A universal planar graph under the minor relation

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Abstract

We construct an infinite planar graph that contains every planar graph as a minor.

1 Introduction

Answering a question of Ulam, Pach [7] proved that there is no 'universal planar graph', i.e. no planar graph that contains every other planar graph as a subgraph. The purpose of this paper is to show that the answer becomes positive if we replace the subgraph relation by the (weaker) minor relation: we shall construct a planar graph G^* that contains every other planar graph as a minor. See [5] for a survey of universality results based on the subgraph relation.

Following a short section on terminology, the construction of G^* is presented in Section 3. Its universality is then proved in two stages. In Section 4 we show that every planar graph G is the minor of some planar graph H of maximum degree at most three. In Section 5 we show how any such H may be embedded in G^* as a minor. In Section 6 finally, we mention a few open problems about minor-universality.

2 Terminology

Our basic notation follows [1]. All the graphs in this paper are countable, and we assume that the edges of plane graphs are polygonal arcs. We impose no restrictions on graph drawings in terms of accumulation points. (Thus, a sequence of points from a plane graph G may converge to any point of the plane, either on or off G.) Universal graphs for such restricted drawings are considered in [6].

Minors for infinite graphs are defined exactly as for finite graphs (see [1]). Note that their *branch sets*, the connected vertex sets to be contracted, may be infinite.

We further assume a fixed orientation of $\mathbb{R}^2$. If C is a cycle embedded in $\mathbb{R}^2$, we denote by $\vec{C}$ (respectively $\overleftarrow{C}$) the cycle C oriented in the direction agreeing with (respectively opposite) this orientation of $\mathbb{R}^2$. We shall also call this direction *clockwise* (respectively *anticlockwise*). Hence, if f is a face in a plane graph, and $\vec{e}$ is an oriented edge on the boundary of f, then f lies on the *right* of $\vec{e}$ or on its *left* (or both) in a natural way. (Think of 'clockwise' as a right turn.) For two vertices v, w on C we write $v\vec{C}w$ for the subpath of C from v to w following the clockwise orientation of C. The path $v\overleftarrow{C}w$ is defined correspondingly.

Whenever $v_1 \cdots v_n v_1$ is a cycle, we put $v_{n+1} := v_1$. The inner face of a plane cycle C will be denoted by $f(C)$. A path in $C \cup f(C)$ that avoids C except possibly in its first and last vertex is said to run *through* $f(C)$.

3 Construction of G^*

Since every finite planar graph is a minor of some large enough finite grid, a first candidate for G^* might be the infinite grid. Unfortunately this does not work: the graph obtained from K^4 by joining to each of its four vertices infinitely many new vertices of degree one is planar but not a minor of the infinite grid. Our plan is to construct G^* inductively, accommodating at each step all the possible ways in which a plane graph to be embedded in G^* as a minor might unfold vertex by vertex.

Let a *cycle of type* n denote a plane cycle of length 2^{n+3} whose vertices are coloured red and blue alternately. We shall construct an infinite sequence $G_0^* \subseteq G_1^* \subseteq \ldots$ of finite plane graphs and, for each G_n^*, a set $\mathcal{C}_n$ of disjoint cycles of type n in G_n^* each bounding an inner face of G_n^*.

Let G_0^* be a drawing of the 8-cycle C^8, colour its vertices red and blue alternately, and let $\mathcal{C}_0$ consist of the cycle G_0^*. Now suppose we have constructed $G_0^*, \ldots, G_n^*$ and $\mathcal{C}_0, \ldots, \mathcal{C}_n$ as desired. In the inner face of each cycle $C =: u_1 \cdots u_{2^{n+3}} u_1$ in $\mathcal{C}_n$ insert a cycle C' of type $n+1$ and 2^{n+3} disjoint paths of length $2^{n+2} - 1$ linking the vertices of C to the blue vertices of C'. Denote the path linking a vertex v on C to C' by P_v and the vertex of P_v on C' by v' (Fig. 1).

Let $C_i := u_i u_{i+1} \cup P_{u_{i+1}} \cup u_i' \vec{C}' u_{i+1}' \cup P_{u_i}$. Each of these cycles C_i bounds an inner face in the plane graph thus obtained. Insert a new cycle C_i' of type $n+1$ in this face, and join its blue vertices bijectively to the vertices of $P_{u_i} \cup P_{u_{i+1}}$, to obtain another plane graph (Fig. 2). Let this graph be G_{n+1}^*.

For each cycle $C \in \mathcal{C}_n$, we call C' the *large cycle of* G_{n+1}^* *inside* C and

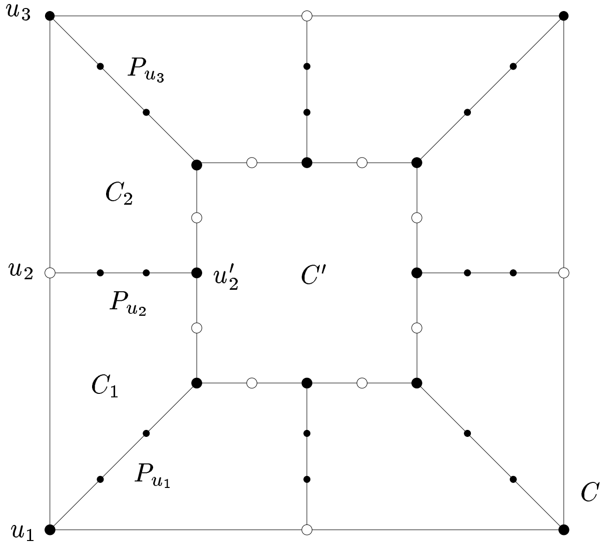

Figure 1: Adding C' for $n = 0$

each C_i' the *small cycle of* G^*_{n+1} *inside* C *preceding* u_{i+1}. Setting

$$\begin{aligned}\mathcal{C}_{n+1} &:= \{C' \mid C \in \mathcal{C}_n\} \cup \{C_i' \mid C \in \mathcal{C}_n;\ i = 1, \ldots, 2^{n+3}\} \\ &= \{D \mid \exists C \in \mathcal{C}_n :\ D \text{ is the large cycle or a small cycle of } G^*_{n+1} \text{ inside } C\}\end{aligned}$$

then completes the induction step of our construction.

Finally, we define the plane graph G^* by setting $G^* := \bigcup_{n=0}^{\infty} G^*_n$.

