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Positive mass theorem for the Yamabe problem on spin manifolds

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Abstract

Let (M,g) be a compact connected spin manifold of dimension $n \geq 3$ whose Yamabe invariant is positive. We assume that (M,g) is locally conformally flat or that $n \in \{3,4,5\}$. According to a positive mass theorem of Witten, the constant term in the asymptotic development of the Green's function of the conformal Laplacian is positive if (M,g) is not conformally equivalent to the sphere. In our article, we will give a proof for this fact which is considerably shorter than previous proofs. Our proof is a modification of Witten's argument, but no analysis on asymtotically flat spaces is needed.

Mathematics Classification: 53C21 (Primary), 58E11, 53C27 (Secondary)

1 Introduction

The positive mass conjecture is a famous and difficult problem with origin in physics. The mass is a Riemannian invariant of an asymptotically flat manifold of dimension $n \geq 3$ and of order $\tau > \frac{n-2}{2}$. The problem consists in proving that the mass is positive if the manifold is not conformally diffeomorphic to $(\mathbb{R}^n, \mathrm{can})$. Good references on this subject are for example [LP87, Her98].

Schoen and Yau [Sch89, SY79] gave a proof if the dimension is at most 7 and Witten [Wit81, Bar86] proved the result if the manifold is spin. The positivity of the mass could be proved in several other particular cases (see e.g. [Sch84]), but the conjecture in its full generality still remains open.

This problem played an important role in geometry because it gave the solution of the Yamabe problem. Namely, let (M,g) be a compact connected Riemannian manifold of dimension $n \geq 3$. In [Yam60] Yamabe attempted to show that there is a metric $\tilde{g}$ conformal to g such that the scalar curvature $Scal_{\tilde{g}}$ of $\tilde{g}$ is constant. However, Trudinger realized that Yamabe's proof contained a serious gap. It was the achievement of many mathematicians to finally solve the problem of finding a conformal metric g with constant scalar curvature. The problem of finding a conformal g with is constant scalar curvature is called the Yamabe problem. As a first step, Trudinger [Tru68] was able to repair the mistake if a conformal invariant named Yamabe invariant is nonpositive. The problem is much more difficult in the case where the Yamabe invariant is positive, which is equivalent to the fact that there is a metric of positive scalar curvature in the conformal class of g. Aubin [Aub76] solved the problem when $n \geq 6$ and M is not locally conformally flat. Then, in [Sch84], Schoen solves completely the Yamabe problem by using the positive mass theorem in the remaining cases. Namely, assume that (M,g) is locally conformally flat or $n \in \{3,4,5\}$. Let

$$L_g = \frac{4(n-1)}{n-2}\Delta_g + Scal_g$$

be the conformal Laplacian of the metric g and $P \in M$. Then, there exists a smooth function Γ, the so-called Green's function of L_g, which is defined on $M - \{P\}$ such that $L_g\Gamma = \delta_P$ in the sense of distributions (see for example [LP87]). Morever, if we note $r = d_g(., P)$, G has the following expansion at P:

$$\Gamma(x) = \frac{1}{4(n-1)\omega_{n-1}\, r^{n-2}} + A + \alpha(x) \qquad \omega_{n-1} = \mathrm{vol}(S^{n-1})$$

where $A \in \mathbb{R}$. In addition, α is a function defined on a neighbourhood of P with $\alpha(0) = 0$ which is smooth on M if (M, g) is locally conformally flat, and which is a Lipschitz function for $n = 3, 4, 5$. Hence, in both cases $\alpha = O(r)$. Schoen has shown in [SY88] that the positivity of A would imply the solution of the Yamabe problem. He also proved that A is a positive multiple of the mass of the asymptotically flat manifold $(M, \Gamma^{\frac{4}{n-2}} g)$. Hence the solution of the Yamabe problem follows from the positive mass theorem in these special cases, which was proven by Schoen and Yau in [SY79, SY88].

In our article, we will give a short proof for the positivity of the constant term A in the development of the Green's function in case that M is spin and locally conformally flat. The statement of this paper is weaker than the results of Witten [Wit81] and Schoen and Yau [Sch89, SY79]. The proof in our paper is inspired by Witten's reasoning, but we could simplify considerably many of the analytic arguments. Witten's argument is based on the construction of test spinor on the stereographic blowup which is both harmonic and asymptotically constant. We show that the Green's function for the Dirac operator on M can be used to construct such a test spinor. In this way, we obtain a very short solution of the Yamabe problem using only elementary and well known facts from analysis on compact manifolds.

