Port-Hamiltonian formulation of nonlinear electrical circuits

H. Gernandt[a], F. E. Haller[b], T. Reis[b], A. J. van der Schaft[c]

[a] *TU Ilmenau, Weimarer Straße 25, 98693 Ilmenau, Germany*
[b] *Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany*
[c] *Bernoulli Institute for Mathematics, CS and AI, University of Groningen, The Netherlands*

Abstract

We consider nonlinear electrical circuits for which we derive a port-Hamiltonian formulation. After recalling a framework for nonlinear port-Hamiltonian systems, we model each circuit component as an individual port-Hamiltonian system. The overall circuit model is then derived by considering a port-Hamiltonian interconnection of the components. We further compare this modelling approach with standard formulations of nonlinear electrical circuits.

Keywords: Port-Hamiltonian system, electrical circuit, graph, Dirac structure, Lagrangian submanifold, resistive relation, differential-algebraic equations
2020 MSC: 34A09, 37J39, 53D12, 93C10, 94C15

1. Introduction

Port-Hamiltonian system models encompass a very large class of nonlinear physical systems [10, 23] and arise from port-based network modelling of complex lumped parameter systems from various physical domains, such as, for instance, mechanical and electrical systems. Modelling by port-Hamiltonian systems has gained a lot of attention, see, for instance, the surveys [10, 22]. Tremendous progress has been recently made in port-Hamiltonian modelling of constrained dynamical systems, which leads to differential-algebraic equations [4, 12, 13, 15, 22]. This enables to apply the framework to modelling of multibody systems with holonomic and non-holonomic constraints as well as electrical circuits. Examples of the latter class has been considered from a port-Hamiltonian point of view in [10, 21, 22, 25]. However, an approach to electrical circuits has been only made for the case where the circuit contains only capacitances and inductances [6]. The recent progress in port-Hamiltonian differential-algebraic equations however allows to treat a by far wider class of electrical circuits. This is exactly the purpose of this article, where we consider a variety of electrical components, such as resistances, capacitances, inductances, diodes, transformers, transistors, current sources and voltage sources from a port-Hamiltonian perspective. Thereafter, we consider the circuit interconnection structure by utilizing the underlying graph of the given electrical circuit. This gives rise to a port-Hamiltonian model, which only incorporates the Kirchhoff laws. Finally, the port-Hamiltonian model of the electrical circuit is obtained by an interconnection with the individual port-Hamiltonian systems representing the components. We will compare the resulting dynamical system with well-known formulations of nonlinear electrical circuits like the (charge/flux-oriented) *modified nodal analysis* and the *modified loop analysis*.

2. Port-Hamiltonian systems and their interconnections

2.1. Port-Hamiltonian DAE systems
We review some basics in port-Hamiltonian differential-algebraic equations (DAEs) from [12, 13]. An important concept is that of the *Dirac structure*, which describes the power preserving energy-routing of the

---

system. In a very general setting, a Dirac structure on a manifold $\mathcal{M}$ is defined [8, Def. 2.2.1] as a certain subbundle of $\mathcal{D} \subset T\mathcal{M} \oplus T^*\mathcal{M}$ (i.e., the direct sum of the tangent bundle and co-tangent bundle of $\mathcal{M}$). It turns out that, even for nonlinear circuits, this general definition is not needed, and we may introduce Dirac structures only for the simple case where $\mathcal{M} = \mathbb{R}^n$ (which gives rise to the identification $T^*\mathbb{R}^n \cong T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$) and $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^n$ is a subspace.

Definition 2.1 (Dirac structure). *A subspace $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^n$ is called a* Dirac structure, *if for all $f, e \in \mathbb{R}^n$ holds*

$$(f, e) \in \mathcal{D} \iff \forall (\hat{f}, \hat{e}) \in \mathcal{D} : \ e^\top \hat{f} + \hat{e}^\top f = 0.$$

We will also write $(f, e) \in \mathcal{D} \subset \mathcal{F} \times \mathcal{E}$, where $\mathcal{F}$ denotes the space of *flows* and $\mathcal{E} = \mathbb{R}^n \cong \mathcal{F}^*$ denotes the space of *efforts*. A useful characterisation of Dirac structures is the following.

Proposition 2.2 ([8, Prop. 1.1.5]). *A subspace $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^n$ is a Dirac structure if, and only if, there exist $K, L \in \mathbb{R}^{n \times n}$ with $KL^\top + LK^\top = 0$ and $\mathrm{rk}\,[K \ \ L] = n$, such that*

$$\mathcal{D} = \left\{ (f, e) \in \mathbb{R}^n \times \mathbb{R}^n \mid Kf + Le = 0 \right\}. \tag{1}$$

Now we introduce a relation describing the energy storage of the system and is called *Lagrange submanifold*. Again, the general definition of Lagrange submanifold as found in [11, p. 568] is not needed for nonlinear circuits. It suffices to consider the case of submanifolds of $\mathbb{R}^n \times \mathbb{R}^n$. Typically, the manifolds are assumed to be smooth. This can however be relaxed, and we may consider less-smooth manifolds for our purposes.

Definition 2.3 (Lagrange submanifold). *A submanifold $\mathcal{L} \subset \mathbb{R}^n \times \mathbb{R}^n$ is called* Lagrange submanifold *of $\mathbb{R}^n \times \mathbb{R}^n$, if for all $x \in \mathcal{L}$ and $(v_1, v_2) \in \mathbb{R}^n \times \mathbb{R}^n$ holds*

$$(v_1, v_2) \in T_x\mathcal{L} \iff \forall (w_1, w_2) \in T_x\mathcal{L} : v_1^\top w_2 - v_2^\top w_1 = 0.$$

Hereby, $T_x\mathcal{L} \subset \mathbb{R}^n \times \mathbb{R}^n$ stands for tangent space of $\mathcal{L}$ at $x \in \mathcal{L}$.

In the following we show that gradient fields induce Lagrange submanifolds.

Proposition 2.4. *Let $Q : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. Then the submanifold consisting of the graph of Q, i.e.,*

$$\mathcal{L}_Q := \{ (x, Q(x)) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \in \mathbb{R}^n \}$$

is a Lagrange submanifold if, and only if, Q is a gradient field. In other words, there exists some twice continuously differentiable function $H : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla H = Q$.

Proof. Using that $\mathbb{R}^n$ is simply connected, the case of smooth Q follows from [11, Prop. 22.12]. The less smooth case follows by a straightforward modification of the proof of [11, Prop. 22.12]. $\qquad\square$

The case where a Lagrangian submanifold is a subspace deserves special attention.

Proposition 2.5 ([12, Prop. 5.2]). *A subspace $\mathcal{L} \subset \mathbb{R}^n \times \mathbb{R}^n$ is a Lagrangian submanifold if, and only if,*

$$\mathcal{L} = \left\{ (f, e) \in \mathbb{R}^n \times \mathbb{R}^n \mid S^\top f = P^\top e \right\}$$

for some matrices $S, P \in \mathbb{R}^{n \times n}$ with $S^\top P = P^\top S$ and $\mathrm{rk}\,[S^\top \ P^\top] = n$.

Another concept needed for port-Hamiltonian systems is that of the *resistive relation*, which represents the internal energy dissipation of the system. It is defined as a relation on the *space of resistive flows* $\mathcal{F}_R$ and *space of resistive efforts* $\mathcal{E}_R$ [10, Sec. 2.4]. In our setting, both $\mathcal{E}_R$ and $\mathcal{F}_R$ will be again $\mathbb{R}^n$.

Definition 2.6 (Resistive relation). *A relation $\mathcal{R} \subset \mathbb{R}^n \times \mathbb{R}^n$ is called* resistive, *if*

$$\forall \ (f_\mathcal{R}, e_\mathcal{R}) \in \mathcal{R} : \ e_\mathcal{R}^\top f_\mathcal{R} \leq 0.$$
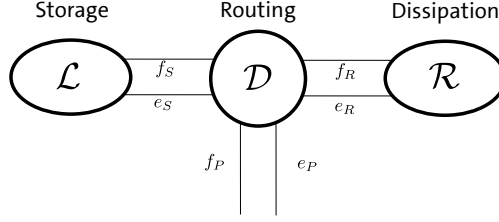
Figure 1: Visual representation of a pH system.

Having defined Dirac structures, Lagrange submanifolds and resistive relations, we are now ready to introduce port-Hamiltonian systems. Again note this class can be defined in a more general setting by using manifolds [10, 13]. We 'boil this down' to the setup which is needed for electrical circuits.

Definition 2.7 (Port-Hamiltonian (pH) system). *Let $n_{\mathcal{L}}, n_{\mathcal{R}}, n_{\mathcal{P}} \in \mathbb{N}_0$ and denote*

$$\mathcal{F}_{\mathcal{L}} = \mathcal{E}_{\mathcal{L}} = \mathbb{R}^{n_{\mathcal{L}}}, \quad \mathcal{F}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}} = \mathbb{R}^{n_{\mathcal{R}}}, \quad \mathcal{F}_{\mathcal{P}} = \mathcal{E}_{\mathcal{P}} = \mathbb{R}^{n_{\mathcal{P}}}.$$

A port-Hamiltonian (pH) system *is a triple $(\mathcal{D}, \mathcal{L}, \mathcal{R})$, where $\mathcal{D} \subset (\mathcal{F}_{\mathcal{L}} \times \mathcal{F}_R \times \mathcal{F}_P) \times (\mathcal{E}_{\mathcal{L}} \times \mathcal{E}_R \times \mathcal{E}_P)$ is a Dirac structure (see Definition 2.1), $\mathcal{L} \subset \mathcal{F}_{\mathcal{L}} \times \mathcal{E}_{\mathcal{L}}$ is a Lagrange submanifold (see Definition 2.3) and $\mathcal{R} \subset \mathcal{F}_{\mathcal{R}} \times \mathcal{E}_{\mathcal{R}}$ a resistive relation (see Definition 2.6).*
The elements of $\mathcal{F}_{\mathcal{L}}, \mathcal{E}_{\mathcal{L}}, \mathcal{F}_{\mathcal{R}}, \mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{P}}, \mathcal{E}_{\mathcal{P}}$ are, accordingly, called the energy-storing flows/efforts, resistive flows/efforts *and* external flows/efforts.
The dynamics *of the pH system are specified by the differential inclusion*

$$(-\tfrac{d}{dt}x(t), f_{\mathcal{R}}(t), f_{\mathcal{P}}(t), e_{\mathcal{L}}(t), e_{\mathcal{R}}(t), e_{\mathcal{P}}(t)) \in \mathcal{D}, \ (x(t), e_{\mathcal{L}}(t)) \in \mathcal{L}, \ (f_{\mathcal{R}}(t), e_{\mathcal{R}}(t)) \in \mathcal{R}.$$

Note that, in this paper, we do not investigate any solvability theory of the resulting equations.

2.2. Interconnection of port-Hamiltonian systems

A key property of pH systems is that this class is closed under power-conserving interconnection. Different methods of how to design such interconnections are for example elucidated in [3, 7, 10, 26]. The interconnection we will be using for the electrical circuits follows the ideas presented in [10]. Interconnection is based on the assumption that each system has two kinds of external flows and efforts, namely specific and to-be-linked ones, where the latter ones are belonging to the same space for each Dirac structure.

