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Cycle-cocycle partitions and faithful cycle covers of locally finite graphs

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Abstract

By a result of Gallai, every finite graph G has a vertex partition into two parts each inducing an element of its cycle space. This fails for infinite graphs if, as usual, the cycle space is defined as the span of the edge sets of finite cycles in G. However we show that, for the adaptation of the cycle space to infinite graphs recently introduced by Diestel and Kühn (which involves infinite cycles as well as finite ones), Gallai's theorem extends to locally finite graphs. Using similar techniques we show that if Seymour's faithful cycle cover conjecture is true for finite graphs then it also holds for locally finite graphs when infinite cyles are allowed in the cover, but not otherwise.

1 Introduction

By a result of Gallai (see Lovász [7]), every finite graph has a 'cycle-cocycle' partition of its edge set induced by a bipartition of its vertex set:

Theorem 1.1. *Every finite graph G admits a vertex partition into (possibly empty) sets V_1, V_2 such that both $E(G[V_1])$ and $E(G[V_2])$ are elements of the cycle space of G.*

As stated above, Gallai's theorem has no obvious extension to infinite graphs. Indeed, when G is infinite, the elements of its (usual) cycle space are still finite sets of edges, so a partition as in Theorem 1.1 does not exist, for instance, when G is an infinite disjoint union of triangles.

One way to deal with this problem is to look for an equivalent reformulation of Theorem 1.1 and extend that. For example:

Theorem 1.2. *Every locally finite graph G admits a vertex partition into (possibly empty) sets V_1, V_2 such that in both $G[V_1]$ and $G[V_2]$ all vertex degrees are even.*

(The proof of Theorem 1.2 is an easy exercise in compactness. It is also an immediate corollary of Theorem 1.4 below.)

However, the requirement that all degrees of a subgraph $H \subseteq G$ should be even is only one equivalent[1] reformulation among many of saying that $E(H)$ lies in the cycle space of G. Another is that H should be an edge-disjoint union of cycles (and isolated vertices). This would be just as meaningful for infinite H,

[1] for finite G

and for locally finite H it implies the even-degree condition but not conversely. (Consider a double ray, which is 2-regular but not a union of cycles.) But with this latter reformulation, Theorem 1.1 no longer extends to infinite graphs:

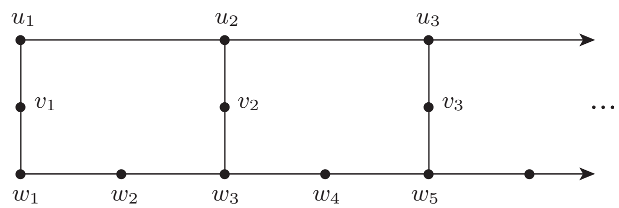

Figure 1: A graph with no bipartition into edge-disjoint unions of cycles

Proposition 1.3. *The graph G shown in Figure 1 has a unique vertex partition into two induced even-degree subgraphs. One of these is edgeless, the other a double ray.*

Proof. Consider any partition (V_1, V_2) of $V(G)$. Note that if two vertices x, y (such as u_1 and w_1) have a common neighbour z (such as v_1) not adjacent to any other vertex, then x and y must lie in the same partition class: otherwise, z would have degree 1 in its partition class. Thus if $u_1 \in V_1$, say, we deduce inductively that $w_1, w_3, w_5, \ldots \in V_1$ and hence also $u_2, u_3, u_4, \ldots \in V_1$. But $u_2, u_3, u_4 \ldots$ must not have degree 3 in $G[V_1]$, so $v_2, v_3, v_4, \ldots \in V_2$. Finally, v_1 lies in V_1 because u_2 does, so inductively $w_2, w_4, \ldots \in V_1$.