The last section shows how to adapt our proof to arbitrary spin manifolds of dimensions 3, 4 and 5. In dimension 3 the proof is completely analogous. However, in dimensions 4 and 5, additional estimates have to be derived in order to get sufficient control on the Green's function of the Dirac operator.

2 The locally conformally flat case

In this section, we will assume that (M, g) is a compact, connected, locally conformally flat spin manifold of dimension $n \geq 3$. The Dirac operator is denoted by D. A spinor ψ is called *harmonic* if $D\psi \equiv 0$. As the solution of the Yamabe problem in the case of non-negative Yamabe invariant follows from [Tru68] we will assume that the Yamabe invariant is positive. Hence, the conformal class contains a positive scalar curvature metric. As $\dim\ker D$ is conformally invariant, we see that $\dim\ker D = 0$. We fix a point $P \in M$. We can assume that g is flat in the ball $B_P(\delta)$ of radius δ around P. Let $(x^1, \ldots, x^n)$ denote local coordinates on $B_P(\delta)$. On $B_P(\delta)$ we trivialize the spinor bundle via parallel transport.

LEMMA 2.1. *Let $\psi_0 \in \Sigma_P M$. Then, there is a harmonic spinor ψ on $M \setminus \{P\}$ satisfying*

$$\psi|_{B_P(\delta)} = \frac{1}{r^{n-1}} \frac{x}{r} \cdot \psi_0 + \theta(x)$$

where $\theta(x)$ is a smooth spinor on $B_P(\delta)$.

It is not hard to see, that $\omega_{n-1}^{-1}\psi$ actually coincides with the Green's function of the Dirac operator.

Proof. Our construction of ψ follows the construction of the Green's function G of the Laplacian in [LP87, Lemma 6.4]. Namely, we take a cut-off function η with support in $B_P(\delta)$ which is equal to 1 near P. We set $\Phi = \eta \frac{1}{r^{n-1}} \frac{x}{r} \cdot \psi_0$ where ψ_0 is constant. The spinor Φ is harmonic on $B_P(\delta) \setminus \{P\}$. We extend Φ to a continuous spinor on $M \setminus \{P\}$. As $D\Phi|_{B_P(\delta)} \equiv 0$, we see that $D\Phi$ extends to a smooth spinor on M. Using the selfadjointness of D together with $\ker D = \{0\}$ we know that there is a smooth spinor θ_1 such that $D\theta_1 = -\Phi$. Obviously, $\psi = \Phi + \theta_1$ is a spinor as claimed. □

We now show that the existence of ψ implies the positivity of A.

THEOREM 2.2. *The mass A satisfies $A \geq 0$. Furthermore, equality holds if and only if (M, g) is conformally equivalent to the standard sphere (S^n, can).*

Proof. We let ψ be given by lemma 2.1. Without loss of generality, we may assume that $|\psi_0| = 1$. Let also $\Gamma = \frac{1}{4(n-1)\omega_{n-1}} G$ be the Green's function for L_g. Using the maximum principle, it is easy to see that G is positive [LP87, Lemma 6.1]. We set

$$\widetilde{g} = G^{\frac{4}{n-2}} g.$$

Using the transformation formula for Scal under conformal changes, we obtain $\mathrm{Scal}_{\widetilde{g}} = 0$. We can identify spinors on $(M \setminus \{P\}, \widetilde{g})$ with spinors $(M \setminus \{P\}, g)$ such that the fiberwise scalar product on spinors is preserved [Hit74, Hij86] . Because of the formula for the conformal change of Dirac operators, the spinor

$$\Psi = G^{-\frac{n-1}{n-2}} \psi$$

is a harmonic spinor on $(M \setminus \{P\}, \widetilde{g})$, i.e. if we write $\widetilde{D}$ for the Dirac operator in the metric $\widetilde{g}$, we have $\widetilde{D}\Psi = 0$. By the Schrödinger-Lichnerowitz formula we have

$$0 = \widetilde{D}^2 \Psi = \widetilde{\nabla}^* \widetilde{\nabla} \Psi + \frac{\mathrm{Scal}_{\widetilde{g}}}{4} \Psi = \widetilde{\nabla}^* \widetilde{\nabla} \Psi.$$