Definition 2.8 (Interconnection of pH systems). *For $i = 1, 2$, let $(\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i)$ be two pH systems with specific flow and effort spaces,*

$$\mathcal{F}_i = \mathcal{F}_{\mathcal{L}i} \times \mathcal{F}_{\mathcal{R}i} \times \mathcal{F}_{\mathcal{P}i} \times \mathcal{F}_{\mathcal{P}\text{link}}, \quad \mathcal{E}_i = \mathcal{F}_{\mathcal{L}i} \times \mathcal{E}_{\mathcal{R}i} \times \mathcal{E}_{\mathcal{P}i} \times \mathcal{E}_{\mathcal{P}\text{link}},$$

which are subdivided into an energy-storing, a resistive, a specific external part, and a to-be-linked part. We define the interconnection of $(\mathcal{D}_1, \mathcal{L}_1, \mathcal{R}_1)$ and $(\mathcal{D}_2, \mathcal{L}_2, \mathcal{R}_2)$,

$$(\mathcal{D}_1, \mathcal{L}_1, \mathcal{R}_1) \circ (\mathcal{D}_2, \mathcal{L}_2, \mathcal{R}_2) := (\mathcal{D}, \mathcal{L}, \mathcal{R}),$$

with respect to $(\mathcal{F}_{\mathcal{P}\text{link}}, \mathcal{E}_{\mathcal{P}\text{link}})$ as the pH system given by

$$\mathcal{D} := \big\{ ((f_{\mathcal{L}1}, f_{\mathcal{L}2}), (f_{\mathcal{R}1}, f_{\mathcal{R}2}), (f_{\mathcal{P}1}, f_{\mathcal{P}2}), (e_{\mathcal{L}1}, e_{\mathcal{L}2}), (e_{\mathcal{R}1}, e_{\mathcal{R}2}), (e_{\mathcal{P}1}, e_{\mathcal{P}2}))$$
$$\mid \exists (f_{\text{link}}, e_{\text{link}}) \in \mathcal{F}_{\text{link}} \times \mathcal{E}_{\text{link}} : (f_{\mathcal{L}1}, f_{\mathcal{R}1}, f_{\mathcal{P}1}, f_{\text{link}}, e_{\mathcal{L}1}, e_{\mathcal{R}1}, e_{\mathcal{P}1}, e_{\text{link}}) \in \mathcal{D}_1$$
$$\wedge (f_{\mathcal{L}2}, f_{\mathcal{R}2}, f_{\mathcal{P}2}, -f_{\text{link}}, e_{\mathcal{L}2}, e_{\mathcal{R}2}, e_{\mathcal{P}2}, e_{\text{link}}) \in \mathcal{D}_2 \big\},$$

and

$$\mathcal{L} = \big\{ ((f_{\mathcal{L}1}, f_{\mathcal{L}2}), (e_{\mathcal{L}1}, e_{\mathcal{L}2})) \in (\mathcal{F}_{\mathcal{L}1} \times \mathcal{F}_{\mathcal{L}2}) \times (\mathcal{E}_{\mathcal{L}1} \times \mathcal{E}_{\mathcal{L}2}) \mid (f_{\mathcal{L}1}, e_{\mathcal{L}1}) \in \mathcal{L}_1 \wedge (f_{\mathcal{L}2}, e_{\mathcal{L}2}) \in \mathcal{L}_2 \big\},$$
$$\mathcal{R} = \big\{ ((f_{\mathcal{R}1}, f_{\mathcal{R}2}), (e_{\mathcal{R}1}, e_{\mathcal{R}2})) \in (\mathcal{F}_{\mathcal{R}1} \times \mathcal{F}_{\mathcal{R}2}) \times (\mathcal{E}_{\mathcal{R}1} \times \mathcal{E}_{\mathcal{R}2}) \mid (f_{\mathcal{R}1}, e_{\mathcal{R}1}) \in \mathcal{R}_1 \wedge (f_{\mathcal{R}2}, e_{\mathcal{R}2}) \in \mathcal{R}_2 \big\}.$$
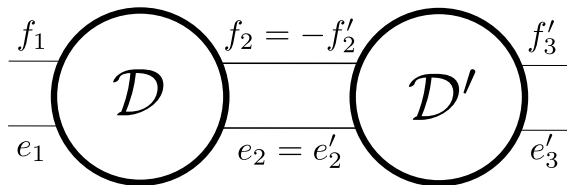
3

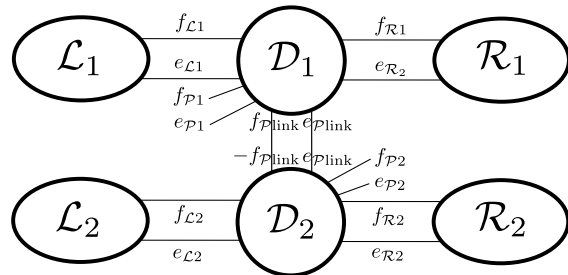Figure 2: Composition of two Dirac structures.

Figure 3: Interconnection of two pH systems.

The above constructed set $\mathcal{D}$ is indeed a Dirac structure [10, Chap. 6]. It is obvious that $\mathcal{L}$ is a Lagrange submanifold and $\mathcal{R}$ is a resistive relation. Hence, the interconnection of pH systems results in a pH system. Next we introduce the Cartesian product of pH systems, which simply means that several coexisting pH systems are united to one pH system. In terms of Definition 2.8, it means that several pH systems are interconnected with trivial linking ports. That is, for pH systems $(\mathcal{D}_1, \mathcal{L}_1, \mathcal{R}_1)$ and $(\mathcal{D}_2, \mathcal{L}_2, \mathcal{R}_2)$ we add artificial and trivial linking ports $\mathcal{F}_{\text{link}} = \mathcal{E}_{\text{link}} = \{0\}$ (which do not affect the dynamic behavior) and interconnect these systems with respect to this trivial port $(\mathcal{F}_{\mathcal{P}\text{link}}, \mathcal{E}_{\mathcal{P}\text{link}})$. A coupling of this kind will be denoted by $(\mathcal{D}_1, \mathcal{L}_1, \mathcal{R}_1) \times (\mathcal{D}_2, \mathcal{L}_2, \mathcal{R}_2)$. We further inductively define

$$\overset{n}{\underset{i=1}{\bigtimes}} (\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i) := \left(\overset{n-1}{\underset{i=1}{\bigtimes}} (\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i) \right) \times (\mathcal{D}_n, \mathcal{L}_n, \mathcal{R}_n).$$

2.3. Port-Hamiltonian systems on graphs

Now we consider interconnections of pH systems, which are defined via graphs [24]. This will lead us to the notions of *Kirchhoff-Dirac structure* and *Kirchhoff-Lagrange manifold*. Later we will show that such interconnections correspond to the Kirchhoff laws in electrical circuits. To this end, we introduce some basic graph theoretical notions from [9].

Definition 2.9 (Graphs and subgraphs). *A* directed graph *is a quadruple* $\mathcal{G} = (V, E, \text{init}, \text{ter})$ *consisting of a* vertex set V, *a* edge set E *and two maps* $\text{init}, \text{ter} : E \to V$ *assigning to each edge* e *an* initial vertex $\text{init}(e)$ *and a* terminal vertex $\text{ter}(e)$. *The edge* e *is said to be* directed *from* $\text{init}(e)$ *to* $\text{ter}(e)$. $\mathcal{G}$ *is said to be* loop-free, *if* $\text{init}(e) \neq \text{ter}(e)$ *for all* $e \in E$. *Let* $V' \subset V$ *and* $E' \subset E$ *with*

$$E' \subset E|_{V'} := \{ e \in E : \text{init}(e) \in V' \wedge \text{ter}(e) \in V' \}.$$

Then the triple $(V', E', \text{init}|_{E'}, \text{ter}|_{E'})$ *is called a* subgraph *of* $\mathcal{G}$. *If* $E' = E|'_V$, *then the subgraph is called the* induced subgraph *on* V'. *If* $V' = V$, *then the subgraph is called* spanning. *Additionally a* proper subgraph *is one where* $E' \neq E$. $\mathcal{G}$ *is called* finite, *if* V *and* E *are finite.*

The notion of a *path* in a directed graph $\mathcal{G} = (V, E, \text{init}, \text{ter})$ is quite descriptive. However, since a path may also go through an edge in reverse direction, we define for each $e \in E$ an additional edge $-e \notin E$ with $\text{init}(-e) = \text{ter}(e)$ and $\text{ter}(-e) = \text{init}(e)$.

Definition 2.10 (Paths, connectivity, cycles, forests and trees). *Let* $\mathcal{G} = (V, E, \text{init}, \text{ter})$ *be a directed finite graph. An* r-tuple $e = (e_1, \ldots, e_r) \in (E \cup -E)^r$ *is called a* path *from* v *to* w, *if*

$$\text{init}(e_1), \ldots, \text{init}(e_r) \text{ are distinct,}$$
$$\text{ter}(e_i) = \text{init}(e_{i+1}) \quad \forall i \in \{1, \ldots, r-1\},$$
$$\text{init}(e_1) = v \wedge \text{ter}(e_r) = w.$$

4

A cycle *is a path from* v *to* v. *Two vertices* v, w *are* connected, *if there is a path from* v *to* w. *This gives is an equivalence relation on the vertex set. The induced subgraph on an equivalence class of connected vertices gives a* component *of the graph. A graph is called* connected, *if there is only one component.*
A subgraph $\mathcal{K} = (V, E', \mathrm{init}|_{E'}, \mathrm{ter}|_{E'})$ *of a directed graph* $\mathcal{G} = (V, E, \mathrm{init}, \mathrm{ter})$ *is called a* spanning forest *in* $\mathcal{G}$, *if* $\mathcal{K}$ *does not contain any cycles and is maximal with this property, that is,* $\mathcal{K}$ *is not a proper subgraph of a subgraph of* $\mathcal{G}$ *which does not contain any cycles. A subgraph* $\mathcal{K}$ *is called* tree, *if it is a forest and connected.*

In the context of electrical circuits, finite and loop-free directed graphs are of major importance. These allow to associate a special matrix [1, Sec. 3.2].

Definition 2.11 (Incidence matrix). *Let* $\mathcal{G} = (V, E, \mathrm{init}, \mathrm{ter})$ *be a finite and loop-free directed graph. Let* $E = \{e_1, \ldots, e_m\}$ *and* $V = \{v_1, \ldots, v_n\}$. *Then the* incidence matrix *of* $\mathcal{G}$ *is* $A_0 \in \mathbb{R}^{n \times m}$ *with*

$$
a_{jk} = \begin{cases} 1 & \mathrm{init}(e_k) = v_j, \\ -1 & \mathrm{ter}(e_k) = v_j, \\ 0 & otherwise. \end{cases}
$$

$\mathcal{G}$ has $k \in \mathbb{N}$ components if, and only if, $\mathrm{rk}\, A_0 = n - k$ [1, p. 140]. This allows to remove up to k rows from A_0 such that a matrix with same rank is obtained. The choice of these to-be-deleted rows has to be done in a special way: One has to choose a row set, which corresponds to a vertex set S that contains at most one vertex per component to $\mathcal{G}$. This deletion plays a crucial role in the following definition of a special Dirac structure and Lagrange submanifold.

