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Subdivisions of K_{r+2} in graphs of average degree at least $r+\varepsilon$ and large but constant girth

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Abstract

We show that for every $\varepsilon > 0$ there exists an $r_0 = r_0(\varepsilon)$ such that for all integers $r \geq r_0$ every graph of average degree at least $r + \varepsilon$ and girth at least 1000 contains a subdivision of K_{r+2}. Combined with a result of Mader this implies that for every $\varepsilon > 0$ there exists an $f(\varepsilon)$ such that for all $r \geq 2$ every graph of average degree at least $r + \varepsilon$ and girth at least $f(\varepsilon)$ contains a subdivision of K_{r+2}. We also prove a more general result concerning subdivisions of arbitrary graphs.

1 Introduction

A classical result of Mader states that for every r there exists a smallest number $d(r)$ such that every graph G of average degree larger than $d(r)$ contains a subdivision of the complete graph K_r on r vertices. Bollobás and Thomason [2] as well as Komlós and Szemerédi [5] independently proved that $d(r) = O(r^2)$. As Jung [4] observed, the complete bipartite graph $K_{s,s}$ with $s = \lfloor r^2/8 \rfloor$ shows that this is the correct order of magnitude. However, Mader [11] showed that if the girth of the graph G is large, then the necessary lower bound on the average degree already suffices:

Theorem 1 (Mader) *For every integer $r \geq 2$ and every $\varepsilon > 0$ there exists an integer $g(r, \varepsilon)$ such that every graph G of average degree at least $r + \varepsilon$ and girth at least $g(r, \varepsilon)$ contains a subdivision of K_{r+2}.*

As there are r-regular graphs of arbitrarily large girth, this result is best possible in the sense that the condition $\varepsilon > 0$ is necessary. Mader's bound on the girth required is quadratic in r and also depends on ε. The main result of this paper is that large but constant girth suffices, provided that r is sufficiently large compared with ε:

Theorem 2 *Let $\varepsilon > 0$ and let r be an integer such that $r \geq \max\{100, 400/\varepsilon^2\}$. Then every graph of average degree at least $r+\varepsilon$ and girth at least 1000 contains a subdivision of K_{r+2}.*

It is easily seen that the condition that r depends on ε is necessary (Proposition 7). The case $\varepsilon = 3/4$ of Theorem 2 answers Question 3.11 of [10] in the affirmative for sufficiently large r (see Corollary 9).

If we combine Theorem 1 with Theorem 2, we obtain a strengthened version of the former where the girth still depends on ε but not on r:

Corollary 3 *For every $\varepsilon > 0$ there exists an integer $f(\varepsilon)$ such that for all $r \geq 2$ every graph G of average degree at least $r + \varepsilon$ and girth at least $f(\varepsilon)$ contains a subdivision of K_{r+2}.*

Indeed, given ε, let $r_0 := \max\{100, 400/\varepsilon^2\}$. Then it suffices to choose $f(\varepsilon)$ larger than both 1000 and $g(r, \varepsilon)$ for all $r \leq r_0$.

Generalizing Theorem 1 to arbitrary graphs, Mader [11] proved the following.

Theorem 4 (Mader) *For every graph H with $\Delta(H) \geq 3$ and every $\varepsilon > 0$ there exists an integer $g(H, \varepsilon)$ such that every graph G of average degree at least $\Delta(H) - 1 + \varepsilon$ and girth at least $g(H, \varepsilon)$ contains a subdivision of H.*

His bound on $g(H, \varepsilon)$ is at least linear in the number of edges of H and also depends on ε. It is easily seen that in contrast to Theorem 1, the bound on the necessary girth must depend on H. Our next result determines the order of magnitude of the girth required for graphs H whose maximum degree is sufficiently large compared with ε.

Theorem 5 *Let $\varepsilon \geq 0$ and let H be a graph with $\Delta(H) \geq \max\{100, 400/\varepsilon^2\}+1$. Then every graph G of average degree at least $\Delta(H) - 1 + \varepsilon$ and girth at least $\frac{1000 \log |H|}{\log(\Delta(H)+1)}$ contains a subdivision of H.*

Note that this immediately implies Theorem 2. Moreover, the bound on the girth in Theorem 5 is best possible up to the value of the constant 1000 [6, Prop. 12]. We prove Theorem 5 by extending techniques of Mader [11].

Let us now mention a few related results. First note that Theorem 2 implies that, for sufficiently large r, every graph of minimum degree r and girth at least 1000 contains a subdivision of K_{r+1}. Based on techniques of Mader [9], we proved in [6] that in fact a girth of 15 already suffices. This implies that the conjecture of Hajós that every graph of chromatic number r contains a subdivision of K_r (which is false in general) is true for all graphs of girth at least 15 and sufficiently large chromatic number.

