LINEAR-QUADRATIC OPTIMAL CONTROL OF DIFFERENTIAL-ALGEBRAIC SYSTEMS: THE INFINITE TIME HORIZON PROBLEM WITH ZERO TERMINAL STATE

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Abstract. In this work we revisit the linear-quadratic optimal control for differential-algebraic systems on the infinite time horizon with zero terminal state. Based on the recently developed Lur'e equation for differential-algebraic equations we obtain new equivalent conditions for feasibility. These are related to the existence of a stabilizing solutions of the Lur'e equation. This approach also allows to determine optimal controls if they exist. In particular, we can characterize regularity of the optimal control problem. The latter refers to existence and uniqueness of optimal controls for any consistent initial condition.

Key words. descriptor systems, differential-algebraic equations, linear-quadratic optimal control, Lur'e equation, Riccati equation, Kalman-Yakubovich-Popov inequality

1. Introduction.

We consider differential-algebraic systems

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + Bu, \tag{1.1}$$

where $E,\, A \in \mathbb{R}^{n\times n}$ are such that the pencil $sE - A \in \mathbb{R}[s]^{n\times n}$ is *regular*, i. e., $\det(sE - A)$ is not the zero polynomial, and $B \in \mathbb{R}^{n\times m}$ (we refer to the end of this section for the notation). For an interval $\mathcal{I} \subset \mathbb{R}$, the $\mathbb{R}^n$- (resp. $\mathbb{R}^m$-) valued functions x and u are called *generalized state* and *input* of the system, respectively. We denote the set of systems (1.1) by $\Sigma_{n,m}$, and we write $[E, A, B] \in \Sigma_{n,m}$. We call $(x, u) : \mathcal{I} \to \mathbb{R}^n \times \mathbb{R}^m$ a *solution of* $[E, A, B]$ on $\mathcal{I}$, if x and u are locally square integrable and (1.1) holds in the weak sense. We further call (x, u) a *solution of* $[E, A, B]$, if it is a solution of $[E, A, B]$ on $\mathbb{R}$.

Note that (x, u) being a solution of implies that Ex is absolutely continuous, hence the evaluation $Ex(0) := (Ex)(0)$ is well-defined. We further consider the vector space of *consistent initial differential variables* of $[E, A, B]$, which is given by

$$\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} := \{x_0 \in \mathbb{R}^n \,:\, \exists\,\text{a solution } (x, u) \text{ of } [E, A, B] \text{ with } Ex(0) = Ex_0\}\,. \tag{1.2}$$

For an interval $\mathcal{I} \subseteq \mathbb{R}$ and matrices $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, and $R = R^\top \in \mathbb{R}^{m\times m}$ we introduce the *cost functional*

$$\mathcal{J}(x, u, \mathcal{I}) := \int_{\mathcal{I}} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^{\!\top} \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau. \tag{1.3}$$

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For a given $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$ we consider the following optimal control problems:

(OC+)

$$\begin{array}{c}\text{Minimize } \mathcal{J}(x,u,\mathbb{R}_{\geq 0})\\ \text{subject to } \frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + Bu \text{ with } Ex(0) = Ex_0 \text{ and } Ex(\infty) = 0.\end{array}$$

(OC−)

$$\begin{array}{c}\text{Minimize } \mathcal{J}(x,u,\mathbb{R}_{\leq 0})\\ \text{subject to } \frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + Bu \text{ with } Ex(0) = Ex_0 \text{ and } Ex(-\infty) = 0.\end{array}$$

Note that the above optimal control problems are also subject to the terminal conditions $Ex(\pm\infty) = 0$ which is a short-hand notation for $\lim_{t\to\pm\infty} Ex(t) = 0$.

This article is devoted to a deep analysis of the above optimal control problems. We will first study *feasibility*. Loosely speaking, this refers to the property that for each $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$ there exists a solution (x,u) of $[E,A,B]$ on $\mathbb{R}_{\geq 0}$ (resp. on $\mathbb{R}_{\leq 0}$) with $Ex(\pm\infty) = 0$ and additionally, the cost functional cannot be made arbitrarily negative. We analyze existence and construction of *optimal controls*, i.e., minimizers of the above optimization problems. Further, we characterize *regularity*, which is a property referring to the existence and uniqueness of optimal controls for any consistent initial condition.

In our approach to the optimal control problems **(OC+)** and **(OC−)**, we present an approach similar to the one of Willems in his seminal article [39]. Namely, our findings are based on quadratic storage functions and certain matrix equations which can be solved for a Hermitian matrix expressing the optimal cost. Precise definitions of the aforementioned concepts will - together with an outline of the results - be presented in the forthcoming section.

Nomenclature. We use the standard notations $\mathbb{N}_0$, $\mathbb{R}$, $\mathbb{C}$, i, $A^\top$, I_n, $0_{m\times n}$ for the natural numbers including zero, the real numbers, the complex numbers, the imaginary unit, the transpose of a matrix, the identity matrix of size $n \times n$, and the zero matrix of size $m \times n$ (subscripts are omitted, if clear from context). The group of invertible $n \times n$ matrices with entries in $\mathbb{R}$ is denoted by $\mathrm{Gl}_n(\mathbb{R})$. Further, the following notation is used throughout this article:

$\mathbb{C}_+$, $\mathbb{C}_-$	the open sets of complex numbers with positive and negative real part, resp.
$\mathbb{R}_{\geq 0}$, $\mathbb{R}_{\leq 0}$	the sets of nonnegative and nonpositive real numbers, resp.
$\mathbb{R}[s]$	the ring of real polynomials
$\mathrm{im}_{\mathcal{R}}\, A$, $\ker_{\mathcal{R}}\, A$, $\mathrm{rank}_{\mathcal{R}}\, A$	the image, kernel, and rank of a matrix $A \in \mathcal{R}^{m\times n}$ over the ring $\mathcal{R}$
$\mathcal{L}^2(\mathcal{I},\mathbb{R}^n)$, $\mathcal{L}^2_{\mathrm{loc}}(\mathcal{I},\mathbb{R}^n)$	the spaces of measurable and (locally) square integrable functions $f : \mathcal{I} \to \mathbb{R}^n$ on the set $\mathcal{I} \subseteq \mathbb{R}$, where functions which agree almost everywhere are identified
$f\vert_{\widetilde{\mathcal{I}}}$	the restriction of $f : \mathcal{I} \to M$ to $\widetilde{\mathcal{I}} \subseteq \mathcal{I}$ (where M is a set)

2. Outline and main concepts. The optimal control problems **(OC+)** and **(OC–)** motivate the introduction of the *value functions* V_+, $V_- : E\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} \to \mathbb{R} \cup \{-\infty, \infty\}$, called the *optimal costs*

$$V_+(Ex_0) = \inf \{\mathcal{J}(x,u,\mathbb{R}_{\geq 0}) : (x,u) \text{ is a solution of } [E,A,B] \text{ on } \mathbb{R}_{\geq 0} \text{ with } Ex(0) = Ex_0 \text{ and } Ex(\infty) = 0\}, \tag{2.1}$$

$$V_-(Ex_0) = -\inf \{\mathcal{J}(x,u,\mathbb{R}_{\leq 0}) : (x,u) \text{ is a solution of } [E,A,B] \text{ on } \mathbb{R}_{\leq 0} \text{ with } Ex(0) = Ex_0 \text{ and } Ex(-\infty) = 0\}, \tag{2.2}$$

Next we define some notions which are, loosely speaking, related to the solvability of the optimal control problems. These concepts are crucial for all considerations in this article.

Definition 2.1 (Feasibility, regularity, optimal control). *Let $[E,A,B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, and $R = R^\top \in \mathbb{R}^{m\times m}$ be given.*

a) The optimal control problem **(OC+)** *(resp.* **(OC–)***) is called* feasible*, if for all $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$ it holds that*

$$-\infty < V_+(Ex_0) < \infty, \quad (\textit{resp. } -\infty < V_-(Ex_0) < \infty).$$

b) A solution (x_*, u_*) of $[E,A,B]$ on $\mathbb{R}_{\geq 0}$ (resp. on $\mathbb{R}_{\leq 0}$) with $Ex(0) = Ex_0$ and $Ex_*(\infty) = 0$ (resp. $Ex_*(-\infty) = 0$) is called an optimal control *for* **(OC+)** *(resp.* **(OC–)***), if*

$$V_+(Ex_0) = \mathcal{J}(x_*, u_*, \mathbb{R}_{\geq 0}) \quad (\textit{resp. } V_-(Ex_0) = \mathcal{J}(x_*, u_*, \mathbb{R}_{\leq 0})).$$

c) The optimal control problem **(OC+)** *(resp.* **(OC–)***) is called* regular*, if for all $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$, there exists a unique optimal control for* **(OC+)** *(resp.* **(OC–)***).*

The key ingredient for our considerations are so-called *storage functions*. This concept has been introduced for ordinary differential equations in [39, 14].

Definition 2.2 (Storage function, dissipation inequality). *Let $[E,A,B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ it given. A continuous function $V : E\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} \to \mathbb{R}$ is called a* storage function*, if it is continuous, $V(0) = 0$, and the cost functional $\mathcal{J}(\cdot,\cdot,\cdot)$ as in (1.3) fulfills the* dissipation inequality

$$\mathcal{J}(x,u,[t_0,t_1]) + V(Ex(t_1)) \geq V(Ex(t_0)) \tag{2.3}$$

for all solutions (x,u) of $[E,A,B]$ and t_0, $t_1 \in \mathbb{R}$ with $t_0 \leq t_1$.

Our special emphasis will be put on *quadratic storage functions*. In this case, there exists some Hermitian matrix $P \in \mathbb{R}^{n\times n}$ such that V attains the form

$$V(Ex_0) = x_0^\top E^\top P E x_0 \quad \forall\, x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}. \tag{2.4}$$

To further characterize quadratic storage functions, we present the notion of the *system space*.

Definition 2.3. *The* system space *of* $[E,A,B]\in\Sigma_{n,m}$ *is the smallest subspace* $\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}\subseteq\mathbb{R}^{n+m}$ *such that*

$$\forall\,\textit{solutions }(x,u)\textit{ of }[E,A,B]:\quad \begin{pmatrix}x(t)\\u(t)\end{pmatrix}\in\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}\textit{ for almost all }t\in\mathbb{R}.$$

A geometric characterization of the system space can be found in [37]. For the theory presented in this article, it is crucial to introduce what we mean be equality and positive semi-definiteness on some subspace.

