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The cycle space of a 3-connected graph is generated by its finite and infinite locally finite peripheral circuits

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Abstract

We extend Tutte's result that in a finite 3-connected graph the cycle space is generated by the peripheral circuits to locally finite graphs. Such a generalization becomes possible by the admission of infinite circuits in the graph compactified by its ends.

1 Introduction

In a finite graph, the edge set of a connected 2-regular subgraph is called a circuit. The set of symmetric differences of circuits constitutes a $\mathbb{F}_2$-vector space, the cycle space. A classical result of Tutte [11] states:

Theorem 1 (Tutte 1963). *Every element of the cycle space of a finite* 3*-connected graph is a sum of peripheral circuits.*

Here, a circuit $C = E(D)$ is *peripheral* if D is an induced non-separating subgraph without isolated vertices. We show that despite obvious counterexamples Tutte's result can be generalized to locally finite graphs by admitting infinite circuits and sums as recently proposed by Diestel and Kühn [4].

Let us look at a simple example due to Halin [7]. Consider the cartesian product of a *double ray* (an infinite 2-way path) with a pentagon (Figure 1). The peripheral circuits of this graph are exactly its 4-circuits. We see that the deletion of all vertices incident with C separates the graph but C is not the sum of any peripheral circuits, so Tutte's theorem fails for this graph.

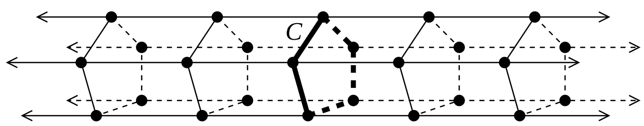

Figure 1: The circuit C is not the finite sum of peripheral circuits

This can be mended, however, by allowing infinite sums. Indeed, C is clearly the (infinite) sum of all the 4-circuits to the left of C (or to the right for that matter).

However, infinite sums of circuits can also produce edge sets of subgraphs such as the double ray shown in Figure 2, which should then also be legitimate

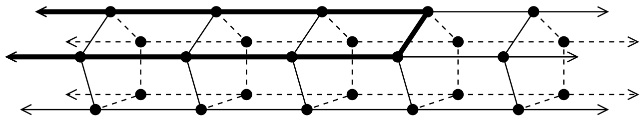

Figure 2: An infinite edge set that arises from an infinite sum of circuits

elements of the cycle space closed under (well-defined) infinite sums. This complicates matters, but not beyond control: the subspace of the edge-space of a locally finite graph G that is generated by (possibly) infinite sums of the (finite) circuits of G has been studied by Diestel and Kühn in [4, 5], who obtained this space as an adaptation of the cycle space to topological *circles* involving the ends of G. (These circles are homeomorphic images of the unit circle in the standard compactification of G by its ends; for example, the double ray in Figure 2 forms an infinite circle together with the *left* end of the graph.) We shall make use of the results in [4, 5] throughout this paper. See also Diestel [2] for an introduction and survey.

We state now our main result:

Theorem 2. *Every element of the cycle space $\mathcal{C}(G)$ of a locally finite 3-connected graph G is a sum of peripheral circuits.*

Infinite circuits and $\mathcal{C}(G)$ will be defined in the next section. Section 3 contains a discussion of the main result. In Section 4 we examine bridges and the overlap graph of a circle in a 3-connected graph. In Section 5 we prove the main lemma for Theorem 2, which then is proved in Section 6.

2 Definitions

In general our notation will be that of [3]. All our graphs will be undirected and simple. Let $G = (V, E)$ be a fixed graph. A 1-way infinite path will be called a *ray*. A subray of a ray will be said to be a *tail* of that ray. Two rays in a graph are *equivalent* if there is no finite vertex set separating them. The resulting equivalence classes are called the *ends* of the graph. The set of ends is denoted by $\Omega(G)$.

Let us define a topology on $G \cup \Omega(G)$; in the case when G is locally finite this will coincide with the *Freudenthal compactification* of G. Let G itself carry the topology of a 1-complex, ie. every edge is homeomorphic to the $[0, 1]$ intervall, and the basic open neighbourhoods of a vertex x are the unions of half open intervalls $[x, z)$, one from every edge $[x, y]$ at x. Next, let us describe the basic open neighbourhoods of the ends. For a finite set $S \subseteq V$ and an end ω there is exactly one component of $G - S$ that contains a tail for every ray in ω. This component will be denoted by $C_G(S, \omega)$ and we say ω *belongs* to $C_G(S, \omega)$. The union of $C_G(S, \omega)$ with all the ends belonging to it is $\overline{C}_G(S, \omega)$. Write $E_G(S, \omega)$ for the set of all edges between S and $C_G(S, \omega)$ and let $\mathring{E}'_G(S, \omega)$ be any union of half-edges $(x, y] \subset e$, one for every $e \in E_G(S, \omega)$, with $x \in \mathring{e}$ and $y \in C_G(S, \omega)$. Then let the basic open neighbourhoods of ω be the sets of the form

$$\hat{C}_G(S, \omega) := \overline{C}_G(S, \omega) \cup \mathring{E}'_G(S, \omega).$$

Denote by $|G|$ the resulting topological space on $G \cup \Omega(G)$. We will freely view subgraphs of G also as subspaces of $|G|$.

For any subset $X \subseteq |G|$, put $V(X) := X \cap V$, and denote by $E(X)$ the set of edges e of G with $e \subseteq X$. For an edge set $Z \subseteq E$, denote by $\overline{Z}$ the closure of $\bigcup Z$ in $|G|$. A continuous image of the unit interval $[0,1]$ is a *topological path.* The images of 0 and 1 are the *endpoints* of the topological path. A homeomorphic image of $[0,1]$ in $|G|$ is called an *arc* in $|G|$. The following lemma can be found in Hall and Spencer [8, p. 208].

Lemma 3. *Every topological path with distinct endpoints x, y in a Hausdorff space X contains an arc between x and y.*

The following two lemmas relate topological connectivity in $|G|$ to graph-theoretic connectivity in G.

Lemma 4. *Every open topologically connected subset of $|G|$ is path-connected.*

The lemma can be proved with standard topological arguments. For closed topologically connected subsets this remains still valid, provided G is locally finite, but not otherwise, see [6].

Lemma 5 (Diestel and Kühn [5]). *Let $A \subseteq |G|$ be an arc between two vertices x and y, and let X be a closed subset of $|G|$ which avoids A. Then G contains an x–y path P with $P \cap X = \emptyset$.*

Having established a topology we may define circuits. First, we call a homeomorphic image C of the unit circle in $|G|$ a *circle.* In [5] it is shown that every edge of which C contains an inner point is completely contained in C. The set $E(C)$ of all these edges is a *circuit.* The edge set $E(C)$ is dense in C (in the sense that $\overline{E(C)} = C$), so a circuit uniquely defines a circle and vice versa. Clearly, this definition of a circuit includes the traditional finite circuits. In contrast, the infinite circuits are the edge set of unions of double rays whose ends fit together nicely. We say a circuit D is *peripheral* if the subgraph $\overline{D} \cap G$ is induced and non-separating.

We now define infinite sums of edge sets. For this, let $(A_i)_{i\in I}$ be a family of edge sets. The family is called *thin* if every edge lies in at most finitely many of the A_i. The sum $\sum_{i\in I} A_i$ of such a thin family is defined to be the set of all edges that appear in exactly an odd number of the A_i. Whenever we talk about sums we will mean sums of thin families.

Now, assume that G is locally finite. Then we define the cycle space $\mathcal{C}(G)$ to be the set of sums of (thin families of) circuits. One of the results of [4, Cor. 11] is that the cycle space of a locally finite graph is closed under taking (infinite) sums. It should be noted that $\mathcal{C}(G)$ is a vector space over $\mathbb{F}_2$.