Let us fix some more notation. Let C_0 be a cycle in $\mathcal{C}_n$, v_0 a vertex on C_0, $k \geq 1$ and C_k a cycle in $\mathcal{C}_{n+k}$. We say that C_k is the *large cycle of* G^*_{n+k} *inside* C_0 if there exist $C_i \in \mathcal{C}_{n+i}$ $(i = 1, \ldots, k-1)$ such that each C_i is the large cycle of G^*_{n+i} inside C_{i-1} $(i = 1, \ldots, k)$. In this case G^* contains a unique path $P^k_{v_0}$ from v_0 to C_k that is the union of C_{i-1}-C_i paths of the form P_v. Indeed, let $P^0_{v_0} := v_0$ and inductively define $P^{i+1}_{v_0} := P^i_{v_0} \cup P_{v_i}$, where v_i is the vertex of $P^i_{v_0}$ on C_i and $0 \leq i \leq k-1$.

For distinct vertices v, w on some cycle $C \in \mathcal{C}_n$ we put $C_{vw} := v\vec{C}wP_ww'\overleftarrow{C}'v'P_vv$, where C' (as in the construction of G^*_{n+1}) is the large cycle of G^*_{n+1} inside C.

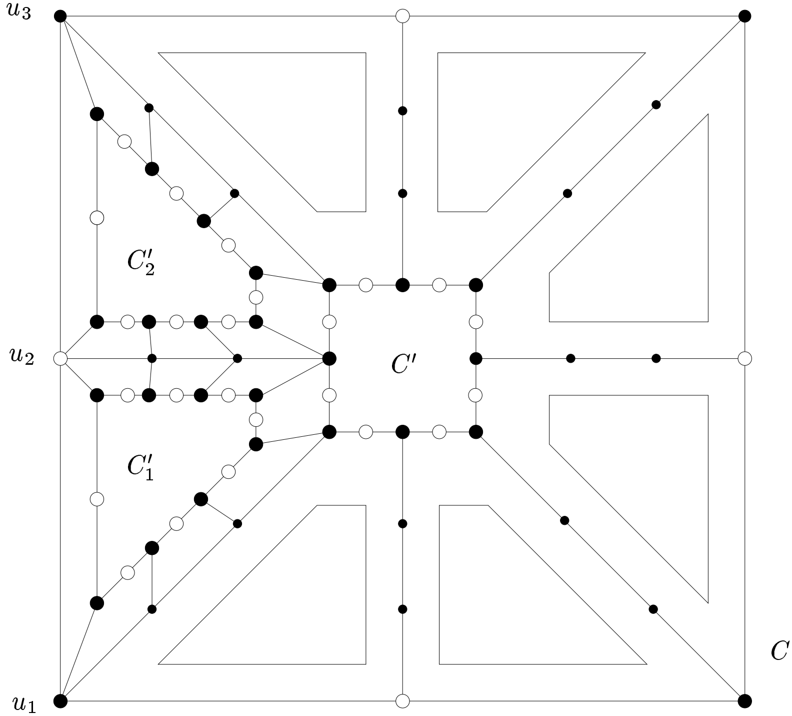

Figure 2: Inserting the cycles C_i' for $n = 0$ to form G_1^*

Lemma 1 *Let $C \in \mathcal{C}_n$, $k \geq 0$ and let $v_k, \ldots, v_0, w_0, \ldots, w_k$ be distinct blue vertices on C in clockwise order. Then G^*_{n+k} contains disjoint paths $P_0, \ldots, P_k$ such that P_i joins v_i to w_i ($i = 0, \ldots, k$), P_0 is a subpath of C, and $P_1, \ldots, P_k$ run through $f(C)$.*

Proof. Let $C_0 := C$, and for $i = 1, \ldots, k$ let C_i be the large cycle of G^*_{n+i} inside C_0. For $i = 0, \ldots, k$ let P_i be the unique v_i-w_i path in $P^i_{v_i} \cup C_i \cup P^i_{w_i}$ whose segment in C_i follows the clockwise orientation of C_i (Fig. 3). □

Lemma 2 *Let v, w be distinct vertices on $C \in \mathcal{C}_n$. Let $P_v =: v_0 \cdots v_{2^{n+2}-1}$ and $P_w =: w_0 \cdots w_{2^{n+2}-1}$ where $v_0 = v$ and $w_0 = w$. Let $k \leq 2^{n+2} - 2$. Then G^*_{n+k} contains disjoint paths $P_1, \ldots, P_k$ running through $f(C_{vw})$, such that for all $i = 1, \ldots, k$:*

(i) P_i joins v_i to w_i;

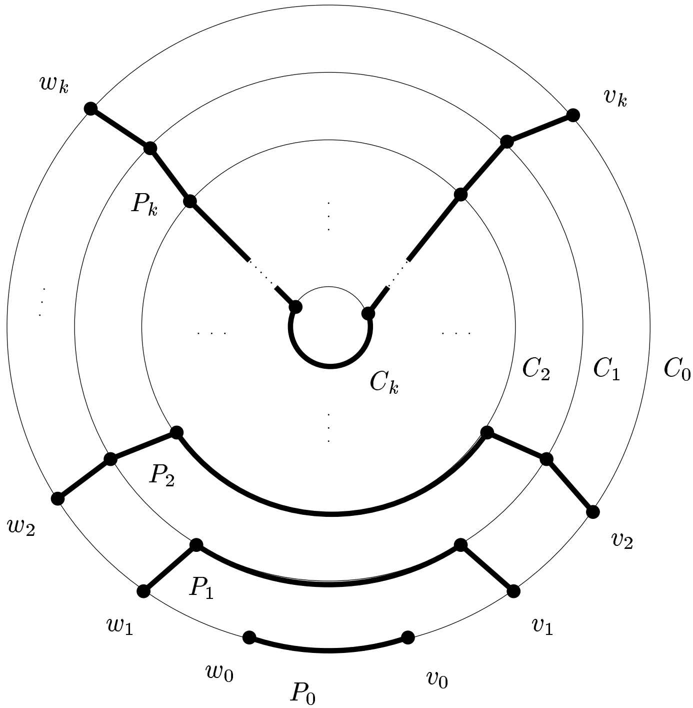

Figure 3: Joining up vertices of $C \in \mathcal{C}_n$ in G^*_{n+k}

(ii) *for every vertex* $u \in v\vec{C}w$*, the path* P_i *meets* $P_u =: u_0 \cdots u_{2^{n+2}-1}$ *exactly in* u_i*.*