Definition 2.4. *Let* $\mathcal{V}\subseteq\mathbb{R}^n$ *be a subspace and* $M,\,N\in\mathbb{R}^{n\times n}$ *be Hermitian matrices. Then we write*

$$\begin{aligned} M=_{\mathcal{V}}N \quad &:\Longleftrightarrow\quad x^\top Mx=x^\top Nx\quad\forall\,x\in\mathcal{V},\\ M\geq_{\mathcal{V}}N \quad &:\Longleftrightarrow\quad x^\top Mx\geq x^\top Nx\quad\forall\,x\in\mathcal{V}.\end{aligned}$$

The previous definitions enable us to introduce the *Kalman-Yakubovich-Popov (KYP) inequality* and the *Lur'e equation.*

Definition 2.5 (Kalman-Yakubovich-Popov inequality, Lur'e equation). *Let* $[E,A,B]\in\Sigma_{n,m}$ *and* $Q=Q^\top\in\mathbb{R}^{n\times n}$, $S\in\mathbb{R}^{n\times m}$, $R=R^\top\in\mathbb{R}^{m\times m}$ *be given. We call* $P\in\mathbb{R}^{n\times n}$ *a solution of the* Kalman-Yakubovich-Popov (KYP) inequality, *if*

$$\begin{bmatrix}A^\top PE+E^\top PA+Q & E^\top PB+S\\ B^\top PE+S^\top & R\end{bmatrix}\geq_{\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}}0,\qquad P=P^\top. \tag{2.5}$$

Further, for some $q\in\mathbb{N}_0$, *we call a triple* $(P,K,L)\in\mathbb{R}^{n\times n}\times\mathbb{R}^{q\times n}\times\mathbb{R}^{q\times m}$ *solution of the* Lur'e equation, *if*

$$\begin{bmatrix}A^\top PE+E^\top PA+Q & E^\top PB+S\\ B^\top PE+S^\top & R\end{bmatrix}=_{\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}}\begin{bmatrix}K^\top\\ L^\top\end{bmatrix}\begin{bmatrix}K & L\end{bmatrix},\qquad P=P^\top \tag{2.6}$$

is satisfied with

$$\operatorname{rank}_{\mathbb{R}[s]}\begin{bmatrix}-sE+A & B\\ K & L\end{bmatrix}=n+q. \tag{2.7}$$

A solution $(P,K,L)\in\mathbb{R}^{n\times n}\times\mathbb{R}^{q\times n}\times\mathbb{R}^{q\times m}$ *of the Lur'e equation is called* stabilizing, *if additionally,*

$$\operatorname{rank}_{\mathbb{C}}\begin{bmatrix}-\lambda E+A & B\\ K & L\end{bmatrix}=n+q\quad\forall\,\lambda\in\mathbb{C}_+, \tag{2.8}$$

and anti-stabilizing, *if additionally,*

$$\operatorname{rank}_{\mathbb{C}}\begin{bmatrix}-\lambda E+A & B\\ K & L\end{bmatrix}=n+q\quad\forall\,\lambda\in\mathbb{C}_-. \tag{2.9}$$

Note that it follows immediately that, if (P,K,L) solves the Lur'e equation, then P is a solution of the KYP inequality.

The Lur'e equation and KYP inequality are crucial for our approach to the linear-quadratic optimal control problem. The main theses of this article are listed at the end of this section. Indeed, we will show that the value functions and optimal controls can be expressed by means of (anti-)stabilizing solutions of the Lur'e equation.

To formulate our main results, we present some notions related controllability and stabilizability of differential-algebraic systems. Algebraic characterizations can be found in [5].

Definition 2.6 (Controllability and stabilizability). *The system* $[E,A,B]\in\Sigma_{n,m}$ *is called*

a) behaviorally stabilizable*, if for all solutions* (x,u) *of* $[E,A,B]$ *on* $\mathbb{R}_{\le 0}$ *there exists a solution* $(\widetilde{x},\widetilde{u})$ *of* $[E,A,B]$ *with* $(\widetilde{x},\widetilde{u})|_{\mathbb{R}_{\le 0}}=(x,u)$ *and*

$$\lim_{t\to\infty}\operatorname*{ess\,sup}_{\tau>t}\|(\widetilde{x}(\tau),\widetilde{u}(\tau))\|=0;$$

b) behaviorally anti-stabilizable*, if for all solutions* (x,u) *of* $[E,A,B]$ *on* $\mathbb{R}_{\ge 0}$ *there exists a solution* $(\widetilde{x},\widetilde{u})$ *of* $[E,A,B]$ *with* $(\widetilde{x},\widetilde{u})|_{\mathbb{R}_{\ge 0}}=(x,u)$ *and*

$$\lim_{t\to-\infty}\operatorname*{ess\,sup}_{\tau<t}\|(\widetilde{x}(\tau),\widetilde{u}(\tau))\|=0;$$

c) behaviorally controllable*, if for all solutions* (x_1,u_1)*,* (x_2,u_2) *of* $[E,A,B]$ *there exists some* $T>0$ *and a solution* (x,u) *of* $[E,A,B]$ *with*

$$(x(t),u(t))=\begin{cases}(x_1(t),u_1(t)) & \text{for } t<0,\\ (x_2(t),u_2(t)) & \text{for } t>T.\end{cases}$$

The main theses of this article are listed below. We first highlight the connection between the solutions of the KYP inequality and storage functions:

S) For a Hermitian matrix $P\in\mathbb{R}^{n\times n}$, V as in (2.4) is a quadratic storage function, if and only if P solves the KYP inequality (2.5), see Theorem 4.3.

Thereafter we will prove the following for the optimal control problem on the positive time axis:

1+) If **(OC+)** is feasible, then the value function V_+ is a quadratic storage function (and thus corresponds to a solution of the KYP inequality). In this case, for all storage functions $V:E\mathcal{V}^{\mathrm{diff}}_{[E,A,B]}\to\mathbb{R}$ it holds that

$$V(Ex_0)\le V_+(Ex_0)\quad\forall\, x_0\in\mathcal{V}^{\mathrm{diff}}_{[E,A,B]},$$

see Theorem 5.2 and Theorem 5.4 a).

2+) The problem **(OC+)** is feasible, if and only if $[E,A,B]$ is behaviorally stabilizable and there exists a storage function, see Theorem 5.4 a).

3+) The problem **(OC+)** is feasible, if and only if $[E,A,B]$ is behaviorally stabilizable and the Lur'e equation has a stabilizing solution, see Theorem 5.7 a).

4+) If (P,K,L) is a stabilizing solution of the Lur'e equation, then (x_*,u_*) is an optimal control for **(OC+)**, if and only if it satisfies the *optimality differential-algebraic equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}E&0\\0&0\end{bmatrix}\begin{pmatrix}x_*\\u_*\end{pmatrix}=\begin{bmatrix}A&B\\K&L\end{bmatrix}\begin{pmatrix}x_*\\u_*\end{pmatrix}\tag{2.10}$$

with $Ex_*(0)=x_0$ and $Ex_*(\infty)=0$, see Theorem 5.7 a).

5+) If (P,K,L) is a stabilizing solution of the Lur'e equation, then the optimal control problem **(OC+)** is regular, if and only if

$$\begin{aligned}(2.11)\qquad \begin{bmatrix} E & 0\\ 0 & 0\end{bmatrix}\cdot \mathcal{V}^{\mathrm{sys}}_{[E,A,B]} + \begin{bmatrix} A & B\\ K & L\end{bmatrix}\cdot \Big((\ker E\times \mathbb{R}^m)\cap \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}\Big)\\ = \begin{bmatrix} E & 0\\ 0 & 0\end{bmatrix}\cdot \mathcal{V}^{\mathrm{sys}}_{[E,A,B]} + \begin{bmatrix} A & B\\ K & L\end{bmatrix}\cdot \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}\end{aligned}$$

and

$$(2.12)\qquad \ker_{\mathbb{C}} \begin{bmatrix} -\mathrm{i}\omega E + A & B\\ K & L\end{bmatrix} = \{0\}\quad \forall\, \omega\in\mathbb{R},$$

see Theorem 5.8 a).

Likewise, we will show the following assertions for optimal control problems on $\mathbb{R}_{\leq 0}$.

1–) If **(OC–)** is feasible, then the value function V_- is a quadratic storage function (and thus corresponds to a solution of the KYP inequality). In this case, for all storage functions $V : E\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} \to \mathbb{R}$ it holds that

$$V_-(Ex_0) \leq V(Ex_0) \quad \forall\, x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]},$$

see Theorem 5.2 and Theorem 5.4 b).

2–) The problem **(OC–)** is feasible, if and only if $[E,A,B]$ is behaviorally anti-stabilizable and there exists a storage function, see Theorem 5.4 b).

3–) The problem **(OC–)** is feasible, if and only if $[E,A,B]$ is behaviorally anti-stabilizable and the Lur'e equation has an anti-stabilizing solution, see Theorem 5.7 b).

4–) If (P,K,L) is an anti-stabilizing solution of the Lur'e equation, then (x_*,u_*) is an optimal control for **(OC–)**, if and only if it satisfies the optimality differential-algebraic equation (2.10) with $Ex_*(0) = Ex_0$ and $Ex_*(-\infty) = 0$, see Theorem 5.7 b).

5–) If (P,K,L) is an anti-stabilizing solution of the Lur'e equation, then the optimal control problem **(OC–)** is regular, if and only if (2.11) and (2.12) are satisfied, see Theorem 5.8 b).

For behaviorally controllable systems, we will show that feasibility of the optimal control problems on positive and negative time axis is equivalent.

1+–) If $[E,A,B]$ is behaviorally controllable, then **(OC+)** is feasible if, and only if **(OC–)** is feasible, see Corollary 5.5.

3. Curiosities in optimal control of differential-algebraic equations. In this part, we present some examples which emphasize the main differences between optimal control of ordinary differential equations and differential-algebraic equations. Whereas in control of ordinary differential equations, positive semi-definiteness of the input weight R is necessary for feasibility of the optimal control problem [39], this is not necessarily true in the case of differential-algebraic equations:

EXAMPLE 3.1. *Feasibility of the optimal control problem* **(OC+)** *does not imply* $R \geq 0$*: Consider the optimal control problem*

$$
\boxed{\begin{array}{c}\textit{Minimize } \mathcal{J}(x,u,\mathbb{R}_{\geq 0}) = \displaystyle\int_0^\infty x_2^2(t) - \tfrac{1}{2}u^2(t)\mathrm{d}t \\ \textit{subject to } \dfrac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{bmatrix}0\\ -1\end{bmatrix}u \\ \textit{with } x_1(0) = x_{01} \textit{ and } x_1(\infty) = 0.\end{array}} \tag{3.1}
$$

For this optimal control problem we have – in the notation of (1.3) *and* **(OC+)** *–* $R = -\frac{1}{2} < 0$*. However, resolving the algebraic constraint* $x_2 = u$ *yields that this optimal control problem is equivalent to*

$$
\boxed{\begin{array}{c}\textit{Minimize } \mathcal{J}(x_1,u,\mathbb{R}_{\geq 0}) = \displaystyle\int_0^\infty \tfrac{1}{2}u^2(t)\mathrm{d}t \\ \textit{subject to } \tfrac{\mathrm{d}}{\mathrm{d}t}x_1 = -x_1 + u \textit{ with } x_1(0) = x_{01} \textit{ and } x_1(\infty) = 0.\end{array}} \tag{3.2}
$$

together with $x_2 = u$*. The optimal control problem* (3.2) *is even regular. Namely, the non-negative cost functional can be made zero by setting* $u_* = 0$*. Then we have indeed* $x_{*,1}(t) = \mathrm{e}^{-t} \cdot x_{01}$ *with* $x_{*,1}(\infty) = 0$*. Therefore, the differential-algebraic optimal control problem* (3.1) *is regular with optimal control* $\left(\left(\begin{smallmatrix}x_{*,1}\\ x_{*,2}\end{smallmatrix}\right), u_*\right)$ *where* $x_{*,1}(t) = \mathrm{e}^{-t} \cdot x_{01}$ *and* $x_{*,2} = u_* = 0$*. In particular,* (3.1) *is a feasible optimal control problem.*

Whereas in control of ordinary differential equations, regularity of the input weight R is necessary for regularity of the optimal control problem [10], this is not necessarily true for differential-algebraic equations, as the following example shows:

EXAMPLE 3.2. *Regularity of the optimal control problem* **(OC+)** *does not imply that* R *is invertible: Consider the optimal control problem*

$$
\boxed{\begin{array}{c}\textit{Minimize } \mathcal{J}(x,u,\mathbb{R}_{\geq 0}) = \displaystyle\int_0^\infty \tfrac{1}{2}x_2^2(t)\mathrm{d}t \\ \textit{subject to } \dfrac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{bmatrix}0\\ -1\end{bmatrix}u \\ \textit{with } x_1(0) = x_{01} \textit{ and } x_1(\infty) = 0.\end{array}} \tag{3.3}
$$

For this optimal control problem we have – in the notation of (1.3) *and* **(OC+)** *–* $R = 0$*. However, resolving the algebraic constraint* $x_2 = u$ *yields that this optimal control problem is again equivalent to* (3.2)*, now together with* $x_2 = u$*. The latter is even a regular optimal control problem with – as previously shown – optimal control* $(x_{*,1}, u_*)$ *with* $x_{*,1}(t) = \mathrm{e}^{-t} \cdot x_{01}$ *and* $u_* = 0$*. Therefore, the differential-algebraic optimal control problem is again regular with optimal control* $\left(\left(\begin{smallmatrix}x_{*,1}\\ x_{*,2}\end{smallmatrix}\right), u_*\right)$ *with* $x_{*,1}(t) = \mathrm{e}^{-t} \cdot x_{01}$ *and* $x_{*,2} = u_* = 0$*.*

Next we show that the *index* of the differential-algebraic equation $[E, A, B]$ does not necessarily cause singularity of the optimal control problem **(OC+)**. In our context, the index is defined by the nilpotency index of the nilpotent matrix N in a *quasi-Weierstraß form*

$$W(sE-A)T = \begin{bmatrix} sI - A_1 & 0 \\ 0 & sN - I \end{bmatrix} \tag{3.4}$$

for some $W, T \in \mathrm{Gl}_n(\mathbb{R})$, see [6].

