ON A CLASS OF NON-HERMITIAN MATRICES WITH POSITIVE DEFINITE SCHUR COMPLEMENTS

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Abstract. Given a positive definite matrix $A \in \mathbb{C}^{n\times n}$ and a Hermitian matrix $D \in \mathbb{C}^{m\times m}$, we characterize under which conditions there exists a strictly contractive matrix $K \in \mathbb{C}^{n\times m}$ such that the non-Hermitian block-matrix

$$\begin{bmatrix} A & -AK \\ K^*A & D \end{bmatrix}$$

has a positive definite Schur complement with respect to its submatrix A. Additionally, we show that K can be chosen such that diagonalizability of the block-matrix is guaranteed and we compute its spectrum. Moreover, we show a connection to the recently developed frame theory for Krein spaces.

1. Introduction

Given a matrix $S \in \mathbb{C}^{(n+m)\times(n+m)}$ assume it is partitioned as

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{m\times n}$ and $D \in \mathbb{C}^{m\times m}$. If A is invertible, then the *Schur complement of A in S* is defined by

$$S_{/A} := D - CA^{-1}B.$$

This terminology is due to Haynsworth [11, 12], but the use of such a construction goes back to Sylvester [15] and Schur [14]. The Schur complement arises, for instance, in the following factorization of the block matrix S:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ CA^{-1} & I_m \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix}\begin{bmatrix} I_n & A^{-1}B \\ 0 & I_m \end{bmatrix}, \tag{1.1}$$

which is due to Aitken [1]; note that I_k denotes the identity matrix of size $k \times k$. It is a common argument in the proof of some well-know results in matrix analysis such as the *Schur determinant formula* [3]:

$$\det(S) = \det(A) \cdot \det(S_{/A}), \tag{1.2}$$

the *Guttman rank additivity formula* [10], and the *Haynsworth inertia additivity formula* [13].

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The Schur complement has been generalized in numerous ways, for example to the case in which A is non-invertible, where generalized inverses can be used to define it. It is a key tool not only in matrix analysis but also in applied fields such as numerical analysis and statistics. For further details see [16].

If S is a Hermitian matrix, then $C = B^*$ and the Schur complement of A in S is $S_{/A} = D - B^*A^{-1}B$. In this particular case (1.1) reads

$$\begin{bmatrix} A & B \\ B^* & D \end{bmatrix} = \begin{bmatrix} I_n & A^{-1}B \\ 0 & I_m \end{bmatrix}^* \begin{bmatrix} A & 0 \\ 0 & D - B^*A^{-1}B \end{bmatrix} \begin{bmatrix} I_n & A^{-1}B \\ 0 & I_m \end{bmatrix},$$

which implies the following well-known criteria to determine the positive definiteness of S: the block-matrix S is positive definite if and only if A and $S_{/A}$ are both positive definite. This equivalence is not true for positive semidefinite matrices, but Albert [2] showed that S is positive semidefinite if and only if A and $S_{/A}$ are both positive semidefinite and $R(B) \subseteq R(A)$, where $R(X)$ stands for the range of a matrix X. Observe that the range inclusion $R(B) \subseteq R(A)$ is equivalent to the existence of a matrix $X \in \mathbb{C}^{n\times m}$ which factorizes B as $B = AX$.

In the present paper, given a positive definite $A \in \mathbb{C}^{n\times n}$ with eigenvalues $0 < \lambda_n \leq \cdots \leq \lambda_1$ and a Hermitian $D \in \mathbb{C}^{m\times m}$ with eigenvalues $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_r \leq 0 < \mu_{r+1} \leq \ldots \leq \mu_m$, we investigate under which conditions there exists a strictly contractive matrix $K \in \mathbb{C}^{n\times m}$ such that

$$S = \begin{bmatrix} A & -AK \\ K^*A & D \end{bmatrix} \tag{1.3}$$

has a positive definite Schur complement $S_{/A}$ with respect to the minor A, that is, under which conditions there exists a strictly contractive matrix $K \in \mathbb{C}^{n\times m}$ such that

$$S_{/A} = D + K^*AK$$

is positive definite.

