Abstract: We consider the differential-algebraic systems obtained by modified nodal analysis of linear RLC circuits from a systems theoretic viewpoint. We derive expressions for the set of consistent initial values and show that the properties of controllability at infinity and impulse controllability do not depend on parameter values but rather on the interconnection structure of the circuit. We further present circuit topological criteria for behavioral stabilizability.

Keywords: electrical circuits, stabilizability, system space, consistent initial values, controllability at infinity, impulse controllability

1. INTRODUCTION

Modified nodal analysis (MNA) is a widely used technique for modelling RLC circuits. It has been first introduced in Ho et al. (1975). It is based on regarding a circuit as a graph, and results in a differential-algebraic model. This model provides a structure which allows a mathematically elegant analysis of essential properties and their physical interpretation. Among these properties is the index, i.e., the order of smoothness of perturbations entering the solution of the differential-algebraic equation, see Lamour et al. (2013); Kunkel and Mehrmann (2006); it is shown in Estévez Schwarz and Tischendorf (2000); Günther and Feldmann (1999a,b), see also Büchle (2007); März et al. (2003); Estévez Schwarz and Lamour (2001); Freund (2005); Reis (2014); Riaza (2006); Takamatsu and Iwata (2010), that the index is not dependent on system parameters (such as values of resistances, capacitances and inductances), but rather on the interconnection structure, i.e., the topology, of the circuit. Further important possible properties of the circuit system are stability and asymptotic stability. Whereas MNA models of RLC circuits are always stable as long as the parameter values of resistances, capacitances and inductances are positive, asymptotic stability requires some further conditions. It is shown in Riaza and Tischendorf (2010, 2007); Riaza (2006) that asymptotical stability is guaranteed, if certain parameter-independent criteria on the circuit interconnection structure are fulfilled. The general idea of these articles is used in Berger and Reis (2014), where topological criteria for asymptotic stability and autonomy of the zero dynamics are presented for the purpose of adaptive tracking control of circuits.

In this article, we analyse further systems theoretic properties of the MNA equations. Besides presenting sufficient topological criteria for behavioral stabilizability, we derive expressions for the system space and the space of consistent initial values, and conclude topological conditions for controllability at infinity and impulse controllability.

In Sec. 2 we present the required tools from graph theory and Sec. 3 collects the basics on RLC circuit models. Sec. 4 and Sec. 5 contain the results on stability and stabilizability of the circuit model and their topological interpretation. Sec. 6 is devoted to the system space of the MNA equations, whereas we specify the space of consistent initial values and give topological conditions for controllability at infinity and impulse controllability in Sec. 7 and Sec. 8.

1.1 Nomenclature

\( \mathbb{N}_0 \) is the set of nonnegative integers, \( \mathbb{R}(s) \) is the field of real rational functions, and \( \mathbb{C}_+, \mathbb{C}_+^* \) are, resp., the open and closed complex right half planes. For a field \( K \), \( K^{n \times m} \) is the set of \( n \times m \) matrices with entries in \( K \). We use \( \text{rk}_K M \), \( \ker_K M \), \( \text{im}_K M \) for the rank, kernel and image of a matrix \( M \) over \( K \). If \( K = \mathbb{R} \), we omit the subindex indicating the underlying field. Further, \( M^\top \) and \( M^* \) resp. stand for the transpose and conjugate transpose of a matrix \( M \), and by writing \( M > 0 \) \( (M \geq 0) \), we mean that the square matrix \( M \) is symmetric positive (semi-)definite. The identity matrix of size \( n \times n \) is denoted by \( I_n \) and the zero matrix of size \( m \times n \) by \( 0_{m,n} \). We omit the subindices, if they are clear from context. \( \mathbb{V}^\perp \) denotes the orthogonal space of a subspace \( \mathbb{V} \subset \mathbb{R}^n \), and we call the matrix \( Z \) a basis matrix of \( \mathbb{V} \), if \( \ker Z = \{0\} \) and \( \text{im} Z = \mathbb{V} \).

2. GRAPH THEORETIC PRELIMINARIES

For the purpose of this article, we consider finite and loop-free directed graphs, see Diestel (2017). We present some basics of graphs and incidence matrices along with some results about the correspondence between the topological structure of a graph and properties of its incidence matrix.
Definition 1. (Graph theoretic concepts). A directed graph is a quadruple \( G = (V, E, \text{init}, \text{ter}) \) consisting of a vertex set \( V \), a edge set \( E \) and two maps \( \text{init}, \text{ter} : E \to V \) assigning to each edge \( e \) an initial vertex \( \text{init}(e) \) and a terminal vertex \( \text{ter}(e) \). The edge \( e \) is said to be directed from \( \text{init}(e) \) to \( \text{ter}(e) \). \( G \) is said to be loop-free, if \( \text{init}(e) \neq \text{ter}(e) \) for all \( e \in E \). Let \( V' \subset V \) and \( E' \subset E \) with
\[
E' \subset E_{|V'}, \text{ where } e \in E : \text{init}(e) \in V' \land \text{ter}(e) \in V'.
\]
Then the triple \( (V', E', \text{init}_{|E'}, \text{ter}_{|E'}) \) is called a subgraph of \( G \). If \( E' = E_{|V'} \), then the subgraph is called the induced subgraph on \( V' \). If \( V' = V \), then the subgraph is called spanning. Additionally a proper subgraph is one where \( E' \neq E \), \( G \) is called finite, if \( V \) and \( E \) are finite.

For each \( e \in E \) define \( -e \notin E \) as an edge with \( \text{init}(-e) = \text{ter}(e) \) and \( \text{ter}(-e) = \text{init}(e) \). Define \( E' \) to be the set which contains all \( e \in E \) and all corresponding \( -e \). An \( r \)-tuple \( e = (e_1, \ldots, e_r) \in E^r \) is called a path from \( v \) to \( w \), if
\[
\text{init}(e_1), \ldots, \text{init}(e_r) \text{ are distinct,}
\]
\[
\text{ter}(e_1) = \text{init}(e_{r+1}) \text{ for } i \in \{1, \ldots, r-1\},
\]
\[
\text{init}(e_r) = v \land \text{ter}(e_r) = w.
\]

A cycle is a path from \( v \) to \( v \). Two vertices \( v, w \) are connected, if there is a path from \( v \) to \( w \). This gives an equivalence relation on the vertex set. A graph is called connected, if there is only one equivalence class.

The induced subgraph on an equivalence class of connected vertices gives a connected component of the graph.