EXAMPLE 3.3. *There are regular optimal control problems with arbitrary index: Consider the system $[E, A, B] \in \Sigma_{n,1}$ with*

$$E = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & & & \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then $A^{-1}(sE-A)$ is in quasi-Weierstraß form, and we obtain that the index of this differential-algebraic equation is n.

We further define a cost functional (1.3) with the matrices $R = 0$, $S = 0$, and $Q \in \mathbb{R}^{n \times n}$ which has the entry 1 at the lower right position and zeros elsewhere. This yields the optimal control problem

$$\boxed{\begin{array}{c} \textit{Minimize } \mathcal{J}(x, u, \mathbb{R}_{\geq 0}) = \displaystyle\int_0^\infty x_n^2(t)\mathrm{d}t \\ \textit{subject to} \quad \begin{array}{rcl} 0 & = & x_1 \quad + u, \\ \frac{\mathrm{d}}{\mathrm{d}t}x_1 & = & -x_1 \quad + x_2, \\ & \vdots & \\ \frac{\mathrm{d}}{\mathrm{d}t}x_{n-1} & = & -x_{n-1} + x_n, \end{array} \\ \textit{with } x_1(0) = x_{01}, \ldots, x_{n-1}(0) = x_{0,n-1} \textit{ and} \\ x_1(\infty) = \ldots = x_{n-1}(\infty) = 0. \end{array}} \tag{3.5}$$

Since u only enters in the algebraic equation $u + x_1 = 0$, we see that the optimal control problem is equivalent to

$$\boxed{\begin{array}{c} \textit{Minimize } \mathcal{J}(x, u, \mathbb{R}_{\geq 0}) = \displaystyle\int_0^\infty x_n^2(t)\mathrm{d}t \\ \textit{subject to} \quad \begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{d}t}x_1 & = & -x_1 \quad + x_2, \\ & \vdots & \\ \frac{\mathrm{d}}{\mathrm{d}t}x_{n-1} & = & -x_{n-1} + x_n, \end{array} \\ \textit{with } x_1(0) = x_{01}, \ldots, x_{n-1}(0) = x_{0,n-1} \textit{ and} \\ x_1(\infty) = \ldots = x_{n-1}(\infty) = 0. \end{array}} \tag{3.6}$$

together with $u = -x_1$. However, since the ordinary differential equation in (3.6) with the additional constraint $x_n = 0$ is asymptotically stable, we obtain that the *nonnegative cost functional $\mathcal{J}(x,u,\mathbb{R}_{\geq 0})$ can indeed be made zero. Thus, an optimal control (x_*,u_*) has to fulfill $x_{*,n}=0$ and the optimal control (x_*,u_*) is uniquely determined by $x_{*,n}=0$, and*

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}x_{*,1} &= -x_{*,1} && +x_{*,2},\\
&\ \vdots\\
\tfrac{\mathrm{d}}{\mathrm{d}t}x_{*,n-2} &= -x_{*,n-2} && +x_{*,n-1},\\
\tfrac{\mathrm{d}}{\mathrm{d}t}x_{*,n-1} &= -x_{*,n-1},
\end{aligned}$$

with the initial conditions $x_{*,1}(0)=x_{01}$, $\ldots$, $x_{*,n-1}(0)=x_{0,n-1}$, and $u_*=-x_{*,1}$. In particular, the optimal control problem is regular.

4. Storage functions and the Kalman-Yakubovich-Popov inequality. Here we present the details on storage functions, the KYP inequality (see Def. 2.2 & Def. 2.5) and their connection.

First we consider the special case where the storage function is differentiable. For this we need an auxiliary result which basically states that we can often restrict to smooth solutions.

Lemma 4.1. *Let $[E,A,B]\in\Sigma_{n,m}$ be given. Then the following holds:*

a) For all $\left(\begin{smallmatrix}x_0\\u_0\end{smallmatrix}\right)\in\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$, there exists some infinitely often differentiable solution (x,u) of $[E,A,B]$ with $x(0)=x_0$ and $u(0)=u_0$. In particular, $x_0\in\mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$.

b) For all $x_0\in\mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$, there exists some infinitely often differentiable solution (x,u) of $[E,A,B]$ with $Ex(0)=Ex_0$. In particular, there exists some $\left(\begin{smallmatrix}x_{01}\\u_{01}\end{smallmatrix}\right)\in\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$ with $Ex_0=Ex_{01}$.

Proof. This follows by an application of [6, Thm. 3.2] to the differential-algebraic equation $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}w(t)=\mathcal{A}w(t)$ with $\mathcal{E}=\begin{bmatrix}E & 0\end{bmatrix}$, $\mathcal{A}=\begin{bmatrix}A & B\end{bmatrix}$, and $w(t)=\left(\begin{smallmatrix}x(t)\\u(t)\end{smallmatrix}\right)$. ◻

Proposition 4.2. *Let $[E,A,B]\in\Sigma_{n,m}$ and $Q=Q^\top\in\mathbb{R}^{n\times n}$, $S\in\mathbb{R}^{n\times m}$, $R=R^\top\in\mathbb{R}^{m\times m}$ be given. Then a differentiable function $V:E\mathcal{V}^{\mathrm{diff}}_{[E,A,B]}\to\mathbb{R}$ with $V(0)=0$ is a storage function, if and only if*

$$\begin{pmatrix}x_0\\u_0\end{pmatrix}^\top\begin{bmatrix}Q & S\\S^\top & R\end{bmatrix}\begin{pmatrix}x_0\\u_0\end{pmatrix}\geq-(\nabla V(Ex_0))^\top(Ax_0+Bu_0)\quad\forall\begin{pmatrix}x_0\\u_0\end{pmatrix}\in\mathcal{V}^{\mathrm{sys}}_{[E,A,B]},\tag{4.1}$$

where $\nabla V(Ex_0)\in\mathbb{R}^n$ denotes the gradient of V in Ex_0.

Proof. First assume that $V:E\mathcal{V}^{\mathrm{diff}}_{[E,A,B]}\to\mathbb{R}$ is a differentiable storage function. Suppose that $\left(\begin{smallmatrix}x_0\\u_0\end{smallmatrix}\right)\in\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$. Then, by Lemma 4.1, there exists some infinitely often differentiable solution (x,u) of $[E,A,B]$ with $x(0)=x_0$ and $u(0)=u_0$. Consequently, the real-valued function $t\mapsto V(Ex(t))$ is differentiable. The dissipation inequality yields that for all $h>0$ we have

$$\frac{1}{h}(V(Ex(h))-V(Ex(0)))\geq-\frac{1}{h}\mathcal{J}(x,u,[0,h])=-\frac{1}{h}\int_0^h\begin{pmatrix}x(t)\\u(t)\end{pmatrix}^\top\begin{bmatrix}Q & S\\S^\top & R\end{bmatrix}\begin{pmatrix}x(t)\\u(t)\end{pmatrix}\mathrm{d}t.$$

Now taking the limit $h \to 0$, we see that the right hand side converges to

$$-\begin{pmatrix} x(0) \\ u(0) \end{pmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{pmatrix} x(0) \\ u(0) \end{pmatrix} = -\begin{pmatrix} x_0 \\ u_0 \end{pmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}.$$

Then (4.1) is a consequence of the fact that the left hand side tends to

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} V(Ex(t))\big|_{t=0} &= (\nabla V(Ex(0)))^\top \tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t)\big|_{t=0} \\ &= (\nabla V(Ex(0)))^\top (Ax(0) + Bu(0)) = (\nabla V(Ex_0))^\top (Ax_0 + Bu_0). \end{aligned}$$

To prove the reverse implication, assume that (4.1) is satisfied. Let (x, u) be a solution of $[E, A, B]$ and let $t_0, t_1 \in \mathbb{R}$ with $t_0 \leq t_1$. Then, by using the chain rule for weak derivatives [29] and the fundamental theorem of calculus for weakly differentiable functions [1, Sec. E3.6] together with $\left(\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right) \in \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$ for almost all $t \in [t_0, t_1]$, we obtain

$$\begin{aligned} V(Ex(t_1)) - V(Ex(t_0)) &= \int_{t_0}^{t_1} \tfrac{\mathrm{d}}{\mathrm{d}t} V(Ex(t))\mathrm{d}t \\ &= \int_{t_0}^{t_1} (\nabla V(Ex(t)))^\top \tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t)\mathrm{d}t \\ &= \int_{t_0}^{t_1} (\nabla V(Ex(t)))^\top (Ax(t) + Bu(t))\mathrm{d}t \\ &\geq -\int_{t_0}^{t_1} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t = -\mathcal{J}(x, u, [t_0, t_1]), \end{aligned}$$

i. e., the dissipation inequality (2.3) is fulfilled. ◻

Next we show that the set of quadratic storage functions corresponds to the set of solutions of the KYP inequality.

THEOREM 4.3. *Let $[E, A, B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ be given. Then the following two statements are equivalent for $P \in \mathbb{R}^{n\times n}$:*

a) It holds that $P = P^\top$ and $V : E\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} \to \mathbb{R}$, $Ex_0 \mapsto x_0^\top E^\top P E x_0$ is a storage function.

b) The matrix P solves the KYP inequality (2.5).

Proof. First note that for $P = P^\top \in \mathbb{R}^{n\times n}$, the function $V : E\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} \to \mathbb{R}$, $Ex_0 \mapsto x_0^\top E^\top P E x_0$ fulfills $\nabla V(Ex_0) = 2E^\top P E x_0$. Thus for all $\left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right) \in \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$ we have

$$\begin{aligned} (\nabla V(Ex_0))^\top (Ax_0 + Bu_0) &= 2x_0^\top E^\top P(Ax_0 + Bu_0) \\ &= \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}^\top \begin{bmatrix} A^\top PE + E^\top PA & E^\top PB \\ B^\top PE & 0 \end{bmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}. \end{aligned} \tag{4.2}$$

Now we show that "a)⇒b)": Assume that $P = P^\top \in \mathbb{R}^{n\times n}$ and that $V : E\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} \to \mathbb{R}$, $Ex_0 \mapsto x_0^\top E^\top P E x_0$ is a storage function. Let $\left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right) \in \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$ be given. By combining (4.2) with Proposition 4.2, we obtain that the KYP inequality (2.5) is satisfied.

Next we show "b)⇒a)": If P fulfills (2.5), then for all $\left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right) \in \mathcal{V}^{\text{sys}}_{[E,A,B]}$ we have

$$\begin{pmatrix} x_0 \\ u_0 \end{pmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} \geq - \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}^\top \begin{bmatrix} A^\top PE + E^\top PA & E^\top PB \\ B^\top PE & 0 \end{bmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix}.$$

By further using that $V : E\mathcal{V}^{\text{diff}}_{[E,A,B]} \to \mathbb{R}$, $Ex_0 \mapsto x_0^\top E^\top PEx_0$ fulfills (4.2), we obtain from Proposition 4.2 that V is a storage function. ∎

Next we show that the right hand side of the KYP inequality can be factored in a special way for which we need the following auxiliary result.