Directly using the definition of the cycle space it may be a bit awkward to identify a given edge set as belonging to the cycle space. Fortunately, Diestel and Kühn provided a more accessible characterization as well. For this, let $\{V_1, V_2\}$ be a partition of the vertex set of a graph G. Then the set of all edges with one endvertex in V_1 and the other in V_2 is called a *cut*; the cut is called a *finite cut* if it consists of finitely many edges.

Theorem 6 (Diestel and Kühn [4]). *Let G be locally finite, and let $Z \subseteq E$. Then $Z \in \mathcal{C}(G)$ if and only if Z meets every finite cut in an even number of edges.*

As circuits are easier to handle than arbitrary elements of the cycle space, it turns out to be convenient that we can always decompose such an element into constituent circuits.

Theorem 7 (Diestel and Kühn [5]). *Every element of the cycle space of G is a disjoint union of circuits in G.*

Furthermore, when dealing with sums or unions of a family $\mathcal{F}$ we will make use of the shorthands $\sum \mathcal{F}$ (resp. $\bigcup \mathcal{F}$) to express the sum $\sum_{F\in\mathcal{F}} F$ (resp. the union $\bigcup_{F\in\mathcal{F}} F$).

A tree T with a distinguished vertex $r \in V(T)$ is called a *rooted tree* with *root* r. For another vertex $t \in V(T)$, the predecessor on the path rTt is called the *parent* of t. The vertices that have t as their parent are the *children* of t. A vertex without children is a *leaf*.

For vertices v and w of a graph G we denote by $d_G(v, w)$ the minimal length of a v–w path. Similarly, for an edge e and a vertex v, $d_G(v, e)$ is the minimal length of a path between v and one of the endvertices of e.

3 Discussion of main result

First, let us briefly motivate the definition of the cycle space $\mathcal{C}(G)$. The cycle space of a finite graph has a number of well-known properties. Among others, these are:

(i) an edge set is an element of the cycle space if and only if it meets every cut in an even number of edges;

(ii) every element of the cycle space is a union of disjoint circuits;

(iii) the cycle space is generated by the fundamental circuits of every spanning tree; and

(iv) Tutte's generating theorem (Theorem 1).

To be useful the cycle space of an infinite graph should retain as many of these properties as possible. For locally finite graphs, (i)–(iv) and a number of others remain true in $\mathcal{C}(G)$ (with some obvious adaptions; for instance, to ensure (iii) we have to forbid spanning trees that contain infinite circuits). Moreover, $\mathcal{C}(G)$ is the smallest cycle space to achieve this. Indeed, suppose $\mathcal{C}'(G)$ is an alternative cycle space. Certainly, $\mathcal{C}'(G)$ should contain all finite circuits, and to ensure that Theorem 1 remains valid in $\mathcal{C}'(G)$, the discussion of Figure 1 seems to imply that at least all sums of finite circuits should lie in $\mathcal{C}'(G)$ too. But then (iii) shows that $\mathcal{C}(G) \subseteq \mathcal{C}'(G)$. See Diestel [2] for more details.

The underlying topology of $|G|$ is Hausdorff. If we drop this requirement and accept that distinct ends may be identified we arrive at other compactifications of locally finite graphs, which have been investigated by Richter and Thomassen [9]. However, a cycle space building on a compactification which allows two ends in a connected graph to be identified will never satisfy (i). Indeed, (the edge set of) a double ray which sends a tail into each of the two ends is a circuit in such a cycle space. But it meets every finite cut separating the two ends in an odd number of edges.

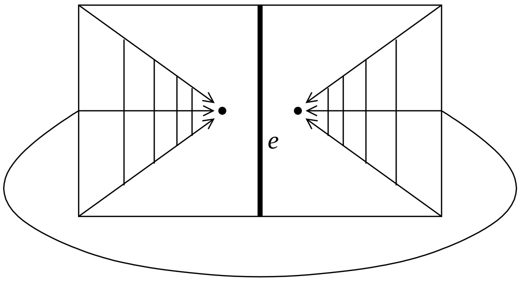

Figure 3: There is no finite peripheral circuit containing the edge e

In the introductory example in Section 1 we were able to generate our given circuit C using only *finite* peripheral circuits. In general, however, we cannot make do without infinite circuits, as Figure 3 demonstrates. The edge e there is not contained in any finite peripheral circuit. Consequently, any circuit containing e cannot be generated by finite peripheral circuits. On the other hand, it is easy to see that e lies on exactly two infinite peripheral circuits (namely the two face boundaries that are incident with e). Note that the graph shown is indeed 3-connected.

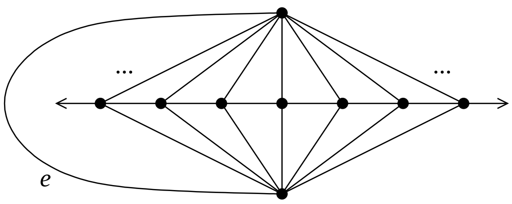

Figure 4: The edge e is not contained in any peripheral circuit

Next, we note that Tutte's result cannot be extended to arbitrarily infinite graphs using the topology as defined in Section 2. A counterexample is shown in Figure 4: for every circuit C containing the edge e, $\overline{C} \cap G$ is separating, and hence C non-peripheral.

Finally, let us give a rough overview of the proof of Theorem 2. Let $Z \in \mathcal{C}(G)$ be given. We fix one of the topological components B of $|G| \setminus \overline{Z}$. By adding peripheral circuits to Z we try to inflate B. More precisely, we want that for the sum Z' there is a topological component $B' \supsetneq B$ of $|G| \setminus \overline{Z'}$. Continuing in this manner, we achieve eventually (after countably many steps) that the inflated B covers all of $|G|$. This is only possible if the resulting sum is the empty set. We have then found a generating set of peripheral circuits for Z. Finding suitable peripheral circuits for the single steps will mostly be the work of Lemma 17.

The components B (in fact, we will be interested in their closures) are also interesting for another reason: We will observe in the next section that for a circle C, the circuit $E(C)$ is peripheral if and only if $|G| \setminus C$ is topologically connected.

4 Bridges and the overlap graph

A key tool in Tutte's proof of Theorem 1 is the concept of a bridge. In a finite graph G, a subgraph B is a bridge of another subgraph H if either B is a chord of H (ie. both its endvertices lie in H) or if B consists of a component K of $G - H$ plus the edges $E(K, H)$ between K and H together with the incident vertices. There is also an alternative way to define bridges by introducing an equivalence relation on $E(G) \setminus E(H)$; see Bondy and Murty [1] for more details on bridges. Our aim in this section is to transport the concept of a bridge to infinite graphs.

Let G be a fixed graph in this section.

Definition 8. *Let Z be an edge set in G. We call the closure B of a topological component of $|G| \setminus \overline{Z}$ a* bridge *of $\overline{Z}$. The points in $B \cap \overline{Z}$ are called the* attachments *of B in $\overline{Z}$.*

For the subgraph $H := \overline{Z} \cap G$, the following can be shown: a set $B \subseteq |G|$ is a bridge of $\overline{Z}$ if and only if it is a chord of H or if there is a component K of $G - H$ such that B is the closure of K plus the edges $E(K, H)$ together with the incident vertices. Thus, our definition coincides with the traditional definition of a bridge in a finite graph. However, we will only need the weaker fact that B, if it is not a chord, contains a whole component of $G - H$ but no other vertices of $G - H$, see (iv) in the next lemma.

The following observation will be used repeatedly: $B \setminus \overline{Z}$ is an open topological component of $|G| \setminus \overline{Z}$, which is thus path-connected, by Lemma 4. As a consequence, two points $x, y \in |G| \setminus \overline{Z}$ are in the same bridge if and only if there is an arc between them that avoids $\overline{Z}$.

Let us prove a number of basic properties.