A spanning subgraph \( K = (V, E, \text{init}, \text{ter}) \) of a directed graph \( G = (V, E, \text{init}, \text{ter}) \) is called a \( K \)-cut of \( G \), if \( G - K = (V, E\setminus E', \text{init}|_{E\setminus E'}, \text{ter}|_{E\setminus E'}) \) has two connected components.

Consider a directed graph \( G \) with spanning subgraph \( K \). We call a subgraph \( L \) of \( G \) a \( K \)-cut, if \( L \) is a cut of \( K \). Further, we call a path in \( G \) to be a \( K \)-cycle, if it is a cycle in \( K \).

If \( K_1 \) and \( K_2 \) are two spanning subgraphs of \( G \), then \( K_1 \cap K_2 \) denotes the spanning subgraph obtained by taking the union of the edges \( K_1 \) and \( K_2 \).

Essential ingredients of the circuit model are incidence matrices.

Definition 2. (Incidence matrix). Let \( G = (V, E, \text{init}, \text{ter}) \) be a finite and loop-free directed graph. Let \( E = \{e_1, \ldots, e_n\} \) and \( V = \{v_1, \ldots, v_k\} \). Then the all-vertex incidence matrix of \( G \) is \( A_0 = (a_{ij}) \in \mathbb{R}^{n_k \times n_e} \) with
\[
a_{ij} = \begin{cases} 1 & \text{if } \text{init}(e_j) = v_i, \\ -1 & \text{if } \text{ter}(e_j) = v_i, \\ 0 & \text{otherwise}. \end{cases}
\]
The rows of \( A_0 \) sum up to zero, so we can delete an arbitrary row to obtain an incidence matrix \( A_0 = (a_{ij}) \in \mathbb{R}^{(n_k-1) \times n_e} \) of \( G \).

Starting with an incidence matrix \( A \) of a finite and loop-free directed graph \( G \), along with a spanning subgraph \( K \) of \( G \), it is possible to obtain an incidence matrix of \( G \) by deleting all of columns corresponding to edges of \( G - K \). By rearranging the columns, it follows that the matrix \( A \) is of the form
\[
A = [A_K A_{G-K}].
\]

Next we collect some auxiliary results on incidence matrices corresponding to subgraphs from Estévez Schwarz and Tischendorf (2000). Note that this reference has wording which slightly differs from ours, as, for instance, cycles are called loops therein. Our notation is oriented by the standard reference Diestel (2017) for graph theory.

Proposition 1. (Estévez Schwarz and Tischendorf, 2000, Lem. 2.1 & 2.3) Let \( G \) be a finite and loop-free connected graph with incidence matrix \( A \). Furthermore let \( K \) be a spanning subgraph, and assume that the incidence matrix is partitioned as in (1). Then the following holds:

(i) \( G \) does not contain any \( K \)-cuts if, and only if, \( \ker A_{G-K} = \{0\} \).

(ii) \( G \) does not contain any \( K \)-cycles if, and only if, \( \ker A_K = \{0\} \).

Let \( G \) be a connected graph with incidence matrix \( A \). Let \( K \) be a spanning subgraph of \( G \), and \( L \) a spanning subgraph of \( K \). Then, as in (1), we can, after possibly rearranging the columns, assume that the incidence matrix of \( G \) reads
\[
A = [A_L A_{K-L} A_{G-K}], \quad A_K = [A_L A_{K-L}].
\]

Proposition 2. ([(Riaza and Tischendorf, 2007, Prop. 4.4 & 4.5]) Let \( G \) be a finite and loop-free connected graph with incidence matrix \( A \). Let \( K \) be a spanning subgraph of \( G \), and \( L \) a spanning subgraph of \( K \). Further assume that the incidence matrix \( A \) of \( G \) is partitioned as in (2). Then the following holds:

(i) \( G \) does not contain \( K \)-cycles except for \( L \)-cycles if, and only if, \( \ker A_{G-K} = \ker A_{G-L} \times \{0\} \).

(ii) \( G \) does not contain \( K \)-cuts except for \( L \)-cuts if, and only if, \( \ker A_{G-K} = \ker A_{G-L} \).

3. CIRCUIT EQUATIONS

The MNA of a linear RLC circuit is given by
\[
\frac{d}{dt} E x(t) = Ax(t) + Bu(t)
\]
with state being composed of vertex potentials, inductive currents, and currents through voltage sources, i.e., \( x = (v_I^T, i^T) \) and input consisting of voltages at voltage sources and currents at current sources, i.e., \( u = (v_E^T, i_E^T)^T \).

The matrices \( E, A, B \) in (3) are given by
\[
sE - A = \begin{bmatrix} sA_LCA_L^T + A_KGA_K^T A_L A_{\nu} \\ -A_K^T A_L \nu L 0 \\ -A_{\nu}^T 0 0 \\ 0 -I_{n_e}\end{bmatrix}, \quad B = \begin{bmatrix} -A_I \ 0 \\ 0 \ 0 \\ 0 \ 0 \end{bmatrix}
\]
where \( s \) has to be regarded as a formal variable. The expression \( sE - A \) is called a matrix pencil. Here, \( G \in \mathbb{R}^{n_k \times n_e}, \ L \in \mathbb{R}^{n_k \times n_k}, \ C \in \mathbb{R}^{n_k \times n_e} \) are the conductance, inductance and capacitance matrix, and
\[
A_K \in \mathbb{R}^{n_k \times n_k}, \quad A_L \in \mathbb{R}^{n_k \times n_k}, \quad A_{\nu} \in \mathbb{R}^{n_k \times n_e}, \quad A_I \in \mathbb{R}^{n_k \times n_k},
\]
are the element-specific incidence matrices with sizes \( n = n_e + n_k + n_v, m = n_k + n_{\nu} \). The matrices \( G, L, C \) contain the parameters of capacitances, resistances, and inductances. Further, \( A_K \) is an incidence matrix of the spanning subgraph consisting of all vertices that contain resistances. Similarly, the incidence matrices \( A_L, A_C, A_{\nu}, A_I \) then resp. correspond to the spanning subgraphs with the edges to inductances, capacitances, voltage and current source. An incidence matrix of the finite and loop-free directed graph modeling the circuit is consequently given by \( A = [A_K A_L A_C A_{\nu} A_I] \). It is also reasonable to assume
Let $C$ be the circuit graph, as any connected component corresponds to a subcircuit which does not physically interact with the remaining components, so one may simply consider the connected components separately. We consider circuits with passive devices. This leads to the assumption that the conductance matrix is dissipative, whereas the inductance and capacitance matrices are positive definite. Altogether, this means
\[\text{rk}[A_R \ A_L \ A_C \ A_V \ A_I] = n_c,\] (5a)
\[\mathbf{g} + \mathbf{g}^\top > 0, \quad \mathbf{L} = \mathbf{L}^\top > 0, \quad \mathbf{C} = \mathbf{C}^\top > 0.\] (5b)

4. REGULARITY AND STABILITY

This section will take a closer look at the properties of the pencil $sE - A$ with matrices as in (4). First we recall some results from Berger and Reis (2014).