Integration over $M \setminus B_P(\epsilon)$, $\epsilon > 0$ and integration by parts yields

$$0 = \int_{M \setminus B_P(\epsilon)} (\widetilde{\nabla}^* \widetilde{\nabla} \Psi, \Psi) dv_{\widetilde{g}} = \int_{M \setminus B_P(\epsilon)} |\widetilde{\nabla} \Psi|^2 dv_{\widetilde{g}} - \int_{S_P(\epsilon)} (\widetilde{\nabla}_{\widetilde{\nu}} \Psi, \Psi) ds_{\widetilde{g}}.$$

Here $S_P(\epsilon)$ denotes the boundary $\partial B_P(\epsilon)$, $\widetilde{\nu}$ is the unit normal vector on $S_P(\epsilon)$ with respect to $\widetilde{g}$ pointing into the ball, and $ds_{\widetilde{g}}$ denotes the riemannian volume element of $S_P(\epsilon)$. Hence, we have proved that

$$\int_{M \setminus B_P(\epsilon)} |\widetilde{\nabla} \Psi|^2 dv_{\widetilde{g}} = \frac{1}{2} \int_{S_P(\epsilon)} \partial_{\widetilde{\nu}} |\Psi|^2 ds_{\widetilde{g}} \tag{2.3}$$

If ε is sufficiently small, we have

$$\widetilde{\nu} = -G^{-\frac{2}{n-2}} \frac{\partial}{\partial r} = -\big(\varepsilon^2 + o(\varepsilon^2)\big) \frac{\partial}{\partial r} \tag{2.4}$$

$$ds_{\widetilde{g}} = G^{\frac{2(n-1)}{n-2}} ds_g = G^{\frac{2(n-1)}{n-2}} \varepsilon^{n-1} ds = \big(\varepsilon^{-(n-1)} + o(\varepsilon^{-(n-1)})\big) ds \tag{2.5}$$

where ds stands for the volume element of (S^{n-1}, can), and

$$|\Psi|^2 = G^{-2\frac{n-1}{n-2}} |\psi|^2 = \left(\frac{1}{r^{n-2}} + 4(n-1)\omega_{n-1} A + r\alpha_1(r) \right)^{-2\frac{n-1}{n-2}} \left| \frac{1}{r^{n-1}} \frac{x}{r} \cdot \psi_0 + \theta(x) \right|^2$$

where α_1 is a smooth function. This gives

$$|\Psi|^2 = (1 + 4(n-1)\omega_{n-1}\, A r^{n-2} + r^{n-1}\alpha_1(r))^{-2\frac{n-1}{n-2}} \times$$
$$\left(1 + 2r^{n-1} Re(< \frac{x}{r} \cdot \psi_0, \theta(x) >) + r^{2(n-1)}|\theta(x)|^2\right)$$

Noting that $\nabla_r(\frac{x}{r}\psi_0) = 0$, we get that on $S_P(\epsilon)$ and for ε small,

$$\partial_r |\Psi|^2 = -8(n-1)^2\omega_{n-1}\, A\varepsilon^{n-3} + o(\varepsilon^{n-3}) \tag{2.6}$$

Plugging (2.4), (2.5) and (2.6) into (2.3), we get that for ε small

$$0 \leq \int_{M\setminus B_P(\epsilon)} |\widetilde{\nabla}\Psi|^2 dv_{\widetilde{g}} = 4(n-1)^2\omega_{n-1}\, A \int_{S^{n-1}} ds + o(1) \tag{2.7}$$
$$= 4(n-1)^2\omega_{n-1}^2 A + o(1) \tag{2.8}$$

This implies that $A \geq 0$.

Now, we assume that $A = 0$. Then, it follows from (2.7) that $\widetilde{\nabla}\Psi = 0$ on $M \setminus \{P\}$ and hence, Ψ is parallel. Since the choice of ψ_0 is arbitrary, we obtain in this way a basis of parallel spinors on $(M \setminus \{P\}, \widetilde{g})$. This implies that $(M \setminus \{P\}, \widetilde{g})$ is flat and hence isometric to euclidean space. Let $I : (M \setminus \{P\}, \widetilde{g}) \to (\mathbb{R}^n, \text{can})$ be an isometry. We define $f(x) = 1 + \|I(x)\|^2/4, \quad x \in M$. Then $M \setminus \{P\}, f^{-2}\widetilde{g} = f^{-2}G^{\frac{4}{n-2}}g$ is isometric to $(S^n \setminus \{N\}, \text{can})$. The function $f^{-2}G^{\frac{4}{n-2}}$ is smooth on $M \setminus \{P\}$ and can be extended continuously to a positive function on M. Hence, M is conformal to (S^n, can).