Thus, V_2 is the independent set $\{v_2, v_3, \ldots\}$, while V_1 consists of the remaining vertices, which span a double ray. □

Our aim in this paper is to show that, despite Proposition 1.3, we can do better than Theorem 1.2. Indeed, we can say more of the double ray $G[V_1]$ in Figure 1 than that its degrees are even: this double ray forms an *infinite cycle* in the cycle space $\mathcal{C}(G)$ introduced for infinite graphs in [2, 4]. (It does so, because its tails converge to the same end of G, which thus 'closes it up'; see Section 2 for formal definitions.) So for that space $\mathcal{C}(G)$, the graph of Figure 1 is no longer a counterexample to Theorem 1.1. And indeed, we have the following extension of Theorem 1.1 to infinite graphs, which implies Theorem 1.2 but is quite a bit stronger:

Theorem 1.4. *For every locally finite graph G there is a partition of $V(G)$ into two (possibly empty) sets V_1, V_2 such that $E(G[V_i]) \in \mathcal{C}(G)$ for both $i = 1, 2$.*

We shall prove Theorem 1.4 in Section 3. In Section 4 we use similar techniques to extend the *cycle double cover conjecture* and Seymour's *faithful cycle cover conjecture* to locally finite graphs: if these conjectures are true for finite graphs, they also hold for locally finite graphs with our notion of an infinite cycle space. (The latter conjecture fails unless infinite cycles are admitted; for the former we have been unable to decide whether infinite cycles are really needed.) In Section 5 we generalize our results to graphs with infinite degrees, as far as this can be reasonably expected.

2 Definitions

The basic terminology we use can be found in [3]. A 1-way infinite path is called a *ray*, a 2-way infinite path is a *double ray*, and the subrays of a ray or double ray are its *tails*. Two rays in a graph $G = (V, E)$ are *equivalent* if no finite set of vertices separates them; the corresponding equivalence classes of rays are the *ends* of G. We denote the set of these ends by $\Omega = \Omega(G)$.

Let us define a topology on G together with its ends; if G is locally finite, then this topology is known as its *Freudenthal compactification*. We begin by endowing G itself (without ends) with the usual topology of a 1-complex. (Thus, every edge is homeomorphic to the real interval $[0, 1]$, and the basic open neighbourhoods of a vertex v are the unions of half-open intervals $[v, z)$, one for every edge e at v with z an inner point of e.) In order to extend this topology to Ω, we take as a basis of open neighbourhoods of a given end $\omega \in \Omega$ the sets of the form

$$C(S, \omega) \cup \Omega(S, \omega) \cup E'(S, \omega)\,,$$

where $S \subseteq V$ is a finite set of vertices, $C(S, \omega)$ is the connected component of $G - S$ in which every ray from ω has a tail, $\Omega(S, \omega)$ is the set of all ends $\omega' \in \Omega$ whose rays have a tail in $C(S, \omega)$, and $E'(S, \omega)$ is any union of half-edges $(z, y]$, one for every S–C edge $e = xy$ of G, with z an inner point of e. Let $|G|$ denote the topological space on the point set $V \cup \Omega \cup \bigcup E$ thus defined. We shall freely view G and its subgraphs either as abstract graphs or as subspaces of $|G|$.

A set $C \subseteq |G|$ is a *circle* if it is homeomorphic to the unit circle. Then C includes every edge of which it contains an inner point, and the graph consisting of these edges and their endvertices is the *cycle* defined by C. Conversely, it is not hard to show [4] that $C \cap G$ is dense in C, so every circle is the closure in $|G|$ of its cycle and hence defined uniquely by it. Note that every finite cycle in G is also a cycle in this sense, but there can also be infinite cycles; see [2] for examples.