**Proposition 3.** Let $E, A \in \mathbb{R}^{n \times n}$ as in (4) and assume that (5) holds. Then there exist invertible $W, T \in \mathbb{R}^{n \times n}$ with
\[W(sE - A)T = \text{diag}(sI - \tilde{A}, sN - I, 0_{n \times n}),\] (6)
where $n_0 \in \mathbb{N}_0$, $N$ is nilpotent with $N^2 = 0$, and $\tilde{A}$ is a square matrix with the property that all its eigenvalues have nonpositive real part. Further, all eigenvalues of $\tilde{A}$ on the imaginary axis are semi-simple (i.e., their respective geometric and algebraic multiplicities coincide). The pencil $sE - A$ further fulfills
\[
\begin{align*}
\ker_{\mathbb{R}(s)}(sE - A) &= \ker_{\mathbb{R}(s)}[A_R \ A_L \ A_C \ A_V]^\top \times \{0\} \times \ker_{\mathbb{R}(s)} A_V, \\
\ker_{\mathbb{R}(s)}(sE - A) &= \ker_{\mathbb{R}(s)}[A_R \ A_L \ A_C \ A_V]^\top \times \{0\} \times \ker_{\mathbb{R}(s)} A_V.
\end{align*}
\] (7)

**Proof.** Since (5) implies $E = E^\top > 0$ and $A + A^\top 
0$, the existence of invertible $W, T \in \mathbb{R}^{n \times n}$ with (6) follows from (Berger and Reis, 2014, Lem. 2.6), whereas (7) is a consequence of (Berger and Reis, 2014, Thm. 4.3). \qed

A direct consequence of Prop. 3 is that
\[
\begin{align*}
\forall \lambda \in \mathbb{C}_+ : \quad & \ker_{\mathbb{C}(\lambda E - A)} = \ker_{\mathbb{C}(\lambda E - A)} (\lambda E - A)^\top \times \{0\} \times \ker_{\mathbb{C}(\lambda E - A)} A_V, \\
\forall \lambda \in \mathbb{C}_+ : \quad & \ker_{\mathbb{C}(\lambda E - A)} (\lambda E - A)^\top \times \{0\} \times \ker_{\mathbb{C}(\lambda E - A)} A_V.
\end{align*}
\] (8)

We further characterize regularity, i.e., the invertibility of $sE - A \in \mathbb{R}(s)^{n \times n}$. Note that regularity translates to the property of a differential-algebraic equation having a solution for all smooth right hand sides, which is moreover unique by specification of the initial condition, see Kunkel and Mehrmann (2006). Prop. 1 and Prop. 3 allow to characterize regularity in terms of the circuit topology.

**Corollary 4.** Let $E, A \in \mathbb{R}^{n \times n}$ as in (4) and assume that (5) holds. Then the pencil $sE - A$ is regular, if and only if, the underlying circuit neither contains $\mathcal{L}$-cycles nor $\mathcal{I}$-cuts; equivalently (by Prop. 1)
\[
\ker [A_R \ A_L \ A_C \ A_V]^\top = \{0\} \cap \ker A_V = \{0\}.
\]

Next we consider generalized eigenvalues of $sE - A$. This is a complex number $\lambda$ with $\text{rk}_{\mathbb{C}} (\lambda E - A) < \text{rk}_{\mathbb{R}(s)} sE - A$. We see from Prop. 3 that all generalized eigenvalues of $sE - A$ have nonpositive real part. In the following we discuss the possible absence of purely imaginary generalized eigenvalues. The absence of generalized eigenvalues on $\mathbb{C}_+$ corresponds to stabilizability of the circuit equation $\frac{d}{dt} Ex(t) = Ax(t)$. The latter refers to the properties that for all $x_0 \in \mathbb{R}^n$ such that there exists a solution $x$ of $\frac{d}{dt} Ex(t) = Ax(t)$ with $Ex(0) = Ex_0$, there also exists a solution $x$ of $\frac{d}{dt} Ex(t) = Ax(t)$ with $Ex(0) = Ex_0$ which vanishes at infinity, see (Berger and Reis, 2013, Sec. 5).

**Proposition 5.** ([Berger and Reis, 2014, Thm. 4.6]) Let $E, A \in \mathbb{R}^{n \times n}$ as in (4) and assume that (5) holds. Then all generalized eigenvalues of $sE - A$ have negative real part, if at least one of the following two assertions holds:

(i) The circuit neither contains $\mathcal{L}$-cycles except for $\mathcal{V}$-cycles, nor $\mathcal{L}$-$\mathcal{I}$-cuts except for $\mathcal{L}$-$\mathcal{I}$-cuts; equivalently (by Prop. 2)
\[
\begin{align*}
\ker [A_R \ A_L \ A_C \ A_V] &= \{0\} \times \ker A_V, \\
\cap \ker [A_R \ A_L \ A_C \ A_V]^\top &= \ker [A_R \ A_C \ A_V]^\top.
\end{align*}
\]

(ii) The circuit neither contains $\mathcal{I}$-cuts except for $\mathcal{I}$-cuts, nor $\mathcal{L}$-$\mathcal{V}$-cycles except for $\mathcal{V}$-$\mathcal{V}$-cycles; equivalently (by Prop. 2)
\[
\begin{align*}
\ker [A_R \ A_L \ A_C \ A_V]^\top &= \ker [A_R \ A_L \ A_C \ A_V]^\top, \\
\cap \ker [A_R \ A_L \ A_C \ A_V] &= \{0\} \times \ker [A_C \ A_V].
\end{align*}
\]

Prop. 5 slightly generalizes (Riaza and Tischendorf, 2007, Thm. 5.2), where regularity (i.e., the absence of $\mathcal{V}$-cycles and $\mathcal{I}$-cuts) is presumed. Now we combine Prop. 3 with Prop. 5 to show a condition for $\ker_{\mathbb{C}} sE - A = \{0\}$ for all $\lambda \in \mathbb{C}_+$. The latter refers to asymptotic stability, i.e., all solutions of $\frac{d}{dt} Ex(t) = Ax(t)$ vanish at infinity.