3 The case of dimensions 3, 4 and 5

In this section, we show that the proof of last section can be adapted when M is not necessarily conformally flat if the dimension is 3, 4 or 5. For simplicity, in this article, we work with the synchronous trivialization, but the same conclusions work with the Gauduchon-Bourguignon-trivialization [AHM03]. Let us assume that (M, g) is an arbitrary connected spin manifold of dimension $n \in \{3, 4, 5\}$. We choose any $P \in M$. We can find a metric $\widetilde{g}$ conformal to g whose Ricci curvature vanishes at P. Hence, we may assume that the Ricci curvature for g vanishes at P. Consider $\rho = (x_1, \cdots, x_n)$ a system of normal coordinates at P defined on U. Let G be the Green's function of the conformal Laplacian L_g.

We trivialize TM and ΣM by synchronous frames e_i and α_i, i.e. by frames which are parallel along radial geodesics. We assume $e_i(P) = \partial/\partial x^i(P)$. Let us introduce a convenient notation for sections in this trivialization. For $v = \sum v^i e_i(P) : U \to T_PM \cong \mathbb{R}^n$ we define $\overline{v} : U \to TM$, $\overline{v}(q) = \sum v^i(q)e_i(q)$, i.e. v is the coordinate presentation for $\overline{v}$. Similarly, for $\psi : U \to \Sigma_P M = \Sigma\mathbb{R}^n$, $\psi = \sum \psi^i \alpha_i(P)$ we write $\overline{\psi}(q) = \sum \psi^i(q)\alpha_i(q)$. In this notation, $\overline{x}$ is the outward radial vector field whose length is the radius. Similarly, we write D for the Dirac operator on flat $\mathbb{R}^n$ and $\overline{D}$ for the Dirac operator on (M, g).

PROPOSITION 3.1. *Let $\psi_0 \in \Sigma_P M$, then there is a spinor $\Psi(\psi_0)$ which is harmonic on $M \setminus P$, and which has in the synchronous trivialization defined above the following expansion at P:*

$$\Psi(\psi_0) = \frac{1}{r^{n-1}}\frac{\bar{x}}{r} \cdot \psi_0 + \Theta_1(x) \text{ if } n = 3$$

$$\Psi(\psi_0) = \frac{1}{r^{n-1}}\frac{\bar{x}}{r} \cdot \psi_0 + \Theta_2(x) \text{ if } n = 4$$

$$\Psi(\psi_0) = \frac{1}{r^{n-1}} \frac{\bar{x}}{r} \cdot \psi_0 + \alpha(x) + \Theta_3(x) \text{ if } n = 5$$

where $\Theta_1 \in C^{0,a}(\mathbb{R}^n)$ *for all* $a \in]0,1[$, *where, for all* $\varepsilon > 0$, $r^\varepsilon \Theta_2$, $r^\varepsilon \Theta_3$, $r^{1+\varepsilon}|\nabla \Theta_2|$ *and* $r^{1+\varepsilon}|\nabla \Theta_3|$ *are continuous on* $\mathbb{R}^n$ *and where* α *is homogeneous of order* -1 *and even near* P.

Remark. As before, the spinor $\omega_{n-1}^{-1}\Psi(\psi_0)$ is the Green's function for the Dirac operator on M. The expansion of $\Psi(\psi_0)$ could be improved but the statement of proposition 3.1 is sufficient to adapt the proof of theorem 2.2.

Proof of positive mass theorem. Using this proposition, the proof of theorem 2.2 can easily be adapted with $\psi = \Psi(\psi_0)$. As one can check, equation (2.7) is still available in dimensions 3, 4 and 5. This is easy to see in dimensions 3 and 4. In dimension 5, we note that since α is even near P and since $\frac{\partial}{\partial r}$ is an odd vector field, we have

$$\int_{S^{n-1}} Re(< \frac{x}{r} \cdot \psi_0, \alpha(x) >) ds = \int_{S^{n-1}} Re(< \frac{x}{r} \cdot \psi_0, \partial_r \alpha(x) >) ds = 0$$

Equation (2.7) easily follows. This proves positive mass theorem. □

We will now give a proof of Proposition 3.1.