Definition 2.12 (Kirchhoff-Dirac structure, Kirchhoff-Lagrange submanifold). *Assume that* $\mathcal{G} = (V, E, \mathrm{init}, \mathrm{ter})$ *is a finite and loop-free directed graph with incidence matrix* $A_0 \in \mathbb{R}^{n \times m}$. *Let* $\mathcal{G}_1, \ldots, \mathcal{G}_k$ *be the components of* $\mathcal{G}$ *and let* $V_1, \ldots, V_k \subset V$ *be the corresponding vertex sets. Let* $S \subset V$ *such that* S *contains at most one vertex form each component, that is*

$$
\forall\, s, s' \in S,\, i \le k: \quad v, v' \in V_i \Rightarrow v = v'. \tag{2}
$$

Let $A \in \mathbb{R}^{(n-k) \times m}$ *be constructed from* $A_0 \in \mathbb{R}^{n \times m}$ *by deleting the rows corresponding to the vertices from* S. *The* Kirchhoff-Dirac structure *of* $\mathcal{G}$ *is*

$$
\mathcal{D}_K^S(\mathcal{G}) := \left\{ (j, i, \phi, u) \in \mathbb{R}^{n-|S|} \times \mathbb{R}^m \times \mathbb{R}^{n-|S|} \times \mathbb{R}^m \;\middle|\; \begin{bmatrix} I & A \\ 0 & 0 \end{bmatrix} \begin{pmatrix} j \\ i \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ A^\top & -I \end{bmatrix} \begin{pmatrix} \phi \\ u \end{pmatrix} = 0 \right\}. \tag{3}
$$

Assume that $S = \{v_1, ..., v_{|S|}\}$ *(which is - by a reordering of the vertices - no loss of generality). Then the* Kirchhoff-Lagrange submanifold *of* $\mathcal{G}$ *with respect to* S *is*

$$
\mathcal{L}_K^S(\mathcal{G}) := \{0\} \times \mathbb{R}^{n-|S|} \subset \mathbb{R}^{n-|S|} \times \mathbb{R}^{n-|S|}. \tag{4}
$$

Remark 2.13. By Proposition 2.2, $\mathcal{D}_K^S(\mathcal{G})$ in (3) is a Dirac structure, whereas Proposition 2.5 implies that $\mathcal{L}_K^S(\mathcal{G})$ in (4) is a Lagrange submanifold of $\mathbb{R}^{n-|S|} \times \mathbb{R}^{n-|S|}$.
The concepts of Definition 2.12 allow to introduce the pH system $(\mathcal{D}_K^S(\mathcal{G}), \mathcal{L}_K^S(\mathcal{G}), \{0\})$ with dynamics

$$
(-\tfrac{d}{dt} q(t), i(t), \phi(t), u(t)) \in \mathcal{D}_K^S(\mathcal{G}), \quad (q(t), \phi(t)) \in \mathcal{L}_K^S(\mathcal{G}). \tag{5}
$$

Then, by the equivalence of $(q(t), \phi(t)) \in \mathcal{L}_K^S(\mathcal{G})$ to $q(t) = 0$ and $\phi(t) \in \mathbb{R}^{n-|S|}$, we see that (5) holds, if, and only if,

$$
q(t) = 0 \,\wedge\, Ai(t) = 0 \,\wedge\, A^\top \phi_2(t) = -u(t).
$$

In particular, $i(t) \in \ker A$ and $u(t) \in \mathrm{im}\, A^\top$. In the context of electrical circuits, this will indeed represent Kirchhoff's current and voltage law [18, Thm. 4.5 & Thm. 4.6]. The choice of S can be interpreted as the

set of *grounded vertices*. The quantities q, i, ϕ and u can respectively be thought as the vertex charges, the edge currents, the vertex potentials, and the edge voltages.

Note that (5) is indeed a pH system. However, this system is of rather pathological nature, since it does not contain any 'true dynamics', as the differential variable q is nulled by the Lagrange submanifold. Note that these 'true dynamics' come into play later on, when we interconnect with dynamic circuit elements like capacitances and inductances.

In the terminology of [24], $\mathcal{D}_K^S(\mathcal{G})$ corresponds to the Kirchhoff-Dirac structure of a graph with $|S| = \emptyset$. Moreover, a Dirac structure similar to (3) has been used in [21], with the main difference that in our present case all nodes are considered to be 'boundary nodes' in the nomenclature of [21].

We briefly present an alternative (slightly less straight-forward) construction of pH systems on graphs, namely by means of cycles instead to vertices. For a given spanning forest $\mathcal{T}$ of a loop-free directed graph $\mathcal{G}$ with n edges, m vertices and k connected components, the minimality property yields that the incorporation of any edge of $\mathcal{G}$ not belonging to $\mathcal{T}$ (called *chord*) results in a subgraph with exactly one cycle. Consequently, the set of edges in the complement of $\mathcal{T}$ in $\mathcal{G}$ leads to a set $\mathcal{C} = \{\mathcal{C}_1, \ldots, \mathcal{C}_{m-n+k}\}$ of cycles, the so-called *fundamental cycles* (see [1, p. 148] & [9, p. 26]). We equip each fundamental cycle with the orientation of its corresponding chord [1, p. 148] and consider the associated *fundamental cycle matrix* $B \in \mathbb{R}^{(m-n+k) \times m}$ which is defined entrywise by (cf. [1, Sec. 3.3])

$$
b_{jl} = \begin{cases} 1 & e_l \in C_j \text{ and the orientations agree,} \\ -1 & e_l \in C_j \text{ and the orientations do not agree,} \\ 0 & \text{otherwise.} \end{cases}
$$

This enables us to introduce the following Dirac structure and Lagrange submanifold

$$
\mathcal{D}_K'(\mathcal{G}) := \left\{ (\varphi, u, \iota, i) \in \mathbb{R}^{m-n+k} \times \mathbb{R}^m \times \mathbb{R}^{m-n+k} \times \mathbb{R}^m \; \middle| \; \begin{bmatrix} I & B \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \varphi \\ u \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ B^\top & -I \end{bmatrix} \begin{pmatrix} \iota \\ i \end{pmatrix} = 0 \right\},
$$
$$
\mathcal{L}_K'(\mathcal{G}) := \{0\} \times \mathbb{R}^{n-m+k},
$$

(6)

which form the pH system $(\mathcal{D}_K'(\mathcal{G}), \mathcal{L}_K'(\mathcal{G}), \{0\})$ with dynamics

$$
(-\tfrac{d}{dt}\psi(t), i(t), \iota(t), u(t)) \in \mathcal{D}_K'(\mathcal{G}), \quad (\psi(t), \iota(t)) \in \mathcal{L}_K'(\mathcal{G}), \tag{7}
$$

from which, analogous to Remark 2.13, one can derive that (7) is equivalent to $\psi(t) = 0$, $Bu(t) = 0$ and $i(t) = B^\top \iota$. Since $\operatorname{im} B = \ker A^\top$ [18, Thm. 4.4], the relations $u(t) \in \ker B = 0$ and $i(t) \in \operatorname{im} B^\top$ respectively represent Kirchhoff's voltage and current law. The quantities ψ, u, ι and i can respectively be thought as the cycle fluxes, the edge voltages, the cycle currents and the edge currents.

3. Electrical circuits as port-Hamiltonian systems

Our essential idea to port-Hamiltonian modelling of electrical circuits is to extend the tuple of voltages across and currents through the edges - in the case where we consider a vertex-based formulation of the Kirchhoff laws - by vertex charges and potentials, and - in the case where we consider a loop-based formulation of the Kirchhoff laws - by cycle fluxes and cycle currents, along with an accordant modelling of the graph interconnection structure by means of the approach in the preceding section. The electrical components are modelled by separate pH systems, and thereafter coupled with the one representing the interconnection structure.

The circuits may be composed of *two-terminal* and *multi-terminal* components. We will speak of ℓ_t-*terminal components*, with $\ell_t \in \mathbb{N}$ denoting the number of terminals [27]. Each ℓ_t-terminal component connects ℓ_t vertices of the electrical circuit through its terminals. For instance, a resistance has two terminals, whereas a transistor has three terminals, and a transformer has four terminals. To regard an electrical circuit as a graph (see Fig. 5), we need to replace the ℓ_t-terminal components by ℓ_p edges between the vertices they
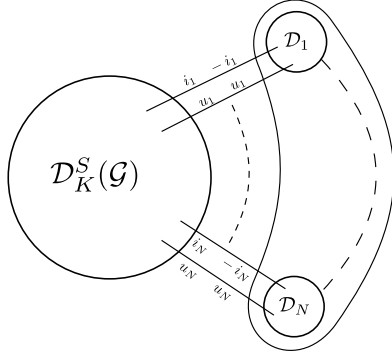
Figure 4: Visual representation of the Dirac structure $\mathcal{D}$ resulting from the interconnection (8).
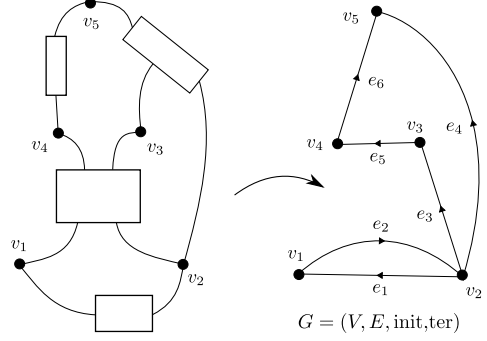

$$G = (V, E, \text{init}, \text{ter})$$

Figure 5: Obtaining the underlying graph of the electrical circuit.

connect, for some $\ell_p \in \mathbb{N}$, which we call the number of ports. Such a device is also called a ℓ_p-*port component*. This replacement is displayed in Fig. 5. The direction assigned to each edge is not a physical restriction but rather a definition of the *positive direction* of the corresponding voltage and current [18]. The physical properties of the electrical components will be reflected by port-Hamiltonian dynamics on these edges. The replacement of an ℓ_t-terminal component by ℓ_p edges between vertices, i.e., by a graph, is subject to physical modelling. For further details on terminals, ports and their relation, we refer to [27].

To be more precise, for $\ell_p, \ell_t \in \mathbb{N}$, an ℓ_t-*terminal component on ℓ_p edges* will be regarded as a pH system $(\mathcal{D}, \mathcal{L}, \mathcal{R})$, where $\mathcal{D} \subset \mathbb{R}^{n_S + n_R + \ell_p} \times \mathbb{R}^{n_S + n_R + \ell_p}$, with $\ell_p = n_S + n_R$ for some $n_S, n_R \in \mathbb{N}_0$. We associate to $\mathcal{D}$ a graph $\mathcal{G} = (V, E, \text{init}, \text{ter})$ with $|V| = \ell_t$ and $|E| = \ell_p$ (cf. Fig. 6). The external flow and effort variables will always represent the current through [18, Def. 3.2] and the voltage along [18, Def. 3.6] the corresponding edges, respectively.