If we only seek ordinary minors instead of topological ones, then the condition on the girth can be relaxed even more. In [8] we prove that for every s there exists a constant $c > 0$ such that every $K_{s,s}$-free graph of average degree at least r already contains a K_t minor for all $t \leq cr^{1+\frac{1}{2(s-1)}}/(\log r)^4$. Furthermore, in [7] we show that for every odd integer g there exists a constant $c > 0$ such that every graph G of girth at least g and average degree at least r contains a minor of average degree at least $cr^{\frac{g+1}{4}}$. A conjecture of Bollobás about the minimal order of graphs of given minimum degree and large girth (which is known to be true for girth 5, 7 and 11) would imply that this is best possible up to the value of the constant c.

2 Terminology

All logarithms in this paper are base e, where e denotes the Euler number. The *length* of a cycle C or a path P is the number of its edges. The *girth* of a graph

G is the length of its shortest cycle and denoted by $g(G)$. The *distance* $d_G(x,y)$ between two vertices x, y of a graph G is the length of the shortest path joining x to y. Given $r \in \mathbb{N}$ and a vertex $x \in G$, the *r-ball $B^r_G(x)$ in G around x* is the subgraph of G induced by all its vertices of distance at most r from x, the *r-shell $S^r_G(x)$ in G around x* is the subgraph induced by all vertices of distance precisely r from x.

We write $e(G)$ for the number of edges of a graph G and $|G|$ for its order. We denote the degree of a vertex $x \in G$ by $d_G(x)$ and the set of its neighbours by $N_G(x)$. Given $r \in \mathbb{N}$, we write $V^{>r}(G)$ for the set of all vertices of degree greater than r in G. We denote by $\delta(G)$ the minimum degree of G, by $\Delta(G)$ its maximum degree and by $d(G) := 2e(G)/|G|$ the average degree of G. Given disjoint $A, B \subseteq V(G)$, we write $e_G(A,B)$ for the number of edges in G between A and B. Similarly, if H and H' are disjoint subgraphs of G, we put $e_G(H,H') := e_G(V(H), V(H'))$. We denote by $N_G(A)$ the set of all those neighbours of vertices in A that lie outside A.

A *subdivision* of a graph G is a graph TG obtained from G by replacing the edges of G with internally disjoint paths. The *branch vertices* of TG are all those vertices that correspond to vertices of G. Given $r \in \mathbb{N}$, we say that a graph G is r-linked if $|G| \geq 2r$ and for every $2r$ distinct vertices $x_1, \dots, x_r$ and $y_1, \dots, y_r$ of G there exist disjoint paths $P_1, \dots, P_r$ such that P_i joins x_i to y_i.

Given a tree T and disjoint subtrees $T_1, \dots, T_k$ of T, we say that T *can be split into* $T_1, \dots, T_k$ if each vertex of T lies in one of the T_i. If T is a subtree of a graph G, we say that T *sends out an edge* e if e joins a vertex of T to a vertex of $G - T$. If $\mathcal{T}$ is a family of subtrees of a graph G, we say that $V(G)$ *can be covered by the trees in* $\mathcal{T}$ if they are disjoint and every vertex of G lies in one of them.

3 Two observations

Let us first observe that the condition that r depends on ε is necessary in Theorem 2 (and thus also in Theorem 5). To do this, we need the following bound on the minimum order of an r-regular graph of large girth which is due to Sauer (who proved a slightly sharper bound, see e.g. [1, Ch. III Thm. 1.4′]).

Theorem 6 *For all integers $r \geq 3$ and all odd $g \geq 3$ there exists an r-regular graph of girth at least g whose order is at most $4(r-1)^{g-2}$.*

Proposition 7 *For all integers $g, r \geq 3$ there exists a graph G of average degree at least $r + \frac{1}{2(r-1)^{2g-3}}$ and girth at least g which does not contain a subdivision of K_{r+2}.*

Proof. Apply Theorem 6 to obtain an r-regular graph G' of girth at least $2g-1$ and order at most $4(r-1)^{2g-3}$. Let x be any vertex on a shortest cycle C in G' and let $y \in C$ be a vertex whose distance from x in C is $g-1$. Then the graph G obtained from G' by adding the edge xy has girth at least g and precisely two of its vertices have degree greater than r. So G cannot contain a subdivision of K_{r+2}. But

$$d(G) \geq \frac{2(e(G')+1)}{|G'|} \geq \frac{r|G'|+2}{|G'|} \geq r + \frac{1}{2(r-1)^{2g-3}},$$

as desired. □

In [10, Question 3.11] Mader asked whether for all $r \geq 4$ every graph G with $r+2 \leq g(G) < \infty$ and more than $\frac{r}{2}(|G|-(r-1))$ edges contains a subdivision of K_{r+1}. These bounds on the average degree and the girth were arrived at by extrapolating from small values of r (see also [9]). We will now use the case $\varepsilon = 3/4$ of Theorem 2 to answer this question in the affirmative for large r (Corollary 9). We will also need the following observation of Tutte.