Proof. We apply induction on the number m of inner vertices of the path $v\vec{C}w$. If $m = 0$ then C_{vw} is one of the cycles C_i from the construction of G^*_{n+1}, and the vertices of $P_v \cup P_w$ are joined bijectively to the blue vertices of $C'_i \in \mathcal{C}_{n+1}$. Applying Lemma 1 to the neighbours of $v_k, \ldots, v_1, w_1, \ldots, w_k$ on C'_i, we may join the v_j to the w_j by paths through $f(C_i)$ as required. The induction step follows from (ii) by concatenating paths. □

4 Degree reduction to $\Delta \leq 3$

Lemma 3 *For every planar graph* G *there exists a connected planar graph* H *such that* $\Delta(H) \leq 3$ *and* G *is a minor of* H*.*

The idea is to replace each vertex of G by a tree of maximum degree at most three, and to join up the leaves of these trees according to the edges of G, retaining planarity. This cannot be done arbitrarily, since we may inadvertently create $K_{3,3}$ minors in this way (Fig. 4). In our proof of the

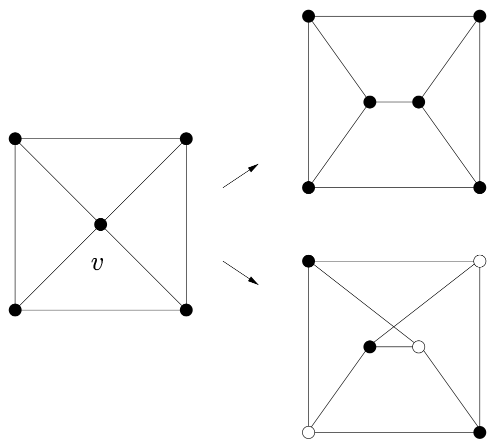

Figure 4: A planar and a non-planar expansion of the vertex v

lemma, we shall therefore construct H as a plane graph, designed with reference to a fixed drawing of G. Thus we assume in the following that G is a plane graph.

We start with some definitions. Let X be a finite plane graph and v a vertex of X, and let $e_1, \ldots, e_k$ be the edges incident with v (in clockwise order). Let $e_{k+1} := e_1$. For all $i = 2, \ldots, k+1$ we call e_i the *successor* of e_{i-1} *at* v and e_{i-1} the *predecessor* of e_i *at* v. (If $k = 1$, then e_1 is its own successor and predecessor.)

Let T be a plane tree with distinct leaves v_1, v_2, v_3. Let $P = x_1 \cdots x_n$ be the v_1-v_2 path in T (so $v_1 = x_1$ and $v_2 = x_n$), and let $P' = y_1 \cdots y_m$ be the v_3-P path in T, with $y_m = x_l$ say. Thus $1 < l < n$. If $x_l x_{l+1}$ is the successor of $y_{m-1} y_m$ at $x_l = y_m$ in the plane graph $P \cup P'$, we say that v_3 lies *between* v_1 and v_2 in T; note that the order of v_1 and v_2 matters here. For distinct leaves $v_1 \neq v_2$ of T, we also say that v_2 lies *between* v_1 and v_1.

Proof of Lemma 3. We may assume that G is connected and has at least two vertices. Indeed, if G is not connected, then adding a new vertex to G and joining it to one vertex in every component of G does not create a K^5 or $K_{3,3}$ minor. Thus, by the infinite version of Kuratowski's theorem the graph obtained is planar, and we may consider this graph instead of G.

Let us fix an infinite sequence $(G_n)_{n=0}^{\infty}$ of finite connected plane subgraphs of G such that

(i) if $G_n = G$ then $G_{n+1} = G$; if not, then G_{n+1} is obtained from G_n either by adding a new edge or by adding a new vertex and joining it to a vertex of G_n;

(ii) $G = \bigcup_{n=0}^{\infty} G_n$;

(iii) $|G_0| = 2$.

We shall inductively construct finite connected plane graphs $H_0 \subseteq H_1 \subseteq \ldots$ together with partitions $\{V_x^n \mid x \in V(G_n)\}$ of $V(H_n)$, satisfying the following conditions for all $n \in \mathbb{N}$:

(a) $V_x^n \subseteq V_x^{n+1}$ for all $x \in G_n$.

(b) For all $x \in G_n$ the subgraph T_x^n of H_n induced by V_x^n is a tree of maximum degree at most three, whose leaves have degree at most two in H_n.

(c) For distinct $x, y \in G_n$, there is a V_x^n-V_y^n edge in H_n if and only if xy is an edge of G_n. This V_x^n-V_y^n edge is unique, and its end v_{xy} in V_x^n is a leaf of T_x^n.

(d) If x_1, x_2 are neighbours of x in G_n (not necessarily distinct), then T_x^n has a leaf between v_{xx_1} and v_{xx_2}.

(e) If $x, x_1, x_2 \in G_n$ and xx_2 is the successor of xx_1 at x in G_n (x_1 and x_2 may coincide), then every leaf of T_x^n between v_{xx_1} and v_{xx_2} has degree one in H_n.

(f) There is a bijection φ_n between the faces of G_n and those of H_n, with the following property: if f is a face of G_n lying on the left of an edge $\overrightarrow{xy}$ of G_n, then $\varphi_n(f)$ lies on the left of $\overrightarrow{v_{xy}v_{yx}}$ in H_n.

Clearly, the graph

$$H := \bigcup_{n=0}^{\infty} H_n$$

will have maximum degree at most three and contain G as a minor (with branch sets $V_x := \bigcup_{n=0}^{\infty} V_x^n$ for all $x \in G$).

Denote the vertices of G_0 by x_1 and x_2. Let H_0 be a drawing of the path $v_1v_2v_3v_4$, and let $V^0_{x_1} := \{v_1, v_2\}$ and $V^0_{x_2} := \{v_3, v_4\}$. Let φ_0 be the obvious bijection.

Now suppose we have defined $H_0, \ldots, H_n$ and $\{V^i_x \mid x \in G_i\}$ for all $i = 0, \ldots, n$, satisfying (a)–(f). We may assume that $G \neq G_n$, for if not let $H_m := H_n$, $V^m_x := V^n_x$ and $\varphi_m := \varphi_n$ for all $m \geq n$ and all vertices $x \in G$.