Lemma 4.4. *Let $\mathcal{V} \subseteq \mathbb{R}^n$ be a subspace and $M \in \mathbb{R}^{n\times n}$ with $M \geq_{\mathcal{V}} 0$. Then there exists some $\ell \in \mathbb{N}_0$ and $K \in \mathbb{R}^{\ell\times n}$ such that $K\mathcal{V} = \mathbb{R}^\ell$ and $M =_{\mathcal{V}} K^\top K$.*

Proof. Let $r := \dim \mathcal{V}$ and $T \in \mathrm{Gl}_n(\mathbb{R})$ be such that $T(\mathbb{R}^r \times \{0\}) = \mathcal{V}$. Partition

$$T^\top MT = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\top & M_{22} \end{bmatrix}$$

with $M_{11} \in \mathbb{R}^{r\times r}$, $M_{12} \in \mathbb{R}^{r\times(n-r)}$, and $M_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$. Then for all $x \in \mathbb{R}^r$ it holds that $T\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) \in \mathcal{V}$, and thus

$$x^\top M_{11}x = \begin{pmatrix} x \\ 0 \end{pmatrix}^\top \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\top & M_{22} \end{bmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}^\top T^\top MT \begin{pmatrix} x \\ 0 \end{pmatrix} \geq 0.$$

As a consequence, $M_{11} \geq 0$. Define $\ell := \operatorname{rank}_{\mathbb{R}} M_{11}$. From the positive semi-definiteness of M_{11}, we obtain that there exists some $K_1 \in \mathbb{R}^{\ell\times r}$ with $M_{11} = K_1^\top K_1$. In particular, K_1 has full row rank. Now define $K := \begin{bmatrix} K_1 & 0 \end{bmatrix} T^{-1}$. Then

$$K\mathcal{V} = \begin{bmatrix} K_1 & 0 \end{bmatrix} T^{-1}\mathcal{V} = \begin{bmatrix} K_1 & 0 \end{bmatrix} (\mathbb{R}^r \times \{0\}) = \operatorname{im}_{\mathbb{R}} K_1 = \mathbb{R}^\ell.$$

Next we show that $M =_{\mathcal{V}} K^\top K$: Assume that $v \in \mathcal{V}$. Then there exists some $x \in \mathbb{R}^r$ with $v = T\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$. Further we have

$$\begin{aligned} \begin{pmatrix} x \\ 0 \end{pmatrix}^\top T^\top MT \begin{pmatrix} x \\ 0 \end{pmatrix} = x^\top M_{11}x = x^\top K_1^\top K_1 x &= \begin{pmatrix} x \\ 0 \end{pmatrix}^\top \begin{bmatrix} K_1 & 0 \end{bmatrix}^\top \begin{bmatrix} K_1 & 0 \end{bmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} \\ &= v^\top (T^{-\top}) \begin{bmatrix} K_1 & 0 \end{bmatrix}^\top \begin{bmatrix} K_1 & 0 \end{bmatrix} T^{-1}v = v^\top K^\top Kv. \end{aligned}$$ ∎

Proposition 4.5. *Let $[E,A,B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ be given. Assume that $P \in \mathbb{R}^{n\times n}$ solves the KYP inequality (2.5). Then there exist $q \in \mathbb{N}_0$, $K \in \mathbb{R}^{q\times n}$ and $L \in \mathbb{R}^{q\times m}$ such that (2.6) and*

$$\begin{bmatrix} K & L \end{bmatrix} \mathcal{V}^{\text{sys}}_{[E,A,B]} = \mathbb{R}^q. \tag{4.3}$$

Further, the dissipation inequality can be reformulated to

$$\begin{aligned} &\mathcal{J}(x,u,[t_0,t_1]) + x(t_1)^\top E^\top PEx(t_1) = x(t_0)^\top E^\top PEx(t_0) + \|Kx+Lu\|^2_{\mathcal{L}^2([t_0,t_1],\mathbb{R}^q)} \\ &\qquad\qquad \forall \text{ solutions } (x,u) \text{ of } [E,A,B] \text{ and } t_0,\, t_1 \in \mathbb{R} \text{ with } t_0 \leq t_1. \end{aligned} \tag{4.4}$$

Proof. The existence of $q \in \mathbb{N}_0$, $K \in \mathbb{R}^{q\times n}$, and $L \in \mathbb{R}^{q\times m}$ such that (2.6) and (4.3) are satisfied follows from Lemma 4.4. The assertion in (4.4) follows by an argumentation as in the proof of statement "b)⇒a)" from Theorem 4.3. ∎

Willems has called $Kx + Lu$ the *dissipation rate* in his article [39] on optimal control of ordinary differential equations. We can conclude the following for the dissipation inequality on the whole positive time axis.

Corollary 4.6. *Let $[E, A, B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ be given. Assume that $P \in \mathbb{R}^{n\times n}$ solves the KYP inequality (2.5) and that further $K \in \mathbb{R}^{q\times n}$ and $L \in \mathbb{R}^{q\times m}$ solve (2.6). Assume that $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$ and let (x, u) be a solution of $[E, A, B]$ with $Ex(0) = Ex_0$ and $Ex(\infty) = 0$ on $\mathbb{R}_{\geq 0}$ be given. Then $\mathcal{J}(x, u, \mathbb{R}_{\geq 0})$ is finite, if and only if $Kx + Lu \in \mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}^q)$. In this case we have*

$$\mathcal{J}(x, u, \mathbb{R}_{\geq 0}) = x_0^\top E^\top P E x_0 + \|Kx + Lu\|^2_{\mathcal{L}^2(\mathbb{R}_{\geq 0},\mathbb{R}^q)}. \tag{4.5}$$

Proof. Let $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$ and (x, u) be a solution of $[E, A, B]$ with $Ex(0) = Ex_0$ and $Ex(\infty) = 0$ be given. Then $x(\infty)^\top E^\top P E x(\infty) = 0$ and we see that, by taking the limit $t \to \infty$, that the left hand side in (4.4) converges, if and only if the right hand side in (4.4) converges. This limiting process further gives rise to equation (4.5). ∎

5. Optimal control with zero terminal condition. In this part we take a closer look at the optimal control problems **(OC+)** and **(OC–)**. We proceed as follows: We first show that, in case of the respective feasibility, the value functions V_+ and V_- are quadratic storage functions. As a consequence of Theorem 4.3, the value functions can be expressed by means of a solution of the KYP inequality (2.5). We will show that the value functions indeed induce special solutions of the KYP inequality, namely the *stabilizing* and *anti-stabilizing solution* of the *Lur'e equation*. The latter type of algebraic matrix equation has been analyzed in detail in [37] from a linear algebraic point of view. Numerical solution for equations of this type has been considered in [35, 36]. Here we will show that feasibility of **(OC+)** (resp. **(OC–)**) is equivalent to the existence of stabilizing (resp. anti-stabilizing) solutions of the Lur'e equation. In other words, we have necessary and sufficient conditions on feasibility of the optimal control problems **(OC+)** and **(OC–)**. The solutions of the Lur'e equations will further be used to characterize regularity and to design optimal controls.

Before we prove that the value functions are quadratic storage functions, we briefly present the connection between (anti-)stabilizability and feasibility of the optimal control problems **(OC+)** and **(OC–)**.

Proposition 5.1. *Let $[E, A, B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ be given. Then the following holds:*

*a) If **(OC+)** is feasible, then $[E, A, B]$ is behaviorally stabilizable,*

*b) If **(OC–)** is feasible, then $[E, A, B]$ is behaviorally anti-stabilizable.*

Proof. To prove statement a), assume that **(OC+)** is feasible. Then $V_+(Ex_0) < \infty$ for all $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$. In particular, the set of all solutions (x, u) of $[E, A, B]$ with $Ex(0) = Ex_0$ and $Ex(\infty) = 0$ is non-empty for all $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$. This gives rise to stabilizability of $[E, A, B]$. The proof of assertion b) is analogous. ∎

Next we show that V_+ and V_- are quadratic storage functions.

Theorem 5.2. *Let $[E, A, B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ be given. Then the following holds:*

*a) If **(OC+)** is feasible, then V_+ is a quadratic storage function.*
*b) If **(OC–)** is feasible, then V_- is a quadratic storage function.*

Proof. It suffices to prove statement a). The second assertion can then be inferred from the fact that replacing E by $-E$ reflects the solutions, that is

$$(5.1)\quad (x(\cdot),u(\cdot)) \text{ is a solution of } [E,A,B] \iff (x(-\cdot),u(-\cdot)) \text{ is a solution of } [-E,A,B],$$

and the fact that V_- is the value function for the optimal cost in **(OC–)**, if and only if $\widetilde{V}_+ := -V_-$ is the value function corresponding to the optimal control problem for the system $[-E,A,B]$ on the positive time axis.

Assume that **(OC+)** is feasible. Then $[E,A,B]$ is stabilizable by Proposition 5.1. For $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$, consider the set of trajectories of $[E,A,B]$ which are square integrable on $\mathbb{R}_{\geq 0}$ and initial differential value x_0, i. e.,

$$\mathfrak{B}_{\mathcal{L}^2}(x_0) := \big\{(x,u) \in \mathcal{L}^2(\mathbb{R}_{\geq 0},\mathbb{R}^n) \times \mathcal{L}^2(\mathbb{R}_{\geq 0},\mathbb{R}^m) \,:\, (x,u) \text{ is a solution of } [E,A,B] \text{ on } \mathbb{R}_{\geq 0} \text{ with } Ex(0) = Ex_0\big\}.$$

Note that $\mathfrak{B}_{\mathcal{L}^2}(x_0)$ is nonempty for all $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$ since $[E,A,B]$ is stabilizable. Consider the functional $\widetilde{V}_+ : E\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} \to \mathbb{R} \cup \{-\infty,\infty\}$ with

$$\widetilde{V}_+(Ex_0) = \inf\left\{\mathcal{J}(x,u,\mathbb{R}_{\geq 0}) \,:\, (x,u) \in \mathfrak{B}_{\mathcal{L}^2}(x_0)\right\}.$$

Step 1: We show that

$$(5.2)\qquad V_+(Ex_0) \leq \widetilde{V}_+(Ex_0) < \infty \quad \forall\, x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}.$$

Assume that $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$. Then, by stabilizability of $[E,A,B]$, there exists some $(x,u) \in \mathfrak{B}_{\mathcal{L}^2}(x_0)$. This gives rise to $\mathcal{J}(x,u,\mathbb{R}_{\geq 0}) \in \mathbb{R}$, whence $\widetilde{V}_+(Ex_0) < \infty$. On the other hand, $(x,u) \in \mathfrak{B}_{\mathcal{L}^2}(x_0)$ implies that x, $\frac{\mathrm{d}}{\mathrm{d}t}(Ex) \in \mathcal{L}^2(\mathbb{R}_{\geq 0},\mathbb{R}^n)$ and therefore, we obtain from [11, Thm. 3], a variant of Barbălat's lemma, that $Ex(\infty) = 0$. Hence, $\widetilde{V}_+(Ex_0)$ is the infimum over a set which is contained in the set whose infimum is $V^+(Ex_0)$. This implies (5.2).