Proposition 8 *Let $g \geq 3$ and $r \geq 2$ be integers. Then every graph G of girth at least g and minimum degree at least r satisfies $|G| \geq (r-1)^{\lfloor \frac{g-1}{2} \rfloor}$.*

Proof. Put $k := \lfloor \frac{g-1}{2} \rfloor$ and let $x \in V(G)$. Then the graph obtained from $B^k_G(x)$ by deleting all edges with both endvertices in $S^k_G(x)$ is a tree in which every vertex that is not a leaf has degree at least r and every leaf has distance precisely k from x. So this tree (and thus also G) has at least $(r-1)^k$ vertices. □

Corollary 9 *There exists an integer r_0 such that for every $r \geq r_0$ every graph G with $r+2 \leq g(G) < \infty$ and more than $\frac{r}{2}(|G|-(r-1))$ edges contains a subdivision of K_{r+1}.*

Proof. We will prove the corollary for $r_0 := 1000$. Let G' be a minimal subgraph of G which satisfies $e(G') > \frac{r}{2}(|G'|-(r-1))$ and $|G'| \geq r+2$. Note that in fact this implies $|G'| \geq r+3$. (Otherwise G' would either be a forest or a cycle of length $r+2$, since $g(G') \geq r+2$. But then $e(G') > \frac{r}{2}((r+2)-(r-1)) = \frac{3r}{2} > r+2$.) Let us now show that $\delta(G') > r/2$. Suppose not and let $x \in G'$ be a vertex of minimum degree. Then $|G'-x| \geq r+2$ and

$$e(G'-x) > \frac{r}{2}(|G'|-(r-1)) - \frac{r}{2} = \frac{r}{2}(|G'-x|-(r-1)),$$

contradicting the minimality of G'.

Since $g(G') \geq r+2 \geq 1002$, Proposition 8 implies that $|G'| \geq (\frac{r}{2}-1)^{500}$. Thus

$$d(G') > r\left(1 - \frac{r-1}{|G'|}\right) \geq r - \frac{1}{4},$$

and so Theorem 2 with $\varepsilon := 3/4$ implies that G' contains a subdivision of K_{r+1}. □

4 Proof of Theorem 5

We now turn to the proof of Theorem 5. We will need the following lemma due to Mader [9, 11]. An explicit proof of the version stated below can be found in [6].

Lemma 10 *Let $c \geq 1$ be an integer and let G be a graph of minimum degree at least $2c$. Then there exist disjoint non-empty sets $A, B \subseteq V(G)$ and a set E of $|B|$ independent A–B edges such that every vertex in B has at least two neighbours in A, $|B| < c$, $N_G(A) \subseteq B$ and so that the graph G^* obtained from $G[A \cup B]$ by contracting the edges in E is $\lceil c/3 \rceil$-connected.*

We also need the following result of Bollobás and Thomason [2].

Theorem 11 *Let $k \geq 1$ be an integer. Then every $22k$-connected graph is k-linked and every graph of average degree at least $44k^2$ contains a subdivision of K_k.*

The other main ingredients of the proof of Theorem 5 are as follows. Firstly, in Lemma 12 we show that we may assume that the vertices of our given graph G can be covered by trees which have small radius, are not too large and all send out many edges to other trees. Secondly, in Lemma 15 we prove that if a subgraph of G has average degree at least $\Delta(H) - 1 + \varepsilon$ and large girth then it contains $|H|$ vertices of degree $\Delta(H)$ which are reasonably far apart. Similar statements also appear in [11]. But the proofs are much easier there since the graph G had larger girth. The proof of Theorem 5 itself then closely follows the lines of Mader. However, the argument is a little simpler since we assume that $\Delta(H)$ is large and the calculations are somewhat different since we have a smaller bound on the girth, so we give the proof for completeness.