In the construction of H_{n+1} we will make use of the following property of H^n. Suppose f is a face of G_n, xx_2 is the successor of xx_1 at x in G_n, and f lies on the left of $\overrightarrow{x_1x}$ (x_1 and x_2 may coincide). Let v be a leaf of T^n_x between v_{xx_1} and v_{xx_2} (which exists by (d)). Then (c),(e) and (f) imply that v has degree one in H_n and lies on the boundary of $\varphi_n(f)$ (Fig. 5).

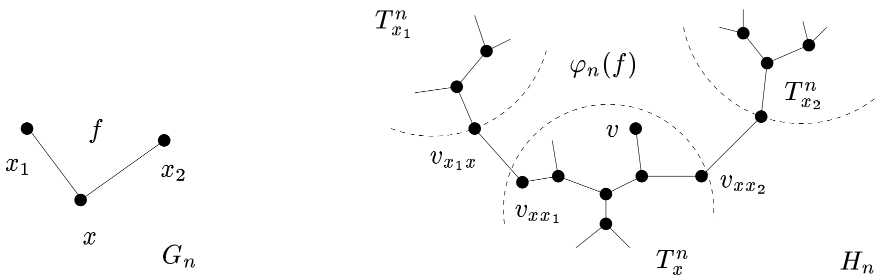

Figure 5: Expanding G_n to H_n

Case 1: G_{n+1} was obtained from G_n by adding an edge $e = xy$ with $x, y \in G_n$.

Then there exist faces f of G_n and f_1, f_2 of G_{n+1} such that $f = f_1 \cup f_2 \cup \mathring{e}$ and f_1 lies on the left of $\overrightarrow{xy}$ in G_{n+1}. (Note that $f_1 \neq f_2$, since G_n is connected.) Let xx_1, yy_1 be the predecessors of xy in G_{n+1} at x and y, respectively, and let xx_2, yy_2 be the successors of xy in G_{n+1} at x and y, respectively. By (d) we can find leaves $v_x \in T^n_x$ and $v_y \in T^n_y$ such that v_x lies between v_{xx_1} and v_{xx_2}, and v_y lies between v_{yy_1} and v_{yy_2} Observe that f lies on the left of both $\overrightarrow{x_1x}$ and $\overrightarrow{y_1y}$. Thus, as noted above, v_x and v_y both have degree one in H_n and lie on the boundary of $\varphi_n(f)$. Join v_x to v_y inside $\varphi_n(f)$ by a path $P = v_xu_1 \cdots u_6v_y$, so as to obtain another plane graph H'_n. Since H_n is connected, H'_n has two faces f'_1 and f'_2 such that $f'_1 \cup f'_2 \cup \mathring{P} = \varphi_n(f)$ and f'_1 lies on the left of $\overrightarrow{u_3u_4}$. Now insert two new vertices u'_1, u'_6 inside f'_1 and two new vertices u'_2, u'_5 inside f'_2. For $i = 1, 2, 5, 6$ join u'_i to u_i so as to obtain another plane graph H_{n+1} (Fig. 6).

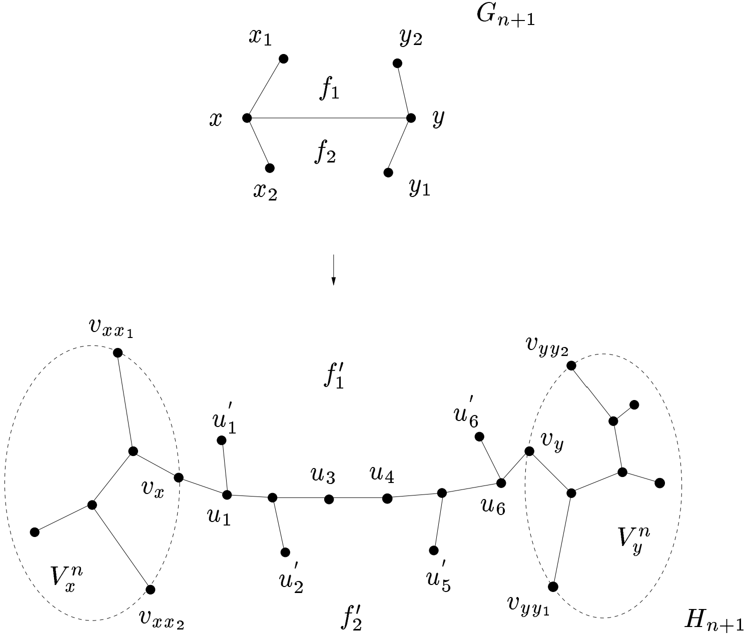

Figure 6: Extending H_n to H_{n+1} in Case 1

For all vertices $z \in G_{n+1}$ let

$$V_z^{n+1} := \begin{cases} V_x^n \cup \{u_1, u_2, u_3, u_1', u_2'\} & \text{if } z = x; \\ V_y^n \cup \{u_4, u_5, u_6, u_5', u_6'\} & \text{if } z = y; \\ V_z^n & \text{otherwise.} \end{cases}$$

Note that every face $g \neq f_1, f_2$ of G_{n+1} is also a face of G_n, and $\varphi_n(g)$ is a face of H_{n+1}. For these faces g we let $\varphi_{n+1}(g) := \varphi_n(g)$, and define $\varphi_{n+1}(f_i)$ as the unique face of H_{n+1} inside f_i' $(i = 1, 2)$.

It remains to check (a)–(f). The conditions (a)–(e) are straightforward. To verify (f), it is sufficent to show that it holds for the faces f_1 and f_2. We consider f_1; the case for f_2 is similar. First note the following. Consider any three vertices $z_0, z_1, z_2 \in G_{n+1}$ such that z_1z_2 is the successor of z_1z_0 at z_1 and f_1 lies on the left of $\overrightarrow{z_0z_1}$. Then f_1 lies on the left also of $\overrightarrow{z_1z_2}$. Suppose that $\varphi_{n+1}(f_1)$ does indeed lie on the left of $\overrightarrow{v_{z_0z_1}v_{z_1z_0}}$. Then the validity of (e) for $n+1$ implies that $\varphi_{n+1}(f_1)$ lies on the left also of $\overrightarrow{v_{z_1z_2}v_{z_2z_1}}$. Second, if f_1 lies on the left of some edge $\overrightarrow{z_1z_2}$, there exist vertices $a_0, \ldots, a_k \in G_{n+1}$ such that $\overrightarrow{a_0a_1} = \overrightarrow{xy}$, $\overrightarrow{a_{k-1}a_k} = \overrightarrow{z_1z_2}$, and for all $i = 1, \ldots, k-1$ the edge a_ia_{i+1} is the successor of $a_{i-1}a_i$ in G_{n+1} at a_i. Thus, by induction, (f) holds for the face f_1, since $\varphi_{n+1}(f_1)$ lies on the left of $\overrightarrow{u_3u_4'} = \overrightarrow{v_{xy}v_{yx}}$ by definition.