Step 2: We show that $\widetilde{V}_+$ is quadratic: To this end, we need to show that for all $\lambda \in \mathbb{R}$ and x_0, x_{01}, $x_{02} \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$ it holds that

$$(5.3a)\qquad \widetilde{V}_+(\lambda \cdot Ex_0) = |\lambda|^2 \cdot \widetilde{V}_+(Ex_0),$$

$$(5.3b)\qquad \widetilde{V}_+(E(x_{01}-x_{02})) + \widetilde{V}_+(E(x_{01}+x_{02})) = 2\cdot\widetilde{V}_+(Ex_{01}) + 2\cdot\widetilde{V}_+(Ex_{02}).$$

An expansion of the products in the integral yields that for all $\lambda \in \mathbb{R}$ and solutions (x_1,u_1), (x_2,u_2) of $[E,A,B]$ with (x_1,u_1), $(x_2,u_2) \in \mathcal{L}^2(\mathbb{R}_{\geq 0},\mathbb{R}^n)\times\mathcal{L}^2(\mathbb{R}_{\geq 0},\mathbb{R}^m)$, the cost function fulfills

$$(5.4a)\qquad \mathcal{J}(\lambda x_1,\lambda u_1,\mathbb{R}_{\geq 0}) = |\lambda|^2 \cdot \mathcal{J}(x_1,u_1,\mathbb{R}_{\geq 0}),$$

$$(5.4b)\qquad 2\cdot\mathcal{J}(x_1,u_1,\mathbb{R}_{\geq 0}) + 2\cdot\mathcal{J}(x_2,u_2,\mathbb{R}_{\geq 0}) = \mathcal{J}(x_1+x_2,u_1+u_2,\mathbb{R}_{\geq 0}) + \mathcal{J}(x_1-x_2,u_1-u_2,\mathbb{R}_{\geq 0}).$$

We first prove (5.3a): We have $\widetilde{V}_+(0) \leq 0$, since $\mathcal{J}(0,0,\mathbb{R}_{\geq 0}) = 0$. On the other hand, the existence of a solution (x,u) of $[E,A,B]$ with $Ex(0) = Ex(\infty) = 0$ with with $\mathcal{J}(x,u,\mathbb{R}_{\geq 0}) < 0$ would imply, by taking scalar multiples of (x,u), that $V_+(0) \leq \widetilde{V}_+(0) = -\infty$. Hence, feasibility of **(OC+)** gives rise to $\widetilde{V}_+(0) = 0$. Thus, we have

$$\widetilde{V}_+(0 \cdot Ex_0) = 0 = |0|^2 \cdot \widetilde{V}_+(Ex_0).$$

Further, for all $\lambda \in \mathbb{R} \setminus \{0\}$, $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$, and $\varepsilon > 0$, the definition of $\widetilde{V}_+$ leads to the existence of some $(x,u) \in \mathfrak{B}_{\mathcal{L}^2}(x_0)$ with

$$\mathcal{J}(x,u,\mathbb{R}_{\geq 0}) \leq \widetilde{V}_+(Ex_0) + \tfrac{\varepsilon}{\lambda^2},$$

and therefore we have

$$\begin{aligned}\widetilde{V}_+(E(\lambda x_0)) &\leq \mathcal{J}(\lambda x, \lambda u, \mathbb{R}_{\geq 0}) = |\lambda|^2 \cdot \mathcal{J}(x,u,\mathbb{R}_{\geq 0}) \\ &\leq |\lambda|^2 \cdot \left(\widetilde{V}^+(Ex_0) + \tfrac{\varepsilon}{|\lambda|^2}\right) = |\lambda|^2 \cdot \widetilde{V}^+(Ex_0) + \varepsilon.\end{aligned}$$

Since the above inequality holds for all $\varepsilon > 0$ it follows that

$$\widetilde{V}_+(E(\lambda x_0)) \leq |\lambda|^2 \cdot \widetilde{V}_+(Ex_0). \tag{5.5}$$

The reverse inequality follows from

$$\widetilde{V}_+(Ex_0) = \widetilde{V}_+\left(E\left(\tfrac{1}{\lambda} \cdot \lambda x_0\right)\right) \overset{(5.5)}{\leq} \tfrac{1}{|\lambda|^2} \cdot \widetilde{V}_+(E(\lambda x_0)).$$

Altogether we obtain that (5.3a) is satisfied.

Next we show (5.3b): Assume that $x_{01}, x_{02} \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$ and $\varepsilon > 0$. The definition of $\widetilde{V}_+$ implies that there exist $(x_1,u_1) \in \mathfrak{B}^{\mathcal{L}^2}(x_{01})$, $(x_2,u_2) \in \mathfrak{B}^{\mathcal{L}^2}(x_{02})$ and

$$\mathcal{J}(x_1,u_1,\mathbb{R}_{\geq 0}) \leq \widetilde{V}_+(Ex_{01}) + \tfrac{\varepsilon}{4}, \quad \mathcal{J}(x_2,u_2,\mathbb{R}_{\geq 0}) \leq \widetilde{V}_+(Ex_{02}) + \tfrac{\varepsilon}{4}. \tag{5.6}$$

Then we obtain

$$\begin{aligned}&\widetilde{V}_+\left(E(x_{01}+x_{02})\right) + \widetilde{V}_+\left(E(x_{01}-x_{02})\right) \\ &\qquad \leq \mathcal{J}(x_1+x_2, u_1+u_2, \mathbb{R}_{\geq 0}) + \mathcal{J}(x_1-x_2, u_1-u_2, \mathbb{R}_{\geq 0}) \\ &\qquad \overset{(5.4b)}{=} 2 \cdot \mathcal{J}(x_1,u_1,\mathbb{R}_{\geq 0}) + 2 \cdot \mathcal{J}(x_2,u_2,\mathbb{R}_{\geq 0}) \\ &\qquad \overset{(5.6)}{\leq} 2 \cdot \widetilde{V}_+(Ex_{01}) + 2 \cdot \widetilde{V}_+(Ex_{02}) + \varepsilon.\end{aligned}$$

Since the above inequality holds for all $\varepsilon > 0$ we have

$$\widetilde{V}_+\left(E(x_{01}+x_{02})\right) + \widetilde{V}_+\left(E(x_{01}-x_{02})\right) \leq 2 \cdot \widetilde{V}^+(Ex_{01}) + 2 \cdot \widetilde{V}_+(Ex_{02}). \tag{5.7}$$

Now we prove the reverse inequality: For $\widetilde{x}_{01} = \frac{1}{2}(x_{01}+x_{02})$ and $\widetilde{x}_{02} = \frac{1}{2}(x_{01}-x_{02})$ we have $\widetilde{x}_{01} + \widetilde{x}_{02} = x_{01}$ and $\widetilde{x}_{01} - \widetilde{x}_{02} = x_{02}$. Then (5.4a) is satisfied due to

$$\begin{aligned}&2 \cdot \widetilde{V}_+(Ex_{01}) + 2 \cdot \widetilde{V}_+(Ex_{02}) \\ &\qquad = 2 \cdot \widetilde{V}_+(E(\widetilde{x}_{01}+\widetilde{x}_{02})) + 2 \cdot \widetilde{V}_+(E(\widetilde{x}_{01}-\widetilde{x}_{02}))\end{aligned}$$

$$
\begin{aligned}
&\overset{(5.7)}{\leq} \; 4 \cdot \widetilde{V}_+(E\widetilde{x}_{01}) + 4 \cdot \widetilde{V}_+(E\widetilde{x}_{02}) \\
&= \; 4 \cdot \widetilde{V}_+\left(E\left(\tfrac{1}{2}(x_{01}+x_{02})\right)\right) + 4 \cdot \widetilde{V}_+\left(E\left(\tfrac{1}{2}(x_{01}-x_{02})\right)\right) \\
&\overset{(5.3a)}{=} \; \widetilde{V}_+(E(x_{01}+x_{02})) + \widetilde{V}_+(E(x_{01}-x_{02})).
\end{aligned}
$$

Step 3: We prove that $\widetilde{V}_+$ is a storage function. Since $\widetilde{V}_+$ is quadratic by Step 2, it is continuous with $\widetilde{V}(0) = 0$. Now assume that $t \geq 0$ and (x, u) be a solution of $[E, A, B]$ with $Ex(0) = Ex_0$. By definition of $\widetilde{V}_+$, there exists some $(\widetilde{x}, \widetilde{u}) \in \mathfrak{B}_{\mathcal{L}^2}(x(t))$ with

$$
\mathcal{J}(\widetilde{x}, \widetilde{u}, \mathbb{R}_{\geq 0}) \leq \widetilde{V}_+(Ex(t)) + \varepsilon. \tag{5.8}
$$

Consider the concatenation $(\overline{x}, \overline{u})$ with $(\overline{x}(\tau), \overline{u}(\tau)) = (x(\tau), u(\tau))$ for all $\tau \in [0, t]$, and $(\overline{x}(\tau), \overline{u}(\tau)) = (\widetilde{x}(\tau - t), \widetilde{u}(\tau - t))$ for all $\tau \in [t, \infty)$. Then $(\overline{x}, \overline{u})$ is a solution of $[E, A, B]$ with $Ex(0) = Ex_0$. In particular, we have $(\overline{x}, \overline{u}) \in \mathfrak{B}_{\mathcal{L}^2}(x_0)$. Then, by using time-invariance, we obtain

$$
\begin{aligned}
\widetilde{V}_+(Ex_0) \leq \mathcal{J}(\overline{x}, \overline{u}, \mathbb{R}_{\geq 0}) \; &= \; \mathcal{J}(\overline{x}, \overline{u}, [0, t]) + \mathcal{J}(\overline{x}, \overline{u}, [t, \infty)) \\
&= \; \mathcal{J}(\overline{x}, \overline{u}, [0, t]) + \mathcal{J}(\overline{x}(\cdot + t), \overline{u}(\cdot + t), \mathbb{R}_{\geq 0}) \\
&= \; \mathcal{J}(x, u, [0, t]) + \mathcal{J}(\widetilde{x}, \widetilde{u}, \mathbb{R}_{\geq 0}) \\
&\overset{(5.8)}{\leq} \; \mathcal{J}(x, u, [0, t]) + \widetilde{V}^+(Ex(t)) + \varepsilon.
\end{aligned}
$$

The result follows now by time-invariance of $[E, A, B]$ and by the fact that $\varepsilon > 0$ can be made arbitrarily small.

Step 4: We show that $\widetilde{V}_+ = V_+$. Assume that $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$. The inequality $V_+(Ex_0) \leq \widetilde{V}_+(Ex_0)$ has already been proven in Step 1. To show the reverse inequality, consider a solution (x, u) of $[E, A, B]$ with $Ex(0) = Ex_0$, $Ex(\infty) = 0$ and $\mathcal{J}(x, u, \mathbb{R}_{\geq 0}) \in \mathbb{R}$. Since by Step 3, $\widetilde{V}_+$ is a storage function, we obtain that for all $t \geq 0$ it holds that

$$
\widetilde{V}_+(Ex_0) - \widetilde{V}_+(Ex(t)) \leq \mathcal{J}(x, u, [0, t]).
$$

Taking the limit $t \to \infty$ and using $Ex(\infty) = 0$ together with the continuity of $\widetilde{V}_+$ and $\widetilde{V}_+(0) = 0$, we obtain

$$
\widetilde{V}_+(Ex_0) \leq \mathcal{J}(x, u, \mathbb{R}_{\geq 0}).
$$

This implies $\widetilde{V}_+(Ex_0) \leq V_+(Ex_0)$. ◻

As an immediate consequence, we have that the value functions define special solutions of the KYP inequality.

Corollary 5.3. *Let $[E, A, B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n \times n}$, $S \in \mathbb{R}^{n \times m}$, $R = R^\top \in \mathbb{R}^{m \times m}$ be given. Then the following holds:*

a) *If **(OC+)** is feasible, then there exists a solution $P_+ \in \mathbb{R}^{n \times n}$ of the KYP inequality (2.5) with $V_+(Ex_0) = x_0^\top E^\top P_+ E x_0$ for all $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$.*

b) *If **(OC−)** is feasible, then there exists a solution $P_- \in \mathbb{R}^{n \times n}$ of the KYP inequality (2.5) with $V_-(Ex_0) = x_0^\top E^\top P_- E x_0$ for all $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$.*

Proof. Assume that **(OC+)** is feasible. Theorem 5.2 implies that there exists some Hermitian $P \in \mathbb{R}^{n\times n}$ with $V_+(Ex_0) = x_0^\top E^\top P_+ Ex_0$ for all $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$. Since further by Theorem 5.2, V_+ is a storage function, Theorem 4.3 then gives rise to the fact that P^+ solves the KYP inequality (2.5). Statement b) can be inferred by the same argumentation. ◻

Now we present some characterizations for feasibility, and we show that V_+ and V_- have a certain extremality condition among all storage functions.