Interest in such non-Hermitian block-matrices arises, for instance, in the recently developed frame theory in Krein spaces, see [6, 8]. There, block-matrices as in (1.3) with a positive definite A, a Hermitian D and a positive definite $S_{/A}$ correspond to so-called J-frame operators, see Section 5 for more details.

In Theorem 3.3 below we show that this special *structured matrix completion problem* has a solution if and only if

$$r \leq n \quad \text{and} \quad \lambda_i + \mu_i > 0 \quad \text{for all } i = 1, \ldots, r.$$

We stress that S is not diagonalizable in general, not even if $S_{/A}$ is positive definite. Under the above conditions, we construct a particular strictly contractive matrix K, which depends on some parameters $\varepsilon_1, \ldots, \varepsilon_r$. In Theorem 4.2 we compute the eigenvalues of the corresponding block matrix S in terms of the eigenvalues of A and D and the parameters $\varepsilon_1, \ldots, \varepsilon_r$. A root locus analysis of the latter reveals that if each ε_i is small enough, then S is diagonalizable and has only (positive) real eigenvalues, although S is non-Hermitian.

2. Preliminaries

Given Hermitian matrices $A, B \in \mathbb{C}^{n\times n}$, several relations between the eigenvalues of A, B and $A+B$ can be obtained. The following result was first proved by Weyl, see e.g. [4].

Theorem 2.1. *Let $A, B \in \mathbb{C}^{n\times n}$ be Hermitian matrices. Then,*

$$\lambda_j^{\downarrow}(A+B) \le \lambda_i^{\downarrow}(A) + \lambda_{j-i+1}^{\downarrow}(B) \quad \text{for } i \le j;$$
$$\lambda_j^{\downarrow}(A+B) \ge \lambda_i^{\downarrow}(A) + \lambda_{j-i+n}^{\downarrow}(B) \quad \text{for } i \ge j;$$

where $\lambda_j^{\downarrow}(C)$ denotes the j-th eigenvalue of C (counted with multiplicities) if they are arranged in nonincreasing order.

Among the numerous consequences of Weyl's inequalities, it is worthwhile to mention that if $A, B \in \mathbb{C}^{n\times n}$ are Hermitian matrices such that $A \le B$ according to Löwner's order, then

$$\lambda_j^{\downarrow}(A) \le \lambda_j^{\downarrow}(B) \quad \text{for } j = 1, \dots, n. \tag{2.1}$$

Another well-known result says that if $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$, then the non-zero eigenvalues of AB and BA are the same (and they have the same multiplicities). Indeed, it is easy to see that

$$\begin{bmatrix} I_m & -A \\ 0 & I_n \end{bmatrix} \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix},$$

and hence the matrices $\left[\begin{smallmatrix} AB & 0 \\ B & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & 0 \\ B & BA \end{smallmatrix}\right]$ are similar. Therefore, they have the same characteristic polynomial

$$p(\lambda) = \lambda^n \det(\lambda I_m - AB) = \lambda^m \det(\lambda I_n - BA), \tag{2.2}$$

and the assertion follows immediately.

We use the above result to prove the following proposition. For $K \in \mathbb{C}^{n\times m}$ we denote by $\|K\|$ the spectral norm of K, i.e., the operator norm induced by the Euclidean vector norm.

Proposition 2.2. *Let $A \in \mathbb{C}^{n\times n}$ be positive definite and $K \in \mathbb{C}^{n\times m}$. Then,*

$$\lambda_j^{\downarrow}(K^*AK) \le \|K\|^2 \lambda_j^{\downarrow}(A) \quad \text{for } j = 1, \dots, \min\{n, m\}.$$

Proof. Since A is positive definite it has a well-defined square root $A^{1/2}$. Then, for all $j = 1, \dots, \min\{n, m\}$,

$$\lambda_j^{\downarrow}(K^*AK) = \lambda_j^{\downarrow}(K^*A^{1/2}A^{1/2}K) \overset{(2.2)}{=} \lambda_j^{\downarrow}(A^{1/2}KK^*A^{1/2}) \le \|K\|^2 \lambda_j^{\downarrow}(A),$$

where the inequality follows from (2.1) because $A^{1/2}KK^*A^{1/2} \le \|K\|^2 A$. □

3. Positive definiteness of the Schur complement

In this section we derive a necessary and sufficient condition for the existence of a strictly contractive matrix K such that the block matrix S in (1.3) has a positive definite Schur complement. Throughout this section we consider the following hypotheses.

