λ is also a right eigenvalue of $\mathbf{C}^{\mathrm{T}}$. □

THEOREM 8.5. *Let $\mathbf{C}^{\mathrm{T}}$ be a companion matrix over $\mathcal{A}$ with an invertible a_0. Then the eigenvalues of $\mathbf{C}^{\mathrm{T}}$ are the zeros of the polynomial p.*

Proof. Follows from the Lemmas 8.3 and 8.4. The fact that $\mathbf{C}$ and $\mathbf{C}^{\mathrm{T}}$ have the same eigenvalues will be shown a little later. □

Since we cannot compute the eigenvalues of the companion matrix $\mathbf{C}^{\mathrm{T}}$ (and $\mathbf{C}$) of order n directly, we use a detour via $2n \times 2n$ complex (for quaternions) or via $2n \times 2n$ real matrices (for algebras $\mathcal{A}$) by using the operators $\imath, \imath_2, \imath_3, \imath_4$. See (3.2) and (8.3) to (8.5). This results in real or complex eigenvalues which define similarity classes which contain a zero of p. The authors of [19] then use Niven's algorithm to find the zeros of p from the eigenvalues. However, usually the eigenvectors are neglected which is justified for the algebra of quaternions, since the only noninvertible element in $\mathbb{H}$ is the zero element. In the definition (8.7) for the eigenvalues of the companion matrix we need the assumption that the corresponding eigenvectors have at least one invertible component. In order to explain the difficulties for problems in $\mathcal{A}$, we will use the following quadratic polynomial p over $\mathbb{H}_{\mathrm{coq}}$ as example:

$$p(z) = z^2 - (\mathbf{i} + \mathbf{j})z + \mathbf{k}. \tag{8.11}$$

Since in all four algebras $\mathbb{H}, \mathbb{H}_{\mathrm{coq}}, \mathbb{H}_{\mathrm{nec}}, \mathbb{H}_{\mathrm{con}}$ we have $\mathbf{ij} = \mathbf{k}$, cf. Table 8.1, we see that $z = \mathbf{j}$ is a zero of p in all four algebras. A separate investigation shows that in $\mathbb{H}$, $\mathbb{H}_{\mathrm{coq}}$ this is the only zero. The eigenvalues of $\imath_2(\mathbf{C}^{\mathrm{T}})$ (and of $\imath_2(\mathbf{C})$ as well) are $\pm 1, \pm\mathbf{i}$. The similarity classes defined by ± 1 are $\{\pm 1\}$. They consist only of the real element $\{\pm 1\}$. However, both real elements do not define zeros of p. The similarity class of $\mathbf{i}$ is defined by all elements $s \in \mathbb{H}_{\mathrm{coq}}$ satisfying $\Re(s) = 0, \mathrm{abs}_2(s) = 1$. Since $\mathrm{abs}_2(\mathbf{j}) = -1$, the element $\mathbf{j}$ does not belong to this similarity class. What are the consequences: The eigenvalues $\pm 1, \pm\mathbf{i}$ are not eigenvalues of the companion matrix, they violate the eigenvector restriction and more generally, the companion matrix $\mathbf{C}^{\mathrm{T}}$ (and $\mathbf{C}$) does not have eigenvalues at all. That this phenomenon exists for the algebras in $\mathcal{A}$ was already established in [11].

Nevertheless, we are not helpless. In another paper, [10] we have described, that pairs of real eigenvalues (or of the roots of the corresponding companion polynomial) can be used to determine the zeros. In the case of the polynomial p defined in (8.11) one finds the zero $\mathbf{j}$ by using the real pair $(1, -1)$. See [10] for details.

9. Appendix: The companion matrices $\mathbf{C}$ and $\mathbf{C}^{\mathrm{T}}$ have the same eigenvalues.

Let $\mathbf{A}$ be an arbitrary square matrix over $\mathcal{A}$. Then, in general, the eigenvalues of $\mathbf{A}$ are different from the eigenvalues of $\mathbf{A}^{\mathrm{T}}$. By $\mathbf{A}^*$ we denote the conjugate transpose of $\mathbf{A}$ or more formally $\mathbf{A}^* = (\mathrm{conj}(\mathbf{A}))^{\mathrm{T}} = \mathrm{conj}(\mathbf{A}^{\mathrm{T}})$.

THEOREM 9.1. *Let $\mathbf{A}$ be an arbitrary square matrix in one of the algebras $\mathbb{H}$, $\mathbb{H}_{\mathrm{coq}}$, $\mathbb{H}_{\mathrm{nec}}$, $\mathbb{H}_{\mathrm{con}}$. Then $\mathbf{A}$ and $\mathbf{A}^*$ have the same eigenvalues.*

Proof. Let λ be an eigenvalue of $\mathbf{A}$. Then $\overline{\lambda}$ is an eigenvalue of $\mathbf{A}^*$. The two eigenvalues λ and $\overline{\lambda}$ belong to the same similarity class. Thus, $\mathbf{A}$ and $\mathbf{A}^*$ have the same eigenvalues in this sense. □

Here we have to remember, that the eigenvalues of algebraic matrices (with elements from $\mathbb{H}$, $\mathbb{H}_{\mathrm{coq}}$, $\mathbb{H}_{\mathrm{nec}}$, $\mathbb{H}_{\mathrm{con}}$) always come in similarity classes. And λ and $\overline{\lambda}$ always belong to the same similarity class. However, we compute the eigenvalues of the algebraic matrices by using the operators $\imath$, $\imath_2$, $\imath_3$, $\imath_4$ which brings the algebraic eigenvalue problem to an eigenvalue problem for real ($\imath_2, \imath_3, \imath_4$) or complex ($\imath$) matrices, only one eigenvalue per similarity class of eigenvalues is computed.