Case 2: G_{n+1} was obtained from G_n by adding a new vertex x and joining it to some vertex $y \in G_n$.

Then there exist faces f of G_n and f_1 of G_{n+1} such that $xy \setminus \{y\} \subseteq f$ and $f_1 = f \setminus xy$. Let yy_1 be the predecessor and yy_2 the successor of xy in G_{n+1} at y. By (d) we can find a leaf v of T_y^n between v_{yy_1} and v_{yy_2}. As before, v has degree one in H_n and lies on the boundary of $\varphi_n(f)$. Insert a new vertex u inside $\varphi_n(f)$ and join it to v by a path $P = uu_1u_2u_3u_4v$ inside $\varphi_n(f)$. Now insert two new vertices u_3', u_4' inside $\varphi_n(f) \setminus P$, and for $i = 3, 4$ join u_i' to u_i, to obtain another plane graph H_{n+1} in which u_3u_3' is the successor of u_2u_3 at u_3 while $u_4'u_4$ is the predecessor of u_4u_3 at u_4 (Fig. 7).

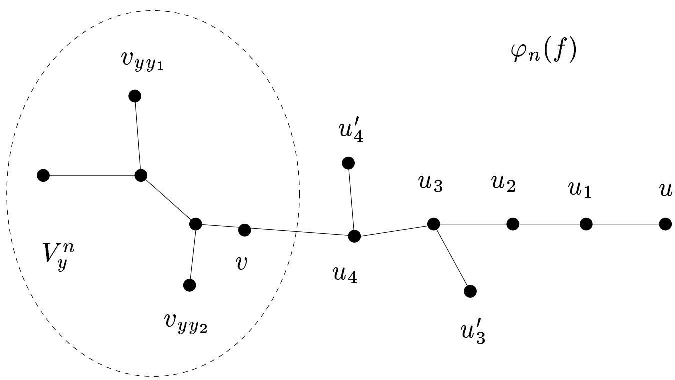

Figure 7: Extending H_n to H_{n+1} in Case 2

For all vertices $z \in G_{n+1}$ let

$$V_z^{n+1} := \begin{cases} \{u, u_1\} & \text{if } z = x; \\ V_y^n \cup \{u_2, u_3, u_4, u_3', u_4'\} & \text{if } z = y; \\ V_z^n & \text{otherwise.} \end{cases}$$

Once more all faces $g \neq f_1$ of G_{n+1} are also faces of G_n, and we let $\varphi_{n+1}(g) := \varphi_n(g)$. Finally, we let $\varphi_{n+1}(f_1)$ be the unique face of H_{n+1} inside $\varphi_n(f)$.

This time all the conditions (a)–(f) are readily checked. □

5 Universality of G^*

Theorem 4 *Every countable planar graph is a minor of G^*.*

Proof. We show that every countably infinite connected plane graph H of maximum degree at most three is a minor of G^*. The theorem then follows by Lemma 3. Subdividing an edge if H is cubic, we may assume that H has a vertex x_0 of degree two.

Note that there exists a sequence $(H^n)_{n=0}^{\infty}$ of finite connected subgraphs of H such that:

(i) $H^0 = \{x_0\}$;

(ii) H^{n+1} is obtained from H^n either by adding a new vertex a and joining it to some vertex of H^n or by adding a new edge between two vertices of H^n;

(iii) H^{n+1} is obtained from H^n by adding a new edge between two vertices $x, y \in H^n$ only if neither x nor y has a neighbour in $H - H^n$;

(iv) $H = \bigcup_{n=0}^{\infty} H^n$.

Our plan is to construct the branch set $V_x \subseteq V(G^*)$ of each vertex $x \in H$ inductively along the construction of G^*, as follows. For all $x \in H$ and all $n \in \mathbb{N}$ we shall define connected vertex sets $V_x^n \subseteq V(G^*)$ such that:

(a) $V_x^n \subseteq V_x^{n+1}$;

(b) $V_x^n \neq \emptyset$ if and only if $x \in H^n$, $V_x^n \cap V_y^n = \emptyset$ if $x \neq y$, and for every edge xy in H^n there is a V_x^n-V_y^n edge in G^*;

(c) there exists $m(n) \geq n$ such that

  (c1) $V_x^n \subseteq V(G^*_{m(n)})$ for every $x \in H^n$;

  (c2) for each vertex $x \in H^n$, the set V_x^n meets a cycle in $\mathcal{C}_{m(n)}$ if and only if $d_{H^n}(x) < d_H(x)$; moreover, if $d_{H^n}(x) < d_H(x)$ then V_x^n meets exactly one cycle $C_x^n \in \mathcal{C}_{m(n)}$, and it does so in a blue singleton v_x^n;

(d) if $P = x \cdots y$ is a nontrivial H^n-path in H (i.e. x, y are the only vertices of P in H^n and no edge of P lies in H^n) then V_x^n and V_y^n meet the same cycle of $\mathcal{C}_{m(n)}$, i.e. $C_x^n = C_y^n$;

(e) if $x_1 \cdots y_1$ and $x_2 \cdots y_2$ are disjoint nontrivial H^n-paths in H with $C^n_{x_1} = C^n_{x_2} =: C$, then $v^n_{x_2}$ and $v^n_{y_2}$ lie in the same component of $C - \{v^n_{x_1}, v^n_{y_1}\}$.
(If $v^n_{x_2}$ and $v^n_{y_2}$ lie in different components of $C - \{v^n_{x_1}, v^n_{y_1}\}$, we say that $x_1 \cdots y_1$ and $x_2 \cdots y_2$ *cross*. Thus, (e) says that there are no crossing H^n-paths.)

Let $V^0_{x_0}$ consist of any blue vertex $v^0_{x_0}$ on G^*_0, and for all vertices $x \in H - H^0$ let $V^0_x := \emptyset$. Then (a)–(e) hold with $m(0) = 0$. Now let $n \geq 0$, and suppose we have defined V^i_x for all vertices $x \in H$ and all $i = 0, \ldots, n$.