Assumption 3.1. Assume that $A \in \mathbb{C}^{n\times n}$ is positive definite and $D \in \mathbb{C}^{m\times m}$ is a Hermitian matrix. Let $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_r \leq 0 < \mu_{r+1} \leq \ldots \leq \mu_m$ denote the eigenvalues of D (counted with multiplicities) arranged in nondecreasing order, and let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$ denote the eigenvalues of A (counted with multiplicities) arranged in nonincreasing order.

First, we record the following important observation.

Lemma 3.2. *Let Assumption 3.1 hold and assume that $r > n$. Then, there is no $K \in \mathbb{C}^{n\times m}$ such that $D + K^*AK$ is positive definite.*

Proof. Let $K \in \mathbb{C}^{n\times m}$ and $\mathcal{S}_1 := \ker(K)$ be the nullspace of K. Consider the subspace $\mathcal{S}_2$ of $\mathbb{C}^m$ spanned by all eigenvectors of D corresponding to non-positive eigenvalues. By Assumption 3.1 we have that $\dim \mathcal{S}_2 = r$ and

$$\dim \mathcal{S}_1 + \dim \mathcal{S}_2 \geq (m-n) + r = m + (r-n) > m.$$

Thus, $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \{0\}$ and for any non-trivial vector $v \in \mathcal{S}_1 \cap \mathcal{S}_2$ we have

$$\langle (D + K^*AK)v, v\rangle = \langle Dv, v\rangle \leq 0.$$

Therefore, $D + K^*AK$ cannot be positive definite. □

In the following result we focus on a special class of matrices K. Recall that $K \in \mathbb{C}^{n\times m}$ is called *strictly contractive*, if its singular values are all smaller than 1. Equivalently, K is strictly contractive if and only if $\|K\| < 1$.

Theorem 3.3. *Let Assumption 3.1 hold. Then, there exists a strictly contractive matrix $K \in \mathbb{C}^{n\times m}$ such that $D + K^*AK$ is positive definite if and only if*

$$r \leq n \quad \textit{and} \quad \lambda_i + \mu_i > 0 \quad \textit{for all } i = 1, \ldots, r. \tag{3.1}$$

Proof. Assume that there exists a strictly contractive matrix $K \in \mathbb{C}^{n\times m}$ such that $D + K^*AK > 0$. By Lemma 3.2, it is necessary that $r \leq n$. On the other hand, by Theorem 2.1,

$$0 < \lambda_m^{\downarrow}(D + K^*AK) \leq \lambda_i^{\downarrow}(D) + \lambda_{m-i+1}^{\downarrow}(K^*AK),$$

for $i = 1, \ldots, m$. In particular, for $i = m-r+1, \ldots, m$ we can combine the above inequalities with Proposition 2.2 and obtain

$$0 < \lambda_i^{\downarrow}(D) + \|K\|^2 \lambda_{m-i+1}^{\downarrow}(A) < \mu_{m-i+1} + \lambda_{m-i+1}.$$

Equivalently, we have that $\mu_j + \lambda_j > 0$ for $j = 1, \ldots, r$.

Conversely, assume that $r \le n$ and $\lambda_i + \mu_i > 0$ for $i = 1, \dots, r$. Then, for each $i = 1, \dots, r$, let $0 < \varepsilon_i < 1$ be such that $\varepsilon_i \lambda_i + \mu_i > 0$ and define $E \in \mathbb{C}^{n\times m}$ by

$$E = \begin{bmatrix} \operatorname{diag}(\sqrt{\varepsilon_1}, \dots, \sqrt{\varepsilon_r}) & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix},$$

where $0_{p,q}$ stands for the null matrix in $\mathbb{C}^{p\times q}$. Further, let $U \in \mathbb{C}^{n\times n}$ and $V \in \mathbb{C}^{m\times m}$ be unitary matrices such that $A = UD_\lambda U^*$ and $D = VD_\mu V^*$, where

$$D_\lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \quad \text{and} \quad D_\mu = \operatorname{diag}(\mu_1, \dots, \mu_m).$$