**Proposition 6.** Let $E, A \in \mathbb{R}^{n \times n}$ as in (4) and assume that (5) holds. Then $\ker_{\mathbb{C}} sE - A = \{0\}$ for all $\lambda \in \mathbb{C}_+$, if at least one of the following two assertions holds:

(i) The circuit neither contains $\mathcal{L}$-cycles, nor $\mathcal{L}$-$\mathcal{I}$-cuts except for $\mathcal{L}$-$\mathcal{I}$-cuts which are no $\mathcal{I}$-cuts; equivalently (by Prop. 1 & Prop. 2)
\[
\begin{align*}
\ker [A_R \ A_L \ A_C \ A_V] &= \{0\}, \\
\cap \ker [A_R \ A_C \ A_V]^\top &= \ker [A_R \ A_C \ A_V]^\top, \\
\cap \ker [A_R \ A_L \ A_C \ A_V]^\top &= \{0\}.
\end{align*}
\]

(ii) The circuit neither contains $\mathcal{I}$-cuts, nor $\mathcal{L}$-$\mathcal{V}$-cycles except for $\mathcal{V}$-$\mathcal{V}$-cycles which are no $\mathcal{V}$-cycles; equivalently (by Prop. 1 & Prop. 2)
\[
\begin{align*}
\ker [A_R \ A_L \ A_C \ A_V]^\top &= \{0\}, \\
\cap \ker [A_L \ A_C \ A_V] &= \{0\} \times \ker [A_C \ A_V], \\
\cap \ker A_V &= \{0\}.
\end{align*}
\]

5. BEHAVIORAL STABILIZABILITY

Loosely speaking, behavioral stabilizability of a differential-algebraic control system (3) means that $x$ can always be asymptotically steered to zero by a suitable choice of the input $u$. More precisely, for any $x_0 \in \mathbb{R}^n$ for which there exists a control $u$ such that a solution $x$ of (3) with initial conditions $Ex(0) = Ex_0$ exists, there especially exists some control $u$ such that a solution $x$ of (3) with initial condition $Ex(0) = Ex_0$ exists which vanishes at infinity. It is proven in (Berger and Reis, 2013, Sec. 5) that this is equivalent to
\[
\forall \lambda \in \mathbb{C}_+ : \quad \ker_{\mathbb{C}} [\lambda E - A \ B] = \ker_{\mathbb{C}} [\lambda E - A \ B].
\] (9)
Now consider the circuit model $E, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}$ as in (4) and assume that (5) holds. Then
\[
\begin{align*}
\text{Prop.} & \quad \im_{\mathbb{R}(s)}[sE - A B] = \im_{\mathbb{R}(s)}(sE - A) + \im_{\mathbb{R}(s)}[B] \\
& \equiv \im_{\mathbb{R}(s)}[A \Lambda_x A_c A_{v'} A_t] \times \mathbb{R}^{n_c} \times \im_{\mathbb{R}(s)}[A] \\
& + \im_{\mathbb{R}(s)}[A] \times \{0\} \times \mathbb{R}^{n_c} \\
& = \im_{\mathbb{R}(s)}[A \Lambda_x A_c A_{v'} A_t] \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_v} (5a) = \mathbb{R}(s)^n.
\end{align*}
\]
Likewise, by using (8), the circuit model (4) with assumption (5) fulfills
\[
\forall \lambda \in \mathbb{C}^+ : \im_{\mathbb{C}}[\lambda E - A B] = \mathbb{C}^n. \tag{10}
\]
As a consequence, the circuit model is behaviorally stabilizable if, and only if, $rk_{\mathbb{C}}[\lambda E - A B] = n$ for all $\lambda \in \mathbb{C}$. This is used in the following result, where we present sufficient conditions for behavior stabilizability in terms of the circuit topology.

**Proposition 7.** Let $E, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}$ as in (4) and assume that (5) holds. Then (3) is behaviorally stabilizable, if at least one of the below two statements holds:

(i) The circuit neither contains $L$-cycles, nor $L_C$-cuts except for $L$-cuts; equivalently (by Prop. 1 & Prop. 2)
\[
\ker A_c = \{0\}, \\
\wedge \ker [A \Lambda_x A_v A_{v'} A_t] = \ker [A \Lambda_x A_{v'} A_t].
\]
(ii) The circuit neither contains $C$-cuts, nor $L_C$-cycles except for $C$-cycles; equivalently (by Prop. 1 & Prop. 2)
\[
\ker [A \Lambda_x A_v A_{v'} A_t] = \{0\}, \\
\wedge \ker [A \Lambda_x A_c] = \{0\} \times \ker A_c.
\]

**Proof.** By the findings prior to this proposition, it suffices to show that the aforementioned topological conditions imply that for all $\omega \in \mathbb{R}$
\[
\ker_{\mathbb{C}} \left[ \omega E^{\top} - A^{\top} \right] = \{0\}. \tag{11}
\]
Let $\omega \in \mathbb{R}$ and $x = (x_1^T, x_2^T, x_3^T)^T \in \ker_{\mathbb{C}} \left[ \omega E^{\top} - A^{\top} \right]$ be partitioned according to the blocks in $E$ and $A$, i.e.,
\[
\begin{bmatrix}
\omega A_c C A_t^\top + A g A_x A_v A_t^\top & A_c A_v^\top \\
-A_t^\top & \omega L_x & 0 & 0 \\
-A_{v'}^\top & 0 & 0 & 0 \\
0 & 0 & 0 & \omega L_v
\end{bmatrix}
\begin{bmatrix}
x_1^T \\
x_2^T \\
x_3^T
\end{bmatrix}
= 0.
\]
This gives $x_3 = 0$, $x_1 \in \ker [A \Lambda_x A_t]$ and
\[
\begin{bmatrix}
\omega A_c C A_t^\top + A g A_x A_v A_t^\top & A_c A_v^\top \\
-A_t^\top & \omega L_x & 0 & 0 \\
-A_{v'}^\top & 0 & 0 & 0 \\
0 & 0 & 0 & \omega L_v
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 0.
\]
A multiplication of the latter equation with $(x_1^T, x_2^T, x_3^T, u_1^T, u_2^T)^T$ partitioned according to the blocks in $[A B]$ as in (4) is in the system space of (3) if, and only if, it satisfies
\[
\begin{align*}
Z_c & = (A \Lambda_x A_v A_{v'} A_t) x_1 + A_c x_2 + A_v x_3 + A_t u_1 = 0, \\
A_v^\top x_1 - u_2 & = 0, \\
Z_{v'} & = \gamma(A \Lambda_x A_v A_{v'} A_t) x_1 = 0.
\end{align*}
\]
Thm. 8 means that a vector $(x_1^T, x_2^T, x_3^T, u_1^T, u_2^T)^T$ partitioned according to the blocks in $[A B]$ as in (4) in the system space of (3) if, and only if, it satisfies
\[
\begin{align*}
Z_c & = (A \Lambda_x A_v A_{v'} A_t) x_1 + A_c x_2 + A_v x_3 + A_t u_1 = 0, \\
A_v^\top x_1 - u_2 & = 0, \\
Z_{v'} & = \gamma(A \Lambda_x A_v A_{v'} A_t) x_1 = 0.
\end{align*}
\]
The remaining part is devoted to the proof of Thm. 8 along with some preparatory results. We first recall a geometric characterization of the system space.