3.1. Electrical circuits as interconnections of port-Hamiltonian systems

Let an electrical circuit consisting of N electrical components $(\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i)_{i \in \{1, \ldots, N\}}$, each with $\ell_{p,i}$ ports, be given, with $N \in \mathbb{N}$ and let $(\mathcal{G}_i)_{i \in \{1, \ldots, N\}} = (V_i, E_i, \text{init}_i, \text{ter}_i)_{i \in \{1, \ldots, N\}}$ be the respective graphs resulting from the physical modelling of the $\ell_{p,i}$-port components (see Fig. 6), where we assume that the edge sets $E_1, \ldots, E_N$ are disjoint. We define the *underlying graph of the circuit* $\mathcal{G}$ (see Fig. 5) as

$$\mathcal{G} = (V, E, \text{init}, \text{ter}) := \left(\bigcup_{i=1}^{N} V_i, \bigcup_{i=1}^{N} E_i, \text{init}, \text{ter} \right),$$

with $\text{init}(e) = \text{init}_i(e)$ and $\text{ter}(e) = \text{ter}_i(e)$ if $e \in E_i$ for some $i \in \{1, ..., N\}$ and let $V = \{v_1, \ldots, v_n\}$, $E = \{e_1, \ldots, e_m\}$ for some $n, m \in \mathbb{N}$. Further, let $A_0 \in \mathbb{R}^{n \times m}$ be the incidence matrix associated to $\mathcal{G}$ and let $S \subset V$ with property (2) represent the vertices grounded in the circuit. We model the dynamics of the electrical circuits as the dynamics of the pH system

$$(\mathcal{D}, \mathcal{L}, \mathcal{R}) := (\mathcal{D}_K^S(\mathcal{G}), \mathcal{L}_K^S(\mathcal{G}), \{0\}) \circ \left(\bigtimes_{i=1}^{N} (\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i) \right), \tag{8}$$

where the interconnection is performed with respect to the flow and effort spaces

$$(\mathcal{F}_{\text{link}}, \mathcal{E}_{\text{link}}) = \left(\bigtimes_{i=1}^{N} \mathbb{R}^{m_{P_i}}, \bigtimes_{i=1}^{N} \mathbb{R}^{m_{P_i}} \right) = (\mathbb{R}^m, \mathbb{R}^m)$$

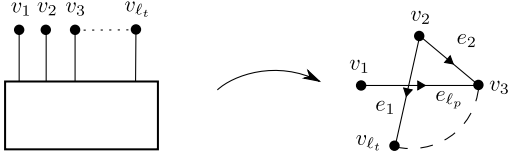corresponding to the port variables associated to the currents and voltages of the ℓ_p-port components.

Figure 6: Replacing an ℓ_t-terminal component by a graph with ℓ_p edges.

Figure 7: Deriving the underlying graph of a capacitance, conductance, ideal diode, PN-junction diode, inductance, resistance, or sources.
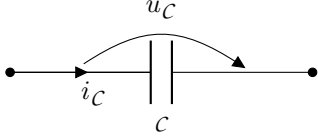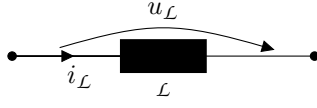

Figure 8: Capacitance: circuit symbol

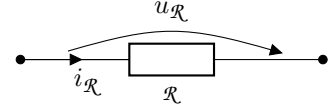Figure 9: Inductance: circuit symbol

Figure 10: Resistance/conductance: circuit symbol

3.2. Physical modelling of circuit components as port-Hamiltonian systems

We present a couple of 'prominent' electrical components from a port-Hamiltonian viewpoint; among them are capacitances, inductances, resistances, diodes, transformers, transistors and sources. Note that this list is by no means complete. In principle, our approach also allows to incorporate components which are modelled by partial differential equations, such as transmission lines and refined models of semiconductor devices. This involves a further generalization of pH systems on infinite-dimensional spaces and particularly leads to the notion of *Stokes-Dirac structure*, see [5, 16, 17].

Throughout this section, i will denote currents and u will denote voltages. An oftentimes used Dirac structure will be, for $\ell_p \in \mathbb{N}$,

$$\mathcal{D}_{\ell_p} = \left\{ \begin{pmatrix} -i \\ i \\ u \\ u \end{pmatrix} \in \mathbb{R}^{4\ell_p} \;\middle|\; i, u \in \mathbb{R}^{\ell_p} \right\}. \tag{9}$$

It can easily verified that this is indeed a Dirac structure. The variable i stands for the vector of currents, whereas u is the vector of voltages in the component. Note that a copy of the voltage and negative of the current vector is required, since it is later on eliminated by the interconnection according to Definition 2.8.

3.2.1. Capacitances

Let $H_C \in C^1(\mathbb{R}^{\ell_p}, \mathbb{R})$. A capacitance with ℓ_p ports is modelled as a pH system $(\mathcal{D}_C, \mathcal{L}_C, \mathcal{R}_C)$, where $\mathcal{D}_C = \mathcal{D}_{\ell_p}$ with $\mathcal{D}_{\ell_p}$ as in (9), $\mathcal{R}_C = \{0\}$, and

$$\mathcal{L}_C = \left\{ (u_C, q_C) \in \mathbb{R}^{2\ell_p} \,\middle|\, q_C = \nabla H_C(u_C) \right\}.$$

The dynamics consequently read

$$(-\tfrac{d}{dt} q_C(t), i_C(t), u_C(t), u_C(t)) \in \mathcal{D}_C, \quad (q_C(t), u_C(t)) \in \mathcal{L}_C.$$

Here, q_C represents the *charge* of the capacitance and the *Hamiltonian* H_C represents the *energy storage function* of the system. From this pH system, one can derive

$$i_C(t) = \tfrac{d}{dt} q_C(t), \quad u_C(t) = \nabla H_C(q_C(t)).$$

If the capacitance has two terminals, then we obtain a conventional capacitance with one port as in Fig. 7.

3.2.2. Inductances

Let $H_L \in C^1(\mathbb{R}^{\ell_p}, \mathbb{R})$. An inductance with ℓ_p ports is modelled as a pH system $(\mathcal{D}_L, \mathcal{L}_L, \mathcal{R}_L)$ with

$$\mathcal{D}_L = \left\{ \begin{pmatrix} -u_L \\ i_L \\ i_L \\ u_L \end{pmatrix} \in \mathbb{R}^{4\ell_p} \;\middle|\; u_L, i_L \in \mathbb{R}^{\ell_p} \right\}$$
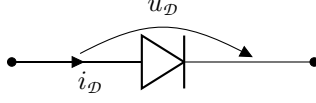
8

Figure 11: Circuit symbol of a diode.

and

$$\mathcal{L}_L = \left\{ (\psi_L, i_L) \in \mathbb{R}^{2\ell_p} \mid i_L = \nabla H_L(\psi_L) \right\}, \quad \mathcal{R}_L = \{0\}.$$

The dynamics are now given by

$$(-\tfrac{d}{dt}\psi_L(t), i_L(t), i_L(t), u_L(t)) \in \mathcal{D}_L, \quad (\psi_L(t), i_L(t)) \in \mathcal{L}_L,$$

Here, ψ_L represents the *magnetic flux* of the inductance and the *Hamiltonian* $H_L \in C^1(\mathbb{R}^{\ell_p}, \mathbb{R})$ represents the *energy storage function* of the system. From this pH system, one can derive

$$u_L(t) = \tfrac{d}{dt}\psi_L(t), \quad i_L(t) = \nabla H_L(\psi_L(t)).$$

If the inductance has two terminals, then we obtain a conventional inductance with one port as in Fig. 7.

3.2.3. Conductances and resistances

Let $\mathcal{R}_{\mathcal{R}} \subset \mathbb{R}^{\ell_p} \times \mathbb{R}^{\ell_p}$ be a resistive relation. Consider the pH system $(\mathcal{D}_{\mathcal{R}}, \mathcal{L}_{\mathcal{R}}, \mathcal{R}_{\mathcal{R}})$, where $\mathcal{D}_{\mathcal{R}} = \mathcal{D}_{\ell_p}$ with $\mathcal{D}_{\ell_p}$ as in (9), $\mathcal{L}_{\mathcal{R}} = \{0\}$. The dynamics are specified by

$$(-i_{\mathcal{R}}(t), i_{\mathcal{R}}(t), u_{\mathcal{R}}(t), u_{\mathcal{R}}(t)) \in \mathcal{D}_{\mathcal{R}}, \quad (-i_{\mathcal{R}}(t), u_{\mathcal{R}}(t)) \in \mathcal{R}_{\mathcal{R}}, \tag{10}$$

If, for some accretive function $g : \mathbb{R}^{\ell_p} \to \mathbb{R}^{\ell_p}$ (that is, $\phi_{\mathcal{R}}^\top g(\phi_{\mathcal{R}}) \geq 0$ for all $\phi_{\mathcal{R}} \in \mathbb{R}^{\ell_p}$), $\mathcal{R}_{\mathcal{R}}$ reads

$$\mathcal{R}_{\mathcal{R}} = \left\{ (-i_{\mathcal{R}}, u_{\mathcal{R}}) \in \mathbb{R}^{2\ell_p} \mid i_{\mathcal{R}} = g(u_{\mathcal{R}}) \right\},$$

then (10) leads to $i_{\mathcal{R}}(t) = g(u_{\mathcal{R}}(t))$. That is, $(\mathcal{D}_{\mathcal{R}}, \mathcal{L}_{\mathcal{R}}, \mathcal{R}_{\mathcal{R}})$ describes a conductance with ℓ_p ports. On the other hand, if for some accretive function $r : \mathbb{R}^{\ell_p} \to \mathbb{R}^{\ell_p}$,

$$\mathcal{R}_{\mathcal{R}} = \left\{ (-i_{\mathcal{R}}, u_{\mathcal{R}}) \in \mathbb{R}^{2\ell_p} \mid u_{\mathcal{R}} = r(i_{\mathcal{R}}) \right\}$$

then (10) leads to $u_{\mathcal{R}}(t) = r(i_{\mathcal{R}}(t))$, i.e. $(\mathcal{D}_{\mathcal{R}}, \mathcal{L}_{\mathcal{R}}, \mathcal{R}_{\mathcal{R}})$ models a resistance with ℓ_p ports.

If the conductance/resistance has two terminals, then we obtain a conventional conductance/resistance with one port as in Fig. 10.

Remark 3.1. Resistances form a pathological case of a pH system, since the underlying Lagrange submanifold is trivial (cf. Remark 2.13). Therefore, the 'dynamics' of the pH system are actually 'statics'. The same holds for the models diodes, transformers and transistors which are discussed in the sequel.