Then, for

$$K := UEV^*, \tag{3.2}$$

it is straightforward to observe that $\|K\| < 1$ and

$$\begin{aligned} D + K^*AK &= V(D_\mu + E^*U^*AUE)V^* = V(D_\mu + E^*D_\lambda E)V^* \\ &= V \begin{bmatrix} \operatorname{diag}(\varepsilon_1\lambda_1 + \mu_1, \dots, \varepsilon_r\lambda_r + \mu_r) & 0_{r,m-r} \\ 0_{m-r,r} & \operatorname{diag}(\mu_{r+1}, \dots, \mu_m) \end{bmatrix} V^* \end{aligned}$$

is a positive definite matrix. □

Remark 3.4. Let Assumption 3.1 hold. Observe that if $\mu_i = 0$ for some $i = 1, \dots, r$, then the condition $\lambda_i + \mu_i > 0$ is automatically fulfilled. Hence, if we assume that dim ker $D = p$, then D has only $r-p$ negative eigenvalues and, in this case, there exists a strictly contractive matrix $K \in \mathbb{C}^{n\times m}$ such that $D + K^*AK$ is positive definite if and only if

$$r \le n \quad \text{and} \quad \lambda_i + \mu_i > 0 \quad \text{for all } i = 1, \dots, r-p.$$

4. Spectrum of the block matrix

Throughout this section, we consider the contraction K constructed in the proof of Theorem 3.3 and investigate the location of the eigenvalues of the block-matrix S in (1.3) for this particular K. The locations depend on the parameters $\varepsilon_1, \dots, \varepsilon_r$ and hence their study resembles a root locus analysis. Before we state the corresponding result we start with a preliminary lemma.

Lemma 4.1. *Let Assumption 3.1 and* (3.1) *hold and set*

$$\alpha_i := \frac{(\lambda_i - \mu_i)^2}{4\lambda_i^2}, \quad i = 1, \dots, r. \tag{4.1}$$

Then we have that

$$0 \le \frac{-\mu_i}{\lambda_i} < \alpha_i < 1, \quad \textit{for all } i = 1, \dots, r.$$

Proof. Given $i = 1, \ldots, r$, by (3.1) we find that $(\lambda_i + \mu_i)^2 > 0$, which implies $(\lambda_i - \mu_i)^2 > -4\mu_i\lambda_i$ and hence

$$\alpha_i > -\frac{\mu_i}{\lambda_i} \geq 0.$$

Furthermore,

$$\lambda_i - \mu_i = -(\lambda_i + \mu_i) + 2\lambda_i < 2\lambda_i,$$

which implies that $\alpha_i < 1$. □

We are now in the position to state the main result of this section.

Theorem 4.2. *Let Assumption 3.1 and* (3.1) *hold. For* $i = 1, \ldots, r$ *choose* $0 < \varepsilon_i < 1$ *such that* $\varepsilon_i\lambda_i + \mu_i > 0$.

If $K \in \mathbb{C}^{n\times m}$ *is the strictly contractive matrix defined in* (3.2) *then the spectrum of the block matrix* $S \in \mathbb{C}^{(n+m)\times(n+m)}$ *given in* (1.3) *consists of the real numbers* $\lambda_{r+1}, \ldots, \lambda_n$, $\mu_{r+1}, \ldots, \mu_m$ *and*

$$\eta_i^{\pm} = \frac{\lambda_i + \mu_i}{2} \pm \lambda_i\sqrt{\alpha_i - \varepsilon_i}, \quad i = 1, \ldots, r, \tag{4.2}$$

where α_i *is given by* (4.1). *Moreover, the following conditions hold:*

a) *if* $0 \leq \frac{-\mu_i}{\lambda_i} < \varepsilon_i < \alpha_i$, *then* $\eta_i^+ > \eta_i^- > 0$;
b) *if* $\alpha_i < \varepsilon_i < 1$, *then* $\eta_i^+ = \overline{\eta_i^-} \in \mathbb{C}\setminus\mathbb{R}$;
c) *if* $\varepsilon_i = \alpha_i$, *then* $\eta_i^+ = \eta_i^- = \frac{1}{2}(\lambda_i + \mu_i)$ *and there exists a Jordan chain of length* 2 *corresponding to this eigenvalue.*