THEOREM 5.4. *Let $[E,A,B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ be given.*

a) The following statements are equivalent:

*i) The problem **(OC+)** is feasible, i. e., $V_+(Ex_0) \in \mathbb{R}$ for all $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$.*

ii) The system $[E,A,B]$ is behaviorally stabilizable and there exists a storage function V.

iii) The system $[E,A,B]$ is behaviorally stabilizable and the KYP inequality (2.5) has a solution $P \in \mathbb{R}^{n\times n}$.

*Further, in case of feasibility of **(OC+)**, all storage functions V fulfill*

$$V(Ex_0) \le V_+(Ex_0) \quad \forall\, x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}. \tag{5.9}$$

b) The following statements are equivalent:

*i) The problem **(OC–)** is feasible, i. e., $V_-(Ex_0) \in \mathbb{R}$ for all $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$.*

ii) The system $[E,A,B]$ is behaviorally anti-stabilizable and there exists a storage function V.

iii) The system $[E,A,B]$ is behaviorally anti-stabilizable and the KYP inequality (2.5) has a solution $P \in \mathbb{R}^{n\times n}$.

*Further, in case of feasibility of **(OC–)**, all storage functions V fulfill*

$$V_-(Ex_0) \le V(Ex_0) \quad \forall\, x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}. \tag{5.10}$$

Proof. Assertion b) can again be inferred from a) by replacing E by $-E$. This follows by using (5.1) and the fact that V is a storage function, if and only if $-V$ is a storage function for the problem in which E is replaced by $-E$.

Hence it suffices to prove a):

"i)⇒iii)": Assume that **(OC+)** is feasible: Then $[E,A,B]$ is behaviorally stabilizable by Proposition 5.1. Further, the existence of a solution of the KYP inequality follows from Corollary 5.3.

"iii)⇒ii)": This follows from Theorem 4.3.

"ii)⇒i)": Assume that $[E,A,B]$ is behaviorally stabilizable and there exists a continuous $V : E\mathcal{V}^{\text{diff}}_{[E,A,B]} \to \mathbb{R}$ with $V(0) = 0$ such that the dissipation inequality (2.3) is satisfied. Assume that $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$. Behavioral stabilizability of $[E,A,B]$ yields the existence of some of a solution (x,u) on $\mathbb{R}_{\ge 0}$ with $Ex(0) = x_0$, $Ex(\infty) = 0$ and $\mathcal{J}(x,u,\mathbb{R}_{\ge 0}) < \infty$, and thus $V(Ex_0) < \infty$. Further, by taking the limit $t \to \infty$, we obtain

$$\mathcal{J}(x,u,\mathbb{R}_{\ge 0}) = \lim_{t\to\infty} \mathcal{J}(x,u,[0,t]) \ge \lim_{t\to\infty} V(Ex(0)) - V(Ex(t)) = V(Ex_0).$$

By taking the infimum over all solutions (x,u) with $Ex(0) = Ex_0$ and $Ex(\infty) = 0$, we obtain that $V_+(Ex_0) \ge V(Ex_0) > -\infty$. This proves feasibility of **(OC+)** as well as the inequality (5.9). ◻

Now we present some consequences of the previous results.

COROLLARY 5.5. *Let $[E, A, B] \in \Sigma_{n,m}$ be behaviorally controllable and let $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ be given. Then the following statements are equivalent:*

*a) The problem **(OC+)** is feasible.*
*b) The problem **(OC–)** is feasible.*
c) There exists a storage function V.
d) The KYP inequality (2.5) has a solution $P \in \mathbb{R}^{n\times n}$.

In the case where the above assertions are valid, we have

$$V_-(Ex_0) \le V(Ex_0) \le V_+(Ex_0) \quad \forall\, x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}.$$

Proof. This follows immediately from Theorem 5.4. ∎

COROLLARY 5.6. *Let $[E, A, B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ be given.*

*a) If **(OC+)** is feasible, then there exists some solution $P_+ \in \mathbb{R}^{n\times n}$ of the KYP inequality (2.5). Further, for all solutions $P \in \mathbb{R}^{n\times n}$ of the KYP inequality (2.5) we have*

$$P \le_{E\mathcal{V}^{\text{diff}}_{[E,A,B]}} P_+. \tag{5.11}$$

*b) If **(OC–)** is feasible, then there exists some solution $P_- \in \mathbb{R}^{n\times n}$ of the KYP inequality (2.5). Further, for all solutions $P \in \mathbb{R}^{n\times n}$ of the KYP inequality (2.5) we have*

$$P_- \le_{E\mathcal{V}^{\text{diff}}_{[E,A,B]}} P. \tag{5.12}$$

Proof. If **(OC+)** is feasible, then Corollary 5.3 implies that there exists a solution $P_+ \in \mathbb{R}^{n\times n}$ of the KYP inequality (2.5) with $V_+(Ex_0) = x_0^\top E^\top P_+ Ex_0$ for all $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$. Assume that $P \in \mathbb{R}^{n\times n}$ is a further solution of the KYP inequality (2.5). Then, by Theorem 4.3, $V : E\mathcal{V}^{\text{diff}}_{[E,A,B]} \to \mathbb{R}$, $x_0 \mapsto V(Ex_0) = x_0^\top E^\top PEx_0$ is a storage function. Then (5.11) can be concluded from (5.9).

The statement for **(OC–)** can again be proven by an analogous argumentation. ∎

We have seen that the value functions are defined by extremal solutions of the KYP inequality. Next we present a further characterization which allows to design optimal controls and to check for regularity of the optimal control problems **(OC+)** and **(OC–)**. We have seen in Proposition 4.5 that the right hand side of the KYP inequality can be factored in a special way. Now we present a certain "specialization" of the KYP inequality which will turn out to be useful for our considerations on optimal control. Namely, we seek some $q \in \mathbb{N}_0$ and a triple $(P, K, L) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ with solves the *Lur'e equation*. That is, (2.6) holds with (2.7). In particular, we show that the stabilizing and anti-stabilizing solutions of the Lur'e equation represent the value functions for **(OC+)** and **(OC–)** and moreover, that they determine the optimal controls. This is done in the following theorem. The condition $Kx_* + Lu_* = 0$ obtained below results in a closed-loop system that is "outer", see [16].

THEOREM 5.7. *Let $[E, A, B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ be given. Then the following statements are satisfied:*

a) The following two statements are equivalent:

i) *The system* $[E,A,B]$ *is behaviorally stabilizable and the Lur'e equation* (2.6) *has a stabilizing solution* $(P,K,L)\in\mathbb{R}^{n\times n}\times\mathbb{R}^{q\times n}\times\mathbb{R}^{q\times m}$.

ii) *The problem* ***(OC+)*** *is feasible.*

In this case, the value functions fulfills $V_+(Ex_0)=x_0^\top E^\top PEx_0$ *for all* $x_0\in\mathcal{V}^{\text{diff}}_{[E,A,B]}$. *Further, a solution* (x_*,u_*) *of* $[E,A,B]$ *on* $\mathbb{R}_{\geq 0}$ *with* $Ex(0)=Ex_0$ *and* $Ex(\infty)=0$ *is an optimal control, if and only if* $Kx_*+Lu_*=0$.

b) *The following two statements are equivalent:*

i) *The system* $[E,A,B]$ *is behaviorally anti-stabilizable and the Lur'e equation* (2.6) *has an anti-stabilizing solution* $(P,K,L)\in\mathbb{R}^{n\times n}\times\mathbb{R}^{q\times n}\times\mathbb{R}^{q\times m}$.

ii) *The problem* ***(OC–)*** *is feasible.*

In this case, the value functions fulfills $V_-(Ex_0)=x_0^\top E^\top PEx_0$ *for all* $x_0\in\mathcal{V}^{\text{diff}}_{[E,A,B]}$. *Further, a solution* (x_*,u_*) *of* $[E,A,B]$ *on* $\mathbb{R}_{\leq 0}$ *with* $Ex(0)=Ex_0$ *and* $Ex(-\infty)=0$ *is an optimal control, if and only if* $Kx_*+Lu_*=0$.

Proof. Again, it suffices to show the statement a):

"i)⇒ii)": Assume that $[E,A,B]$ is behaviorally stabilizable and that $(P,K,L)\in\mathbb{R}^{n\times n}\times\mathbb{R}^{q\times n}\times\mathbb{R}^{q\times m}$ is a stabilizing solution of the Lur'e equation. In particular, P solves the KYP inequality (2.5). Then the feasibility of **(OC+)** follows from Theorem 5.4.

"ii)⇒i)": Assume that **(OC+)** is feasible. Then $[E,A,B]$ is behaviorally stabilizable by Proposition 5.1. Next we show that the Lur'e equation has a stabilizing solution. By Theorem 5.2, feasibility of **(OC+)** implies that the value function V_+ is a quadratic storage function. In other words, there exists some Hermitian $P\in\mathbb{R}^{n\times n}$ such that $V_+(Ex_0)=x_0^\top E^\top PEx_0$ for all $x_0\in\mathcal{V}^{\text{diff}}_{[E,A,B]}$. Theorem 4.3 then gives rise to the fact that P solves the KYP inequality (2.5), and by Proposition 4.5, we obtain that there exist $q\in\mathbb{N}_0$, $K\in\mathbb{R}^{q\times n}$, $L\in\mathbb{R}^{q\times m}$ such that $\begin{bmatrix}K & L\end{bmatrix}\mathcal{V}^{\text{sys}}_{[E,A,B]}=\mathbb{R}^q$ and (2.6) is satisfied. Now we show that (P,K,L) is a stabilizing solution of the Lur'e equation. To this end, we have to prove that (2.8) is fulfilled. According to [16, Thm. 6.6 a)], this is the case if the following two statements are valid:

1) If for $y_0\in\mathbb{R}^q$ we have $y_0^\top(Kx+Lu)\equiv 0$ for all solutions (x,u) of $[E,A,B]$, then $y_0=0$.
2) For all $x_0\in\mathcal{V}^{\text{diff}}_{[E,A,B]}$ and $\varepsilon>0$, there exists a solution (x,u) of $[E,A,B]$ on $\mathbb{R}_{\geq 0}$ with $Ex(0)=Ex_0$, $Ex(\infty)=0$ and $\|Kx+Lu\|_{\mathcal{L}^2(\mathbb{R}_{\geq 0},\mathbb{R}^q)}<\varepsilon$.

First we show 1): Assume that for $y_0\in\mathbb{R}^q$ we have $y_0^\top(Kx+Lu)\equiv 0$ for all solutions (x,u) of $[E,A,B]$. Since $\begin{bmatrix}K & L\end{bmatrix}\mathcal{V}^{\text{sys}}_{[E,A,B]}=\mathbb{R}^q$, there exists some $(x_0,u_0)\in\mathcal{V}^{\text{sys}}_{[E,A,B]}$ with $Kx_0+Lu_0=y_0$. By Lemma 4.1, there exists some infinitely often differentiable solution (x,u) of $[E,A,B]$ with $x(0)=x_0$ and $u(0)=u_0$. Then $0=y_0^\top(Kx(0)+Lu(0))=y_0^\top(Kx_0+Lu_0)=y_0^\top y_0=\|y_0\|^2$, which implies $y_0=0$.

Next we show 2): Suppose that $x_0\in\mathcal{V}^{\text{diff}}_{[E,A,B]}$ and $\varepsilon>0$. By definition of the value function, there exists some solution (x,u) of $[E,A,B]$ on $\mathbb{R}_{\geq 0}$ with $Ex(0)=Ex_0$, $Ex(\infty)=0$ and $\mathcal{J}(x,u,\mathbb{R}_{\geq 0})-V_+(Ex_0)<\varepsilon^2$. On the other hand, by using $V_+(Ex_0)=x_0^\top E^\top PEx_0$ and Corollary 4.6, we obtain

$$\|Kx+Lu\|^2_{\mathcal{L}^2(\mathbb{R}_{\geq 0},\mathbb{R}^q)}=\mathcal{J}(x,u,\mathbb{R}_{\geq 0})-x_0^\top E^\top PEx_0=\mathcal{J}(x,u,\mathbb{R}_{\geq 0})-V_+(Ex_0)<\varepsilon^2,$$

and the proof of "ii)⇒i)" is complete.