Let $\mathbf{C}$ be the companion matrix, as defined in (2.4). Then

$$\mathbf{C}^* = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \dots & \dots & & \ddots & \dots \\ 0 & 0 & \dots & 0 & 1 \\ -\overline{a_0} & -\overline{a_1} & \dots & -\overline{a_{n-2}} & -\overline{a_{n-1}} \end{bmatrix}. \tag{9.1}$$

THEOREM 9.2. *The eigenvalues of* $\mathbf{C}^*$ *are the zeros of the polynomial*

$$\tilde{p}(z) = \sum_{j=0}^{n} \overline{a_j} z^j, \quad a_n = 1,\ a_0 \text{ invertible}.$$

Proof. Follows directly from Theorem 8.5. ◻

LEMMA 9.3. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial over* $\mathcal{A}$ *with* a_0 *invertible and* $a_n = 1$. *Let* q *be an invertible element in* $\mathcal{A}$ *and* $\tilde{p}(z) = \sum_{j=0}^{n} (q^{-1} a_j q)\, z^j$ *be another polynomial. Then* p *and* $\tilde{p}$ *have similar zeros.*

Proof. The companion polynomials of p and of $\tilde{p}$ are identical and, thus, have the same roots. The zeros of p and of $\tilde{p}$ are in the similarity classes of the roots. Thus, the zeros must be similar. ◻

THEOREM 9.4. *The eigenvalues of the three matrices* $\mathbf{C}$, $\mathbf{C}^{\mathrm{T}}$, $\mathbf{C}^*$ *are the same.*

Proof. The eigenvalues of $\mathbf{C}$ and $\mathbf{C}^*$ are the same. And Lemma 9.3 implies that the matrices $\mathbf{C}^{\mathrm{T}}$, $\mathbf{C}^*$. also have the same eigenvalues. ◻

We will present a very simple example.

EXAMPLE 9.5. Let

$$p_1(z) = z - a, \quad p_2(z) = z - \overline{a}, \quad a \in \mathcal{A},\ a \text{ invertible}.$$

It is clear that a, $\overline{a}$ are the zeros of p_1, p_2, respectively. The companion polynomials of p_1 and of p_2 are both $c(z) = z^2 - 2\Re(a)z + \mathrm{abs}_2(a)$. The roots of c are $\Re(a) \pm \sqrt{\Re(a)^2 - \mathrm{abs}_2(a)}$. In order to find the zeros of p_1, p_2 one has to apply Niven's algorithm. For polynomials of degree one, it is simply

$$p_j(z) = q \cdot 0 + R_0(u,v) + R_1(u,v)z,\ j = 1, 2.$$

Thus, $R_0 = -a$ for p_1 and $R_0 = -\overline{a}$ for p_2 and $R_1 = 1$ in both cases. The zeros are

$$-R_1^{-1} R_0 = \begin{cases} a & \text{for } p_1 \\ \overline{a} & \text{for } p_2 \end{cases}.$$

Computational details are in [10, 17].

The eigenvalues of the companion matrices $\mathbf{C}_1 = [a]$, $\mathbf{C}_2 = [\overline{a}] = \mathbf{C}_1^*$ are $a, \overline{a}$, repectively. However, since $[a]$, $[\overline{a}]$ are similar, the eigenvalues of $\mathbf{C}_1$, $\mathbf{C}_2$ may be regarded as the same. If we apply one of the operations $\imath_2, \imath_3, \imath_4$ to $\mathbf{C}_1$ and $\mathbf{C}_2$ we obtain two 2×2 real matrices which have the same eigenvalues which are determined by the roots of the above c.

All in all we have shown that in all four algebras the Niven algorithm applied to the roots of the companion polynomial produces all zeros of the underlying polynomial p, [10], and also Niven's algorithm applied to the eigenvalues of the companion matrix does the same. See [17]. Therefore, the Proof in Section 7 is valid in all algebras: The companion polynomial c is the characteristic polynomial of $\imath(\mathbf{C})$ and of $\imath_k(\mathbf{C})$, $k = 2, 3, 4$, where $\mathbf{C}$ is the companion matrix with respect to $\mathbb{H}$, $\mathbb{H}_{\mathrm{coq}}$, $\mathbb{H}_{\mathrm{nec}}$, $\mathbb{H}_{\mathrm{con}}$.

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