Additionally, if $\varepsilon_i \neq \alpha_i$ *for all* $i = 1, \ldots, r$, *then* S *is diagonalizable.*

Proof. First note that by Lemma 4.1 the ranges for ε_i in the cases a) and b) are non-empty. Using the notation from the proof of Theorem 3.3 we obtain

$$\begin{aligned} S = \begin{bmatrix} A & -AK \\ K^*A & D \end{bmatrix} &= \begin{bmatrix} UD_\lambda U^* & -UD_\lambda EV^* \\ VE^*D_\lambda U^* & VD_\mu V^* \end{bmatrix} = \\ &= \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} D_\lambda & -B \\ B^* & D_\mu \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}^* = W \begin{bmatrix} D_\lambda & -B \\ B^* & D_\mu \end{bmatrix} W^*, \end{aligned}$$

where $B \in \mathbb{C}^{n\times m}$ is given by

$$B := D_\lambda E = \begin{bmatrix} \operatorname{diag}(\lambda_1\sqrt{\varepsilon_1}, \ldots, \lambda_r\sqrt{\varepsilon_r}) & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix},$$

and $W := \left[\begin{smallmatrix} U & 0 \\ 0 & V \end{smallmatrix}\right] \in \mathbb{C}^{(n+m)\times(n+m)}$ is unitary. Then, if $\{e_1, \ldots, e_{n+m}\}$ denotes the standard basis of $\mathbb{C}^{n+m}$, it is easy to see that

$$\begin{aligned} SWe_i &= \lambda_i We_i \quad && \text{for } i = r+1, \ldots, n, \\ \text{and} \quad SWe_j &= \mu_{j-n} We_j \quad && \text{for } j = n+r+1, \ldots, n+m, \end{aligned} \tag{4.3}$$

which yields that $\lambda_{r+1}, \ldots, \lambda_n$ and $\mu_{r+1}, \ldots, \mu_m$ are eigenvalues of S.

Now, define the following $r \times r$ diagonal matrices:

$$F_\lambda := \operatorname{diag}(\lambda_1, \ldots, \lambda_r), \qquad F_\mu := \operatorname{diag}(\mu_1, \ldots, \mu_r),$$
$$G := \operatorname{diag}(\lambda_1\sqrt{\varepsilon_1}, \ldots, \lambda_r\sqrt{\varepsilon_r}),$$

and observe that the remaining $2r$ eigenvalues of S coincide with the spectrum of the submatrix $\tilde{S}$ of W^*SW given by

$$\tilde{S} := \begin{bmatrix} F_\lambda & -G \\ G & F_\mu \end{bmatrix}.$$

In order to calculate the eigenvalues of $\tilde{S}$, we make use of the Schur determinant formula (1.2), by which the characteristic polynomial of $\tilde{S}$ is given by

$$q(\eta) = \det(\tilde{S} - \eta I_{2r}) = \det(F_\mu - \eta I_r)\det\left((\tilde{S} - \eta I_{2r})_{/(F_\mu - \eta I_r)}\right).$$

Since the matrix $(\tilde{S} - \eta I_{2r})_{/(F_\mu-\eta I_r)} = (F_\lambda - \eta I_r) + G(F_\mu - \eta I_r)^{-1}G$ is diagonal and has the form

$$\begin{bmatrix} \lambda_1 - \eta + \varepsilon_1\frac{\lambda_1^2}{\mu_1-\eta} & 0 & \ldots & 0 \\ 0 & \lambda_2 - \eta + \varepsilon_2\frac{\lambda_2^2}{\mu_2-\eta} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_r - \eta + \varepsilon_r\frac{\lambda_r^2}{\mu_r-\eta} \end{bmatrix},$$

we have that

$$\begin{aligned} q(\eta) &= \prod_{i=1}^r(\mu_i - \eta)\prod_{i=1}^r\left(\lambda_i - \eta + \frac{\varepsilon_i\lambda_i^2}{\mu_i - \eta}\right) \\ &= \prod_{i=1}^r\left((\mu_i - \eta)(\lambda_i - \eta) + \varepsilon_i\lambda_i^2\right). \end{aligned}$$