3.2.4. Ideal and PN-junction diodes

An *ideal diode* is modelled as a two-terminal component $(\mathcal{D}_{\mathcal{D}}, \mathcal{L}_{\mathcal{D}}, \mathcal{R}_{\mathcal{D}})$ with one port (see Fig. 7), and dynamics

$$(-i_{\mathcal{D}}(t), i_{\mathcal{D}}(t), u_{\mathcal{D}}(t), u_{\mathcal{D}}(t)) \in \mathcal{D}_{\mathcal{D}}, \quad (j_{\mathcal{D}}(t), \phi_{\mathcal{D}}(t)) \in \mathcal{R}_{\mathcal{D}},$$

where $\mathcal{D}_{\mathcal{D}} = \mathcal{D}_1$ with $\mathcal{D}_1$ as defined in (9), $\mathcal{L}_{\mathcal{D}} = \{0\}$ and

$$\mathcal{R}_{\mathcal{D}} = \left\{ (-i_{\mathcal{D}}, u_{\mathcal{D}}) \in \mathbb{R}^2 \mid i_{\mathcal{D}} u_{\mathcal{D}} = 0 \wedge i_{\mathcal{D}} \leq 0 \wedge u_{\mathcal{D}} \leq 0 \right\}.$$

From this pH system, one can derive that

$$(i_{\mathcal{D}}(t), u_{\mathcal{D}}(t)) \in (\{0\} \times \mathbb{R}_{\leq 0}) \cup (\mathbb{R}_{\geq 0} \times \{0\}).$$
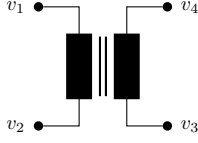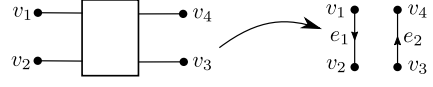
9

Figure 12: Circuit symbol of a transformer.

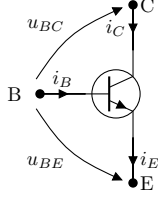Figure 13: Deriving the underlying graph of a transformer.
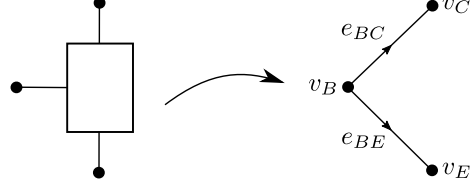

Figure 14: Circuit symbol of a NPN transistor.

Figure 15: Deriving the underlying graph of an NPN transistor.

A *PN-junction diode* is modelled as a one-port component $(\mathcal{D}_{\mathcal{D}}, \mathcal{L}_{\mathcal{D}}, \mathcal{R}_{\mathcal{D}})$ with $\mathcal{D}_{\mathcal{D}}$ and $\mathcal{L}_{\mathcal{D}}$ as for the ideal diode, and the resistive relation is, for some constants $a, b > 0$, given by

$$\mathcal{R}_{\mathcal{D}} = \left\{ (-i_{\mathcal{D}}, u_{\mathcal{D}}) \in \mathbb{R}^2 \mid i_{\mathcal{D}} = a \left(e^{\frac{u_{\mathcal{D}}}{b}} - 1 \right) \right\}.$$

From the dynamics of this pH system, one can derive the characteristic equation [14, Eq. (39.46)]

$$i_{\mathcal{D}}(t) = a \left(e^{\frac{u_{\mathcal{D}}(t)}{b}} - 1 \right).$$

The PN-junction diode serves as an approximation for an ideal diode. In a certain sense, the behavior of a PN-junction diode indeed tends to that of the ideal diode, if $b \to 0$.

3.2.5. Transformers

A transformer is modelled as a four-terminal component with two ports, see Fig. 13. It is described by the pH system $(\mathcal{D}_{\mathcal{T}}, \mathcal{L}_{\mathcal{T}}, \mathcal{R}_{\mathcal{T}})$, where we use the Dirac structure $\mathcal{D}_{\mathcal{T}} = \mathcal{D}_2$ with $\mathcal{D}_2$ as defined in (9) and trivial Lagrange submanifold $\mathcal{L}_{\mathcal{T}} = \{0\}$. The dynamics are given by

$$(-i_{\mathcal{T}1}(t), -i_{\mathcal{T}2}(t), i_{\mathcal{T}1}(t), i_{\mathcal{T}2}(t), u_{\mathcal{T}1}(t), u_{\mathcal{T}2}(t), u_{\mathcal{T}1}(t), u_{\mathcal{T}2}(t)) \in \mathcal{D}_{\mathcal{T}},$$
$$(-i_{\mathcal{T}1}(t), -i_{\mathcal{T}2}(t), u_{\mathcal{T}1}(t), u_{\mathcal{T}2}(t)) \in \mathcal{R}_{\mathcal{T}},$$

with, for some $T \in \mathbb{R}$,

$$\mathcal{R}_{\mathcal{T}} = \left\{ (-i_{\mathcal{T}1}, -i_{\mathcal{T}2}, u_{\mathcal{T}1}, u_{\mathcal{T}2}) \in \mathbb{R}^4 \mid T i_{\mathcal{T}1} = -i_{\mathcal{T}2}, \ u_{\mathcal{T}1} = T u_{\mathcal{T}2} \right\}.$$

From this pH system, one can derive $T i_{\mathcal{T}1}(t) = -i_{\mathcal{T}2}(t)$ and $u_{\mathcal{T}1}(t) = T u_{\mathcal{T}2}(t)$, which means that a transformer is a power-conserving component.

3.2.6. NPN transistors

A transistor is a component with three terminals, which are called *emitter*, *basis* and *collector*. We replace this by a graph with two edges, which are respectively located are between basis and collector, and basis and emitter, see Fig. 15. The behavior of a transistor *of type NPN* is often modelled by the *Ebers-Moll model*

[20, Eqs. (5.26) & (5.27)], which can, in a certain voltage and current range around zero, be summarized by the equations

$$i_C(t) = i_S\left(e^{\frac{u_{BE}(t)}{V_T}} - 1\right) - \frac{i_S}{\alpha_R}\left(e^{\frac{u_{BC}(t)}{V_T}} - 1\right),$$

$$i_E(t) = \frac{i_S}{\alpha_F}\left(e^{\frac{u_{BE}(t)}{V_T}} - 1\right) - i_S\left(e^{\frac{u_{BC}(t)}{V_T}} - 1\right),$$

(11)

for some constants $\alpha_F \in \left[\frac{50}{51}, \frac{1000}{1001}\right]$, $\alpha_R \in \left[\frac{1}{100}, \frac{1}{2}\right]$, $i_S \in [10^{-15}, 10^{-12}]$, $V_T \approx \frac{1}{40}$ [20, pp. 382-394]. Hereby, $i_C(t)$, $i_E(t)$, $u_{BE}(t)$, $u_{BC}(t)$ respectively denote the collector current, emitter current, basis-emitter voltage and basis collector voltage. Note that, by the Kirchhoff laws, the basis current fulfills $i_B(t) = i_E(t) - i_C(t)$ and the collector emitter voltage is given by $u_{CE}(t) = u_{BE}(t) - u_{BC}(t)$. We model an NPN transistor as a 'resistive' two-port component $(\mathcal{D}_{\mathcal{N}}, \mathcal{L}_{\mathcal{N}}, \mathcal{R}_{\mathcal{N}})$ on two edges, where $\mathcal{D}_{\mathcal{N}} = \mathcal{D}_2$ with $\mathcal{D}_2$ as defined in (9), $\mathcal{L}_{\mathcal{N}} = \{0\}$ and

$$\mathcal{R}_{\mathcal{N}} = \left\{ (i_C, -i_E, u_{BC}, u_{BE}) \in \mathbb{R}^4 \,\middle|\, \begin{matrix} i_C = i_S\left(e^{\frac{u_{BE}}{V_T}} - 1\right) - \frac{i_S}{\alpha_R}\left(e^{\frac{u_{BC}}{V_T}} - 1\right), \\ i_E = \frac{i_S}{\alpha_F}\left(e^{\frac{u_{BE}}{V_T}} - 1\right) - i_S\left(e^{\frac{u_{BC}}{V_T}} - 1\right), \end{matrix} \right\} \cap U_0,$$

where $U_0 \subset \mathbb{R}^4$ is a neighborhood of the origin. The dynamics of the system read

$$(i_C(t), -i_E(t), -i_C(t), i_E(t), u_{BC}(t), u_{BE}(t), u_{BC}(t), u_{BE}(t)) \in \mathcal{D}_{\mathcal{N}}, \ (i_C(t), -i_E(t), u_{BC}(t), u_{BE}(t)) \in \mathcal{R}_{\mathcal{N}},$$

which implies (11), at least as long as $(i_C(t), -i_E(t), u_{BC}(t), u_{BE}(t)) \in U_0$. Note that we have provided the collector current $i_C(t)$ with another sign, since it is - in contrast to the emitter current and the basis-emitter current - directed contrarily to the basis-collector current.

Note that, if we choose $U_0 = \mathbb{R}^4$, then the relation $\mathcal{R}_{\mathcal{N}}$ is *not* resistive, since for there may exist quadruples $(i_C, -i_E, u_{BC}, u_{BE}) \in \mathcal{R}_{\mathcal{N}}$ holds $i_C u_{BC} - i_E u_{BE} > 0$. However, we can show that $\mathcal{R}_{\mathcal{N}}$ is resistive for a suitable neighborhood $U_0 \subset \mathbb{R}^4$ of the origin. This can be seen as follows: Since for $(u_{BC}, u_{BE}) \in (\mathbb{R}\setminus\{0\})^2$ holds

$$\begin{pmatrix} u_{BC} \\ u_{BE} \end{pmatrix}^\top \begin{pmatrix} -\frac{i_S}{\alpha_R}\left(e^{\frac{u_{BC}}{V_T}} - 1\right) + i_S\left(e^{\frac{u_{BE}}{V_T}} - 1\right) \\ i_S\left(e^{\frac{u_{BC}}{V_T}} - 1\right) - \frac{i_S}{\alpha_F}\left(e^{\frac{u_{BE}}{V_T}} - 1\right) \end{pmatrix}$$

$$= \begin{pmatrix} u_{BC} \\ u_{BE} \end{pmatrix}^\top \underbrace{\left[\begin{matrix} -\frac{i_S}{\alpha_R u_{BC}}\left(e^{\frac{u_{BC}}{V_T}} - 1\right) & \frac{i_S}{u_{BE}}\left(e^{\frac{u_{BE}}{V_T}} - 1\right) \\ \frac{i_S}{u_{BC}}\left(e^{\frac{u_{BC}}{V_T}} - 1\right) & -\frac{i_S}{\alpha_F u_{BE}}\left(e^{\frac{u_{BE}}{V_T}} - 1\right) \end{matrix} \right]}_{=:A(u_{BC}, u_{BE})} \begin{pmatrix} u_{BC} \\ u_{BE} \end{pmatrix}.$$

Namely, by using that $A(\cdot, \cdot)$ has a continuous extension to $\mathbb{R}^2$ with

$$A(0,0) = \frac{i_S}{V_T} \cdot \left[\begin{matrix} -\frac{1}{\alpha_R} & 1 \\ 1 & -\frac{1}{\alpha_F} \end{matrix} \right].$$

By $\alpha_F \in \left[\frac{50}{51}, \frac{1000}{1001}\right]$, $\alpha_R \in \left[\frac{1}{100}, \frac{1}{2}\right]$, we have $\alpha_F \cdot \alpha_R < 1$, which leads to negative definiteness of $A(0,0) = \frac{1}{2}(A(0,0) + A(0,0)^\top)$. The continuity of $(u_{BC}, u_{BE}) \mapsto \frac{1}{2}(A(u_{BC}, u_{BE}) + A(u_{BC}, u_{BE})^\top)$ implies that there exists some neighborhood $U_0 \subset \mathbb{R}^4$ such that this function takes values in the cone of negative definite matrices on U_0. This consequences that, by taking this neighborhood U_0, $\mathcal{R}_{\mathcal{N}}$ is a resistive relation.