Lemma 9. *Let Z be an edge set in G, and let B be a bridge of $\overline{Z}$. Let x be an attachment of B. Then:*

(i) *x is a vertex or an end;*

(ii) *if x is an end then every neighbourhood of x contains attachments of B that are vertices;*

(iii) *every edge of which B contains an inner point lies entirely in B; and*

(iv) *either B is a chord of $\overline{Z}$ (ie. B is an edge whose endvertices lie in $V(\overline{Z})$) or the subgraph $(B \cap G) - V(\overline{Z})$ is non-empty and connected.*

Proof. (i) Suppose x is an inner point of an edge e. Then $x \in \overline{Z}$ implies $e \in Z$, and hence there is a neighbourhood $U \subseteq \overline{Z}$ of x. Thus, U is disjoint from $B \setminus \overline{Z}$, which is a contradiction to that x lies on the boundary of $B \setminus \overline{Z}$.

Let us prove (iii) before (ii): Let e be an edge of which B contains an inner point. Then the interior of e is disjoint from $\overline{Z}$, as otherwise $e \in Z$. Thus, $e \subseteq B$.

(ii) Consider any basic open neighbourhood $\hat{C}(S, x)$. As $x \in \overline{Z}$ there is a vertex $v_Z \in V(\overline{Z} \cap C(S, x))$. Similarily, let us find a vertex $v_B \in V(B \cap C(S, x))$. Since $\hat{C}(S, x) \cap (B \setminus \overline{Z})$ is open, it contains an inner point of an edge e. Then, $e \subseteq B$, by (ii), and hence B contains both endvertices. One of these lies also in

$\hat{C}(S,x)$; take that to be v_B. Now, the first vertex on a v_B–v_Z path through the connected subgraph $C(S,x)$ that lies in $\overline{Z}$ is an attachment of B.

(iv) First, suppose that $B \setminus \overline{Z}$ consists only of inner points of edges. Take two of those, x and y, and consider an arc in $B \setminus \overline{Z}$ between them (which exists by Lemma 4). Then, the arc has to lie completely in a single edge, and hence x and y are inner points of the same edge. Thus, B is a chord of $\overline{Z}$.

Second, if $B \setminus \overline{Z}$ contains an end then it also contains an open neighbourhood of that end. Consequently, $K := (B \cap G) - V(\overline{Z})$ is non-empty. As K is a subset of the path-connected set $B \setminus \overline{Z}$ there is an arc between any two vertices of K which avoids $\overline{Z}$. Then, Lemma 5 yields also a path between them that is disjoint from $\overline{Z}$. Therefore, K is connected. □

A consequence of (iv) is that a bridge is either a chord or it has at least k attachments when G is k-connected. The next lemma provides the main reason why we are interested in bridges.

Lemma 10. *Let G be 3-connected, and consider a circle $C \subseteq |G|$. Then the circuit $E(C)$ is peripheral if and only if C has at most a single bridge.*

Proof. First, let $E(C)$ be peripheral. By Lemma 9 (iv), every bridge has a vertex in $G - V(C)$. Since $G - V(C)$ is connected, there is thus only one bridge of C.

Conversely, let C have only a single bridge B. Then B cannot be a chord as in this case at least one of the vertices of C has degree 2 in G, and by Lemma 9 (iv), $G - V(C)$ is connected. □

Consider a circle C in $|G|$ with a bridge B. If G is finite the attachments of B divide C into edge-disjoint paths, called the residual arcs of B (as long as B has at least two attachments). Our aim is to reproduce this situation in an infinite graph as closely as possible. So let G be an arbitrary graph, and define a *residual arc* of B in C to be the closure of a topological component of $C \setminus B$. Note that if B has at least two attachments every residual arc is indeed an arc (if not then the circle C itself is a residual arc, and it is the only one). As in the finite case, any two residual arcs meet at most in their attachments. However, C does not have to be the union of its residual arcs: consider an end $\omega \in C \cap B$ against which attachments of B converge from both sides on C. Then ω does not lie in any residual arc. Consequently, (iii) fails for ends in the following lemma:

Lemma 11. *Let G be 2-connected, and let $C \subseteq |G|$ be a circle with a bridge B. Then:*

(i) *the endpoints of a residual arc L of B in C are attachments of B;*

(ii) *for a point $x \in C \setminus B$ there is exactly one residual arc L of B in C containing x; and*

(iii) *for a vertex $v \in V(B \cap C)$ there are exactly two residual arcs of B in C with v as an endpoint.*

Proof. (i) The endpoints of L lie in B.

(ii) C is the union of the topological components of $C \setminus B$ and $C \cap B$, the set of attachments. As x is not an attachment it lies in exactly one of the components of $C \setminus B$, and therefore in exactly one residual arc.

(iii) Pick in each of the two with v incident edges in C an inner point, and let L_1 and L_2 be the two residual arcs containing either one of them (as B has at least two attachments no residual arc can contain both). Clearly, both L_1 and L_2 have v as endpoint. $\square$

Definition 12. *Let $C \subseteq |G|$ be a circle with two bridges B and B'.*

(i) *We say that B* avoids *B' if there is a residual arc of B that contains all attachments of B'. Otherwise, they* overlap.

(ii) *B and B' are called* skew *if C contains four points v, v', w, w' in that cyclic order such that v, w are attachments of B, and v', w' attachments of B'.*

Note that if B avoids B' then B' also avoids B. Indeed, let L be a residual arc of B in which all attachments of B' lie. Then, let $x \in C \setminus L$ be any point, and let L' be the residual arc of B' containing x (Lemma 11). As the endpoints of L' are attachments of B' they lie in L. Now, if there is an attachment y of B with $y \notin L'$, then it lies in the interior of L, which is impossible for a residual arc of B. Consequently, avoiding and overlapping are symmetric relations.

Clearly, if two bridges are skew, they overlap. On the other hand, it is not difficult to prove that in a 3-connected graph two overlapping bridges are either skew or 3-equivalent, ie. they both have only three attachments and those are the same. However, we will not make use of this result.

For an edge set Z we define the *overlap graph* of $\overline{Z}$ in G as the graph on the bridges of $\overline{Z}$ where two bridges are adjacent if they overlap. In contrast, let the *skew-overlap graph* of $\overline{Z}$ in G be the graph on the same vertex set such that two bridges are adjacent if they are skew. Clearly, the skew-overlap graph is a subgraph of the overlap graph. See Thomassen [10] for more details on skew-overlap graphs.

It is easy to see that if G is 3-connected then it is impossible for a bridge of a circle to avoid all other bridges (unless there is only one). Thus, the overlap graph of a circle cannot have trivial components. It turns out, even more is true: there is only a single component at all. This will be the main result of this section, which we shall prove with the help of the following lemma.

Lemma 13. *Let G be 3-connected, and let C be a circle in $|G|$. Let K be a connected subgraph in the skew-overlap graph of C in G, and let B be a bridge of C. If there are four points u, u', v, v' appearing in that cyclic order on C such that u and v are each an attachment of a bridge in $V(K)$ and such that u', v' are attachments of B, then B is skew to some bridge in $V(K)$.*

Proof. Pick a bridge $B_u \in V(K)$ for which u is an attachment, and respectively a bridge $B_v \in V(K)$ with attachment v. Denote by L_u the topological component of $C \setminus \{u', v'\}$ that contains u, and by L_v the other one (which contains v). We may assume that B_u has no attachments in L_v as otherwise B_u would clearly be skew to B. Consider a B_u–B_v path P in K (ie. a sequence of consecutively skew bridges). Let B' be the first bridge on P to have attachments in L_v. As B' is skew to its predecessor on P, which has all its attachments in $L_u \cup \{u', v'\}$, B' must have an attachment in L_u. As it has also one in L_v it is skew to B. $\square$

Lemma 14. *If G is 3-connected then for every circle $C \subseteq |G|$ the overlap graph of C in G is connected.*

Proof. First, observe that if C is a triangle then all bridges have $V(C)$ as the set of attachments and are thus mutually overlapping. So we may assume that $|V(C)| \geq 4$.