Thus, $\eta \in \mathbb{C}$ is a root of $q(\eta)$ if and only if

$$\eta^2 - (\lambda_i + \mu_i)\eta + \lambda_i(\mu_i + \varepsilon_i\lambda_i) = 0$$

for some $i \in \{1, \ldots, r\}$. This leads to the following eigenvalues of $\tilde{S}$:

$$\eta_i^\pm = \frac{\lambda_i + \mu_i}{2} \pm \frac{1}{2}\sqrt{(\lambda_i - \mu_i)^2 - 4\varepsilon_i\lambda_i^2} \tag{4.4}$$

for $i = 1, \ldots, r$. Hence, (4.2) follows and statement b) holds. For statement a) we additionally observe that if $\varepsilon_i > \frac{-\mu_i}{\lambda_i}$ then

$$\eta_i^- > \tfrac{1}{2}(\lambda_i + \mu_i) - \tfrac{1}{2}\sqrt{(\lambda_i - \mu_i)^2 + 4\lambda_i\mu_i} = 0.$$

To prove c), assume that $\varepsilon_i = \alpha_i$ for some $i \in \{1, \dots, r\}$. Since $\eta_i^+ = \eta_i^- = \frac{1}{2}(\lambda_i + \mu_i)$ and $\sqrt{\varepsilon_i} = \frac{\lambda_i - \mu_i}{2\lambda_i}$, it is straightforward to compute that

$$\left(\tilde{S} - \tfrac{1}{2}(\lambda_i + \mu_i) I_{2r}\right) \begin{pmatrix} \left(1 + \frac{2}{\lambda_i - \mu_i}\right) f_i \\ f_i \end{pmatrix} = \begin{pmatrix} f_i \\ f_i \end{pmatrix},$$

$$\left(\tilde{S} - \tfrac{1}{2}(\lambda_i + \mu_i) I_{2r}\right) \begin{pmatrix} f_i \\ f_i \end{pmatrix} = 0,$$

using the standard basis $\{f_1, \dots, f_r\}$ of $\mathbb{C}^r$. The vectors above form a Jordan chain of length 2 of $\tilde{S}$ corresponding to the eigenvalue $\frac{1}{2}(\lambda_i + \mu_i)$. Hence, a Jordan chain of S can be constructed corresponding to the eigenvalue $\frac{1}{2}(\lambda_i + \mu_i)$ can also be constructed.

Finally, assume that $\varepsilon_i \neq \alpha_i$ for all $i = 1, \dots, r$. In this case, the space $\mathbb{C}^{n+m}$ has a basis consisting of eigenvectors of S. Indeed, this follows from (4.3) together with

$$\left(\tilde{S} - \eta_i^+ I_{2r}\right) \begin{pmatrix} f_i \\ -\frac{\lambda_i \sqrt{\varepsilon_i}}{\mu_i - \eta_i^+} f_i \end{pmatrix} = 0, \quad \left(\tilde{S} - \eta_i^- I_{2r}\right) \begin{pmatrix} f_i \\ -\frac{\lambda_i \sqrt{\varepsilon_i}}{\mu_i - \eta_i^-} f_i \end{pmatrix} = 0$$

for $i = 1, \dots, r$. □

We emphasize that if for all $i = 1, \dots, r$ the parameter ε_i in Theorem 4.2 is chosen such that *a*) holds, then the block matrix S in (1.3) is diagonalizable and has only positive eigenvalues. This is possible because of Lemma 4.1.