The following two claims will be used to establish condition (e) in the induction step for the case that H^{n+1} was obtained from H^n by adding a new vertex a and joining it to some vertex x of H^n (Case 1 below). Suppose that x has another neighbour $b \neq a$ in H (which may or may not lie in H^n), such that xb is not an edge of H^n. Since x has degree at most three in H (degree two if $x = x_0$), and x has at least one neighbour in H^n (unless $n = 0$), any such b is unique.

Claim 1. Suppose that $b \in H^n$. Let $\mathcal{P}$ be the set of H^n-paths P in H that start with the edge xa. Let $z(P)$ denote the last vertex of P. Then either $v^n_{z(P)}$ lies in $v^n_b \vec{C}^n_x v^n_x$ for all paths $P \in \mathcal{P}$ (and we say that *a belongs between b and x on C^n_x*) or $v^n_{z(P)}$ lies in $v^n_x \vec{C}^n_x v^n_b$ for all paths $P \in \mathcal{P}$ (and we say that *a belongs between x and b on C^n_x*).

Proof of Claim 1. Suppose the claim is false. Then we can find $P, P' \in \mathcal{P}$ such that $\{v^n_b, v^n_x\}$ separates $v^n_{z(P)}$ from $v^n_{z(P')}$ in C^n_x. Then $P \cup P'$ contains an H^n-path $z(P) \cdots z(P')$ that crosses xb, contradicting (e).

Claim 2. Suppose that $b \in H - H^n$. Let $\mathcal{P}$ be the set of pairs (P_1, P_2) of H^n-paths in H of the form $P_1 = xa \cdots z(P_1)$ and $P_2 = xb \cdots z(P_2)$, and such that P_1 and P_2 meet only in x. Then either $v^n_{z(P_1)}$ lies in $v^n_{z(P_2)} \vec{C}^n_x v^n_x$ for every pair $(P_1, P_2) \in \mathcal{P}$ (and we say that *a belongs between b and x on C^n_x*), or $v^n_{z(P_1)}$ lies in $v^n_x \vec{C}^n_x v^n_{z(P_2)}$ for every pair $(P_1, P_2) \in \mathcal{P}$ (and we say that *a belongs between x and b on C^n_x*).

Proof of Claim 2. Suppose the claim is false. Let $C := C^n_x$ and $v := v^n_x$, and choose $(P_1, P_2), (P_3, P_4) \in \mathcal{P}$ so that, with $z_i := z(P_i)$ and $v_i := v^n_{z_i}$ for $i = 1, \ldots, 4$,

(α) $v_1 \in v_2 \vec{C} v$ while $v_3 \in v \vec{C} v_4$;

(β) subject to (α), $|v_1\vec{C}v| + |v\vec{C}v_2| + |v\vec{C}v_3| + |v_4\vec{C}v|$ is minimum.

Let us show the following:

(1) *If v_4 lies in $v_1\vec{C}v$ then $z_1 = z_4$.*

Suppose not and let w be the first vertex of z_4P_4b which also lies in P_2. If z_4P_4w avoids P_1, then $z_2P_2wP_4z_4$ and P_1 are crossing H^n-paths contradicting (e). So z_4P_4w has a (first) vertex w' on P_1. But now $(xP_1w'P_4z_4, P_2), (P_3, P_4)$ would have been a better choice than $(P_1, P_2), (P_3, P_4)$.

In the same way one can show:

(2) *If v_1 lies in $v_4\vec{C}v$ then $z_1 = z_4$.*

Taken together, (1) and (2) imply that $z_1 = z_4$. Similarly it follows that $z_2 = z_3$. Let P be a z_1-z_2 path in H^n and $P' = x\cdots y$ an x-P path in H^n (Fig. 8). Note that $x \neq y$ since $d_H(x) \leq 3$. Since $z_1 \neq z_2$, y cannot be equal

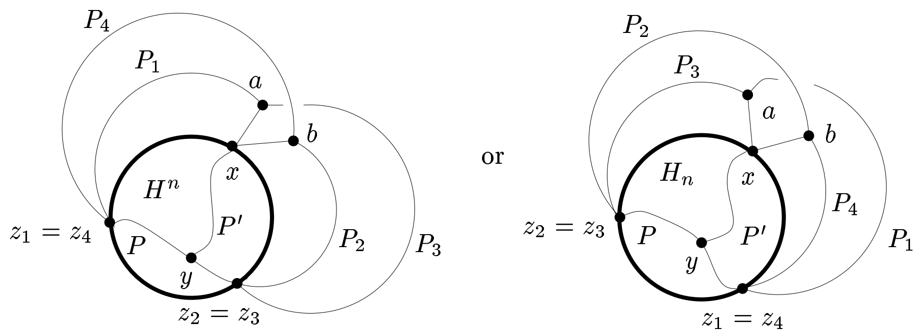

Figure 8: The proof of Claim 2

to both z_1 and z_2. We assume that $y \neq z_2$; the case of $y \neq z_1$ is analogous. Now either the cycle $z_1P_4xP'yPz_1$ separates a from $z_2 = z_3$ or the cycle $z_1P_1xP'yPz_1$ separates b from $z_2 = z_3$. This contradicts either the fact that aP_3z_3 avoids $z_1P_4xP'yPz_1$ or the fact that bP_2z_2 avoids $z_1P_1xP'yPz_1$.

Case 1: H^{n+1} was obtained from H^n by adding a new vertex a and joining it to $x \in H^n$.