Second, let K be a component of the skew-overlap graph. Suppose there is a vertex w in $V(C)$ which is not incident with any bridge in K. Let $\mathcal{A}$ be the set of points of C that are attachments for some bridge in $V(K)$. Fix an $a \in \mathcal{A}$, and let $A_1, A_2 \subseteq C$ be arcs from w to a such that $A_1 \cup A_2 = C$. Denote by x_i the first point on A_i in the closed set $\overline{\mathcal{A}}$ (more precisely, if $\sigma_i : [0,1] \to A_i$ is a homeomorphism with $\sigma_i(0) = w$ then let x_i be such that $\sigma_i^{-1}(x_i)$ is minimal under all points in $A_i \cap \overline{\mathcal{A}}$). Note that $w \neq x_i$, as $w \notin \mathcal{A}$, and as w is a vertex. Then $C \setminus \{x_1, x_2\}$ has two topological components, one, L_w, which is disjoint from $\mathcal{A}$ and another, $L_{\mathcal{A}}$, for which $\mathcal{A} \subseteq \overline{L_{\mathcal{A}}}$. Observe that both contain vertices. Indeed, L_w contains w. If $|\mathcal{A}| \geq 3$, then $L_{\mathcal{A}}$ clearly contains vertices (note Lemma 9 (ii)). If $\mathcal{A}$ has cardinality two, then K consists of a single chord with endvertices x_1 and x_2. But then x_1 and x_2 cannot be adjacent in C.

As G is 3-connected there is a path in $G - \{x_1, x_2\}$ from a vertex u' in $V(L_w)$ to a vertex v' in $V(L_{\mathcal{A}})$ which meets C only in $\{u', v'\}$. The path is contained in a bridge of C; denote that bridge by B. Now, we can easily find $u, v \in \mathcal{A}$ such that u, u', v, v' appear in that cyclic order on C. Indeed, if we cannot choose x_1 for u, then $x_1 \in \overline{A} \setminus A$. Consequently, every neighbourhood of x_1 contains elements of $\mathcal{A}$, and thus a neighbourhood that is disjoint from u' and v' yields a suitable $u \in \mathcal{A}$. The same holds for x_2. Lemma 13 yields a bridge in K which is skew to B. Hence $B \in V(K)$, a contradiction since the attachment u' of B lies in L_w, which is disjoint from $\mathcal{A}$.

Finally, we show that every bridge B of C lies in $V(K)$. As C is not a triangle, B has two attachments u', v' which are not adjacent vertices in C. Thus, we find two other vertices u, v such that u, u', v, v' appear in that cyclic order on C. As u and v are each an attachment of some bridge in $V(K)$, Lemma 13 shows $B \in V(K)$. □

So far, the results were true for arbitrary graphs. For the final lemma of this section, which we will need in the next one, G has to be locally finite.

Lemma 15. *Assume that G is locally finite, and consider a circle $C \subseteq |G|$ with bridge B. Let ω be an end which is an attachment of B in C. Then $B \setminus C$ contains a ray of ω.*

For the proof we need a simple lemma which can be found in [4]:

Lemma 16. *Let U be an infinite set of vertices in a connected locally finite graph H. Then there exists a ray $R \subseteq H$ for which H contains an infinite set of disjoint R–U paths.*

Proof of Lemma 15. By Lemma 9 (ii), there is a sequence $x_1, x_2, \ldots \subseteq V(G)$ of attachments of B converging against ω. Let U be the set of neighbours of the x_i in $K := (B \cap G) - V(C)$. Applying Lemma 15 to the graph K (which is connected by Lemma 9 (iv)) and the set U, we obtain a ray R and infinitely many disjoint R–U paths in $K \subseteq B \setminus C$. Then every neighbourhood of ω contains a tail of R, and thus $R \in \omega$. □

5 Locally generating a circuit

In this section we prove the following lemma, which will later be used in the induction step for the proof of our main result. For a vertex v, denote by $E(v)$ the set of incident edges.

Lemma 17. *Consider a circle $C \subseteq |G|$ for a locally finite 3-connected graph G. Let B be a bridge of C, and let v be a vertex in $V(C \cap B)$. Then there are peripheral circuits $D_1, \ldots, D_m$ which are disjoint from $E(B)$ and which satisfy*

$$\sum_{i=1}^{m} D_i \cap E(v) = E(C) \cap E(v).$$

Apart from being used in the proof of Theorem 2, the lemma may serve as an indicator that the theorem itself is not unreasonable. Indeed, at the very least one should be able to find a peripheral circuit for any given edge—and this is in fact the case according to the lemma.

Lemma 17 and its proof are inspired by a result of Tutte [11, (2.2)]. For the remainder of this section let us work in $|G|$ for a fixed 3-connected locally finite graph G.

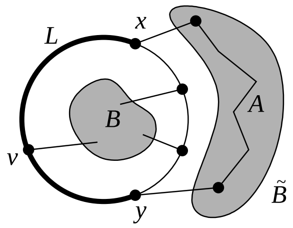

Figure 5: The extension step

Consider a circle $C \subseteq |G|$ with a bridge B and a vertex $v \in V(C \cap B)$. Let a bridge $\tilde{B}$ overlapping B be given, and let L be a residual arc of $\tilde{B}$ in C that meets v. Denote by x and y the two endpoints of L. Our aim is to replace the arc $(C \setminus L) \cup \{x, y\}$ by an arc A through $\tilde{B}$ which is internally disjoint from C; see Figure 5. For this, let us find a topological path P_x from x to a point $x' \in B \setminus C$ that meets C only in x, and analogously P_y and y'. Then, as $B \setminus C$ is path-connected (by Lemma 4) there is a topological path $P \subseteq B \setminus C$ from x' to y'. Finally, Lemma 3 yields an arc $A \subseteq P_x \cup P \cup P_y$ between x and y.

So how do we find x' and P_x (respectively y' and P_y)? If x is a vertex we may simply take any edge $[x, z]$ in $\tilde{B}$ incident with x, and put $P_x := [x, x']$ where x' is any inner point of the edge. Observe, that we may choose freely the edge $[x, z]$ (as long as it lies in $\tilde{B}$), and that $[x, z] \subseteq A$. So assume that x is an end. Lemma 15 yields then a ray $R \subseteq B \setminus C$ of x. Let x' be the starting vertex of R and put $P_x := R \cup \{x\}$.

As a result, $C' := L \cup A$ is a circle that contains v and that has a bridge $B' \supseteq B$. We say (C', B') is *gained* from (C, B) through the *extension step* $(\tilde{B}, L, v)$.

More precisely, the following basic properties hold:

Lemma 18. *Let C be a circle with a bridge B, and let $v \in V(C \cap B)$. If (C', B') is gained from (C, B) by the extension step $(\tilde{B}, L, v)$ then:*

(i) $C' \subseteq L \cup (\tilde{B} \setminus C)$;

(ii) $(B \cup C) \setminus L \subseteq B' \setminus C'$; *and*

(iii) $E(C \cup B) \subseteq E(C' \cup B')$.

Proof. Assertion (i) is true by definition of A. For (ii), note that $B \setminus L \subseteq B' \setminus L$, and that because of (i), B meets C' only in L. Hence $B \setminus L \subseteq B' \setminus C'$. Also, since B and $\tilde{B}$ are overlapping, B cannot have all its attachments in L. Thus, there is an attachment of B in the connected set $C \setminus L$, which is disjoint from C', by (i), and therefore becomes a part of $B' \supseteq B$. This shows (ii). Finally, for (iii), note that $C = L \cup (C \setminus L) \subseteq C' \cup B'$. □

If $\tilde{B}$ contains v then there exists a second residual arc L' of $\tilde{B}$ in C meeting v, by Lemma 11 (iii). In that case we find an arc A' from v to the other endpoint y' of L' which meets C only in $\{v, y'\}$ and which uses the same e edge incident with v as A (as noted above). Then $C'' := L' \cup A'$ is a circuit with a bridge B'' so that (C'', B'') is gained from (C, B) by the extension step $(\tilde{B}, L', v)$. We call (C'', B'') a *twin* of (C', B') with respect to (C, B). We see that the following holds

$$E(C) \cap E(v) = (E(C') + E(C'')) \cap E(v). \qquad (*)$$

The main idea in the proof of Lemma 17 is the following: given a circle C_1 with a bridge B_1 we try to inflate the bridge B_1. More precisely, we will construct a sequence $(C_1, B_1), (C_2, B_2), \ldots$ of circle-bridge pairs such that (C_i, B_i) is gained from its predecessor by an extension step. Then, we have $B_i \supseteq B_{i-1}$ and our aim is to do the extension in such a way that eventually B_i grows so big that it is the only bridge left. Clearly, the corresponding circuit is then peripheral. Unfortunately, this sequential approach may be insufficient. Rather, it is sometimes necessary to perform two alternative extension steps simultaneously. This parallel approach is captured in the concept of an extension tree we shall now introduce.