The strategy of the proof of Theorem 5 is as follows. We will use Lemma 12 to cover the vertex set of G with a family $\mathcal{T}$ of disjoint trees. As G has large girth, there are no multiple edges between these trees. So the graph G' obtained from G by contracting each tree in $\mathcal{T}$ has large minimum degree and thus contains a highly connected subgraph G''. By Theorem 11, G'' is highly linked and so we may link suitable disjoint stars in G'' to obtain a subdivision of H. This corresponds to a subdivision of H in G (and not only to H as an ordinary minor) if each of the stars in G'' corresponds to a subdivided star in G. But such stars may not exist in G''. So instead of moving to a highly connected subgraph as above, we will apply Lemma 10 to G' to find sets A and B as described there. The aim then is to link disjoint stars in the graph G^* defined in Lemma 10 to obtain a subdivision of H in G^*. Again, each of these stars must correspond to a subdivided star in G to ensure that this subdivision corresponds to one in G. Lemma 15 will be used to show that $G'[A \cup B]$ contains a set X' of $|H|$ vertices such that each vertex $x' \in X'$ corresponds to a tree in $\mathcal{T}$ which contains a vertex x of degree at least $\Delta(H)$ and such that the distance between every two vertices from X' is reasonably large. The images in G^* of the vertices in X' will be the centres of the stars in G^* and so the vertices x will become the branch vertices of our subdivision of H in G.

Lemma 12 *Let h, r, s be integers such that $h > r \geq 44$ and $s \geq 3 \log h / \log(r/3)$. Suppose that G is a graph of minimum degree at least $r/2$ and girth at least $8s+2$. Then either G contains a subdivision of K_h or else $V(G)$ can be covered by disjoint induced subtrees of G of order at most $(r/3)^{3s}$ and radius at most $4s$ such that each of these trees sends out at least $(r/3)^s$ edges.*

For the proof of this lemma we will need the following two simple propositions.

Proposition 13 *Let G be a graph of minimum degree at least 3. Suppose that T is an induced subtree of G and $\Delta \geq \Delta(T) \geq 1$. If $|T| > \Delta^2$, then T can be split into disjoint subtrees such that each of these trees has at most Δ^2 vertices and sends out at least Δ edges.*

Proof. Let us first show that T can be split into disjoint subtrees such that each of them has between $\Delta - 1$ and Δ^2 vertices. Orient each edge e of T towards the larger component of $T - e$, breaking ties arbitrarily. Consider a sink $x \in T$ and let T' be the largest component of $T - x$. Then $|T'| > \Delta - 1$, since $d_T(x) \leq \Delta$ and $|T| > \Delta^2$. By definition of x, $|T - T'| \geq |T'| > \Delta - 1$. If necessary, we may continue in this fashion to split T' and $T - T'$ into trees of the required order.

To complete the proof of the proposition, it suffices to show that every subtree T' of T sends out at least $|T'| + 2$ edges. But as T (and thus T') is induced in G, we have

$$2e(T') + e(T', G - T') = \sum_{x \in T'} d_G(x) \geq 3|T'|.$$

Therefore $e(T', G - T') \geq |T'| + 2$, as required. □

Proposition 14 *If G is a graph such that $d(F) \leq d$ for every subgraph F of G, then the edges of G can be oriented in such a way that the outdegree of every vertex is at most d.*

Proof. By induction on $|G|$. Let x be a vertex of minimum degree in G, orient all the (at most d) edges incident to x away from x and apply the induction hypothesis to $G - x$. □

Proof of Lemma 12. Let X be the set of all vertices of G of degree at least $(r/3)^s$. Let A be the set of all those vertices in $R := G - X$ that have at least two neighbours in X. Let B be the set of all those vertices of $R - A$ that have distance at most $2s$ from A in R and put $C := V(R) \setminus (A \cup B)$. If C is non-empty, let Z be a maximal set of vertices in C such that the distance in R between every two of them is at least $2s + 1$. Thus the balls $B_R^s(z)$ are disjoint for different $z \in Z$. Extend these balls to disjoint connected subgraphs of R by first adding every vertex of R of distance $s + 1$ from Z to one of the $B_R^s(z)$ to which is it adjacent. Then add each vertex of distance $s+2$ from Z to one of the subgraphs constructed in the previous step to which it is adjacent. Continue in this fashion until each vertex of R of distance at most $2s$ from Z

lies in one of the subgraphs and let T'_z denote the subgraph of R obtained from $B^s_R(z)$. By the choice of Z, each vertex in C has distance at most $2s$ from Z in R and thus it lies in some T'_z. Let D be the set of all those vertices of R that lie in some T'_z. As each vertex of T'_z has distance at most $2s$ from z in T'_z, we have $D \cap A = \emptyset$.