Call a family $(D_i)_{i\in I}$ of subsets of E *thin* if no vertex of G is incident with an edge in D_i for infinitely many i. (Thus in particular, no edge lies in more than finitely many D_i.) Let the *sum* $\sum_{i\in I} D_i$ of this family be the set of all edges that lie in D_i for an odd number of indices i, and let the *cycle space* $\mathcal{C}(G)$ of G be the set of all sums of (thin families of) edge sets of cycles, finite or infinite. Symmetric difference as addition makes $\mathcal{C}(G)$ into an $\mathbb{F}_2$ vector space, which coincides with the usual cycle space of G when G is finite. We remark that $\mathcal{C}(G)$ is closed also under taking infinite thin sums [4, 5], which is not obvious from the definitions.

As with finite graphs, elements of $\mathcal{C}(G)$ can be decomposed into cycles:

Theorem 2.1. [5] *Every element of the cycle space of a graph is the edge-disjoint union of cycles.*

3 Cycle-cocycle partitions

The purpose of this section is to prove Theorem 1.4. This proof will also serve as a model for other proofs later in the paper, which will refer to this proof and skip the corresponding details.

Our proof of Theorem 1.4 will be a compactness proof, but we shall need a non-trivial lemma from [4] to make this possible. Recall that while Theorem 1.2 has a straightforward compactness proof, the naïve extension of Theorem 1.1 to locally finite graphs does not (and is in fact false). The reason is, roughly speaking, that having all degrees even is a 'local' property of finite subsets $S \subseteq V(G)$ (one that S will satisfy in every large enough induced subgraph or in none), while inducing part of an element of the usual cycle space (based on finite cycles) is not: the sequence of finite cycles $C_n = P_n + e_n$, for example, where the $P_n = v_{-n}v_{-(n-1)} \dots v_{n-1}v_n$ are nested paths and e_n is the edge $v_{-n}v_n$, 'tends' for $n \to \infty$ to the double ray $D = \dots v_{-1}v_0v_1 \dots$ whose edge set does not lie in the usual cycle space of $\bigcup C_n$. However, D is an infinite cycle in $\bigcup C_n$, and more generally it turns out that all such 'limits' of finite cycles in a graph G are elements of $\mathcal{C}(G)$ (though not necessarily single infinite cycles).

The following result from [4] makes this precise by providing a characterization of the elements of $\mathcal{C}(G)$ among all the subsets of $E(G)$ that is 'local' in the above sense.

Lemma 3.1. [4] *Let G be a locally finite graph. Then the following statements are equivalent for every $Z \subseteq E(G)$:*

(i) *$Z \in \mathcal{C}(G)$; and*

(ii) *$|F \cap Z|$ is even for every finite cut F of G.*

We shall cast our compactness proof in terms of König's infinity lemma (see [3]), which we restate:

Lemma 3.2. *Let $W_1, W_2, \dots$ be an infinite sequence of disjoint non-empty finite sets, and let H be a graph on their union. For every $n \geq 2$ assume that every vertex in W_n has a neighbour in W_{n-1}. Then H contains a ray $v_1v_2 \dots$ with $v_n \in W_n$ for all n.*

Proof of Theorem 1.4. By treating the components of G separately, we may assume that G is connected. Let $v_1, v_2, \dots$ be an enumeration of $V(G)$. For $n \in \mathbb{N}$ put $S_n := \{v_1, \dots, v_n\}$, and define W_n as the set of all quadruples $(V_1, V_2, \mathcal{E}_1, \mathcal{E}_2)$ such that

(i) (V_1, V_2) is a partition of S_n into two (possibly empty) sets; and

(ii) for $i = 1, 2$, $\mathcal{E}_i$ is a partition of $E(G[V_i])$ such that for each $E \in \mathcal{E}_i$ there is a finite cycle $C \subseteq G - V_{3-i}$ with $E(C \cap G[V_i]) = E$.

Each set W_n is clearly finite. It is non-empty by Theorem 1.1 applied to $G[S_n]$; recall that every element of the cycle space of a finite graph is a disjoint union of edge sets of cycles, which we can take as the partition sets for $\mathcal{E}_1$ and $\mathcal{E}_2$.