Definition 19. *Let C be a circle with a bridge B, and consider a vertex $v \in V(C \cap B)$. Let T be a finite rooted tree with root r, and let there be mappings*

$$\begin{aligned} C_T : V(T) &\rightarrow \{C' \subseteq |G| \,:\, C' \textit{ is a circle}\} \\ B_T : V(T) &\rightarrow \{B' \subseteq |G| \,:\, B' \textit{ is a bridge of a circle}\} \end{aligned}$$

satisfying the following:

(i) *for $w \in V(T)$, $B_T(w)$ is a bridge of the circle $C_T(w)$;*

(ii) *$C_T(r) = C$ and $B_T(r) = B$;*

(iii) *let $p \in V(T)$ be the parent of $w \in V(T)$. Then there is a bridge $\tilde{B}$ overlapping $B_T(p)$ and a residual arc L of $\tilde{B}$ in $C_T(p)$ meeting v such that $(C_T(w), B_T(w))$ is gained from $(C_T(p), B_T(p))$ by the extension step $(\tilde{B}, L, v)$; and*

(iv) *let* $p \in V(T)$ *have a child* u *such that* $(C_T(u), B_T(u))$ *has a twin with respect to* $(C_T(p), B_T(p))$. *Then* p *has exactly one other child and that is mapped on such a twin.*

If all these conditions are satisfied we call T *or, more formally,* (T, r, C_T, B_T), *an* extension tree *with parameters* (C, B, v).

Note that, firstly, (iii) and (iv) imply that a vertex in an extension tree has at most two children. Secondly, deleting all descendants of a given vertex and restricting the mappings to the remaining vertices will yield another extension tree with the same parameters. And finally, the converse operation leads to an extension tree too: let $(T_1, r, C_{T_1}, B_{T_1})$ be an extension tree with parameters (C, B, v), l a leaf of T_1 and $(T_2, l, C_{T_2}, B_{T_2})$ an extension tree with parameters $(C_{T_1}(l), B_{T_1}(l), v)$. Then the tree $T := T_1 \cup T_2$ with root r together with the mappings C_T, B_T is an extension tree with parameters (C, B, v), where C_T and B_T are induced by the corresponding mappings on T_1 and T_2.

Lemma 20. *Let* T *be an extension tree with parameters* (C, B, v). *For every vertex* $t \in V(T)$ *the circuit* $E(C_T(t))$ *is disjoint from* $E(B)$.

Proof. Induction on the depth of t—note that if p is the parent of t we have $B_T(p) \subseteq B_T(t)$, and that $B_T(p) \cap C_T(t) \subseteq B_T(t) \cap C_T(t)$ is a set of attachments, hence a set of vertices and ends, by Lemma 9 (i). □

The next lemma is the reason why we have introduced extension trees at all, instead of employing a sequential algorithm: the circles associated with the leaves sum to precisely the edges in C at v.

Lemma 21. *Let* T *be an extension tree with parameters* (C, B, v). *Then the following holds*

$$\sum_{l \text{ leaf of } T} E(C_T(l)) \cap E(v) = E(C) \cap E(v).$$

Proof. Induction on $|V(T)|$ and the equation $(*)$. □

Recall that we want to inflate the bridge B so that eventually it is the only one left. To achieve this, we have to ensure that B grows in a relatively controlled way. In particular, we need to be able to perform the extension steps in such a way that after finitely many steps our favourite bridge $\tilde{B}$ can be used for the next step (if it is not already contained in the inflated bridge).

Lemma 22. *Let* C *be a circle with bridge* B, *and let* $v \in V(C \cap B)$. *For a bridge* $\tilde{B}$ *of* C *there is an extension tree* T *with parameters* (C, B, v) *such that for every leaf* l *of* T *holds that*

$$\textit{either } \tilde{B} \subseteq B_T(l) \textit{ or } \tilde{B} \textit{ is a bridge of } C_T(l) \textit{ overlapping } B_T(l).$$

Proof. By Lemma 14, there is a B–$\tilde{B}$ path P in the overlap graph of C in G. We do an induction on the length of such a path.

If P is trivial we have $B = \tilde{B}$, and if P has length one then B and $\tilde{B}$ are overlapping. In both cases we are done. So assume that P has length $k-1$, which is at least two. Let $P = K_1 \dots K_k$, where $K_1 = B$ and $K_k = \tilde{B}$. We define an extension tree T' with parameters (C, B, v) as follows. For the root r we map $C_{T'}(r) := C$ and $B_{T'}(r) := B$. Now, let L be a residual arc of the bridge K_2 in C that meets v, and let (C', B') be gained from (C, B) by the extension step (K_2, L, v). We assign a vertex c' to be a child of r and map $C_{T'}(c') := C'$ and $B_{T'}(c') := B'$. Should (C', B') have a twin (C'', B'') with respect to r, we let r have a second child c'' and define the mappings accordingly. The resulting tree T' is an extension tree.

Consider the child c'. The bridge K_3 overlaps K_2 and has therefore, by definition, an attachment in $C \backslash L$. As $K_3 \cap C' \subseteq K_3 \cap C$ this implies together with $C \backslash L \subseteq B' \setminus C'$ (by Lemma 18 (ii)) that $K_3 \subseteq B'$. Therefore, the greatest index $i \leq k$ with $K_i \subseteq B'$ is at least three. Any bridge K_j with a greater index j must necessarily have all its vertices of attachement in L and is thus still a bridge of C'. Also, if for $j, j' > i$, K_j and $K_{j'}$ are overlapping as bridges of C then they are overlapping as bridges of C' as well (as this is decided on L). If $j = k$ then $\tilde{B} \subseteq B'$, and we define $T_1 := \emptyset$. Otherwise, $B' K_{i+1} \ldots K_k$ is a path in the overlap graph of C'. As it is shorter than P, induction yields an extension tree T_1 with parameters (C', B', v) such that the associated bridge of every leaf either contains $\tilde{B}$ or overlaps it. Now, if r has only one child the tree $T := T' \cup T_1$ with root r satisfies the assertion. Otherwise, we obtain in a similar way an extension tree T_2 with parameters (C'', B'', v) and $T := T' \cup T_1 \cup T_2$ is the desired tree. $\square$

If we can find an extension tree for which the circuit $E(C_T(l))$ for every leaf l is peripheral, we are done, as Lemmas 20 and 21 demonstrate. We introduce a measure of how "far" the leaves of an extension tree are from being peripheral.

Let T be an extension tree with parameters (C, B, v). For a vertex t of T we define

$$\rho(t) := \sup\{N \in \mathbb{Z} : \forall e \in E(G),\, d_G(v, e) \leq N \Rightarrow e \in E(C_T(t) \cup B_T(t))\}$$

where we admit ∞. This definition ensures that every edge e with $d_G(v, e) \leq \rho(t)$ lies in $E(C_T(t) \cup B_T(t))$. If $\rho(t) = \infty$ then every edge is contained in $E(C_T(t) \cup B_T(t))$, and $E(C_T(t))$ is therefore a peripheral circuit. With that we define

$$\rho(T) := \min\{\rho(l) : l \text{ is leaf of } T\}.$$

Thus, rewriting our statement above, if we can find an extension tree with $\rho(T) = \infty$, we are done. Unfortunately, this is not always possible.