3.2.7. Current and voltage sources

The sources of the electrical circuit represent the ports of the system, that is points at which physical interaction of the electrical circuit with the environment happens. We may distinguish two types of sources:
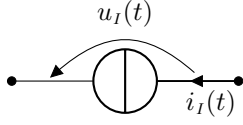
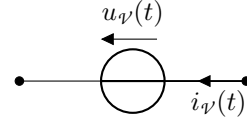Figure 16: Circuit symbol of a current source.
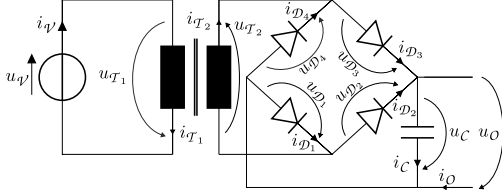

Figure 17: Circuit symbol of a voltage source.
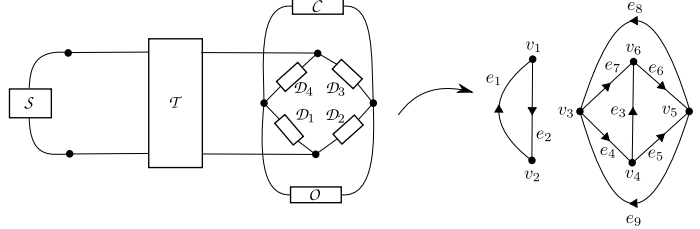

Figure 18: AC/DC converter circuit

Figure 19: Obtaining the underlying graph of the AC/DC converter

current sources and *voltage sources*, see Fig. 16 and Fig. 17. The name indicates which physical variable is controlled or influenced by the environment. This variable is also denoted as *input*, while the other is denoted as *output*. However, this distinction is not relevant for the *geometrical* formulation of pH systems (cf. [15]). We unite both classes under the term *sources*. These have two terminals, and, consequently, one port (see Fig. 7). Sources are modelled as a pH system $(\mathcal{D}_\mathcal{S}, \mathcal{L}_\mathcal{S}, \mathcal{R}_\mathcal{S})$, where the Dirac structure is $\mathcal{D}_\mathcal{S} = \mathcal{D}_1$ with $\mathcal{D}_1$ as defined in (9), and the Lagrange submanifold and resistive relation are trivial, i.e., $\mathcal{L}_\mathcal{S} = \mathcal{R}_\mathcal{S} = \{0\}$. The dynamics are

$$(-i_\mathcal{S}(t), i_\mathcal{S}(t), u_\mathcal{S}(t), u_\mathcal{S}(t)) \in \mathcal{D}_\mathcal{S}.$$

Example 3.2 (AC/DC converter). We illustrate our methodology by considering an AC/DC converter, which we model by the electrical circuit shown in Fig. 18. The AC/DC converter consists of a source $\mathcal{S} = (\mathcal{D}_\mathcal{S}, \mathcal{L}_\mathcal{S}, \mathcal{R}_\mathcal{S})$, a transformer $\mathcal{T} = (\mathcal{D}_\mathcal{T}, \mathcal{L}_\mathcal{T}, \mathcal{R}_\mathcal{T})$, four PN-junction diodes $\mathcal{I}_i = (\mathcal{D}_{\mathcal{I}_i}, \mathcal{L}_{\mathcal{I}_i}, \mathcal{R}_{\mathcal{I}_i})$ for $i \in \{1, ..., 4\}$, a capacitor $\mathcal{C} = (\mathcal{D}_\mathcal{C}, \mathcal{L}_\mathcal{C}, \mathcal{R}_\mathcal{C})$, and a 'sink' $\mathcal{O} = (\mathcal{D}_\mathcal{O}, \mathcal{L}_\mathcal{O}, \mathcal{R}_\mathcal{O})$ (modelled like a source), which are connected by the vertices $v_1, ..., v_6$ as shown in Fig. 19. The circuit graph $\mathcal{G} = (V, E, \text{init}, \text{ter})$ with $V = \{v_1, ..., v_6\}$ and $E = \{e_1, ..., e_9\}$ has two components, and we ground the nodes below the voltage source and the capacitance, i.e., we choose $S = \{v_2, v_3\}$. Let $A \in \mathbb{R}^{4 \times 9}$ be obtained from the incidence matrix of $\mathcal{G}$ by deleting the rows corresponding to the grounded nodes. We arrive at a pH system $(\mathcal{D}, \mathcal{L}, \mathcal{R})$ as in (8), whose dynamics read

$$\left(\frac{d}{dt} \begin{pmatrix} -q_1 \\ -q_4 \\ -q_5 \\ -q_6 \\ -q_\mathcal{C} \end{pmatrix}, \begin{pmatrix} -i_{\mathcal{T}1} \\ -i_{\mathcal{T}2} \\ -i_{\mathcal{D}1} \\ -i_{\mathcal{D}2} \\ -i_{\mathcal{D}3} \\ -i_{\mathcal{D}4} \end{pmatrix}, \begin{pmatrix} -i_\mathcal{V} \\ -i_\mathcal{O} \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ u_\mathcal{C} \end{pmatrix}, \begin{pmatrix} u_{\mathcal{T}1} \\ u_{\mathcal{T}2} \\ u_{\mathcal{D}1} \\ u_{\mathcal{D}2} \\ u_{\mathcal{D}3} \\ u_{\mathcal{D}4} \end{pmatrix}, \begin{pmatrix} u_\mathcal{V} \\ u_\mathcal{O} \end{pmatrix} \right) \in \mathcal{D},$$

$$\left(\begin{pmatrix} -q_1 \\ -q_4 \\ -q_5 \\ -q_6 \\ -q_\mathcal{C} \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ u_\mathcal{C} \end{pmatrix} \right) \in \mathcal{L}, \qquad \left(\begin{pmatrix} -i_{\mathcal{T}1} \\ -i_{\mathcal{T}2} \\ -i_{\mathcal{D}1} \\ -i_{\mathcal{D}2} \\ -i_{\mathcal{D}3} \\ -i_{\mathcal{D}4} \end{pmatrix}, \begin{pmatrix} u_{\mathcal{T}1} \\ u_{\mathcal{T}2} \\ u_{\mathcal{D}1} \\ u_{\mathcal{D}2} \\ u_{\mathcal{D}3} \\ u_{\mathcal{D}4} \end{pmatrix} \right) \in \mathcal{R}.$$

4. Comparison with other formulations of electrical circuits

With the attention electrical circuits attracted over the past decades, quite a bunch of 'standard formulations' of the dynamics have emerged. An overview of popular models in the context of DAEs is found in [19]. We compare for certain electrical circuits the dynamics of our port-Hamiltonian modelling (8) with other equations used in the modelling of electrical circuits.

4.1. The (charge/flux-oriented) modified nodal analysis

Let an electrical circuit consisting of conductances, inductances, capacitances and sources. Let

$$(\mathcal{D}_{\mathcal{R}_i}, \mathcal{L}_{\mathcal{R}_i}, \mathcal{R}_{\mathcal{R}_i})_{i \in \{1,\dots,l_{\mathcal{R}}\}}, \qquad (\mathcal{D}_{L_i}, \mathcal{L}_{L_i}, \mathcal{R}_{L_i})_{i \in \{1,\dots,l_L\}},$$
$$(\mathcal{D}_{C_i}, \mathcal{L}_{C_i}, \mathcal{R}_{C_i})_{i \in \{1,\dots,l_C\}}, \qquad (\mathcal{D}_{S_i}, \mathcal{L}_{S_i}, \mathcal{R}_{S_i})_{i \in \{1,\dots,l_S\}}.$$

be the pH systems modelling the components as derived in Section 3.1. Let $\ell_{p,\mathcal{R}i}$ be the number of ports of the component modelled by $(\mathcal{D}_{\mathcal{R}_i}, \mathcal{L}_{\mathcal{R}_i}, \mathcal{R}_{\mathcal{R}_i})$, and let $\ell_{p,Li}$ and $\ell_{p,Ci}$ be analogously defined. Moreover, let

$$m_{\mathcal{R}} = \sum_{i=1}^{l_{\mathcal{R}}} \ell_{p,\mathcal{R}i}, \quad m_L = \sum_{i=1}^{l_{\mathcal{R}}} \ell_{p,Li}, \quad m_C = \sum_{i=1}^{l_C} \ell_{p,\mathcal{R}i}, \quad m_S = l_S,$$

and introduce

$$i_{\mathcal{R}} = \begin{pmatrix} i_{\mathcal{R}1} \\ \vdots \\ i_{\mathcal{R}m_{\mathcal{R}}} \end{pmatrix}, \; i_L = \begin{pmatrix} i_{L1} \\ \vdots \\ i_{Lm_L} \end{pmatrix}, \; i_C = \begin{pmatrix} i_{C1} \\ \vdots \\ i_{Cm_C} \end{pmatrix}, \; i_S = \begin{pmatrix} i_{S1} \\ \vdots \\ i_{Sm_S} \end{pmatrix}, \; i = \begin{pmatrix} i_{\mathcal{R}} \\ i_L \\ i_C \\ i_S \end{pmatrix},$$

and analogous notations for $u_{\mathcal{R}}$, u_L, u_C, u_S, u, as well as

$$q_C = \begin{pmatrix} q_{C1} \\ \vdots \\ q_{Cm_C} \end{pmatrix}, \quad \psi_L = \begin{pmatrix} \psi_{L1} \\ \vdots \\ \psi_{Lm_L} \end{pmatrix}, \quad g(u_{\mathcal{R}}) = \begin{pmatrix} g_1(u_{\mathcal{R}1}) \\ \vdots \\ g_{m_{\mathcal{R}}}(u_{\mathcal{R}m_{\mathcal{R}}}) \end{pmatrix},$$

$$H_C(q_C) = \sum_{i=1}^{m_C} H_{Ci}(q_{Ci}), \quad H_L(\psi_L) = \sum_{i=1}^{m_L} H_{Li}(\psi_{Li}).$$