For all vertices $a \in A$ choose disjoint trees $T'_a \subseteq R$ with $a \in T'_a$ such that they cover $A \cup B$ and so that each vertex of T'_a has distance at most $2s$ from a in T'_a. As $g(G) \geq 4s+2$, each T'_a is an induced subtree of G. Let T_a be the component of $T'_a - D$ containing a. Given $a \in A$, consider a component L of $T'_a - D$ which is distinct from T_a. Let v_L denote the unique neighbour of L that lies on the a–L path in T'_a. So $v_L \in D$ and every vertex of L has distance at most $2s$ from v_L in T'_a. Add L (together with the v_L–L edge in T'_a) to the unique T'_z with $z \in Z$ that contains v_L. Carry this out for all such components L and every $a \in A$. Let T_z be the connected subgraph of R obtained from T'_z in this way. Then each vertex of T_z has distance at most $4s$ from z in T_z. As $g(G) \geq 8s+2$, this implies that T_z is an induced subtree of G. Moreover, as T_z contains $B^s_R(z) \subseteq R - A$, it has at least $(r/2-2)^s$ leaves and so T_z sends out at least $(r/2-2)^{s+1} \geq (r/3)^s$ edges. As the maximum degree of T_z is less than $(r/3)^s$, we may apply Proposition 13 to split each T_z with $|T_z| > (r/3)^{2s}$ into subtrees of order at most $(r/3)^{2s}$ such that each of them sends out at least $(r/3)^s$ edges. Let A^* denote the set of all those vertices $a \in A$ for which $|T_a| \leq (r/3)^{2s}$ and split all the T_a with $a \in A \setminus A^*$ similarly. Let $\mathcal{T}$ be the family consisting of all the trees thus obtained from the T_v with $v \in (A \setminus A^*) \cup Z$. So the trees in $\mathcal{T}$ form a suitable covering of all vertices except those in $X \cup \bigcup_{a \in A^*} V(T_a) \subseteq X \cup A \cup B$ and it remains to deal with these vertices. (In particular, we are already done if $X = \emptyset$.)

For each $a \in A^*$ choose two of its neighbours in X, x^1_a and x^2_a say. As $g(G) > 4$, the sets $\{x^1_a, x^2_a\}$ are distinct for different vertices $a \in A^*$. Consider the auxiliary graph F on X in which two vertices $x, y \in X$ are joined by an edge if there exists a vertex $a \in A^*$ such that $x = x^1_a$ and $y = x^2_a$. Note that we may assume that every subgraph of F has average degree less than $44h^2$. Indeed, if $d(F') \geq 44h^2$ for some $F' \subseteq F$ then, by Theorem 11, F' (and thus also G) would contain a subdivision of K_h. So Proposition 14 implies that the edges of F can be oriented in such a way that the outdegree of every vertex of F is at most $44h^2$. Thus we may partition the vertices in A^* into disjoint sets A^*_x ($x \in X$) of size at most $44h^2$ such that each vertex in A^*_x is joined to x.

For all $x \in X$, let T_x be the subtree of G which consists of x together with the T_a for all $a \in A^*_x$ and all edges between x and A^*_x. Then each vertex of T_x has distance at most $2s+1$ from x in T_x. As $g(G) \geq 4s+4$, it follows that each T_x must be an induced subtree of G. As $|T_a| \leq (r/3)^{2s}$ for all $a \in A^*_x$, the order of T_x is at most

$$1 + 44h^2 \cdot \left(\frac{r}{3}\right)^{2s} \leq h^3 \left(\frac{r}{3}\right)^{2s} \leq \left(\frac{r}{3}\right)^{3s}.$$

Moreover, using $d_G(x) \geq (r/3)^s$, it is easily seen that T_x sends out at least $(r/3)^s$ edges. So together the trees T_x ($x \in X$) and the trees in $\mathcal{T}$ are as desired. $\square$

Lemma 15 *Let $\varepsilon > 0$ and let g, h, k, r be integers such that $h > r \geq \max\{100, 100/\varepsilon^2\}$ and $g \geq \max\{24\lceil \log h/\log r\rceil + 1, 4k+11\}$. Suppose that G is a graph of girth g and average degree at least $r + \varepsilon$. Then there are at least h vertices in $V^{>r}(G)$ such that the distance between every two of them is greater than k.*

For the proof of Lemma 15 we need the following two easy propositions. A proof of the first one can be found in [3, Prop. 1.2.2].