Definition. Let $\alpha \in \Gamma_{\mathbb{R}^n \setminus \{0\}}(\Sigma M)$ be a smooth spinor defined on $\mathbb{R}^n \setminus \{0\}$. For $k \in \mathbb{R}$, we say that α is *homogeneous of order* k if $\alpha(sx) = s^k \alpha(x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and all $s > 0$. This is equivalent to $\partial_r \alpha = k\alpha$.

PROPOSITION 3.2. *Let* $k \in]-n, -1[$ *and* α *be homogeneous of order* k. *Then, there exists* β $(k+1)$*-homogen such that* $D(\beta) = \alpha$.

Proof. Let α be a homogeneous form of order k. Recall that $\Gamma_D := \frac{x}{\omega_{n-1} r^n} \cdot$ is the Green's function for the Dirac operator on $\mathbb{R}^n$. We define $\beta := \Gamma_D * \alpha$ in the sense of distributions, i.e.

$$\langle \beta, \varphi \rangle(x) = \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \left\langle \alpha, \frac{y-x}{\omega_{n-1}|y-x|^n} \cdot \varphi \right\rangle dy$$

for all test spinors φ on $\mathbb{R}^n$. Since $k \in]-n, -1[$, the integral is well defined. Moreover, $D(\beta) = \alpha$ and a simple change of variables shows that β is homogeneous of order $k+1$.

LEMMA 3.3 (Regularity Lemma). *Let* α *be a smooth spinor on* $\mathbb{R}^n - \{0\}$. *We assume that* $D(\alpha) = O(\frac{1}{r})$ *and* $\partial_i D(\alpha) = 0(\frac{1}{r^2})$ *as* $r \to 0$. *Then, for all* $\varepsilon > 0$, $r^\varepsilon \alpha$ *and* $r^{1+\varepsilon}|\nabla \alpha|$ *extends continuously to* $\mathbb{R}^n$.

Proof. As the statement is local, we can assume for simplicity that α vanishes outside a ball $B_0(R)$ around 0. Since $D(\alpha) \in L^q(\mathbb{R}^n)$ for all $q < n$, we get from regularity theory that $\alpha \in H_1^q(\mathbb{R}^n)$ for all $q < n$. Sobolev embedding theorem then implies that $\alpha \in L^q(\mathbb{R}^n)$ for all $q > 1$. Moreover, we have

$$|D(r^\varepsilon \alpha)| \leq \varepsilon r^{\varepsilon - 1}|\alpha| + O(r^{\varepsilon - 1})$$

Using Hölder inequality, we see that $D(r^\varepsilon \alpha) \in L^q(\mathbb{R}^n)$ for some $q > n$. By regularity theorem, we have $r^\varepsilon \alpha \in H_1^q(\mathbb{R}^n)$ and by Sobolev embedding theorem, $r^\varepsilon \alpha \in C^0(\mathbb{R}^n)$. This proves the first part of lemma 3.3. For the second part, we apply the same argument twice: a calculation yields on $\mathbb{R}^n - \{0\}$:

$$|D(r^{1+\varepsilon} \partial_i \alpha)| \leq (1+\varepsilon) r^\varepsilon |\partial_i \alpha| + O(r^{\varepsilon - 1}) \tag{3.4}$$

In the same way, we have:

$$|D(r^\varepsilon \partial_i \alpha)| \leq \varepsilon r^{\varepsilon-1}|\partial_i \alpha| + O(r^{\varepsilon-2}) \tag{3.5}$$