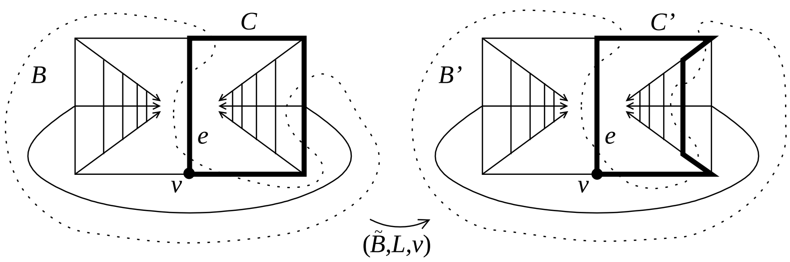

Figure 6: An extension step

Let us have a look at a concrete example. Consider the circle C on the left in Figure 6. It has two bridges, the bridge B' indicated in the figure, and another one $\tilde{B}$ consisting of the infinite ladder in the interior face of C. We have $\rho = 0$ for the circle-bridge pair (C, B) as there are edges incident with a neighbour of v that lie in $\tilde{B}$. We push up ρ by performing an extension step through $\tilde{B}$, the result of which is seen on the right of Figure 6. Indeed, for the circle-bridge pair (C', B') we obtain $\rho = 1$. Note that C' is formed by replacing $C \setminus L$ (where L is the residual arc of $\tilde{B}$ which contains v) with a finite path A using a rung of the infinite ladder that comprises $\tilde{B}$. Now, the next extension step might use a similar arc A' along another one of the rungs of the infinite ladder, and so on. Then, all the subsequently gained circles will be finite, and consequently, we will never reach an extension tree for which $\rho(T) = \infty$, since there is no finite peripheral circuit containing e.

Lemma 23, however, shows that we can achieve the next best thing, namely finding a sequence of nested extension trees with strictly increasing $\rho(T)$. First, we make the notion of nested extension trees more precise.

To keep notation simple we will just write $T \subseteq T'$ for two extension trees T and T' while tacitly assuming that both trees are extension trees with the same parameters and the same root and that the mappings C_T and B_T of T are induced by the corresponding mappings of T'.

So let $(T_n)_{n\in\mathbb{N}}$ be a family of extension trees with parameters (C, B, v) such that $T_n \subseteq T_{n+1}$ for all $n \in \mathbb{N}$. Then we call the family an *extension family* with *parameters* (C, B, v).

Lemma 23. *Let C be a circle with a bridge B, and let $v \in V(C \cap B)$. Assume that there is no extension tree T with parameters (C, B, v) and $\rho(T) = \infty$. Then there is an extension family $(T_n)_{n\in\mathbb{N}}$ with parameters (C, B, v) so that*

$$\rho(T_1) < \rho(T_2) < \dots$$

Proof. We will inductively construct nested extension trees $T_1, \dots, T_n$ with $\rho(T_1) < \dots < \rho(T_n)$. For T_1, take any extension tree with parameters (C, B, v). For $n > 1$, let $T_1, \dots, T_n$ be already constructed.

Setting $d := \rho(T_n) + 1$ (note that $\rho(T_n) < \infty$) we put

$$m(t) := |\{e \in E(G)\setminus E(C_T(t) \cup B_T(t)) \,:\, d_G(v, e) \le d\}|$$

for a vertex t of an extension tree T, and define

$$m(T) := \max_{l \text{ leaf of } T} m(l).$$

Observe, that since G is locally finite, $m(T)$ is always finite (and is, in particular, defined). We see that $m(T) = 0$ implies $\rho(T) \ge d$. Thus, our task is to find an extension tree T with $T \supseteq T_n$ and $m(T) = 0$. For this, it suffices to establish that

> *for each leaf p of T_n there is an extension tree T_p with root p and parameters $(C_{T_p}(p), B_{T_p}(p), v)$ such that $m(T_p) < m(T_n)$.*

Indeed, the union T of all those T_p and T_n is an extension tree (with parameters (C, B, v)) with $T \supseteq T_n$ and $m(T) < m(T_n)$. An induction argument then allows us to find a T_{n+1} with $m(T_{n+1}) = 0$.

To establish the claim, consider an edge $e \notin E(C_{T_n}(p) \cup B_{T_n}(p))$ with $d_G(v,e) \leq d$. Thus e contributes to $m(p)$. Denote by $\tilde{B}$ the bridge of $C_{T_n}(p)$ containing e. With Lemma 22 we find an extension tree T' with parameters $(C_{T_n}(p), B_{T_n}(p), v)$ such that the associated bridge of every leaf of T' either contains $\tilde{B}$ or overlaps it.

Let l be a leaf of T'. If $\tilde{B} \subseteq B_{T'}(l)$ then e is contained in $B_{T'}(l)$ as well, and we obtain $m(l) < m(p)$. So assume $\tilde{B}$ and $B_{T'}(l)$ to be overlapping. Let L be a residual arc of $\tilde{B}$ in $C_{T'}(l)$ containing v, and let (C', B') be gained from $(C_{T'}(l), B_{T'}(l))$ by the extension step $(\tilde{B}, L, v)$. Assume for the moment that (C', B') has no twin with respect to l. Denote by T the extension tree obtained from T' by adding a child c to l and mapping $C_T(c) := C'$ and $B_T(c) := B'$. Observe, that e has a vertex in common with $C_T(l)$, say the vertex w (every edge with lesser distance to v is contained in $C_T(l) \cup B_T(l)$). Should w be contained in $C_T(l) \backslash L$, we have by Lemma 18 (ii), $e \subseteq B_T(c)$, leading to $m(c) < m(p)$. Thus, we have to deal with the case that w is one of the two endpoints of L (being an attachment, w cannot be contained in the interior of L). The w–$\tilde{B}$ edge f that is used to construct $C_T(c)$ clearly has the same distance to v as e. Consequently, f contributes to $m(p)$—but not to $m(c)$. Again, this leads to $m(c) < m(p)$. Should c have a twin with respect to l we extend T' in a similar way for that twin. Modifying T' in this way for each leaf l we arrive at the desired extension tree T_p. □

Let $(T_n)_{n\in\mathbb{N}}$ be an extension family with parameters (C, B, v). The union $T := \bigcup_{n\geq 1} T_n$ is then an infinite rooted tree. We extend the mappings C_{T_n} and B_{T_n} of the T_n to mappings C_T and B_T of T in the natural way. T will be called an *infinite extension tree* with *parameters* (C, B, v). To distinguish clearly between these infinite extension trees and the extension trees defined earlier (in Definition 19) we shall speak of *finite* extension trees when the latter ones are meant.

By Lemma 18 (iii), $\rho(p) \leq \rho(c)$ holds for a child c of a vertex p in an extension tree. Thus, in the extension family (T_n) the sequence $\rho(T_n)$ is monotonically non-decreasing, and hence $\rho(T) := \lim_{n\mapsto\infty} \rho(T_n)$ is well defined (if we admit ∞).

With this terminology Lemma 23 asserts that if we cannot find a finite extension tree of which all the (associated circuits of the) leaves are peripheral, then there is an infinite extension tree T with $\rho(T) = \infty$. Being infinite, T has rays starting in the root vertex. These rays play a similar role as the leaves, and indeed we may extract a peripheral circuit from each of these rays.

Lemma 24. *Let C be a circle with a bridge B, and let $v \in V(C \cap B)$. Let T be an infinite extension tree with parameters (C, B, v) and $\rho(T) = \infty$. Consider a ray $c_1c_2\ldots$ in T starting in the root vertex of T. Then there is a peripheral circuit D which is disjoint from $E(B)$ and for which there is an $M \in \mathbb{N}$ such that $D \cap E(v) = E(C_T(c_n)) \cap E(v)$ for every $n \geq M$.*

Proof. Let (T_n) be the T defining extension family. We may assume that $\rho(T_1) < \rho(T_2) < \ldots$. For $n \in \mathbb{N}$, put $C_n := C_T(c_n)$, $B_n := B_T(c_n)$ and $V_n := \{w \in V(G) : d_G(v,w) \leq n\}$.