Let $m(n+1) := m(n) + 1$. For all cycles C of the form C_z^n let C_z^{n+1} be the large cycle of $G^*_{m(n+1)}$ inside C. For all vertices $z \in H^n$ let

$$V_z^{n+1} := \begin{cases} V_z^n \cup V(P_{v_z^n}) & \text{if } d_{H^{n+1}}(z) < d_H(z); \\ V_z^n & \text{otherwise.} \end{cases}$$

Suppose first that x has a neighbour $b \neq a$ in H (which may or may not lie in H^n) such that xb is not an edge of H^n. Recall that b is unique, so either Claim 1 or Claim 2 applies (but not both). If a belongs between x and b, let u be the first (red) vertex on $\vec{C}_x^n$ after v_x^n. If a belongs between b and x, or if x has no such neighbour b in H, let u be the first (red) vertex on $\overleftarrow{C}_x^n$ after v_x^n. Hence, u is adjacent to v_x^n. Define

$$V_a^{n+1} := \begin{cases} \{u\} & \text{if } d_{H^{n+1}}(a) = d_H(a) \\ V(P_u) & \text{otherwise,} \end{cases}$$

and for each $z \in H - H^{n+1}$ let

$$V_z^{n+1} := \emptyset.$$

It remains to check that the sets V_z^{n+1} satisfy (a)–(e) for all $z \in H$. The conditions (a)–(d) are straightforward. For (e) note that for all $z \in H^n$ the cyclic order of the vertices v_z^{n+1} on any given cycle $C_y^{n+1} \in \mathcal{C}_{m(n+1)}$ agrees with the cyclic order of the corresponding vertices v_z^n on C_y^n. This together with the validity of (e) for n implies that crossing paths P, P' must be such that P (say) ends in a (otherwise P, P' cross also for n) and P' ends in x (otherwise P' crosses Pax for n). Let $P =: a \cdots z$ and $P' =: xb \cdots z'$ (where possibly $b = z'$). If a belongs between x and b then both v_a^{n+1}, v_z^{n+1} lie in $v_x^{n+1} \vec{C}_x^{n+1} v_{z'}^{n+1}$ (irrespective of whether $b \in H^n$), i.e. P and P' do not cross. If not, then a belongs between b and x, so both v_a^{n+1}, v_z^{n+1} lie in $v_{z'}^{n+1} \vec{C}_x^{n+1} v_x^{n+1}$, and again P and P' do not cross.

Case 2. H^{n+1} was obtained from H^n by adding an edge xy, for some vertices x, y of H^n.

By (d), V_x^n and V_y^n meet the same cycle of $\mathcal{C}_{m(n)}$, i.e. $C_x^n = C_y^n =: C$. Let s_0 be the red neighbour of v_x^n in $v_x^n \vec{C} v_y^n$ and t_0 the red neighbour of v_y^n in $v_y^n \vec{C} v_x^n$. Let $s_1, \ldots, s_k$ be the inner vertices of $v_x^n \vec{C} v_y^n$ of the form v_z^n (in order). Similarly, let $t_1, \ldots, t_l$ be the inner vertices of $v_y^n \vec{C} v_x^n$ of the form v_z^n (in order). Let s_{k+1} be the red neighbour of v_y^n in $v_x^n \vec{C} v_y^n$ and t_{l+1} the red neighbour of v_x^n in $v_y^n \vec{C} v_x^n$.

Now let

$$m(n+1) := \max\{m(n) + k + 2, m(n) + l + 2\}$$

and

$$N := m(n+1) - m(n).$$

For all vertices $z \in H^n$ such that V_z^n does not meet C, extend V_z^n by setting

$$V_z^{n+1} := \begin{cases} V_z^n \cup V(P_{v_z^n}^N) & \text{if } d_{H^{n+1}}(z) < d_H(z); \\ V_z^n & \text{otherwise.} \end{cases}$$

Note that $k+1 \leq n \leq m(n) \leq 2^{m(n)+2} - 2$ by (c). Thus we may apply Lemma 2 to obtain disjoint P_{s_0}-$P_{s_{k+1}}$ paths $P_1, \ldots, P_{k+1}$. Let $P_0 := s_0$. For each $i = 0, \ldots, k+1$, let $R(s_i)$ be the path through $f(C_{v_x^n v_y^n})$ that starts at s_i, follows P_{s_i} until it hits P_i, then follows P_i to P_{s_0}, then traverses the unique edge pq from its vertex p on P_{s_0} to the small cycle of $G^*_{m(n)+1}$ inside C preceding s_0, and finishes with the path P_q^{N-1} to end on, say, $C'_{s_0} \in \mathcal{C}_{m(n+1)}$ (Fig. 9). Similarly define C'_{t_0} and, for all $i = 0, \ldots, l+1$, disjoint t_i-C'_{t_0}

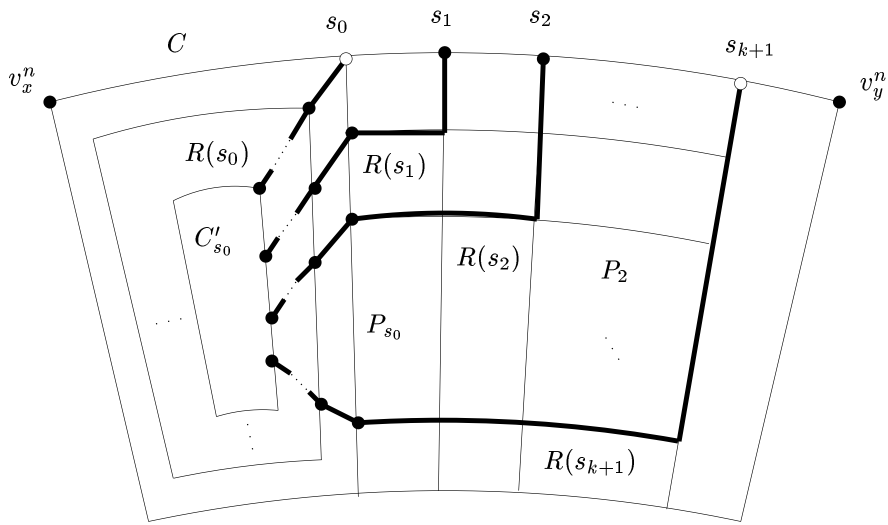

Figure 9: Joining the s_i to C'_{s_0} by paths $R(s_i)$

paths $R(t_i)$ through $f(C_{v_y^n v_x^n})$.