**Lemma 9.** (Reis et al., 2015, Prop. 3.3) Let $E, A \in \mathbb{R}^{k \times n}$ and $B \in \mathbb{R}^{k \times m}$. Consider the sequence $(\mathbb{V}_i)_{i \in \mathbb{N}_0}$ of subspaces of $\mathbb{R}^{n+m}$ with $\mathbb{V}_0 = \mathbb{R}^{n+m}$ and
\[
\mathbb{V}_{i+1} = \{ (x, u) \in \mathbb{R}^{n+m} : A x + B u \in [E 0], \mathbb{V}_i \} \forall i \in \mathbb{N}_0.
\]
Then $\mathbb{V}_i \supset \mathbb{V}_{i+1}$ for all $i \in \mathbb{N}_0$. Further, there exists some $i_0 \in \mathbb{N}_0$ $\mathbb{V}_{i_0} = \mathbb{V}_{i+1}$ for some $i_0 \in \mathbb{N}_0$. Then the system space of (3) is $\mathbb{V}_{i_0}$.

**Remark 1.** Consider the matrices $A = [A B] \in \mathbb{R}^{n \times (n+m)}$, $E = [E 0] \in \mathbb{R}^{n \times (n+m)}$. Then $\mathbb{V}_{i+1}$ is the preimage of $\mathbb{E} \mathbb{V}_i$ under $A$, i.e., $\mathbb{V}_{i+1} = A^{-1}(\mathbb{E} \mathbb{V}_i)$.

To determine the system space, we advance some helpful results.

**Lemma 10.** (Basile and Marro, 1992, Property 3.1.3)) Let $M \in \mathbb{R}^{k \times l}$ and $V \subset \mathbb{R}^k$ a subspace. Then
\[
(M^\top V)^\perp = M^{-1}(V^\perp).
\]
By taking $V = \mathbb{R}^k$, Lem. 10 implies
\[
\im M^\top = (\ker M)^\perp \forall M \in \mathbb{R}^{k \times l}. \tag{13}
\]

**Lemma 11.** Let $E, A \in \mathbb{R}^{k \times n}$, $B \in \mathbb{R}^{k \times m}$ and consider the sequence $(\mathbb{V}_i)$ as in Lem. 9. Then $(\mathbb{V}_i) := (\mathbb{V}^\perp)$ fulfills $\mathbb{V}_0 = \{0\}$ and
\[
\mathbb{W}_{i+1} = [A_B] \cdot \left( \begin{bmatrix} E^\top \\ 0 \end{bmatrix} \right)^{-1} \mathbb{W}_i \forall i \in \mathbb{N}_0. \tag{14}
\]

6. **SYSTEM SPACE**

A useful space to understand differential-algebraic systems is the **system space**, which is the minimal subspace $V \subset \mathbb{R}^{n+m}$ in which all solutions $(x(t), u(t))^\top$ of (3) evolve pointwisely. This space plays a crucial role, for instance in optimal control and dissipativity analysis of differential-algebraic systems, see Reis and Voigt (2015, 2019).

The main result in this section is an expression for the system space of the MNA equations (4).

**Theorem 8.** Let $E, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}$ as in (4) and assume that (5) holds. Let $Z_c$ and $Z_{v'}$ be basis matrices of $\ker A_{v'}$ and, resp., $\ker [A_c A_{c} A_v A_{v'} A_t]$. Then the system space of (3) is given by
\[
\ker \begin{bmatrix}
Z_c A_c A_v A_{v'} A_t \\
A_v^\top \\
Z_{v'} A_c A_v A_{v'} A_{v'} A_t \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
-1
\end{bmatrix}.
\]

The remaining part is devoted to the proof of Thm. 8 along with some preparatory results. We first recall a geometric characterization of the system space.
Proof. We prove the statement via induction on $i$. The induction start $i = 0$ is fulfilled by $W_0 = \{0\}$. For the induction step, assume that $i \in \mathbb{N}_0$ with $\mathcal{W}_i = W_i^\perp$. Then

$$W_{i+1} = [A B]^{-1} ([E 0] \cdot \mathcal{W}_i) = [A B]^{-1} ([E 0] \cdot W_i)$$

Lem. 10

$$[A B]^{-1} \left( \begin{bmatrix} E^T & 0 \end{bmatrix} \right) \downarrow = W_{i+1}$$

Lem. 10

$$\left( \begin{bmatrix} A^T & B^T \end{bmatrix} \right) \downarrow = W_{i+1}$$

\(\square\)

Lemma 12. Consider an electrical circuit with incidence and partition matrices $Z, W$. We prove the statement via induction on $i$. We have $\exists \text{ a basis matrix } \mathcal{V}_n$. Let $Z$ and $Z_{\mathcal{V}_n}$ be basis matrices of $\ker A_{\mathcal{V}_n}$ and, resp., $\ker [A \ A_{\mathcal{V}} A_{\mathcal{V}} A_{\mathcal{V}} A_{\mathcal{V}}]$. Then there exists a basis matrix $Z_{\mathcal{V}_n - c}$ of $\ker [A \ A_{\mathcal{V}} A_{\mathcal{V}} A_{\mathcal{V}} A_{\mathcal{V}}] Z_c$ such that $Z_{\mathcal{V}_n} = Z_c Z_{\mathcal{V}_n - c}$.

Proof. We have $Z_{\mathcal{V}_n} \subseteq Z_c$ by definition. Hence there exists a matrix $Z_{\mathcal{V}_n - c}$ of $Z_{\mathcal{V}_n} = Z_c Z_{\mathcal{V}_n - c}$. Then $\ker Z_{\mathcal{V}_n} = \{0\}$ implies $\ker Z_{\mathcal{V}_n - c} = \{0\}$. Then, with $k := \dim \ker A_{\mathcal{V}_n}$, the result follows from

$$\ker Z_{\mathcal{V}_n - c} \subseteq \ker Z_{\mathcal{V}_n} = \{0\} \iff \ker [A \ A_{\mathcal{V}} A_{\mathcal{V}} A_{\mathcal{V}} A_{\mathcal{V}}] Z_c = 0$$

\(\square\)

Now we present a proof of Thm. 8. In doing so, we use the subspace iteration in Lem. 9. Instead of a direct calculation, we determine the orthogonal space via Lem. 11.