(i) First, we claim that

$$\textit{for all } m \in \mathbb{N} \textit{ there is a } N_m \in \mathbb{N} \textit{ such that } C_n \cap G[V_m] = C_{N_m} \cap G[V_m] \textit{ for all } n \geq N_m. \tag{1}$$

Then, putting

$$Z := \bigcup_{m \in \mathbb{N}} E(C_{N_m} \cap G[V_m]),$$

we see that

$$Z \cap E(v) = E(C_n) \cap E(v) \text{ for all } n \geq N_1 =: M \tag{2}$$

(note that for $m = 1$ all neighbours of v and v itself lie in $G[V_m]$). In addition, observe that every vertex of G is either incident with at most two of the edges in Z.

So let us prove (1). For $m \in \mathbb{N}$, assume $N_1, \ldots, N_{m-1}$ to be already defined. With $N \in \mathbb{N}$ such that $m \leq \rho(T_N)$ we observe that

$$C_{n+1} \cap G[V_m] \subseteq C_n \cap G[V_m] \subseteq C_N \cap G[V_m] \tag{3}$$

holds for any $n \geq N$. Indeed, note that $m \leq \rho(T_N)$ implies $G[V_m] \subseteq B_N \cup C_N$. Let (C_{N+1}, B_{N+1}) be gained by the extension step $(\tilde{B}, L, v)$. Then, $\tilde{B} \cap G[V_m] \subseteq \tilde{B} \cap (B_N \cup C_N) \subseteq \tilde{B} \cap C_N$. Hence, $\tilde{B} \setminus C_N$ is disjoint from $G[V_m]$. By Lemma 18 (i), we obtain

$$C_{N+1} \cap G[V_m] \subseteq (L \cup (\tilde{B} \setminus C_N)) \cap G[V_m] \subseteq C_N \cap G[V_m].$$

Inductively, (3) follows.

Since $C_n \cap G[V_m]$ is finite for each $n \geq N$, it follows from (3) that there is an N_m from which point on $C_n \cap G[V_m]$ does not change. This shows (1).

(ii) Next, consider a finite cut F of G. Choose m large enough for $F \subseteq E(G[V_m])$. We obtain

$$F \cap Z = (F \cap E(G[V_m])) \cap Z = F \cap (Z \cap E(G[V_m])) = F \cap E(C_{N_m}),$$

where the last equality holds because of (1). Since $E(C_{N_m}) \in \mathcal{C}(G)$, $F \cap E(C_{N_m})$ is an even set, by Theorem 6. Therefore, by the other direction of Theorem 6, $Z \in \mathcal{C}(G)$. Consequently, it is, by Theorem 7, a disjoint union of circuits. Exactly one of these circuits, which we denote by D, is incident with v (since v is incident with exactly two edges of Z). Thus, (2) yields $D \cap E(v) = E(C_n) \cap E(v)$ for all $n \geq M$, as desired.

By Lemma 20, all the circuits $E(C_n)$ are disjoint from $E(B)$, so this holds for D as well.

(iii) Finally, what remains is to show that D is peripheral. We start with the claim that $\overline{Z}$ has a single bridge B_Z. Let us show that any two points $x, y \in |G| \setminus \overline{Z}$ lie in the same bridge. As every bridge is either a chord or contains vertices, by Lemma 9 (iv), we may assume that neither of x, y is an end. Thus, both lie in some edge and we may choose m large enough so that at the same time $x, y \notin C_{N_m}$ and that the distance from v to any edge x or y is incident with is at most $\rho(T_{N_m})$. Clearly, this choice of m implies $x, y \in B_{N_m} \setminus C_{N_m}$. Thus, there is an arc A with endpoints x and y in the, by Lemma 4, path-connected set $B_{N_m} \setminus C_{N_m}$. As $B_{N_n} \supseteq B_{N_m}$ for every $n \geq m$, it follows that $A \subseteq |G| \setminus C_{N_n}$, and therefore $A \subseteq |G| \setminus \overline{Z}$. Thus, $\overline{Z}$ has only one bridge B_Z.

As a subset of $\overline{Z}$, $\overline{D}$ has a bridge $B_D \supseteq B_Z$. Note that $\overline{Z}$ has no chords as otherwise the only bridge B_Z is a chord, which implies that there is a vertex in G with degree two (since every vertex is incident with at most two of the edges in Z). Then also $\overline{D}$ has no chords, and if every vertex in $|G| \setminus \overline{D}$ lies in $B_D \setminus \overline{D}$

then $\overline{D}$ has only a single bridge, by Lemma 9 (iv), and D is a peripheral circuit (Lemma 10).

So consider a vertex $v \in |G| \setminus \overline{D}$. If $v \notin \overline{Z}$ then $v \in B_Z \setminus \overline{Z} \subseteq B_D \setminus \overline{D}$. Thus, let $v \in \overline{Z}$. Then v has a neighbour $w \notin \overline{Z}$ (otherwise the edge vw is a chord), which consequently lies in $B_D \setminus \overline{D}$. Hence, $v \in B_D \setminus \overline{D}$ too. $\square$

We have seen that the desired peripheral circuits may be obtained from the leaves and rays (starting in the root vertex) of a suitable extension tree. However, we want only finitely many peripheral circuits, but an infinite tree can clearly have more leaves and rays. Fortunately, this cannot happen in an extension tree:

Lemma 25. *Let T_∞ be an infinite extension tree. Then T_∞ has only finitely many leaves and only finitely many rays starting in its root vertex.*

Proof. Let T_∞ have the parameters (C, B, v). To establish the assertion it suffices to show that there is a $N \in \mathbb{N}$ such that for any finite extension tree T with parameters (C, B, v) the number of leaves is bounded by N. For a vertex t of T, let $k(t)$ be the number of edges of G incident with v that are not contained in $C_T(t) \cup B_T(t)$. We prove by induction on $|V(T)|$ that

$$\sum_{l \text{ leaf of } T} 2^{k(l)} \leq 2^{k(r)},$$

where r denotes the root vertex of T. Note that every leaf is counted in the sum, as $2^{k(l)} \geq 1$, and that the righthand side is the same for all extension trees with parameters (C, B, v).

Clearly, the inequality holds for trivial trees. So assume $|V(T)| > 1$. Then, we find a vertex p of T all of whose children are leaves. Deleting all these children leads to an extension tree T' with fewer vertices than T. If p has only a single child c, we have $k(c) \leq k(p)$ by Lemma 18 (iii). Now, assume that p has two children c and d. Let $(C_T(c), B_T(c))$ be gained from its parent by the extension step $(\tilde{B}, L, v)$. Consider the edge $f \in E(C_T(c)) \setminus E(C_T(p))$ incident with v (such an edge exists as $(C_T(c), B_T(c))$ has a twin). We see that $f \in E(C_T(c) \cup B_T(c))$ but $f \notin E(C_T(p) \cup B_T(p))$, leading to $k(c) < k(p)$. By symmetry this holds for d as well, yielding $2^{k(c)} + 2^{k(d)} \leq 2^{k(p)}$. Summing over all leaves of T we obtain

$$\begin{aligned}
\sum_{l \text{ leaf of } T} 2^{k(l)} &\leq \sum_{\substack{l \text{ leaf of } T, \\ \text{no child of } p}} 2^{k(l)} + 2^{k(p)} \\
&= \sum_{l' \text{ leaf of } T'} 2^{k(l')} \\
&\leq 2^{k(r)},
\end{aligned}$$

where the last inequality is because of the induction hypothesis. $\square$

We can now put the pieces together.

Proof of Lemma 17. If there is a finite extension tree T with parameters (C, B, v) such that $\rho(T) = \infty$ then we are done, by Lemmas 20 and 21.