Let us define a graph H on $\bigcup_{n=1}^{\infty} W_n$. For $n \geq 2$, let $(V_1, V_2, \mathcal{E}_1, \mathcal{E}_2) \in W_n$ be adjacent to $(V_1', V_2', \mathcal{E}_1', \mathcal{E}_2') \in W_{n-1}$ if and only if, for both $i = 1, 2$,

(iii) $V_i' \subseteq V_i$;

(iv) for each $E' \in \mathcal{E}_i'$ there is an $E \in \mathcal{E}_i$ such that $E \cap E(G[V_i']) = E'$.

Observe that for $n \geq 2$ every vertex in W_n has a neighbour in W_{n-1}.

By the infinity lemma, there is a ray $v_1 v_2 \dots$ in H with $(V_1^n, V_2^n, \mathcal{E}_1^n, \mathcal{E}_2^n) := v_n \in W_n$ for all n. Clearly, $V_1 := \bigcup_{n=1}^{\infty} V_1^n$ and $V_2 := \bigcup_{n=1}^{\infty} V_2^n$ form a partition of $V(G)$. By (iv), there is for every non-empty element E_i^n of a set $\mathcal{E}_i^n$ a unique ascending chain $E_i^n \subseteq E_i^{n+1} \subseteq \dots$ with $E_i^m \in \mathcal{E}_i^m$ for all $m \geq n$. Let $\mathcal{E}_i$ be the set consisting of the unions of such ascending chains, $i = 1, 2$. By (ii), the sets in $\mathcal{E}_i$ are disjoint and cover all of $E(G[V_i])$. Thus, $\mathcal{E}_i$ is a partition of $E(G[V_i])$.

We shall use Lemma 3.1 to show that all the sets $E \in \mathcal{E}_1 \cup \mathcal{E}_2$ are elements of $\mathcal{C}(G)$; since disjoint unions are thin sums, this will imply that $\bigcup \mathcal{E}_1$ and $\bigcup \mathcal{E}_2$ too are elements of $\mathcal{C}(G)$. Let E be given, and write $E_n := E \cap E(G[V_i^n])$ for each n.

Consider a finite cut F of G. Choose n large enough that $F \subseteq E(G[S_n])$. By (ii), there is a finite cycle $C \subseteq G - V_{3-i}^n$ with $E(C \cap G[V_i^n]) = E_n$. Then

$$F \cap E = F \cap E(G[S_n]) \cap E = F \cap E_n = F \cap E(C \cap G[V_i^n]) = F \cap E(C).$$

Since C is a cycle, the last intersection is even. Hence $E \in \mathcal{C}(G)$ by Lemma 3.1, as desired. $\square$

4 Faithful cycle covers

Another problem concerning cycles is the well-known cycle double cover conjecture, which states that every bridgeless finite graph has a cycle double cover. (A *cycle double cover* of a graph G is a family of cycles such that each edge of G lies on exactly two of those cycles.) Using the same techniques as in the proof of Theorem 1.4 one can show that if the cycle double cover conjecture is true for finite graphs then it also holds for locally finite graphs, possibly with infinite cycles. However, we have been unable to construct an example where infinite cycles are really needed.

This is different for the following related conjecture of Seymour, which extends with infinite cycles but fails with finite cycles only. For a graph G and a map $p : E(G) \to \mathbb{N}\ (\ni 0)$ a *faithful cycle cover* of (G, p) is a family of cycles such that every edge $e \in G$ lies on exactly $p(e)$ of those cycles. Such a map p is *admissible* if $p(F) = \sum_{f \in F} p(f)$ is even and $p(e) \leq p(F)/2$ for every finite cut F and every edge $e \in F$. We call p *even* if all its values $p(e)$ are even numbers. If (G, p) is to have a faithful cycle cover, then obviously p has to be admissible, and we shall see below that for some G it has to be even. Since the constant map with value 2 is admissible for bridgeless graphs, the following *faithful cycle cover conjecture* extends the cycle double cover conjecture:

Conjecture 4.1 (Seymour [8]). *Let G be a finite graph, and p an even admissible map. Then (G, p) has a faithful cycle cover.*

Unlike the cycle double cover conjecture, Conjecture 4.1 fails for locally finite graphs unless we allow infinite cycles. Here is a simple example. Let G be the double (= two-way infinite) ladder, and let p assign 0 to every rung and 2 to all the other edges. By our current definition of admissibility (which requires $p(e) \leq p(F)/2$ only for finite cuts F), the function p is admissible. But G contains no finite cycle that avoids all rungs, so (G, p) has no faithful cover consisting of finite cycles. (It does, however, have a faithful cover consisting of two copies of the infinite cycle spanned by its 2-edges.)

The above example is no longer a counterexample to the infinite analogue of Conjecture 4.1 if we require of an admissible map p that it satisfies $p(e) \leq p(F)/2$ also for infinite cuts F (and edges $e \in F$): if e is any edge with $p(e) = 2$ and R is a maximal ray in the subgraph of $G - e$ spanned by all its remaining 2-edges, then e and the 0-edges of G incident with R form an infinite cut F such that $p(e) = p(F)$. Thus, p is no longer admissible, and we no longer have a contradiction.

Our next example, however, shows that strengthening the definition of 'admissible' as above is not enough to make Conjecture 4.1 extendable to locally finite graphs, ie. that infinite cycles will be needed even then. Consider the ladder G shown in Figure 2 and the admissible map $p : E(G) \to \mathbb{N}$ defined by $p(e_i) = p(e'_i) = 2i$ and $p(f_i) = 2$ for all i. (Since $p(e) > 0$ for all e, we trivially have $p(e) \leq p(F)/2$ also for infinite cuts F.) Suppose there is a faithful cycle cover which contains a finite cycle D. Obviously, D contains exactly two rungs f_m, f_n, with $m < n$, say. Let $\mathcal{C}$ be the subfamily of the cover consisting of those cycles which pass through the edge e_n. Each but at most one (which might go through f_n) of the cycles in $\mathcal{C}$ must use the edge e_{n-1}. Thus, at least $|\mathcal{C}| - 1 = 2n - 1$ cycles of the cover meet the edge e_{n-1}, contradicting $p(e_{n-1}) = 2n - 2$. Therefore, the only faithful cycle cover that (G, p) can have (and which is easily seen to exist) must be one consisting of infinite cycles.

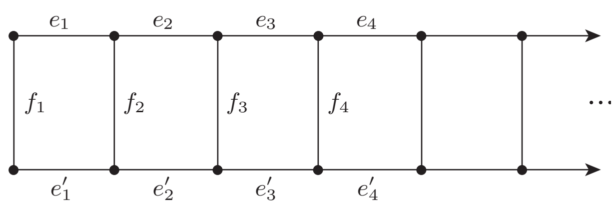

Figure 2: The unique faithful cycle cover consists of infinite cycles only

As soon as we allow infinite cycles, however, Conjecture 4.1 does extend to locally finite graphs:

Theorem 4.2. *Let G be a locally finite graph and $p : E(G) \to \mathbb{N}$ an even admissible map. If Conjecture 4.1 is true then (G, p) has a faithful cycle cover.*

Proof. We sketch how the proof of Theorem 1.4 has to be amended for Theorem 4.2. As before, we may assume that G is connected. Let $v_1, v_2, \ldots$ be an enumeration of its vertices, and put $G_n := G[\{v_1, \ldots, v_n\}]$. We define W_n as the set of all families $\mathcal{E}$ of edge sets $E \subseteq E(G_n)$ such that

(i) every edge $e \in G_n$ lies in exactly $p(e)$ members of $\mathcal{E}$; and

(ii) for every $E \in \mathcal{E}$ there is a finite cycle $C \subseteq G$ with $E(C \cap G_n) = E$.