Proposition 16 *Every graph G with at least one edge contains a subgraph of average degree at least $d(G)$ and minimum degree greater than $d(G)/2$.*

Proposition 17 *Let $k \geq 1$ be an integer and suppose that G is a non-empty graph of girth at least $2k+1$. Then $d(G) \leq 2|G|^{1/k} + 2 \leq 4|G|^{1/k}$.*

Proof. Clearly, we may assume that $d(G) > 4$. Apply Proposition 16 to obtain a subgraph F of G of minimum degree $r > d(G)/2$. Proposition 8 implies that

$$|G| \geq |F| \geq (r-1)^k \geq \lfloor d(G)/2 \rfloor^k,$$

as required. □

Proof of Lemma 15. Suppose not. Then for some $\ell < h$ there are distinct vertices $x_1, \dots, x_\ell$ in $V^{>r}(G)$ such that $V^{>r}(G)$ is contained in $B^k_G(x_1) \cup \dots \cup B^k_G(x_\ell)$. For all $i = 1, \dots, \ell$, choose a connected subgraph T_{x_i} of $B^{k+1}_G(x_i)$ such that $x_i \in T_{x_i}$, the T_{x_i} are disjoint for distinct i, each vertex of T_{x_i} has distance at most $k+1$ from x_i in T_{x_i} and such that

$$V(T_{x_1}) \cup \dots \cup V(T_{x_\ell}) = V(B^{k+1}_G(x_1)) \cup \dots \cup V(B^{k+1}_G(x_\ell)) =: X.$$

Let $Y := N_G(X)$. Extend the T_{x_i} to disjoint connected subgraphs $T'_{x_1}, \dots, T'_{x_\ell}$ by adding each vertex $y \in Y$ to some tree T_{x_i} adjacent to y. So each vertex of T'_{x_i} has distance at most $k+2$ from x_i in T'_{x_i}. Since $g(G) \geq 2k+6$, each T'_{x_i} is an induced subtree of G.

Both $G[X]$ and $G[X \cup Y]$ have average degree at most $4r^{1/4}$. $\qquad (*)$

To prove that $d(G[X \cup Y]) \leq 4r^{1/4}$, let us first assume that $|T'_{x_i}| \leq h^2$ for all $i = 1, \dots, \ell$. As $g(G) \geq 24\lceil \log h/\log r\rceil + 1$, Proposition 17 implies that

$$d(G[X \cup Y]) \leq 4|X \cup Y|^{\frac{\log r}{12 \log h}} \leq 4(h \cdot h^2)^{\frac{\log r}{12 \log h}} = 4r^{1/4}.$$

Thus we may assume that $|T'_{x_i}| \geq h^2$ for some i. So $|X \cup Y| \geq h^2$. As $g(G) \geq 4k+11$, at most one edge of G joins a given pair T'_{x_i}, T'_{x_j}. Hence

$$d(G[X \cup Y]) \leq 2\,\frac{\binom{h}{2} + \sum_{i=1}^{\ell} e(T'_{x_i})}{|X \cup Y|} \leq \frac{h^2 + 2|X \cup Y|}{|X \cup Y|} \leq 1 + 2 < 4r^{1/4}.$$

The proof for $G[X]$ is exactly the same except that we consider the T_{x_i} instead of the T'_{x_i}. This completes the proof of $(*)$.

Case 1. $e(X, Y) \le (r - 4r^{1/4})|X|$.
Recalling that every vertex of $G - X$ has degree at most r in G, we find

$$\begin{aligned} d(G)|G| &= \sum_{v \in G} d_G(v) = \sum_{v \in X} d_{G[X]}(v) + \sum_{v \in G-X} d_G(v) + e(X, Y) \\ &\le 4r^{1/4}|X| + r|G - X| + (r - 4r^{1/4})|X| = r|G|, \end{aligned}$$

a contradiction to our assumption on G.

Case 2. $e(X, Y) > (r - 4r^{1/4})|X|$.
As $d(G[X \cup Y]) \le 4r^{1/4}$, we have

$$2r^{1/4}|X \cup Y| \ge e(G[X \cup Y]) \ge e(X, Y) > (r - 4r^{1/4})|X|,$$

and thus

$$|Y| > \left(\frac{r^{3/4}}{2} - 3\right)|X| \ge \frac{4r^{3/4}}{10}|X| \ge \frac{4r^{1/4}}{\varepsilon}|X|. \tag{1}$$

As by definition, X contains all vertices in $V^{>r}(G)$ together with all their neighbours, we have $e(X, Y) \le r|X|$. Therefore,

$$\begin{aligned} d(G)|G| &= \sum_{v \in G} d_G(v) = \sum_{v \in X} d_{G[X]}(v) + \sum_{v \in G-X} d_G(v) + e(X, Y) \\ &\le 4r^{1/4}|X| + r|G - X| + r|X| \overset{(1)}{<} \varepsilon|Y| + r|G| < (r + \varepsilon)|G|, \end{aligned}$$

which is again a contradiction to our assumption on G. □

We will also need the following observation [11, Lemma 2.2].