Now we prove the remaining statement: Assume that $(P,K,L)\in\mathbb{R}^{n\times n}\times\mathbb{R}^{q\times n}\times\mathbb{R}^{q\times m}$ is a stabilizing solution of the Lur'e equation (2.6) and $[E,A,B]$ is behaviorally stabilizable. Assume further that $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$. Then the value function fulfills $x_0 E^\top P E x_0 \le V_+(Ex_0)$ by Theorem 5.4. To prove that also $x_0 E^\top P E x_0 \ge V_+(Ex_0)$, let $\varepsilon > 0$. Since (P, K, L) is a stabilizing solution, we know that (2.8) is satisfied. Then, by [16, Thm. 6.6], there exists a solution (x, u) of $[E, A, B]$ on $\mathbb{R}_{\ge 0}$ with $Ex(0) = Ex_0$, $Ex(\infty) = 0$ and $\|Kx + Lu\|^2_{\mathcal{L}^2(\mathbb{R}_{\ge 0},\mathbb{R}^q)} < \varepsilon$. By using Corollary 4.6, we obtain that this trajectory fulfills

$$\mathcal{J}(x, u, \mathbb{R}_{\ge 0}) = x_0^\top E^\top P E x_0 + \|Kx + Lu\|^2_{\mathcal{L}^2(\mathbb{R}_{\ge 0},\mathbb{R}^q)} < x_0^\top E^\top P E x_0 + \varepsilon,$$

and thus $x_0^\top E^\top P E x_0 \ge V_+(Ex_0)$.

Finally we show that a solution (x_*, u_*) of $[E, A, B]$ on $\mathbb{R}_{\ge 0}$ with $Ex(0) = Ex_0$ and $Ex(\infty) = 0$ is an optimal control, if and only if $Kx_* + Lu_* = 0$: If a solution (x_*, u_*) of $[E, A, B]$ on $\mathbb{R}_{\ge 0}$ with $Ex(0) = Ex_0$ and $Ex(\infty) = 0$ is an optimal control, then $V_+(Ex_0) = \mathcal{J}(x_*, u_* \mathbb{R}_{\ge 0})$. Then by using $V_+(Ex_0) = x_0^\top E^\top P E x_0$ and Corollary 4.6, we obtain

$$x_0^\top E^\top P E x_0 = V_+(Ex_0) = \mathcal{J}(x_*, u_*, \mathbb{R}_{\ge 0}) = x_0^\top E^\top P E x_0 + \|Kx_* + Lu_*\|^2_{\mathcal{L}^2(\mathbb{R}_{\ge 0},\mathbb{R}^q)},$$

and thus $Kx_* + Lu_* = 0$. On the other hand, by the same argumentation, we see that $x_0^\top E^\top P E x_0 = V_+(Ex_0) = \mathcal{J}(x_*, u_*, \mathbb{R}_{\ge 0})$, if $Kx_* + Lu_* = 0$ is satisfied. ☐

We have seen in Theorem 5.7 that a solution (x_*, u_*) of $[E, A, B]$ on $\mathbb{R}_{\ge 0}$ with $Ex(0) = Ex_0$ and $Ex(\infty) = 0$ is an optimal control, if and only if it fulfills the differential-algebraic equation (2.10). As a consequence, regularity corresponds to the unique solvability of the differential-algebraic equation (2.10) for all $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$. This is characterized in the following theorem.

Theorem 5.8. *Let $[E, A, B] \in \Sigma_{n,m}$ and $Q = Q^\top \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$, $R = R^\top \in \mathbb{R}^{m\times m}$ be given.*

- *a) If **(OC+)** is feasible and $(P, K, L) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ is a stabilizing solution of the Lur'e equation (2.6), then the following two statements are equivalent:*
  - *i) The problem **(OC+)** is regular.*
  - *ii) The conditions (2.11) and (2.12) are satisfied.*
- *b) If **(OC−)** is feasible and $(P, K, L) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ is an anti-stabilizing solution of the Lur'e equation (2.6), then the following two statements are equivalent:*
  - *i) The problem **(OC−)** is regular.*
  - *ii) The conditions (2.11) and (2.12) are satisfied.*

Proof. As before, it suffices to show statement a). Since Theorem 5.7 implies that optimal controls are exactly those elements of the behavior which fulfill (2.10), we know that i) is equivalent to

- i') For all $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$, the differential-algebraic equation (2.10) has a unique solution.

The rest of the proof proceeds in several steps.

Step 1: We show that i') implies

$$\ker_{\mathbb{R}[s]} \begin{bmatrix} -sE + A & B \\ K & L \end{bmatrix} = \{0\}. \tag{5.13}$$

Assuming the opposite, then [5, Cor. 5.2] implies that the solution set of

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x \\ u \end{pmatrix} = \begin{bmatrix} A & B \\ K & L \end{bmatrix}\begin{pmatrix} x \\ u \end{pmatrix} \tag{5.14}$$

defines a non-autonomous system in the sense of [33, Def. 3.2.1]. Since, by [33, Thm. 5.2.14], the solution set of (5.14) can be represented as a direct sum of an autonomous and a behaviorally controllable part, the latter is nontrivial. Then we can conclude from [33, Thm. 5.2.14] that there exists some nontrivial solution (x, u) of (5.14) with $(x, u)|_{\mathbb{R}_{\leq 0}} = 0$ and $(x, u)|_{[1,\infty)} = 0$. This is a contradiction to i').

Step 2: We show that i') implies (2.12). Aiming for a contradiction, assume that i') is satisfied and that $\ker_{\mathbb{C}} \left[\begin{smallmatrix} -\mathrm{i}\omega E + A & B \\ K & L \end{smallmatrix}\right] \neq \{0\}$ for some $\omega \in \mathbb{R}$. Let $\left(\begin{smallmatrix} x_{c0} \\ u_{c0} \end{smallmatrix}\right) \in \mathbb{C}^n \times \mathbb{C}^m \setminus \{(0,0)\}$ be an element of this nullspace. Then $\left(\begin{smallmatrix} x_c(t) \\ u_c(t) \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_{c0} \\ u_{c0} \end{smallmatrix}\right) \cdot \mathrm{e}^{\mathrm{i}\omega t}$ is a solution of the complex differential-algebraic equation (5.14). Since E, A, B, K and L are real, we have that the component- and pointwise imaginary part

$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} := \operatorname{Im}\left(\begin{pmatrix} x_c(t) \\ u_c(t) \end{pmatrix}\right) = \begin{pmatrix} \operatorname{Re}(x_{c0})\sin(\omega t) + \operatorname{Im}(x_{c0})\cos(\omega t) \\ \operatorname{Re}(u_{c0})\sin(\omega t) + \operatorname{Im}(u_{c0})\cos(\omega t) \end{pmatrix}$$

solves the real differential-algebraic equation (5.14). By (5.13) and [5, Cor. 5.2], this is moreover the unique solution of (5.14) with $Ex(0) = Ex_0$ for $x_0 := \operatorname{Im}(x_{c0})$. The limit $Ex(\infty)$ does not exist. However, the optimal control with $Ex(0) = Ex_0$ and $Ex(\infty) = 0$ should satisfy (5.14). This is again a contradiction.

Step 3: Let $V \in \mathbb{R}^{(n+m)\times k}$ be a matrix with full column rank and $\operatorname{im}_{\mathbb{R}} V = \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$. We show that (2.11) is equivalent to

$$\operatorname{im}_{\mathbb{R}} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} V + \begin{bmatrix} A & B \\ K & L \end{bmatrix} V \cdot \ker_{\mathbb{R}}\left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} V\right) = \operatorname{im}_{\mathbb{R}} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} V + \operatorname{im}_{\mathbb{R}} \begin{bmatrix} A & B \\ K & L \end{bmatrix} V. \tag{5.15}$$

By using $\operatorname{im}_{\mathbb{R}} V = \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$, we see that (2.11) is satisfied, if we can show that

$$V \cdot \ker_{\mathbb{R}}\left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} V\right) = (\ker_{\mathbb{R}} E \times \mathbb{R}^m) \cap \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}.$$

To show "$\subseteq$", assume that $\left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right) \in V \cdot \ker_{\mathbb{R}}\left(\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right] V\right)$. Then there exists some $w_0 \in \mathbb{R}^k$ with $Vw_0 = \left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right)$ and $\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right] Vw_0 = 0$. This gives rise to

$$\begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = Vw_0 \in (\ker_{\mathbb{R}} E \times \mathbb{R}^m) \cap \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}.$$

For the inclusion "$\supseteq$", assume that $\left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right) \in (\ker_{\mathbb{R}} E \times \mathbb{R}^m) \cap \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$. Then, by $\left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right) \in \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$, there exists some $w_0 \in \mathbb{R}^k$ with $Vw_0 = \left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right)$. By $\left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right) \in (\ker_{\mathbb{R}} E \times \mathbb{R}^m)$, we obtain

$$0 = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} Vw_0 = 0,$$

and thus

$$\begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = Vw_0 \in V \cdot \ker_{\mathbb{R}}\left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} V\right).$$

Step 4: Let $V \in \mathbb{R}^{(n+m)\times k}$ be a matrix with full column rank and $\operatorname{im}_{\mathbb{R}} V = \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$.

We show that $w \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^k)$ fulfills

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} Vw = \begin{bmatrix} A & B \\ K & L \end{bmatrix} Vw, \tag{5.16}$$

if and only if $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) := Vw$ fulfills (5.14): If w fulfills (5.16), then $(x,u) = Vw$ clearly satisfies (5.14). On the other hand, if (x,u) fulfills (5.14), then (x,u) is a solution of $[E,A,B]$, and thus $\left(\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right) \in \mathcal{V}^{\mathrm{sys}}_{[E,A,B]} = \mathrm{im}_{\mathbb{R}}\, V$ for almost all $t \in \mathbb{R}$. Then there exists some $w \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^k)$ such that $\left(\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right) = Vw(t)$ for almost all $t \in \mathbb{R}$. Plugging this into (5.14), we obtain (5.16).

Step 5: We show that i') implies (2.11): Assume that i') holds and let $V \in \mathbb{R}^{(n+m)\times k}$ be a matrix with full column rank and $\mathrm{im}_{\mathbb{R}}\, V = \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$. We first show that for all $w_0 \in \mathbb{R}^k$ there exists a $w \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^k)$ such that (5.16) is fulfilled with

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} Vw(0) = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} Vw_0. \tag{5.17}$$

Assume that $w_0 \in \mathbb{R}^k$ is given. Then, by Lemma 4.1, for $\left(\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right) := Vw_0 \in \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$ there exists some infinitely often differentiable solution (x,u) of $[E,A,B]$ with $x(0) = x_0$ and $u(0) = u_0$. As a consequence, $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$ and i') implies that the differential-algebraic equation (5.14) has a unique solution with $Ex(0) = Ex_0$. By the result from Step 4, we obtain that there exists some $w \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^k)$ such that (5.16) is fulfilled with $Vw = \left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right)$. We further have

$$\begin{aligned} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} Vw_0 = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = \begin{pmatrix} Ex_0 \\ 0 \end{pmatrix} & \\ = \begin{pmatrix} Ex(0) \\ 0 \end{pmatrix} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x(0) \\ u(0) \end{pmatrix} &= \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} Vw(0). \end{aligned}$$

We have shown that the differential-algebraic equation (5.16) with initial condition (5.17) has a solution for all $w_0 \in \mathbb{R}^n$. By using [5, Cor. 4.3], we obtain that (5.15) is satisfied. Then, by the result from Step 3, (2.11) is fulfilled.