So assume otherwise. By Lemma 23 there is an extension family (T_n) with parameters (C, B, v) such that $\rho(T_1) < \rho(T_2) < \ldots$. Hence, we obtain an infinite extension tree $T := \cup_{n\geq 1} T_n$ with parameters (C, B, v) and with $\rho(T) = \infty$. The number of leaves of T is, because of Lemma 25, finite. We denote those leaves by $l_1, \ldots, l_k$ and put $D_1 := E(C_T(l_1)), \ldots, D_k := E(C_T(l_k))$. The circuits D_i are by definition peripheral and, by Lemma 20, disjoint from $E(B)$. Lemma 25 also ensures that there are only finitely many rays starting in the root vertex; let these be $R_{k+1}, \ldots, R_m$. With Lemma 24 we obtain a peripheral circuit D_i from each of the rays R_i. In addition, for each $i \in \{k+1, \ldots, m\}$, D_i is disjoint from $E(B)$ and there is a vertex t_i on the ray R_i such that $D_i \cap E(v) = E(C_T(t)) \cap E(v)$ for every vertex t on the tail $t_i R_i$.

Choose N large enough so that T_N contains all the leaves l_i from T, and all the vertices t_i. The set of leaves of T_N is then

$$\{l_1, \ldots, l_k, t'_{k+1}, \ldots, t'_m\},$$

where t'_i is a vertex on the tail $t_i R_i$. Together with Lemma 21 we obtain

$$\begin{aligned} E(C) \cap E(v) &= \sum_{l \text{ leaf of } T_N} E(C_{T_N}(l)) \cap E(v) \\ &= \sum_{i=1}^{k} E(C_{T_N}(l_i)) \cap E(v) + \sum_{i=k+1}^{m} E(C_{T_N}(t'_i)) \cap E(v) \\ &= \sum_{i=1}^{m} D_i \cap E(v). \end{aligned}$$

□

6 Generating the cycle space

Theorem 2 will be proved by induction, where Lemma 17 provides the essential part of the induction step. However, the lemma is only applicable to circuits, but we are dealing with arbitrary elements of the cycle space. In order to overcome this, we strengthen Lemma 17:

Lemma 26. *Let G be a locally finite and 3-connected graph. Let $Z \in \mathcal{C}(G)$, let B be a bridge of $\overline{Z}$, and let $v \in V(B)$ be a vertex. Then there are peripheral circuits $D_1, \ldots, D_m$ each of which is disjoint from $E(B)$ such that for $Z' := Z + \sum_{i=1}^{m} D_i$ holds*

(i) *$\overline{Z'}$ leaves a bridge $B' \supseteq B$; and*

(ii) *$E(v) \subseteq E(B')$.*

Proof. By Theorem 7, Z is a union of disjoint circuits; denote by $C_1, \ldots, C_n$ those of these circuits which are incident with v. Since $C_i \subseteq Z$, the circle $\overline{C_i}$ leaves a bridge B_i containing B. Therefore, applying Lemma 17 to $\overline{C_i}$ with bridge B_i yields peripheral circuits $D_{i1}, \ldots, D_{im_i}$ each of which is disjoint from $E(B_i) \supseteq E(B)$ and for which hold

$$\sum_{j=1}^{m_i} D_{ij} \cap E(v) = C_i \cap E(v)$$

(for $i = 1, \dots, n$). Summing up, we arrive at

$$\sum_{i=1}^{n} \sum_{j=1}^{m_i} D_{ij} \cap E(v) = \sum_{i=1}^{n} C_i \cap E(v) = Z \cap E(v). \tag{4}$$

Next, observe that $\overline{D_{ij}}$ is disjoint from $B \setminus \overline{Z}$, for each i, j. Indeed, $B \setminus \overline{Z}$ cannot contain any inner points of edges in D_{ij}, as D_{ij} is disjoint from $E(B)$ (note Lemma 9 (iii)). Since $B \setminus \overline{Z}$ is open this leads to $(B \setminus \overline{Z}) \cap \overline{D_{ij}} = \emptyset$.

Consequently, the connected set $B \setminus \overline{Z}$ is contained in a bridge of $\overline{D_{ij}}$ and then also in a bridge of $\overline{Z''}$, for $Z'' := Z \cup \bigcup_{i,j} D_{ij}$. Now, for $Z' := Z + \sum_{i=1}^{n} \sum_{j=1}^{m_i} D_{ij}$, $Z' \subseteq Z''$ implies that $\overline{Z'}$ has a bridge $B' \supseteq B$.

Finally, (4) shows that v is not incident with any edge in Z'. Together with $v \in V(B)$ this leads to $E(v) \subseteq E(B')$, as required. $\square$

We now prove our main result, which we restate.

Theorem 2. *Every element of the cycle space $\mathcal{C}(G)$ of a locally finite 3-connected graph G is a sum of peripheral circuits.*

Proof. We begin by proving the following statement.

> *Let $Z \in \mathcal{C}(G)$, and let B be a bridge of $\overline{Z}$. Then Z is the sum of a thin family $\mathcal{D}$ of peripheral circuits.* (5)

Fix a vertex b in B, and let $\{e_1, e_2, \dots\}$ be an enumeration of the edge set of G such that $d_G(b, e_i) < d_G(b, e_j)$ implies $i < j$. We will obtain $\mathcal{D}$ as the union of inductively constructed finite sets of peripheral circuits. More formally, for all $n \in \mathbb{N}$ we inductively show the existence of finite sets $\mathcal{D}_n$ of peripheral circuits and of bridges B_n of $\overline{Z_n}$, where $Z_n := Z + \sum \mathcal{D}_n$, satisfying

(i) $B_{n-1} \subseteq B_n$;

(ii) $\mathcal{D}_{n-1} \subseteq \mathcal{D}_n$;

(iii) every $D \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$ is disjoint from $E(B_{n-1})$; and

(iv) $\{e_1, \dots, e_n\} \subseteq E(B_n)$,

where $\mathcal{D}_0 := \emptyset$, $B_0 := B$ and $Z_0 := Z$.

For $n \in \mathbb{N}$, let $\mathcal{D}_{n-1}$ and B_{n-1} be already constructed. We claim that e_n is incident with a vertex v in B_{n-1}. If $d_G(b, e_n) = 0$ this is obvious, so let there be an edge e_i adjacent to e_n with strictly lesser distance to b. By the choice of the enumeration we obtain $i < n$ and in turn $e_i \subseteq E(B_{n-1})$. Hence, the vertex v incident with both edges e_i and e_n is contained in B_{n-1} as well.

By applying Lemma 26 to Z_{n-1}, bridge B_{n-1} and vertex v we obtain peripheral circuits $D_1, \dots, D_m$ each of which is disjoint from $E(B_{n-1})$. In addition, $\overline{Z_n}$ leaves a bridge $B_n \supseteq B_{n-1}$ such that $E(v) \subseteq E(B_n)$. Since $e_n \in E(v)$ the conditions (i)–(iv) are clearly satified by putting $\mathcal{D}_n := \mathcal{D}_{n-1} \cup \bigcup_{i=1}^{m} D_i$.

We claim that $\mathcal{D} := \bigcup_{n \geq 1} \mathcal{D}_n$ satisfies the assertion of the claim (5). To see this, consider an edge e_n of G. First note that because of the conditions (iii) and (iv) the edge e_n may only be used by the finitely many circuits in $\mathcal{D}_n$ and by none other in $\mathcal{D}$—proving $\mathcal{D}$ to be a thin family. Furthermore, (iv) implies that

$$e_n \notin Z + \sum \mathcal{D}_n \text{ and hence } e_n \notin Z + \sum \mathcal{D}$$

by the preceding argument. Thus, (5) is established.

Now, if for $Z \in \mathcal{C}(G)$, $\overline{Z}$ leaves a bridge, (5) guarantees that Z is the sum of peripheral circuits. If that is not the case, let C be any circuit of G. Then, both $\overline{C}$ and $\overline{Z+C}$ have at least one bridge and the statement (5) may be applied to each of them. Clearly, the union of the two generating sets of peripheral circuit is a generating set of Z. $\square$

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