Lemma 18 *Let $\delta \ge 3$ and $g \ge 12$ be integers. Let A be a non-empty set of vertices of a graph G of minimum degree δ and girth at least g such that $|N_G(A)| < \delta$. Then $|A| \ge (\delta - 1)^{g/4}$.*

For convenience, instead of proving Theorem 5 directly, we prove the following slightly more technical statement.

Theorem 19 *Given $\varepsilon > 0$ and a graph H with $r := \Delta(H) - 1 \ge \max\{100, 100/\varepsilon^2\}$, let $s := \lceil 3 \log |H| / \log(r/3) \rceil$. Then every graph G of average degree at least $r + 2\varepsilon$ and girth at least $192s + 31$ contains a subdivision of H.*

Let us first check that Theorem 19 indeed implies Theorem 5.

Proof of Theorem 5. Put $r := \Delta(H) - 1$. We have to show that the integer s defined in Theorem 19 satisfies $192s + 31 \le 1000 \log |H| / \log(r + 2)$. First note that $r/3 \ge (r + 2)^{3/4}$ for all $r \ge 100$. So

$$s \le \frac{3 \log |H|}{\log((r + 2)^{3/4})} + 1 = 4 \frac{\log |H|}{\log(r + 2)} + 1.$$

Since $\log |H| / \log(r + 2) \ge 1$, this yields the required inequality. □

Proof of Theorem 19. First note that we may assume that $\varepsilon \leq 1$. Put $h := |H|$. By replacing G with a suitable subgraph, we may assume that every proper subgraph of G has average degree less than $r + 2\varepsilon$. Thus in particular the minimum degree of G is at least $r/2$. Apply Lemma 12 to G. Clearly, we may assume that the lemma returns a family $\mathcal{T}$ of disjoint induced subtrees of G covering $V(G)$ as described there. Let G' be the graph obtained from G by contracting each tree in $\mathcal{T}$. As each tree in $\mathcal{T}$ has radius at most $4s$ and $g(G) \geq 16s + 3$, a given pair of trees $T_1, T_2 \in \mathcal{T}$ is joined by at most one edge of G. Since each tree $T \in \mathcal{T}$ sends out at least $(r/3)^s$ edges, it follows that the minimum degree of G' is at least $(r/3)^s$. Moreover, we have

$$g(G') \geq \frac{g(G)}{8s+1} \geq 24.$$

For every vertex $v \in G'$, let T_v denote the tree in $\mathcal{T}$ that contracts to v. Given a set V of vertices of G', let $\widetilde{V}$ denote the set of all those vertices of G that lie in a tree T_v with $v \in V$. Apply Lemma 10 with $c := \lfloor (r/3)^s/2 \rfloor$ to G' to find sets $A, B \subseteq V(G')$ and a set E of $|B|$ independent A–B edges as described there. We will now show that $d(G[\widetilde{A} \cup \widetilde{B}]) \geq r + \varepsilon$. So suppose on the contrary that

$$e(G[\widetilde{A} \cup \widetilde{B}]) < \left(\frac{r}{2} + \frac{\varepsilon}{2}\right) |\widetilde{A} \cup \widetilde{B}|. \tag{2}$$

The minimality of G implies that

$$e(G - \widetilde{A}) < \left(\frac{r}{2} + \varepsilon\right) |G - \widetilde{A}|. \tag{3}$$

As $N_{G'}(A) \subseteq B$ and thus $N_G(\widetilde{A}) \subseteq \widetilde{B}$, adding (2) and (3) gives

$$e(G) \leq e(G[\widetilde{A} \cup \widetilde{B}]) + e(G - \widetilde{A}) < \left(\frac{r}{2} + \varepsilon\right) |G| - \frac{\varepsilon}{2}|\widetilde{A}| + \left(\frac{r}{2} + \frac{\varepsilon}{2}\right) |\widetilde{B}|. \tag{4}$$

As every tree in $\mathcal{T}$ has at most $(r/3)^{3s}$ vertices, we have $|\widetilde{B}| \leq (r/3)^{3s}|B| \leq (r/3)^{4s}$. On the other hand, Lemma 18 implies that

$$|\widetilde{A}| \geq |A| \geq \left(\left(\frac{r}{3}\right)^s - 1\right)^{g(G')/4} \geq \left(\left(\frac{r}{3}\right)^s - 1\right)^6 \geq \left(\frac{r}{3}\right)^{5s}.$$