For all $i = 1, \ldots, n$ we have $D(\partial_i \alpha) = \partial_i D\alpha = O(r^{-2})$ and $D(\partial_i \alpha) \in L^q(\mathbb{R}^n)$ for all $q < \frac{n}{2}$. Using regularity theorem and then Sobolev embedding theorem, we get that $\partial_i \alpha \in H_1^q(\mathbb{R}^n)$ for all $q < \frac{n}{2}$ and that $\partial_i \alpha \in L^q(\mathbb{R}^n)$ for all $q < n$. The Hölder inequality implies that there is a $q > \frac{n}{2}$, close to $\frac{n}{2}$, such that $r^{\varepsilon-1}|\partial_i \alpha| \in L^q(\mathbb{R}^n)$. Together with (3.5), this shows that $D(r^\varepsilon \partial_i \alpha) \in L^q(\mathbb{R}^n)$ for some $q > \frac{n}{2}$. By regularity and Sobolev theorems, we obtain that $r^\varepsilon \partial_i \alpha \in L^q(\mathbb{R}^n)$ for some $q > n$. Now using (3.4), we obtain $|D(r^{1+\varepsilon} \partial_i \alpha)| \in L^q(\mathbb{R}^n)$ for some $q > n$. Applying again regularity and Sobolev theorems, we get that $r^{1+\varepsilon} \partial_i \alpha \in C^0(\mathbb{R}^n)$. This proves that $r^{1+\varepsilon}|\nabla \alpha| \in C^0(\mathbb{R}^n)$. □

Proof of Proposition 3.1.
Standard formulas in normal coordinates yield for any spinor ψ

$$\overline{D}(\overline{\psi}) = \overline{D\psi} + \frac{1}{4} \sum_{ijk} \Gamma_{ij}^k e_i \cdot e_j \cdot e_k \cdot \overline{\psi} \tag{3.6}$$

with $\Gamma_{ij}^k = g(\nabla_{e_i} e_j, e_k) = \sum_l \frac{1}{2} x^l g(R(e_l, e_i)e_j, e_k) + O(r^2)$. The Bianchi identity implies that

$$\sum_{\substack{i,j,k \\ i \neq j \neq k \neq i}} \Gamma_{ij}^k e_i \cdot e_j \cdot e_k = 0.$$

Hence,

$$\sum_{ijk} \Gamma_{ij}^k e_i \cdot e_j \cdot e_k = \sum_l x^l \mathrm{Ric}_{lk}(P) e_k + O(r^2) = O(r^2) \tag{3.7}$$

Let η be a cut-off function equal to 1 in a neighborhood of P in M, and supported in the normal neighborhood. Let ψ_0 be a constant spinor on $\mathbb{R}^n$. We define ψ on $U - \{0\}$ by

$$\overline{\psi} = \frac{\eta}{r^{n-1}} \frac{\overline{x}}{r} \cdot \overline{\psi}_0$$

on $M - \{P\}$. Then, ψ is harmonic near 0 (see locally conformally flat case) and by (3.6), we have near P

$$\overline{D}(\overline{\psi}) = \frac{1}{4r^{n-1}} \sum_{ijk} \Gamma_{ij}^k e_i \cdot e_j \cdot e_k \cdot \frac{\overline{x}}{r} \cdot \overline{\psi}_0.$$

Writing the Taylor development of $\sum_{ijk} \Gamma_{ij}^k e_i \cdot e_j \cdot e_k$ at order 3 in 0, we see by (3.7) that we can write $\overline{D}(\overline{\psi})$ as a sum of a spinor γ which is homogeneous of order $3 - n$ and a spinor γ', smooth on $\mathbb{R}^n - \{0\}$, which satisfies $\gamma' = 0(r^{4-n})$ and $|\nabla \gamma'| = 0(r^{3-n})$.

$$\overline{D}(\overline{\psi}) = \gamma + \gamma'$$

If $n = 3$, $\gamma + \gamma' \in L^q(\mathbb{R}^n)$ for all $q > 1$. Let η be a compactly supported function such that $\eta|_U \equiv 1$ on a neighborhood U of 0. Then $\Theta := \Gamma_D * (\eta(\gamma + \gamma'))$ is a smooth spinor on $\mathbb{R}^n - \{0\}$ such that $D(\Theta) = \gamma + \gamma'$ near 0. Clearly, Θ is smooth on $\mathbb{R}^n - \{0\}$. By regularity theorem and Sobolev embedding theorem, we get that $\Theta \in H_1^q(\mathbb{R}^n)$ for all $q > 1$ and that $\Theta \in C^{0,a}(\mathbb{R}^n)$ for all $a \in (0, 1)$. Moreover, we can multiply Θ by a cut-off function and assume that Θ is supported in a small neighborhood of 0.