Step 6: We show that (2.12) implies that for all solutions (x,u) of (5.14) satisfy $Ex(\infty) = 0$: Equation (2.12) implies that $\mathrm{rank}_{\mathbb{R}[s]} \left[\begin{smallmatrix} -sE+A & B \\ K & L \end{smallmatrix}\right] = n+m$. On the other hand, since (P,K,L) is a solution of the Lur'e equation, we have (2.7) which implies $q = m$. This fact together with (2.12) and since (P,K,L) is a stabilizing solution of the Lur'e equation, implies that

$$\ker_{\mathbb{C}} \begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = \{0\} \quad \forall \lambda \in \mathbb{C} \setminus \mathbb{C}_-.$$

Then we obtain from [33, Thm. 7.2.2] that all solutions (x,u) of (5.14) fulfill $Ex(\infty) = 0$.

Step 7: We show that (2.11) implies that for all $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$ there exists a solution (x,u) of (5.14) with $Ex(0) = Ex_0$: Assume that (2.11) is satisfied and let $x_0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$. Then, by Lemma 4.1, there exists some $\left(\begin{smallmatrix} x_{01} \\ u_{01} \end{smallmatrix}\right) \in \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$ with $Ex_0 = Ex_{01}$. Let $w_0 \in \mathbb{R}^k$ with $\left(\begin{smallmatrix} x_{01} \\ u_{01} \end{smallmatrix}\right) = Vw_0$. Since, by Step 3, (2.11) implies (5.15), [5, Cor. 4.3] implies that there exists a solution $w \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^k)$ of the differential-algebraic equation (5.16) with initial condition (5.17). Then, by Step 4, we obtain that $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) := Vw$ fulfills (5.14). Further, we have

$$\begin{aligned} Ex_0 = Ex_{01} = \begin{bmatrix} E & 0 \end{bmatrix} \begin{pmatrix} x_{01} \\ u_{01} \end{pmatrix} = \begin{bmatrix} E & 0 \end{bmatrix} Vw_0 \\ = \begin{bmatrix} E & 0 \end{bmatrix} Vw(0) = \begin{bmatrix} E & 0 \end{bmatrix} \begin{pmatrix} x(0) \\ u(0) \end{pmatrix} = Ex(0). \end{aligned}$$

Step 8: We deduce the overall statement: By the initial statement in this proof, it suffices two prove the equivalence between i') and (2.11), (2.12). We obtain from Step 2 that i') implies (2.12) and from Step 5 that i') implies (2.11). This yields the implication "i)⇒ii)".

Now we show the converse. By Step 6, we obtain that, if (2.12) is satisfied, then all solutions (x, u) of (5.14) fulfill $Ex(\infty) = 0$. If (2.11) is fulfilled, then by Step 7, for all $x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$ there exists a solution (x, u) of (5.14) with $Ex(0) = Ex_0$. To deduce that (2.11), (2.12) imply i'), it suffices to justify that the prescription of the initial condition $Ex(0) = Ex_0$ yields a unique solution of (x, u) of (5.14). This is however a consequence of [5, Cor. 5.2] and (5.13) proven in Step 1. □

6. Notes and references. In this section we discuss the relation of our new results to work that was previously done. Our approach via storage and value functions is motivated by Jan C. Willems' article [39], where the optimal control problems discussed here have been introduced for systems governed by ordinary differential equations. Feasibility conditions in terms of the solvability of the KYP inequality, an algebraic Riccati inequality, and the algebraic Riccati equation have been developed under the additional assumption of controllability. In order to solve the optimal control problem (in the case where R is invertible, which is for instance the case if it is regular) one employs the *algebraic Riccati equation (ARE)* [39, 28]

$$A^\top X + XA - \big(XB + S\big)R^{-1}\big(B^\top X + S^\top\big) + Q = 0, \quad X = X^\top. \tag{6.1}$$

In the literature there exist various attempts to generalize the KYP inequality and the ARE to differential-algebraic systems. Many works focus on the case that the $\mathcal{L}_2$-norm of an output $y = Cx + Du$ is minimized and therefore, there is a focus on the special case

$$\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} = \begin{bmatrix} C^\top \\ D^\top \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \geq 0. \tag{6.2}$$

In this case, the $P = 0$ solves KYP inequality. In [13], the optimal control problem with the additional assumption (6.2) is approached by considering a generalized KYP inequality

$$\begin{bmatrix} A^\top PE + E^\top PA + Q & E^\top PB + S \\ B^\top PE + S^\top & R \end{bmatrix} \geq 0, \quad P = P^\top. \tag{6.3}$$

In [13, Thm. 4.7] it is shown that if the system $[E, A, B]$ is impulse controllable and behaviorally stabilizable, then there exists a maximal solution of (6.3) such that we obtain

$$V_+(Ex_0) = x_0^\top E^\top PEx_0 \geq 0 \quad \forall\, x_0 \in \mathbb{R}^n.$$

However, in the case that (6.2) is not satisfied, it is possible that the KYP inequality (6.3) has no solution even if the linear-quadratic optimal control problem is feasible. The latter has been observed in [12] in the context of passive systems. This is due to the fact that the inequality (6.3) is not restricted to the system space. On the other hand, existence and uniqueness results for optimal controls with positive semi-definite cost functional are presented in [13]. These conditions are based on rank conditions for the pencil $\left[\begin{smallmatrix} -sE+A & B \\ C & D \end{smallmatrix}\right]$.

Another approach which was originally designed for behavior systems is presented in [8, 7]. There, also specializations to differential-algebraic systems are given, however under the additional assumption that $[E, A, B]$ is completely controllable (a much stronger condition than behavioral controllability, see [5]). These considerations are based on the linear matrix inequality

$$\begin{bmatrix} A^\top H + H^\top A + Q & A^\top J + H^\top B + S \\ B^\top H + J^\top A + S^\top & B^\top J + J^\top B + R \end{bmatrix} \geq 0, \quad E^\top H = H^\top E, \quad E^\top J = 0. \tag{6.4}$$

which has to be solved for a pair $(H, J) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$. It is shown that for impulse controllable systems, $P = H$ is a solution of (2.5), if and only if there exists a matrix J such that (H, J) solves (6.4), see also [37].

Other approaches are based on generalizations of the ARE to differential-algebraic equations. One possibility is presented in [25, 26], where

$$A^\top X + X^\top A - (X^\top B + S)R^{-1}(B^\top X + S^\top) + Q = 0, \quad E^\top X = X^\top E \tag{6.5}$$

is considered under the assumptions (6.2) and $R > 0$. It has been proven that

$$V_+(Ex_0) = x_0^\top E^\top X E x_0 \geq 0 \quad \forall\, x_0 \in \mathbb{R}^n,$$

where X is a stabilizing solution of (6.5), meaning that $sE - (A - BR^{-1}(B^\top X + S^\top))$ is of index at most one and all its eigenvalues are contained in $\mathbb{C}_-$. By [9], a necessary condition for the existence of such a solution X is impulse controllability. On the other hand, a sufficient condition for the existence of a solution under the above mentioned assumptions is

$$\operatorname{rank}_{\mathbb{C}} \begin{bmatrix} -\mathrm{i}\omega E + A & B \\ Q & S \\ S^\top & R \end{bmatrix} = n + m \quad \forall\, \omega \in \mathbb{R},$$

which has been shown in [17]. In [37] it is shown that if X is a stabilizing solution of (6.5), then (X, K, L) with $K = R^{-1/2}(B^\top X + S^\top)$ and $L = R^{1/2}$ is a stabilizing solution of the Lur'e equation (2.6).

A further solvability analysis of this type of equation is given in [19, 18] where the *generalized ARE*

$$A^\top X + X^\top A + Q + X^\top R X = 0, \quad E^\top X = X^\top E$$

is considered for E, A, Q, $R \in \mathbb{R}^{n\times n}$ with $Q = Q^\top$ and $R = R^\top$. A solution $X \in \mathbb{R}^{n\times n}$ is called *stabilizing*, if the pencil $-sE + A + RX$ has index at most one and all its eigenvalues are in $\mathbb{C}_-$, which requires impulse controllability of $[E, A, R] \in \Sigma_{n,n}$. It is proven in [19] that solvability of the generalized ARE requires the solvability of a so-called *quadratic matrix equation*. Moreover, in [18] stabilizing solutions are constructed using deflating subspaces of *Hamiltonian matrix pencils*.

In [30], the optimal control problem **(OC+)** for systems of index at most one with $R \geq 0$, $Q \geq 0$, and $S = 0$ is studied. In the case $R > 0$, the value function can be again expressed by the stabilizing solution X (i.,e., the pencil $sE - A + BR^{-1}B^\top X$ has index at most one and all its eigenvalues are in $\mathbb{C}_-$) of the *generalized ARE*

$$A^\top XE + E^\top XA - E^\top XBR^{-1}B^\top XE + Q = 0, \quad X = X^\top. \tag{6.6}$$

Again, in [37] it is discussed that if $X \in \mathbb{R}^{n\times n}$ is a stabilizing solution of (6.6), then (X, K, L) with $K = R^{-1/2}(B^\top X + S^\top)$ and $L = R^{1/2}$ is a stabilizing solution of the Lur'e equation (2.6).

One of the disadvantages of the generalization of the ARE is the need for invertibility of R, which is neither necessary for feasibility nor for regularity of the optimal control problem, see Sec. 3. If **(OC+)** is regular, then it is possible to transform the system $[E, A, B] \in \Sigma_{n,m}$ by certain feedback transformations to so-called SVD coordinates and then extract a regular optimal control problem governed by an ordinary differential equation (see [4] and [32]). Such transformations however require impulse controllability of the system.

An alternative approach to optimal control of differential-algebraic equations with scalar input has recently been published [15]. The key ingredient of this approach is an a priori transformation to quasi-Weierstraß form (3.4) leading to an equivalent optimal control problem for ordinary differential equations.

Boundary value problems for the solution of linear-quadratic optimal control problems and the associated *even matrix pencils* have also been studied intensively in the literature. In [32], the problem of constructing solutions is mainly considered from the numerical point of view. The spectral structure of these pencils for the case $E = I_n$ and their relation to the Lur'e equation are considered in [34], whereas [38, 37] extend this analysis to the case of differential-algebraic systems. Moreover, in [38] also feasibility of the optimal control problems as well as existence and uniqueness of optimal controls have been studied for impulse controllable systems. For the latter, equivalent conditions have been given in terms of the spectrum of the matrix pencil $\left[\begin{smallmatrix} -sE+A & B \\ K & L \end{smallmatrix}\right]$.

To complete the literature review we briefly discuss some generalizations into the direction of time-varying and nonlinear differential-algebraic equations. In [27], linear-quadratic optimal control problems for time-varying differential-algebraic equations and time-varying weights Q, S, and R in the cost functional are treated. Then a time-varying boundary value problem is constructed. Necessary and sufficient conditions for feasibility of the optimal control problems are derived via an inherent Hamiltonian ODE system which is obtained by applying certain projectors to the boundary value problem. We also refer to [2, 3] where differential-algebraic equations of index two are considered.

A different approach for time-varying optimal control problems has been developed in [23, 21, 22]. In [22] two optimality boundary value problems are considered, one constructed from the original system and another one based on a so-called *strangeness-free* formulation of the differential-algebraic system (which corresponds to impulse controllability in our context). Then the solvability conditions and solutions of both boundary value problems are studied and they are related to each other. Moreover, in the recent works [24, 31], structured global condensed forms for the optimality system are derived which allow to analyze its properties.

Control problems subject to nonlinear differential-algebraic equations are mainly treated in the works by Kunkel and Mehrmann, see for instance [20, 21]. The optimal control problem is approached in [21] by using local linearizations of the nonlinear equation which usually result in time-varying linear differential-algebraic equations and allow the application of the previously mentioned techniques.

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