The sets W_n are not empty. Indeed, consider the multigraph obtained by contracting the components of $G - G_n$ to one vertex each, keeping parallel edges but deleting loops. Subdividing the parallel edges we obtain a bridgeless finite graph G'_n. The map p induces an even and admissible map on G'_n, for which there is a faithful cycle cover by assumption. It is easy to see that the corresponding edges in G satisfy (i) and (ii).

The rest of the proof is analogous to that of Theorem 1.4: applying the infinity lemma to an auxiliary graph H, we obtain a family of elements of the cycle space such that every edge e lies on exactly $p(e)$ members of this family. By Theorem 2.1, we can modify this into a faithful cover consisting of single cycles. Therefore, if the faithful cycle cover conjecture holds for finite graphs, it is also true for locally finite graphs. □

Conjecture 4.1 requires p to be even, and indeed if p is allowed to assume odd values the conjecture becomes false: take the Petersen graph, and give p the value 2 on a perfect matching and 1 on all other edges. The following result, whose finite version is a theorem of Alspach, Goddyn and Zhang [1], can be proved like Theorem 4.2.

Theorem 4.3. *Let G be a locally finite graph not containing the Petersen graph as a minor, and let $p: E(G) \to \mathbb{N}$ be any admissible map (even or not). Then (G, p) has a faithful cycle cover.*

5 Graphs with infinite degrees

Theorem 1.4 does not extend to arbitrary graphs with vertices of infinite degree. For example, consider the graph G obtained by joining a vertex v_0 to every vertex of a ray $R := v_1v_2v_3\dots$. Suppose there is a partition as in Theorem 1.4, and assume that $v_0 \in V_1$. By the definition of thin sums, no element of $\mathcal{C}(G)$ can have infinitely many edges incident with v_0. So there is a maximal $n \geq 0$ with $v_n \in V_1$. But then v_{n+1} has degree 1 in $G[V_2]$, a contradiction.

The problem here is that no element of the cycle space is allowed to have a vertex of infinite degree. And indeed, if we weaken our concept of infinite sums, forbidding only those where some *edge* lies in infinitely many of the summands (ie. making no restrictions on vertices), our counterexample ceases to be one: putting $V_1 := \{v_3, v_6, v_9, \dots\}$, the set $V_2 := V \setminus V_1$ induces an element of the cycle space. Of course, there was a good reason for forbidding these sums: summing up the triangles $v_0v_1v_2v_0, v_0v_2v_3v_0, v_0v_3v_4v_0, \dots$ yields the ray v_0v_1R, which should then also be a member of the cycle space. But this is not unreasonable: as v_0 cannot be separated finitely from the ray R, this ray may be seen as converging to v_0.[2] And if we adjust our topology accordingly by identifying v_0 with the end containing R, the ray v_0v_1R becomes a cycle as desired.

This means we no longer view the graph G with the standard topology (the Freudenthal compactification), but with the identification topology ITOP, which we now define. A vertex v of G *dominates* an end ω of G if there is ray $R \in \omega$ and an infinite set of v–R paths that meet pairwise only in v. In order not to identify vertices, we require the following for G:

$$\textit{Every end of } G \textit{ is dominated by at most one vertex.} \tag{1}$$

[2]This can be made precise. An alternative way to define TOP, equivalent for locally finite G, is to take as basic open sets the components C of $G - F$ for all finite *edge* sets $F \subseteq E$, including the ends belonging to C and open half-edges from F incident with C. Then TOP cannot separate a vertex v_0 from an end ω to whose rays it sends an infinite fan as above, ie. TOP fails to be Hausdorff. But identifying v_0 with ω makes the rays of ω converge to v_0, and $|G|$ will be Hausdorff again. Similarly, our requirement (2) below can be seen as a condition ensuring that TOP can be made Hausdorff without vertex identifications that would change G.