If $n = 4$, we have $\gamma + \gamma' = 0(\frac{1}{r})$ and $|\nabla(\gamma + \gamma')| = 0(\frac{1}{r^2})$. As in dimension 3, we can find Θ, a smooth spinor on $\mathbb{R}^n - \{0\}$ such that $D(\Theta) = \gamma + \gamma'$ near 0. We fix $\epsilon \in (0, 1)$. By the regularity lemma 3.3 we get $r^\varepsilon \Theta \in C^0(\mathbb{R}^n)$ and $r^{1+\varepsilon}|\nabla\Theta| \in C^0(\mathbb{R}^n)$.

If $n = 5$, by proposition 3.2, we can find a spinor α homogeneous of order -1 such that $D(\alpha) = \gamma$. Moreover, $\gamma' = 0(\frac{1}{r})$ and $|\nabla\gamma'| = 0(\frac{1}{r^2})$. Proceeding as in dimension 3 and using lemma 3.3, we can find a spinor f smooth on $\mathbb{R}^n - \{0\}$ such that $D(f) = \gamma'$ near 0, such that $r^\varepsilon f \in C^0(\mathbb{R}^n)$ and such that $r^{1+\varepsilon}|\nabla f| \in C^0(\mathbb{R}^n)$. We set $\Theta = \eta(\alpha + f)$ where η is a cut-off function.

Now, for all dimensions, we set

$$\varphi = \psi - \Theta$$

By (3.6), we have

$$\overline{D}\overline{\varphi} = \overline{D}(\overline{\psi}) - \overline{D\Theta} - \frac{1}{4}\sum_{ijk} \Gamma_{ij}^k e_i \cdot e_j \cdot e_k \cdot \overline{\Theta}$$

Using (3.7) and the fact that $\overline{D}(\overline{\psi}) - \overline{D\Theta} = 0$, we get that $\overline{D}\overline{\varphi} = O(r)$ and hence is of class $C^\infty(M - \{P\}) \cap C^{0,1}(M)$. As a consequence, there exists $\overline{\varphi}' \in \Gamma(\Sigma M)$ of class $C^\infty(M - \{P\}) \cap C^{1,1}(M)$ such that $\overline{D}\overline{\varphi}' = \overline{D}\overline{\varphi}$. We now set $\Psi(\psi_0) = \overline{\varphi} - \overline{\varphi}'$. Proposition 3.1 follows. □

References

[AHM03] B. Ammann, E. Humbert, B. Morel. Preprint

[Aub76] T. Aubin. Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. *J. Math. Pur. Appl., IX. Ser.*, 55:269–296, 1976.

[Bar86] R. Bartnik. The mass of an asymptotically flat manifold. *Commun. Pure Appl. Math.*, 39:661–693, 1986.

[Her98] M. Herzlich. Théorèmes de masse positive. (Positive mass theorems). In *Séminaire de théorie spectrale et géométrie. Année 1997-1998. St. Martin D'Hères: Université de Grenoble I, Institut Fourier, Sémin. Théor. Spectrale Géom., Chambéry-Grenoble. 16, 107-126* . 1998.

[Hij86] O. Hijazi A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing Spinors. *Comm. Math. Phys.*, 104: 151–162, 1986.

[Hit74] N. Hitchin. Harmonic spinors. *Adv. Math.*, 14:1–55, 1974.

[LP87] J. M. Lee and T. H. Parker. The Yamabe problem. *Bull. Am. Math. Soc., New Ser.*, 17:37–91, 1987.

[Sch84] R. Schoen. Conformal deformation of a Riemannian metric to constant scalar curvature. *J. Diff. Geom.*, 20:479–495, 1984.

[Sch89] R. M. Schoen. Variational theory for the total scalar curvature functional for Riemannian metrics and related. In *Topics in calculus of variations, Lect. 2nd Sess., Montecatini/Italy 1987, Lect. Notes Math. 1365, 120-154* . 1989.

[SY79] R. Schoen and S.-T. Yau. On the proof of the positive mass conjecture in general relativity. *Comm. Math. Phys.*, 65:45–76, 1979.

[SY88] R. Schoen and S.-T. Yau. Conformally flat manifolds, Kleinian groups and scalar curvature. *Invent. Math.*, 92(1):47–71, 1988.

[Tru68] N.S. Trudinger. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. *Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser.*, 22:265–274, 1968.

[Wit81] E. Witten. A new proof of the positive energy theorem. *Commun. Math. Phys.*, 80(3):381–402, 1981.

[Yam60] H. Yamabe. On a deformation of Riemannian structures on compact manifolds. *Osaka Math. J.*, 12:21–37, 1960.

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