Further, let $\mathcal{G} = (V, E, \text{init}, \text{ter})$ be the graph induced by the electrical circuit with $|V| = n$ and $|E| = m$. Let S be the set of grounded vertices (cf. Definition 2.12), and let $A \in \mathbb{R}^{(n-|S|) \times m}$ be obtained from the incidence matrix of $\mathcal{G}$ by deleting the rows corresponding to the vertices in S. By a suitable reordering, we may sort into edges to the specific components, i.e.,

$$A = \begin{bmatrix} A_{\mathcal{R}} & A_L & A_C & A_S \end{bmatrix},$$

where the columns of $A_{\mathcal{R}} \in \mathbb{R}^{(n-|S|) \times m_{\mathcal{R}}}$, $A_L \in \mathbb{R}^{(n-|S|) \times m_L}$, $A_C \in \mathbb{R}^{(n-|S|) \times m_C}$ and $A_S \in \mathbb{R}^{(n-|S|) \times m_S}$ respectively represent the edges corresponding to conductances, inductances, capacitances and sources. For the representation of the port-Hamiltonian dynamics of the electrical circuit, first note that the Dirac structure of the pH system

$$\underset{i=1}{\overset{m_{\mathcal{R}}}{\times}} (\mathcal{D}_{\mathcal{R}_i}, \mathcal{L}_{\mathcal{R}_i}, \mathcal{R}_{\mathcal{R}_i}) \times \underset{i=1}{\overset{m_L}{\times}} (\mathcal{D}_{L_i}, \mathcal{L}_{L_i}, \mathcal{R}_{L_i}) \times \underset{i=1}{\overset{m_C}{\times}} (\mathcal{D}_{C_i}, \mathcal{L}_{C_i}, \mathcal{R}_{C_i}) \times \underset{i=1}{\overset{m_S}{\times}} (\mathcal{D}_{S_i}, \mathcal{L}_{S_i}, \mathcal{R}_{S_i})$$

is given by

$$
\mathcal{D}_{\mathrm{prod}} = \Bigg\{ \big(-i_L, -i_C, -i_{\mathcal{R}}, i_{\mathcal{R}}, i_L, i_C, -i_S, i_S, i_L, u_C, u_{\mathcal{R}}, u_{\mathcal{R}}, u_L, u_C, u_S, u_S\big) \in \mathbb{R}^{2m} \times \mathbb{R}^{2m} \mid
$$
$$
i_L, u_L \in \mathbb{R}^{m_L}, \ i_C, u_C \in \mathbb{R}^{m_C}, \ i_{\mathcal{R}}, u_{\mathcal{R}} \in \mathbb{R}^{m_{\mathcal{R}}}, \ i_S, u_S \in \mathbb{R}^{m_S} \Bigg\}
$$

and

$$
\mathcal{D}_K^S(\mathcal{G}) = \Bigg\{ \big(j, i_{\mathcal{R}}, i_L, i_C, i_S, \phi, u_{\mathcal{R}}, u_L, u_C, u_S\big) \in \mathbb{R}^{n-|S|} \times \mathbb{R}^m \times \mathbb{R}^{n-|S|} \times \mathbb{R}^m \mid
$$
$$
\begin{bmatrix} I & A_{\mathcal{R}} & A_L & A_C & A_S \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} j \\ i_{\mathcal{R}} \\ i_L \\ i_C \\ i_S \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -A_{\mathcal{R}}^\top & I & 0 & 0 & 0 \\ -A_L^\top & 0 & I & 0 & 0 \\ -A_C^\top & 0 & 0 & I & 0 \\ -A_S^\top & 0 & 0 & 0 & I \end{bmatrix} \begin{pmatrix} \phi \\ u_{\mathcal{R}} \\ u_L \\ u_C \\ u_S \end{pmatrix} = 0 \Bigg\}.
$$

It follows that the Dirac structure of

$$
(\mathcal{D}_K^S(\mathcal{G}), \mathcal{L}_K^S(\mathcal{G}), \{0\}) \circ \left(\mathop{\text{\huge$\times$}}_{i=1}^N (\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i) \right)
$$

is given by

$$
\mathcal{D} = \Bigg\{ \big(j, -u_L, -i_C, -i_{\mathcal{R}}, -i_S, \phi, i_L, u_C, u_{\mathcal{R}}, u_S\big) \in \mathbb{R}^{n-|S|} \times \mathbb{R}^m \times \mathbb{R}^{n-|S|} \times \mathbb{R}^m \mid
$$
$$
\begin{bmatrix} I & 0 & A_C & A_{\mathcal{R}} & A_S \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} j \\ -u_L \\ -i_C \\ -i_{\mathcal{R}} \\ -i_S \end{pmatrix} + \begin{bmatrix} 0 & -A_L & 0 & 0 & 0 \\ -A_{\mathcal{R}}^\top & 0 & 0 & I & 0 \\ -A_L^\top & 0 & 0 & 0 & 0 \\ -A_C^\top & 0 & I & 0 & 0 \\ -A_S^\top & 0 & 0 & 0 & I \end{bmatrix} \begin{pmatrix} \phi \\ i_L \\ u_C \\ u_{\mathcal{R}} \\ u_S \end{pmatrix} = 0 \Bigg\}, \tag{12a}
$$

whereas the Lagrange submanifold and resistive relation read

$$
\mathcal{L} = \Bigg\{ \big(q, \psi_L, q_C, \phi, i_L, u_C\big) \in \mathbb{R}^{n-|S|} \times \mathbb{R}^{m_L} \times \mathbb{R}^{m_C} \times \mathbb{R}^{n-|S|} \times \mathbb{R}^{m_L} \times \mathbb{R}^{m_C} \mid
$$
$$
q = 0 \wedge i_L = \nabla H_L(\psi_L) \wedge u_C = \nabla H_C(q_C) \Bigg\}, \tag{12b}
$$

$$
\mathcal{R} = \Big\{ \big(-i_{\mathcal{R}}, u_{\mathcal{R}}\big) \in \mathbb{R}^{m_{\mathcal{R}}} \times \mathbb{R}^{m_{\mathcal{R}}} \mid i_{\mathcal{R}} = g(u_{\mathcal{R}}) \Big\}. \tag{12c}
$$

The triple $(\mathcal{D}, \mathcal{L}, \mathcal{R})$ with $\mathcal{D}$, $\mathcal{L}$ and $\mathcal{R}$ as in (12) is the port-Hamiltonian representation of a circuit with conductances, inductances, capacitances and sources in a compact form. The dynamics of $(\mathcal{D}, \mathcal{L}, \mathcal{R})$ read

$$
(-\tfrac{d}{dt}q(t), -\tfrac{d}{dt}\psi_L(t), -\tfrac{d}{dt}q_C(t), -i_{\mathcal{R}}(t), -i_S(t), \phi(t), i_L(t), u_C(t), u_{\mathcal{R}}(t), u_S(t)) \in \mathcal{D},
$$
$$
(q(t), \psi_L(t), q_C(t), \phi(t), i_L(t), u_C(t)) \in \mathcal{L}, \quad (-i_{\mathcal{R}}(t), e_{\mathcal{R}}(t)) \in \mathcal{R},
$$

which is equivalent to

$$
\begin{bmatrix} I & 0 & A_C & A_{\mathcal{R}} & A_S \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} -\tfrac{d}{dt}q(t) \\ -\tfrac{d}{dt}\psi_L(t) \\ -\tfrac{d}{dt}q_C(t) \\ -i_{\mathcal{R}}(t) \\ -i_S(t) \end{pmatrix} + \begin{bmatrix} 0 & -A_L & 0 & 0 & 0 \\ -A_{\mathcal{R}}^\top & 0 & 0 & I & 0 \\ -A_L^\top & 0 & 0 & 0 & 0 \\ -A_C^\top & 0 & I & 0 & 0 \\ -A_S^\top & 0 & 0 & 0 & I \end{bmatrix} \begin{pmatrix} \phi(t) \\ i_L(t) \\ u_C(t) \\ u_{\mathcal{R}}(t) \\ u_S(t) \end{pmatrix} = 0,
$$

$$
q(t) = 0, \ \ i_L(t) = \nabla H_L(\psi_L(t)), \ \ u_C(t) = \nabla H_C(q_C(t)), \ \ i_{\mathcal{R}}(t) = g(u_{\mathcal{R}}(t)).
$$

Plugging in the latter relations, we obtain

$$A_{\mathcal{C}} \tfrac{d}{dt} q_{\mathcal{C}}(t) + A_{\mathcal{R}} g(A_{\mathcal{R}}^{\top} \phi(t)) + A_{\mathcal{L}} i_{\mathcal{L}}(t) + A_{\mathcal{S}} i_{\mathcal{S}}(t) = 0,$$
$$-A_{\mathcal{L}}^{\top} \phi(t) + \tfrac{d}{dt} \psi_{\mathcal{L}}(t) = 0,$$
$$-A_{\mathcal{S}}^{\top} \phi(t) + u_{\mathcal{S}}(t) = 0, \tag{13}$$
$$A_{\mathcal{C}}^{\top} \phi(t) - \nabla H_{\mathcal{C}}(q_{\mathcal{C}}(t)) = 0,$$
$$i_{\mathcal{L}}(t) - \nabla H_{\mathcal{L}}(\psi(t)) = 0.$$

If we additionally assume that $\nabla H_{\mathcal{C}} \in C^1(\mathbb{R}^{m_{\mathcal{C}}}, \mathbb{R}^{m_{\mathcal{C}}})$, $\nabla H_{\mathcal{L}} \in C^1(\mathbb{R}^{m_{\mathcal{L}}}, \mathbb{R}^{m_{\mathcal{L}}})$ are homeomorphisms, we can introduce the inverse functions $Q_{\mathcal{C}} := (\nabla H_{\mathcal{C}})^{-1} \in C(\mathbb{R}^{m_{\mathcal{C}}}, \mathbb{R}^{m_{\mathcal{C}}})$, $\Psi_{\mathcal{L}} := (\nabla H_{\mathcal{L}})^{-1} \in C(\mathbb{R}^{m_{\mathcal{L}}}, \mathbb{R}^{m_{\mathcal{L}}})$. Then (13) leads to $q_{\mathcal{C}}(t) = Q_{\mathcal{C}}(u_{\mathcal{C}}(t))$ and $\psi_{\mathcal{L}}(t) = \Psi_{\mathcal{L}}(i_{\mathcal{L}}(t))$. Further decomposing

$$A_{\mathcal{S}} = \begin{bmatrix} A_I & A_{\mathcal{V}} \end{bmatrix}, \quad u_{\mathcal{S}} = \begin{pmatrix} u_I \\ u_{\mathcal{V}} \end{pmatrix}, \quad i_{\mathcal{S}} = \begin{pmatrix} i_I \\ i_{\mathcal{V}} \end{pmatrix}$$

into edges, voltages and currents to current and voltage sources, we see that (13) leads to the so-called *charge/flux-oriented modified nodal analysis* [2, Eq. (3.21)]

$$A_{\mathcal{C}} \tfrac{d}{dt} q_{\mathcal{C}}(t) + A_{\mathcal{R}} g(u_{\mathcal{R}}(t)) + A_{\mathcal{L}} i_{\mathcal{L}}(t) + A_I i_I(t) + A_{\mathcal{V}} i_{\mathcal{V}}(t) = 0,$$
$$-A_{\mathcal{L}}^{\top} \phi(t) + \tfrac{d}{dt} \psi_{\mathcal{L}}(t) = 0,$$
$$-A_{\mathcal{V}}^{\top} \phi(t) + u_{\mathcal{V}}(t) = 0, \tag{MNA c/f}$$
$$q_{\mathcal{C}}(t) - Q_{\mathcal{C}}(A_{\mathcal{C}}^{\top} \phi(t)) = 0,$$
$$\psi_{\mathcal{L}}(t) - \Psi_{\mathcal{L}}(i_{\mathcal{L}}(t)) = 0.$$