Proof of Thm. 8. Let $\{W_i\}$ be a sequence of subspaces as in Lem. 11. For $i \in \mathbb{N}_0$, define

$$Z_i := \left[ \begin{bmatrix} E^T & 0 \end{bmatrix} \right]^{-1} W_i.$$

Then $W_i = \left[ \begin{bmatrix} A^T & B^T \end{bmatrix} \right] \cdot Z_i$ for all $i \in \mathbb{N}_0$ with $i \geq 1$. Further, let $Z_{\mathcal{V}_n - c}$ be a basis matrix of $\ker [A \ A_{\mathcal{V}} A_{\mathcal{V}} A_{\mathcal{V}} A_{\mathcal{V}}] Z_c$, such that $Z_{\mathcal{V}_n} = Z_c Z_{\mathcal{V}_n - c}$ (which exists by Lem. 12).

Step 1: We determine $W_1$. By $W_0 = \{0\}$ and $E = E^T$, we have $W_1 = \ker E$. By incorporating $\mathcal{V} > 0$, we obtain the latter space equals to $\ker Z_c \times \ker E$, and thus

$$W_1 = \left[ \begin{bmatrix} A^T & B^T \end{bmatrix} \right] \cdot \ker E = \left[ \begin{bmatrix} A_{\mathcal{V}} \mathcal{A}^T Z_c & A_{\mathcal{V}} \mathcal{A} \mathcal{A}^T Z_c \end{bmatrix} \right] \subset Z_c Z_{\mathcal{V}_n} \mathcal{V}_n$$

Step 2: We show that $Z_2$ fulfills

$$Z_2 = \left[ \begin{bmatrix} Z_{\mathcal{V}_n}^\perp & 0 \end{bmatrix} \right] \cdot Z_{\mathcal{V}_n} \subset W_2$$

\(\subset\)

Let $z$ be in the space on the right hand side of (15), and partition $z = (z_1 \ z_2 \ z_3)^T$ with $z_1 \in \mathbb{R}^n \setminus Z_{\mathcal{V}_n}$, $z_2 \in \mathbb{R}^n \setminus Z_{\mathcal{V}_n}$, and $z_3 \in \mathbb{R}^n \setminus Z_{\mathcal{V}_n}$. Then there exist vectors $v_1, v_2$ with $z_1 = Z_c v_1$ and $z_2 = Z_{\mathcal{V}_n} v_2$. By using $A_{\mathcal{V}} \mathcal{A}^T Z_c = A_{\mathcal{V}} \mathcal{A} \mathcal{A}^T Z_{\mathcal{V}_n} v_2$, we obtain

$$W_2 = \left[ \begin{bmatrix} A_{\mathcal{V}} \mathcal{A}^T Z_c & A_{\mathcal{V}} \mathcal{A} \mathcal{A}^T Z_c \end{bmatrix} \right] \subset Z_c Z_{\mathcal{V}_n} \mathcal{V}_n$$

Thus, $W_2 \subset W_1$. We obtain from (16) that there exist vectors $w_1, w_2$ with

$$\left( \begin{bmatrix} A_{\mathcal{V}} \mathcal{A}^T Z_c \ w_2 \end{bmatrix} \right) \in W_1 = \left[ \begin{bmatrix} A_{\mathcal{V}} \mathcal{A}^T Z_c & A_{\mathcal{V}} \mathcal{A} \mathcal{A}^T Z_c \end{bmatrix} \right] \subset W_2$$

Hence $w_2 = 0$ and $A_{\mathcal{V}} \mathcal{A}^T z_1 = A_{\mathcal{V}} \mathcal{A} \mathcal{A}^T Z_{\mathcal{V}_n} v_2$, and a multiplication with $Z_{\mathcal{V}_n}^T$ results in $0 = Z_{\mathcal{V}_n}^T A_{\mathcal{V}} \mathcal{A} (\mathcal{G} + \mathcal{G}^T) A_{\mathcal{V}} \mathcal{A} Z_{\mathcal{V}_n} v_2$. Then $\mathcal{G} + \mathcal{G}^T > 0$ gives $A_{\mathcal{V}} \mathcal{A} Z_{\mathcal{V}_n} v_2 = 0$. Thus $w_1 = \ker [A \ A_{\mathcal{V}} A_{\mathcal{V}} A_{\mathcal{V}} A_{\mathcal{V}}] Z_c = \ker Z_{\mathcal{V}_n - c}$. That is, $w_1 = Z_{\mathcal{V}_n - c} v_2$ for a vector $y$, and (17) leads to

$$\left( \begin{bmatrix} A_{\mathcal{V}} \mathcal{A}^T z_1 \ y \end{bmatrix} \right) \subset Z_{\mathcal{V}_n - c} v_2$$

Thus $z_1 \in \ker A_{\mathcal{V}} \mathcal{A}^T$ and $z_2 \in \ker Z_{\mathcal{V}_n - c}$.

Step 3: We conclude that

$$W_2 = \left[ \begin{bmatrix} A^T & B^T \end{bmatrix} \right] Z_2$$

Step 4: We show that $Z_3 \subset Z_2$. Let $z = (z_1 \ z_2 \ z_3)^T \in Z_3$ with $z_1 \in \mathbb{R}^n \setminus Z_{\mathcal{V}_n}, z_2 \in \mathbb{R}^n, z_3 \in \mathbb{R}^n \setminus Z_{\mathcal{V}_n}$. Then $z \in W_2$ together with Step 3, leads to the existence of vectors $w_1, w_2, w_3$ with

$$\left( \begin{bmatrix} A_{\mathcal{V}} \mathcal{A}^T z_1 \ y \end{bmatrix} \right) \subset Z_{\mathcal{V}_n - c} v_2$$

Then $Z_{\mathcal{V}_n - c} v_2 = 0$ by a multiplication of the first row with $Z_{\mathcal{V}_n - c} v_2$, and $\mathcal{L} > 0$ gives $A_{\mathcal{V}} \mathcal{A} Z_{\mathcal{V}_n - c} v_2 = 0$. Altogether, we have

$$\left( \begin{bmatrix} A_{\mathcal{V}} \mathcal{A}^T z_1 \ y \end{bmatrix} \right) \subset Z_{\mathcal{V}_n - c} v_2$$

This is exactly the situation in (17), and we can follow the argumentation in Step 2 to conclude $z \in Z_2$. 