For all vertices $z \in H^n$ such that $z \neq x, y$ and V_z^n meets C, define

$$V_z^{n+1} := V_z^n \cup V(R(v_z^n)).$$

Note that if $d_{H^{n+1}}(x) < d_H(x)$, then H has a unique vertex y' such that x is adjacent to y' in H but not in H^{n+1}. Then $y' \in H^n$ by condition (iii) on the choice of the sequence $(H^n)_{n=0}^{\infty}$, and $v_{y'}^n \in C$ by (d). Extend V_x^n by setting

$$V_x^{n+1} := \begin{cases} V_x^n \cup V(P_{v_x^n}) & \text{if } d_{H^{n+1}}(x) = d_H(x); \\ V_x^n \cup V(P_{v_x^n} \cup R(s_0)) & \text{if } d_{H^{n+1}}(x) < d_H(x) \text{ and } v_{y'}^n \in \{s_1, \dots, s_k\}; \\ V_x^n \cup V(P_{v_x^n} \cup R(t_{l+1})) & \text{if } d_{H^{n+1}}(x) < d_H(x) \text{ and } v_{y'}^n \in \{t_1, \dots, t_l\}. \end{cases}$$

Let C' be the large cycle of $G^*_{m(n)+1}$ inside C, and let W denote the set of inner vertices of the subpath of C' between the endpoints of $P_{v_y^n}$ and $P_{v_x^n}$ that follows the clockwise orientation of C'. As before denote by x' the vertex adjacent to y in H but not in H^{n+1}, if it exists. Define

$$V_y^{n+1} := \begin{cases} V_y^n \cup W \cup V(P_{v_y^n}) & \text{if } d_{H^{n+1}}(y) = d_H(y); \\ V_y^n \cup W \cup V(P_{v_y^n} \cup R(t_0)) & \text{if } d_{H^{n+1}}(y) < d_H(y) \text{ and } v_{x'}^n \in \{t_1, \dots, t_l\}; \\ V_y^n \cup W \cup V(P_{v_y^n} \cup R(s_{k+1})) & \text{if } d_{H^{n+1}}(y) < d_H(y) \text{ and } v_{x'}^n \in \{s_1, \dots, s_k\}. \end{cases}$$

(Fig. 10.) For all vertices $z \in H - H^{n+1}$ let $V_z^{n+1} := \emptyset$. Again, the sets V_z^{n+1}

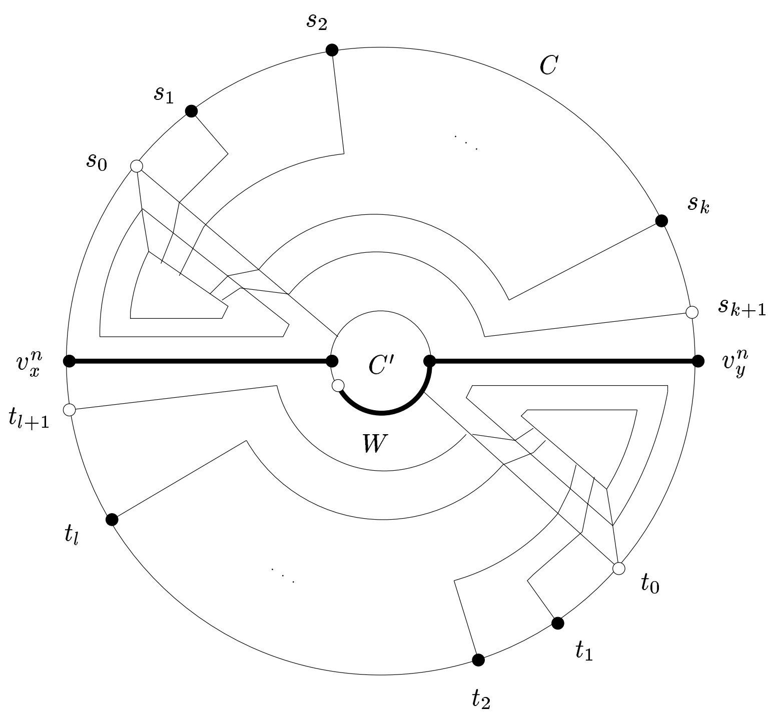

Figure 10: Accommodating the edge xy

satisfy (a)–(e). Indeed, checking the conditions (a)–(c) is straightforward. Any counterexample to (d) would be an H^n-path crossing xy, contradicting (e) for n. Finally, (e) for $n+1$ follows from (e) for n since the cyclic order of the vertices v_z^{n+1} on C'_{s_0} and on C'_{t_0} reflects that of the corresponding vertices v_z^n on C (and similarly for the other cycles in $\mathcal{C}_{m(n+1)}$).

For all vertices $x \in H$ let $V_x := \bigcup_{n=0}^{\infty} V_x^n$. By (a), these sets V_x inherit connectedness in G^* from the sets V_x^n. Hence by (b), H is a minor of G^* with branch sets V_x, for all $x \in H$. $\square$

6 Open Problems

As the construction of our universal graph G^* shows at once, G^* is locally finite; indeed $\Delta(G^*) = 8$. Thus if G is a planar graph with a vertex x of infinite degree, the branch set of x in any embedding of G in G^* as a minor will be infinite. One obvious question that we have not addressed is whether this can be avoided:

Problem 5 *Is there a planar graph G^* such that every planar graph can be embedded in G^* as a minor with finite branch sets?*

Another obvious strengthening of Theorem 4 would be to ask for a universal planar graph with respect to the topological minor relation:

Problem 6 *Is there a planar graph G^* that contains a subdivision of every planar graph as a subgraph?*

Abstracting from planarity, one might ask which other minor-closed graph properties have a (countable) universal graph. For example:

Problem 7 *For which graphs X is there a countable graph G^* without an X minor such that every countable graph without an X minor is a minor of G^*?*

For the complete graphs $X = K^3$ and $X = K^4$, there are graphs without an X minor that are universal for this property even with respect to the subgraph relation. Indeed, the $\aleph_0$-regular tree contains every countable graph without a K^3 minor as a subgraph, and a universal graph for K^4 can be obtained by recursively pasting triangles together along edges (see [1, Prop. 8.3.1]). For $n \geq 5$, there is no (countable) subgraph-universal graph without a K^n minor [2], but we do not know whether these classes have minor-universal elements. By a result of Halin [4], the edge-maximal graphs without a K^5 minor are precisely those that have a certain tree-decomposition into countable maximal planar graphs and copies of the Wagner graph W (see [1]); thus, Theorem 4 may help in the construction of any minor-universal graph for $X = K^5$. Similarly, the graphs without a $K^{\aleph_0}$ minor have a characterization by their tree-decompositions that involves planar graphs [3], so here too Theorem 4 might conceivably be of help.

Interestingly, the sphere stands out in that for no other closed surface S does the class of graphs embeddable in S have a minor-universal element (C. Thomassen, personal communication). Indeed, any such graph G^* would contain a cycle C^* whose deletion reduces the Euler genus of G^*. Then every minor of G^* can be embedded in a smaller surface than S after the deletion of at most $|C^*|$ vertices. This, however, will not be the case for every graph embeddable in S.

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