Example 4.3. We illustrate Theorem 4.2 with a simple example. Let $n = m = 1$, $D = [0]$ and $A = [a]$ with $a > 0$. Then $r = 1$ and choosing K as in (3.2) with $0 < \varepsilon < 1$ gives $K = [\sqrt{\varepsilon}]$. In this case $\alpha = \frac{1}{4}$.

By Theorem 4.2, for $\varepsilon = \frac{1}{4}$ there is a Jordan chain of length 2 corresponding to the only eigenvalue $\frac{a}{2}$, and indeed we find that

$$\begin{pmatrix} \frac{1}{a} \\ \frac{-1}{a} \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

form a Jordan chain of S, hence S is not diagonalizable.

On the other hand, for $\varepsilon \neq \frac{1}{4}$ the block matrix S has eigenvalues $\eta^+ = \frac{a}{2} + a\sqrt{\frac{1}{4} - \varepsilon}$ and $\eta^- = \frac{a}{2} - a\sqrt{\frac{1}{4} - \varepsilon}$. They are positive if $\varepsilon < \frac{1}{4}$, and they are non-real if $\frac{1}{4} < \varepsilon < 1$. In these last two cases S is diagonalizable.

5. Application to J-frame operators

In this section, we exploit Theorems 3.3 and 4.2 to investigate whether a block matrix S as in (1.3) represents a so-called J-frame operator and when it is similar to a Hermitian matrix. In the following we briefly recall the concept of J-frame operators, which arose in [6, 8] in the context of frame theory in Krein spaces.

In a finite-dimensional setting, every indefinite inner product space is a (finite-dimensional) Krein space, see [9]. A map $[\cdot,\cdot] : \mathbb{C}^k \times \mathbb{C}^k \to \mathbb{C}$ is called an indefinite inner product in $\mathbb{C}^k$, if it is a non-degenerate Hermitian sesquilinear form. The indefinite inner product allows a classification of vectors: $x \in \mathbb{C}^k$ is called positive if $[x,x] > 0$, negative if $[x,x] < 0$ and neutral if $[x,x] = 0$. Also, a subspace $\mathcal{L}$ of $\mathbb{C}^k$ is positive if every $x \in \mathcal{L} \setminus \{0\}$ is a positive vector. Negative and neutral subspaces are defined analogously. A positive (negative) subspace of maximal dimension will be called maximal positive (maximal negative, respectively).

It is well-known that there exists a Gramian (or Gram matrix) $G \in \mathbb{C}^{k\times k}$, which is invertible and represents $[\cdot,\cdot]$ in terms of the usual inner product in $\mathbb{C}^k$, i.e., $[x,y] = \langle Gx, y\rangle$ for all $x, y \in \mathbb{C}^k$. The positive (resp. negative) index of inertia of $[\cdot,\cdot]$ is the number of positive (resp. negative) eigenvalues of the Gramian G, and it equals the dimension of any maximal positive (resp. negative) subspace of $\mathbb{C}^k$. It is clear that the sum of the inertia indices equals the dimension of the space.

A finite family of vectors $\mathcal{F} = \{f_i\}_{i=1}^q$ in $\mathbb{C}^k$ is a *frame for* $\mathbb{C}^k$, if

$$\operatorname{span}(\{f_i\}_{i=1}^q) = \mathbb{C}^k,$$

see e.g. [5] and the references therein. Roughly speaking, a J-frame is a frame, which is compatible with the indefinite inner product $[\cdot,\cdot]$.

Definition 5.1. Let $(\mathbb{C}^k, [\cdot,\cdot])$ be an indefinite inner product space. Then, a frame $\mathcal{F} = \{f_i\}_{i=1}^q$ in $\mathbb{C}^k$ is called a *J-frame for* $\mathbb{C}^k$, if

$$\begin{aligned} \mathcal{M}_+ &:= \operatorname{span}\{\ f \in \mathcal{F} \ \mid\ [f,f] \geq 0\ \} \\ \text{and} \quad \mathcal{M}_- &:= \operatorname{span}\{\ f \in \mathcal{F} \ \mid\ [f,f] < 0\ \} \end{aligned}$$

are a maximal positive and a maximal negative subspace of $\mathbb{C}^k$, respectively.