Step 5: We conclude the statement of Thm. 8. We have $W_3 = \left[ \begin{bmatrix} A^T \ B^T \end{bmatrix} \right] Z_3 \subset \left[ \begin{bmatrix} A^T \ B^T \end{bmatrix} \right] Z_2 = W_2$ by Step 4. Thus, by Lem. 11, $W_2 = W_2^\perp \subset W_2^\perp = V_3$, whence, by Lem. 9, the system space reads $V_2 = W_2^\perp$. Now using Step 3, we obtain
\[ Y_2 = W_2 = \begin{pmatrix} \begin{bmatrix} A_G A_L^T Z_C & A_V & A_L L^{-1} A_L^T Z_{\mathcal{V}^n} \\ A_L^T Z_C & 0 & 0 \\ A_L^T Z_V & 0 & 0 \\ Z_{\mathcal{V}^n} A_L L^{-1} A_L^T \\ 0 & 0 & -I_{n_V} \\ Z_{\mathcal{V}^n} A_L L^{-1} A_L^T \\ \end{bmatrix} \end{pmatrix} \] 

which completes the proof.

7. CONSISTENT INITIAL VALUES AND CONTROLLABILITY AT INFINITY

Here we analyze the space of consistent initial values, which is the space of all \( x_0 \in \mathbb{R}^n \) for which there exists some control \( u \) for which there is a weakly differentiable solution \( x \) of (3) with initial condition \( x(0) = x_0 \). If this space is the entire \( \mathbb{R}^n \), then the system (3) is called \textit{controllable at infinity}. It is proven in (Berger and Reis, 2013, Sec. 5) that controllability at infinity is equivalent to \( \ker[E \, A \, B] = \{0\} \) for \( E, A \in \mathbb{R}^{n \times n} \). For \( E, A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \) as in the circuit model (4) with assumption (5), we can conclude from (10) that \( \ker[E \, A \, B] = \{0\} \), hence the analysis of controllability at infinity for MNA reduces to check whether \( \ker[E \, A \, B] = \{0\} \). By using \( c > 0 \), \( \mathcal{L} > 0 \), we obtain that \( \ker[A \, C \, A_1] = \{0\} \), and we summarize these findings in the following result.

**Proposition 13.** Let \( E, A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \) as in (4) and assume that (5) holds. Then the system (3) is controllable at infinity if, and only if, the underlying circuit does not contain any \( \mathcal{RLV} \)-cuts; equivalently (by Prop. 1)

\[ \ker[A \, C \, A_1] = \{0\} \]

It can be concluded from (Reis and Voigt, 2019, Lem. 3.7) that the system space \( \mathcal{V}_{\mathcal{Y}} \) and the space \( \mathcal{V}_{\mathcal{Y}^c} \) of consistent initial values of the system (3) fulfil the identity

\[ \mathcal{V}_{\mathcal{Y}} = \{ x \in \mathbb{R}^n : \exists u \in \mathbb{R}^m \text{ s.t. } (\begin{array}{c} x \\ u \end{array}) \in \mathcal{V}_{\mathcal{Y}^c} \} \]

(19)

This identity is the essential ingredient in the proof of the following result which contains an expression of the space of consistent initial values for the MNA system.

**Theorem 14.** Let \( E, A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \) as in (4) and assume that (5) holds. Let \( Z_{\mathcal{V}^n} \) and \( Z_C \) be basis matrices of \( \ker[A_G A_L^T A_1^T] \) and, resp., \( \ker[A \, C \, A_1] \). Then the space of consistent initial values (3) is given by

\[ \ker\begin{bmatrix} Z_{\mathcal{V}^n} Z_C L^{-1} A_L^T \\ Z_C A_G A_L^T Z_C \\ Z_C A_L L^{-1} A_L^T \end{bmatrix} \]

**Proof.** Analogous to Lem. 12, there exists a basis matrix \( Z_{\mathcal{V}^n-c} \) of \( \ker[A_L^T Z_C] \) such that \( Z_C = Z_{\mathcal{V}^n-c} \).

\[ \text{“C”: Let } x = (x_1, x_2, x_3)^T \text{ with } x_1 \in \mathbb{R}^n, \quad x_2 \in \mathbb{R}^m, \quad x_3 \in \mathbb{R}^{n_V} \text{ be a consistent initial value. Then by (19), there exist } u_1 \in \mathbb{R}^n, \quad u_2 \in \mathbb{R}^{n_V} \text{ such that for } u = (\begin{array}{c} u_1 \\ u_2 \end{array}) \text{ holds that } (\begin{array}{c} x \\ u_2 \end{array}) \text{ is in the system space of (3). Then Thm. 8 gives} \]

\[ Z_{\mathcal{V}^n-c} (A_G A_L^T x_1 + A_L x_2 + A_V x_3 + A_1 u_1) = 0, \]

\[ Z_{\mathcal{V}^n-c} (A_G A_L^T x_1 + A_L x_2 + A_V x_3 + A_1 u_1) = 0. \]

Then a multiplication of the first equation with \( Z_{\mathcal{V}^n-c} \) gives

\[ Z_{\mathcal{V}^n-c} Z_C (A_G A_L^T x_1 + A_L x_2 + A_V x_3 + A_1 u_1) = 0, \]

\[ Z_{\mathcal{V}^n-c} Z_C = 0. \]

8. CONSISTENT INITIAL DIFFERENTIAL VALUES AND IMPULSE CONTROLLABILITY

We now consider another type of initialization, namely (3) with initial condition \( \dot{x}(0) = E x_0 \). \( x_0 \in \mathbb{R}^n \) is called a consistent initial differential value, if there exists a control \( u \) for which a solution \( x \) of (3) with initial condition \( \dot{x}(0) = E x_0 \) exists. If this space equals to \( \mathbb{R}^n \), then the system (3) is called \textit{impulse controllable}. It is proven in (Berger and Reis, 2013, Sec. 5) that impulse controllability is equivalent to \( \ker[E \, A \, Z_2] = \ker[E \, A \, B] \) for some (and hence any) basis matrix \( Z \) of \( \ker E \). By using that the circuit model (4) with assumption (5) has the property \( \ker[E \, A \, B] = \{0\} \), it is impulse controllable if, and only if, \( \ker[E \, A \, Z_2] = \{0\} \). By using that \( \mathcal{L} > 0 \) and \( \ker E \), we obtain that a basis matrix of \( \ker E \) is given by \( Z = \text{diag}(Z_C, 0, I) \), where \( Z_C \) is a basis matrix of \( \ker A_L^T \). Then

\[ \ker[E \, A \, Z_2] = \ker\begin{bmatrix} A_C A_L^T Z_C & A_V A_1 & A_1 \\ 0 & \mathcal{L} & -A_L Z_C \\ 0 & 0 & 0 \\ 0 & 0 & -I_{n_V} \end{bmatrix} \]

\[ = \ker[A \, C \, A_1] + n_c + n_V. \]