If we additionally assume that $Q_{\mathcal{C}} \in C(\mathbb{R}^{m_{\mathcal{C}}}, \mathbb{R}^{m_{\mathcal{C}}})$ and $\Psi_{\mathcal{L}} \in C^1(\mathbb{R}^{m_{\mathcal{L}}}, \mathbb{R}^{m_{\mathcal{L}}})$, then we can, by denoting the Jacobians by $\mathcal{C}(u_{\mathcal{C}}) = \tfrac{d}{du_{\mathcal{C}}} Q_{\mathcal{C}}(u_{\mathcal{C}})$ and $\mathcal{L}(i_{\mathcal{L}}) = \tfrac{d}{di_{\mathcal{L}}} \Psi_{\mathcal{L}}(i_{\mathcal{L}})$, reformulate (MNA c/f) to obtain the *modified nodal analysis* [18, Eq. (52)]

$$A_{\mathcal{C}} \mathcal{C}(A_{\mathcal{C}}^{\top} \phi(t)) A_{\mathcal{C}}^{\top} \tfrac{d}{dt} \phi(t) + A_{\mathcal{R}} g(A_{\mathcal{R}}^{\top} \phi(t)) + A_{\mathcal{L}} i_{\mathcal{L}}(t) + A_I i_I(t) + A_{\mathcal{V}} i_{\mathcal{V}}(t) = 0,$$
$$-A_{\mathcal{L}}^{\top} \phi(t) + \mathcal{L}(i_{\mathcal{L}}(t)) \tfrac{d}{dt} i_{\mathcal{L}}(t) = 0, \tag{MNA}$$
$$-A_{\mathcal{V}}^{\top} \phi(t) + u_{\mathcal{V}}(t) = 0.$$

Note that, if $H_{\mathcal{C}} \in C^2(\mathbb{R}^{m_{\mathcal{C}}}, \mathbb{R})$, $H_{\mathcal{L}} \in C^2(\mathbb{R}^{m_{\mathcal{L}}}, \mathbb{R})$, then $\mathcal{C}(u_{\mathcal{C}})$ and $\mathcal{L}(i_{\mathcal{L}})$ are, respectively, the inverses of the Hessians of $H_{\mathcal{C}}$ and $H_{\mathcal{L}}$ at $Q_{\mathcal{C}}(u_{\mathcal{C}})$ and $\Psi_{\mathcal{L}}(i_{\mathcal{L}})$.

4.2. The (charge/flux-oriented) modified loop analysis

We present an alternative modelling involving the pH system $(\mathcal{D}'_K(\mathcal{G}), \mathcal{L}'_K(\mathcal{G}), \{0\})$ with $\mathcal{D}'_K(\mathcal{G})$ and $\mathcal{L}'_K(\mathcal{G})$ as in (6). That is, the loops in the underlying graph structure is now taken to model the Kirchhoff laws. First note that the external flows and efforts variables in the pH system $(\mathcal{D}^S_K(\mathcal{G}), \mathcal{L}^S_K(\mathcal{G}), \{0\})$ in Remark 2.13 are, respectively, the current and the voltage of the components, while the external flows and efforts variables in $(\mathcal{D}'_K(\mathcal{G}), \mathcal{L}'_K(\mathcal{G}), \{0\})$ are, respectively, the voltage and the current of the components. This means that in order to obtain a pH system $(\mathcal{D}', \mathcal{L}', \mathcal{R}')$ describing the circuit dynamics by performing an interconnection of $(\mathcal{D}'_K(\mathcal{G}), \mathcal{L}'_K(\mathcal{G}), \{0\})$ with $N \in \mathbb{N}$ electrical components $(\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i)_{i \in \{1,\dots,N\}}$, i.e.,

$$(\mathcal{D}', \mathcal{L}', \mathcal{R}') := (\mathcal{D}'_K(\mathcal{G}), \mathcal{L}'_K(\mathcal{G}), \{0\}) \circ \left(\underset{i=1}{\overset{N}{\times}} (\mathcal{D}_i, \mathcal{L}_i, \mathcal{R}_i) \right),$$

we have to adjust the definition of the components by interchanging the role of the effort and flow variables, which is possible by an argument similar to one in Remark 2.13. Given an electrical circuit consisting of

resistances, inductances, capacitances and sources, it can, completely analogous to Section 4.1, be shown that the dynamics of $(\mathcal{D}', \mathcal{L}', \mathcal{R}')$ lead, under certain additional invertibility and smoothness assumptions on the functions representing capacitances and inductances, to the *modified loop analysis* [18, Eq. (53)]

$$B_L \mathcal{L}(B_L^\top \iota(t)) B_L^\top \tfrac{d}{dt} \iota(t) + B_\mathcal{R} r(B_\mathcal{R}^\top \iota(t)) + B_\mathcal{C} u_\mathcal{C}(t) + B_I u_I(t) + B_\mathcal{V} u_\mathcal{V}(t) = 0,$$
$$-B_\mathcal{C}^\top \iota(t) + \mathcal{C}(u_\mathcal{C}(t)) \tfrac{d}{dt} u_\mathcal{C}(t) = 0,$$
$$-B_I^\top \iota(t) + i_I(t) = 0.$$

References

[1] B. Andrásfai. *Graph Theory: Flows, Matrices.* Taylor & Francis, New York London, 1991.

[2] S. Bächle. *Numerical Solution of Differential-Algebraic Systems Arising in Circuit Simulation.* PhD thesis, Fakultät II - Mathematik und Naturwissenschaften, Technische Universität Berlin, Berlin, Germany, 2007.

[3] M. Barbero-Liñán, H. Cendra, E. García-Toraño Andrés, and D. Martín de Diego. New insights in the geometry and interconnection of port-Hamiltonian systems. *J. Phys. A*, 51(37):375201 (30 p.), 2018.

[4] C. Beattie, V. Mehrmann, H. Xu, and H. Zwart. Linear port-Hamiltonian descriptor systems. *Math. Control Signals Syst.*, 30(4), 2018. Article: 17.

[5] J. Behrndt, M. Kurula, A.J. van der Schaft, and H. Zwart. Dirac structures and their composition on Hilbert spaces. *J. Math. Anal. Appl.*, 372(2):402–422, 2010.

[6] P.C. Breedveld, B. Maschke, and A.J. van der Schaft. An intrinsic hamiltonian formulation of the dynamics of LC-circuits. *IEEE Trans. Circuits Syst. I. Regul. Pap.*, 42(2):73–82, 1995.

[7] J. Cervera, A.J. van der Schaft, and A. Baños. Interconnection of port-Hamiltonian systems and composition of dirac structures. *Automatica*, 43(2):212–225, 2007.

[8] T.J. Courant. Dirac manifolds. *Trans. Amer. Math. Soc.*, 319:631–661, 1990.

[9] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Berlin, 5th edition, 2017.

[10] D. Jeltsema and A.J. van der Schaft. Port-Hamiltonian systems theory: An introductory overview. *Foundations and Trends in Systems and Control*, 1(2-3):173–387, 2014.

[11] J.M. Lee. *Introduction to Smooth Manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, 2nd edition, 2012.

[12] B. Maschke and A.J. van der Schaft. Generalized port-Hamiltonian DAE systems. *Systems & Control Letters*, 121:31–37, 2018.

[13] B. Maschke and A.J. van der Schaft. Dirac and Lagrange algebraic constraints in nonlinear port-Hamiltonian systems. Technical report, 2019.

[14] W. Mathis and A. Reibiger. *Küpfmüller Theoretische Elektrotechnik.* Springer Vieweg, Berlin, 20th edition, 2017.

[15] C. Mehl, V. Mehrmann, and M. Wojtylak. Linear algebra properties of dissipative Hamiltonian descriptor systems. *SIAM J. Matrix Anal. Appl.*, 39(3):1489–1519, 2018.

[16] C. Melchiorri and A.J. van der Schaft. Port Hamiltonian formulation of infinite dimensional systems - i. modeling. In *43rd IEEE Conference on Decision and Control, December 14-17, 2004, Atlantis, Paradise Island, Bahamas*, 2004.

[17] C. Melchiorri and A.J. van der Schaft. Port Hamiltonian formulation of infinite dimensional systems - ii. boundary control by interconnection. In *43rd IEEE Conference on Decision and Control, December 14-17, 2004, Atlantis, Paradise Island, Bahamas*, 2004.

[18] T. Reis. Mathematical modeling and analysis of nonlinear time-invariant RLC circuits. In P. Benner, R. Findeisen, D. Flockerzi, U. Reichl, and K. Sundmacher, editors, *Large-Scale Networks in Engineering and Life Sciences*, Modeling and Simulation in Science, Engineering and Technology, pages 125–198. Birkhäuser, Basel, 2014.

[19] R. Riaza. DAEs in circuit modelling: A survey. In Achim Ilchmann and Timo Reis, editors, *Surveys in Differential-Algebraic Equations I*, Differential-Algebraic Equations Forum, pages 97–136. Springer, Berlin-Heidelberg, 2013.

[20] A.S. Sedra and K.C. Smith. *Microelectronic Circuits.* Oxford University Press, New York, 5th edition, 2004.

[21] A.J. van der Schaft. Characterization and partial synthesis of the behavior of resistive circuits at their terminals. *Systems & Control Letters*, 59(7):423–428, 2010.

[22] A.J. van der Schaft. Port-Hamiltonian differential-algebraic systems. In A. Ilchmann and T. Reis, editors, *Surveys in Differential-Algebraic Equations I*, Differential-Algebraic Equations Forum, pages 173–226. Springer, Berlin Heidelberg, 2013.

[23] A.J. van der Schaft. L_2-Gain and Passivity Techniques in Nonlinear Control. Lecture Notes in Control and Information Sciences. Springer, London, 3rd edition, 2017.

[24] A.J. van der Schaft and B. Maschke. Port-Hamiltonian systems on graphs. *SIAM J. Control Optim.*, 51(2):906–937, 2013.

[25] A. Venkatraman and A. van der Schaft. Energy shaping of port-hamiltonian systems by using alternate passive input-output pairs. *Eur. J. Control*, 16(6):665–677, 2010.

[26] A. Venkatraman and A. van der Schaft. Interconnections of port-Hamiltonian systems: generating new passive outputs and feedback stabilization. *IFAC Proceedings Volumes*, 43(14):605–610, 2010.

[27] J.C. Willems. Terminals and ports. *IEEE Circuits and Systems Magazine*, 10(4):8–16, 2010.