If $[\cdot,\cdot]$ is an indefinite inner product with positive and negative index of inertia n and m, respectively, then the maximality of $\mathcal{M}_+$ and $\mathcal{M}_-$ is equivalent to

$$\dim \mathcal{M}_+ = n \quad \text{and} \quad \dim \mathcal{M}_- = m.$$

Note that if $\mathcal{F}$ is a J-frame for $\mathbb{C}^k$, then there are no (non-trivial) $f \in \mathcal{F}$ with $[f,f] = 0$.

Given a J-frame $\mathcal{F} = \{f_i\}_{i=1}^q$ for $\mathbb{C}^k$, its associated J-frame operator $S : \mathbb{C}^k \to \mathbb{C}^k$ is defined by

$$Sf = \sum_{i=1}^q \sigma_i\,[f, f_i]\, f_i,$$

where $\sigma_i = \operatorname{sgn}[f_i, f_i]$ is the signature of the vector f_i. S is an invertible symmetric operator with respect to $[\cdot,\cdot]$, i.e.,

$$[Sf, g] = [f, Sg] \quad \text{for all} \quad f, g \in \mathbb{C}^k.$$

Its relevance follows from the indefinite sampling-reconstruction formula: given an arbitrary $f \in \mathbb{C}^k$,

$$f = \sum_{i=1}^{q} \sigma_i \left[f, S^{-1} f_i\right] f_i = \sum_{i=1}^{q} \sigma_i \left[f, f_i\right] S^{-1} f_i.$$

In the following, we aim to apply the results from Sections 3 and 4, hence we restrict ourselves to the following inner product on $\mathbb{C}^k = \mathbb{C}^{n+m}$,

$$[(x_1, \ldots, x_{n+m}), (y_1, \ldots, y_{n+m})] = \sum_{i=1}^{n} x_i \overline{y_i} - \sum_{j=1}^{m} x_{n+j} \overline{y_{n+j}}.$$

In [6, Theorem 3.1] a criterion was provided to determine if an (invertible) symmetric operator is a J-frame operator. In our setting it says that an invertible operator S in $(\mathbb{C}^k, [\cdot,\cdot])$, which is symmetric with respect to $[\cdot,\cdot]$, is a J-frame operator if and only if there exists a basis of $\mathbb{C}^k$ such that S can be represented as a block-matrix

$$S = \begin{bmatrix} A & -AK \\ K^*A & D \end{bmatrix}, \tag{5.1}$$

where $A \in \mathbb{C}^{n\times n}$ is positive definite, $K \in \mathbb{C}^{n\times m}$ is strictly contractive, and $D \in \mathbb{C}^{m\times m}$ is a Hermitian matrix such that $D + K^*AK$ is also positive definite. Any block-matrix $S \in \mathbb{C}^{(n+m)\times(n+m)}$ of the form (5.1), which satisfies these conditions will be called *J-frame matrix*.

Therefore, Theorem 3.3 can be restated in the following way.

Theorem 5.2. *Let $A \in \mathbb{C}^{n\times n}$ and $D \in \mathbb{C}^{m\times m}$ be matrices satisfying Assumption 3.1. Then there exists $K \in \mathbb{C}^{n\times m}$ with $\|K\| < 1$ such that S as in* (5.1) *is a J-frame matrix if and only if*

$$r \le n \quad \text{and} \quad \lambda_i + \mu_i > 0 \quad \text{for } i = 1, \ldots, r.$$

We mention that the study of the spectral properties of a J-frame operator is quite recent, see [6, 7]. In the case of J-frame matrices, for given A and D, we always find conditions such that a strictly contractive K exists which turns S into a matrix similar to a Hermitian one. The following result is a direct consequence of Theorem 4.2 and Lemma 4.1.

Theorem 5.3. *Let Assumption 3.1 and* (3.1) *hold. Then, there exists a strictly contractive matrix K such that the matrix S given in* (5.1) *is a J-frame matrix which is similar to a Hermitian matrix. In this case, all eigenvalues of S are positive and there exists a basis of $\mathbb{C}^{n+m}$ consisting of eigenvectors of S.*

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