If \( \ker[A \, C \, A_1] \neq \{0\} \), (22) implies \( \ker[E \, A \, B] < n \). Conversely, if \( \ker[A \, C \, A_1] \neq \{0\} \) and \( x_1 \in \ker[A \, C \, A_1] \), then \( x_1 \in \ker A_C \), i.e., \( x_1 = Z_{c1} \) for a vector \( z_1 \), and thus \( Z_C A_G A_L^T Z_C z_1 = 0 \). Then \( \mathcal{G} + \mathcal{G} > 0 \) leads to \( A_L^T x_1 = A_L^T Z_C z_1 = 0 \), whence \( x_1 \in \ker[A \, C \, A_1] \). We summarize these findings in the following result.

**Proposition 15.** Let \( E, A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \) as in (4) and assume that (5) holds. Then the system (3) is impulse controllable if, and only if, the underlying circuit does not contain any \( \mathcal{L} \)-cuts; equivalently (by Prop. 1)

\[ \ker[A \, C \, A_1] = \{0\} \]

It is shown in (Berger and Reis, 2013, Lem. 2.3) that the space \( \mathcal{V}_{\mathcal{Y}^c} \) of consistent initial values and the space \( \mathcal{V}_{\mathcal{Y}^c} \) of consistent initial differential values of the system (3) fulfill

\[ \mathcal{V}_{\mathcal{Y}^c} = \mathcal{V}_{\mathcal{Y}^c} + \ker E. \]
This identity is the essential ingredient in the proof of the following result on the space of consistent initial differential values for the MNA system. We will make use of the following preparatory result.

**Lemma 16.** For any subspace \( V \subset \mathbb{R}^E \) and \( M \in \mathbb{R}^{k \times l} \) holds
\[
M^{-1}(MV) = V + \ker M.
\]

**Proof.** “\( \subset \)” Let \( x \in M^{-1}(MV) \). Then \( Mx = My \) for some \( y \in \mathbb{R}^E \), whence \( x = (x - y) + y \in \ker M + V \).

“\( \supset \)” Let \( x \in V + \ker M \), i.e., \( x = y + e \) for some \( v \in V \) and \( e \in \ker M \). Thus \( Mx = My \in \ker(MV) \). \( \square \)

**Theorem 17.** Let \( E, A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \) as in (4) and assume that (5) holds. Let \( Z_{\mathcal{CVI}} \) be a basis matrix of \( \ker [A_L A_K A_K^T A_{CV}]^T \). Then the space of consistent initial differential values of (3) is given by
\[
\ker [0 Z_{\mathcal{CVI}} A_L 0].
\]

**Proof.** Let \( Z_{\mathcal{CVI}} \) be a basis matrix of \( \ker [A_L A_K A_K^T A_{CV}]^T \). Analogous to Lem. 12, there exists a basis matrix \( Z_{\mathcal{CVI}}^{-1} \) of \( \ker [A_K A_K^T]^T \), such that \( Z_{\mathcal{CVI}}^{-1} A_K Z_{\mathcal{CVI}} = Z_{\mathcal{CVI}}^{-1} A_K Z_{\mathcal{CVI}} \).

**Step 1:** We show that the space \( \mathcal{V}_{\text{diff}} \) of consistent initial differential values fulfills
\[
E^{-1} \mathcal{V}_{\text{diff}} = \ker A_L^T \times \mathbb{L}\cdot -A_L^T Z_{\mathcal{CVI}} \times \mathbb{R}^n \quad (24)
\]

“\( \subset \)” Let \( x = (x_1^T x_2^T x_3^T)^T \in E^{-1} \mathcal{V}_{\text{diff}} \) with \( x_1 \in \mathbb{R}^n \), \( x_2 \in \mathbb{R}^n \), \( x_3 \in \mathbb{R}^n \). Now using Thm. 14 together with (13), we obtain that there exist vectors \( z_1, z_2 \) with
\[
\begin{pmatrix}
A_L A_C A L_{x_1} \\
A_L A_C A L_{x_2}
\end{pmatrix} =
\begin{pmatrix}
A_L A_C A L_{x_1} + A_K \theta A_L Z_{\mathcal{CVI}} z_2 \\
A_L A_C A L_{x_2} + A_K \theta A_L Z_{\mathcal{CVI}} z_2
\end{pmatrix}.
\]

A multiplication of the first equation in (25) with \( Z_{\mathcal{CVI}}^T \) yields \( Z_{\mathcal{CVI}}^T E^{-1} \mathcal{V}_{\text{diff}} = 0 \). Then the positivity of \( \mathcal{V} \) now follows rise to \( A_L^T Z_{\mathcal{CVI}} z_1 = 0 \). It follows that \( Z_{\mathcal{CVI}} z_1 \in \ker A_L^T \).

**Step 2:** We conclude from Step 1 that
\[
E^{-1} \mathcal{V}_{\text{diff}} = \ker [0 Z_{\mathcal{CVI}} A_L 0].
\]

**Step 3:** We conclude the statement of Thm. 17: By using the symmetry of \( E \), we obtain
\[
\ker [0 Z_{\mathcal{CVI}} A_L 0] \quad \text{Step 2} \quad (E^{-1} \mathcal{V}_{\text{diff}})^+ \quad \text{Lem. 16} \quad E^{-1} (E^{-1} \mathcal{V}_{\text{init}})^+ \quad \text{Lem. 10} \quad \mathcal{V}_{\text{init}} + \ker E (E^{-1} \mathcal{V}_{\text{diff}}). \quad \square
\]

In the case where the circuit does not contain any \( L \)-cuts, we can conclude from Prop. 1 that \( Z_{\mathcal{CVI}} \) is trivial, i.e., it has zero columns. As a consequence, we also obtain from Thm. 17 that in the case of absence of \( L \)-cuts, any vector in \( \mathbb{R}^n \) is a consistent initial differential value for the MNA system (cf. Prop. 15).

**REFERENCES**


Lamour, R., Márz, R., and Tischendorf, C. (2013). Differential Algebraic Equations: A Projector Based Analysis,