Together with (4) and the fact that $\varepsilon \geq 10/r^{1/2}$ this shows

$$e(G) < \left(\frac{r}{2} + \varepsilon\right) |G| - \frac{5(r/3)^{5s}}{r^{1/2}} + r \cdot (r/3)^{4s} \leq \left(\frac{r}{2} + \varepsilon\right) |G|,$$

which contradicts our assumption on G. This proves that $d(G[\widetilde{A} \cup \widetilde{B}]) \geq r + \varepsilon$.

Let $\widetilde{G}$ be the subgraph of $G[\widetilde{A} \cup \widetilde{B}]$ obtained by successively deleting all vertices of degree at most one. So $\delta(\widetilde{G}) \geq 2$ and $d(\widetilde{G}) \geq r + \varepsilon$. For every vertex $b \in B$, let T'_b be the minimal subtree of T_b which contains all vertices of T_b that send an edge to $\widetilde{A} \cup (\widetilde{B} \setminus V(T_b))$. As b has at least two neighbours in A (Lemma 10), T'_b is non-empty and each vertex of T'_b has degree at least two in $G[\widetilde{A} \cup \bigcup_{b' \in B} V(T'_{b'})]$. As $N_G(\widetilde{A}) \subseteq \widetilde{B}$, every vertex in $\widetilde{A}$ has degree at least $r/2$

in $G[\widetilde{A} \cup \bigcup_{b' \in B} V(T'_{b'})]$. Thus $\widetilde{G}$ consists of precisely all the T_a with $a \in A$, all the T'_b with $b \in B$ and all edges between these trees.

Put $T'_a := T_a$ for all $a \in A$. So the trees T'_v with $v \in A \cup B$ cover $V(\widetilde{G})$. As $g(\widetilde{G}) \geq 4 \cdot (48s + 5) + 11$, we may apply Lemma 15 with $k := 48s + 5$ to $\widetilde{G}$ to find vertices $x_1, \dots, x_h$ in $V^{>r}(\widetilde{G})$ which have distance at least $48s + 6$ from each other in $\widetilde{G}$. Let $v_i \in A \cup B$ be such that $x_i \in T'_{v_i}$. Let S_i be a subdivided star in $\widetilde{G}$ with centre x_i and $r + 1$ leaves such that each leaf is a neighbour of T'_{v_i} and every other vertex of S_i lies in T'_{v_i}. (Such an S_i exists since $d_{\widetilde{G}}(x_i) > r$ and $\delta(\widetilde{G}) \geq 2$.) If some leaf y of S_i lies in a tree T'_v such that $v_i v \in E$, then extend S_i by adding a path running from y through T'_v to a neighbour of T'_v which does not lie in T'_{v_i}. Note that the graph obtained from $\widetilde{G}$ by contracting each T'_v with $v \in A \cup B$ is precisely $G'[A \cup B]$. As $g(\widetilde{G}) \geq 24s + 4$, the leaves of (the extended) S_i lie in different T'_v, and so each S_i corresponds to a subdivided star S'_i in $G'[A \cup B]$ with centre v_i and $r + 1$ leaves. As $d_G(x_i, x_j) \geq 48s + 6$, the S'_i are all disjoint and furthermore no edge from E joins distinct S'_i. Let G^* be the graph obtained from $G'[A \cup B]$ by contracting the edges in E. Then each S'_i corresponds to a star S^*_i in G^* with $r + 1$ leaves and all these S^*_i are disjoint. As G^* is $\lceil c/3 \rceil$-connected (Lemma 10) and

$$\frac{c}{3} \geq \left(\frac{r}{3}\right)^s \cdot \frac{1}{6} - 1 \geq \frac{h^3}{6} - 1 \geq h^2 \cdot \frac{100}{6} - h^2 \geq 22\binom{h}{2} + h,$$

the graph obtained from G^* by deleting the centres of the S^*_i is $\binom{h}{2}$-linked (Theorem 11). Thus in G^* we may link the leaves of the S^*_i to obtain a subdivision of H whose branch vertices are the centres of the S^*_i. By construction, this corresponds to a subdivision of H in G with branch vertices $x_1, \dots, x_h$. □

We remark that with more elaborate calculations the constants in Theorems 2 and 5 can be improved a little. Also the constant in the bound on the girth can be reduced further at the expense of raising the bound on $\Delta(H)$ (and vice versa).

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