Now call a vertex *equivalent* to all the ends it dominates, and call two ends *equivalent* if they are dominated by the same vertex. This clearly defines an equivalence relation on the space $|G|$. Denote the corresponding quotient space by $\tilde{G}$, and its (quotient) topology by ITOP. See [6] for more details on ITOP.

To obtain a cycle space which retains the natural properties of the cycle space of a locally finite graph, we have to impose another restriction on our graph G. Indeed, consider two vertices x and y that are linked by infinitely many independent paths. Then we can generate each of these paths P as a sum of cycles, so P should be in our cycle space. To avoid this, we require the following:

$$\textit{No two vertices of } G \textit{ are joined by infinitely many independent paths.} \quad (2)$$

Note that (2) implies (1). As before, we define as *cycles* those subgraphs of G whose closure in $\tilde{G}$ is homeomorphic to the unit circle, and the *cycle space* $\mathcal{C}(\tilde{G})$ of $\tilde{G}$ is defined as the span of all sums of cycles such that no edge appears in infinitely many of the summands. For the rest of this section, we assume that the graphs G we consider satisfy (2), and that all cycles are defined with respect to $\tilde{G}$.

Our main result now extends to graphs with infinite degrees, as follows:

Theorem 5.1. *Let G be a graph satisfying (2). Then there is a partition of $V(G)$ into two (possibly empty) sets V_1, V_2 such that $E(G[V_i]) \in \mathcal{C}(\tilde{G})$ for both $i = 1, 2$.*

For the proof of Theorem 5.1 we may assume G to be 2-connected, because the cycle space of a graph is the direct product of the cycle spaces of its blocks. (Recall that vertices are now allowed to lie in infinitely many summands as long as no edge does.) Then G is countable [6]. We now proceed exactly as in the proof of Theorem 1.4, except that instead of Lemma 3.1 we use the following analogous result:

Lemma 5.2. [6] *Let G be a graph satisfying (2). Then its cycle space $\mathcal{C}(\tilde{G})$ consists of precisely those sets of edges that meet every finite cut in an even number of edges.*

Our results of Section 4 can also be extended to graphs with infinite degrees, but we require the following strengthening[3] of (2):

$$\textit{No two vertices of } G \textit{ are joined by infinitely many edge-disjoint paths.} \quad (3)$$

Theorem 5.3. *Let G be a graph satisfying (3), and let $p : E(G) \to \mathbb{N}$ be an even admissible map. If Conjecture 4.1 is true then (G, p) has a faithful cycle cover.*

Proof. Consider a block B of G. Every cut of B is a cut of G, so the restriction of p to B is an even admissible map on B. As the cycle space of G is the direct product of the cycle spaces of the blocks of G, we may therefore assume G to be 2-connected. (Note that p assigns zero to bridges, so we need not cover these.) Then G is countable [6].

So consider an enumeration $v_1, v_2, \ldots$ of $V(G)$, and put $G_n := G[\{v_1, \ldots, v_n\}]$. We define W_n as the set of all families $\mathcal{E}$ of edge sets $E \subseteq E(G_n)$ such that

[3] This is indeed stronger than (2), see [6].

(i) every edge $e \in G_n$ lies in exactly $p(e)$ members of $\mathcal{E}$; and

(ii) for every $E \in \mathcal{E}$ there is a finite cycle $C \subseteq G$ with $E(C \cap G_n) = E$.

Let us show that the sets W_n are not empty. First note that the set $\mathcal{K}$ of components of $G - G_n$ is finite. Indeed, since G is 2-connected, every component of $G - G_n$ has two different neighbours in the finite set $V(G_n)$. So if $|\mathcal{K}| = \infty$ then there are two vertices $u, v \in G_n$ which are both adjacent to infinitely many $C \in \mathcal{K}$. Then there are infinitely many independent paths linking u to v, a contradiction to (3).

Next, consider a component $C \in \mathcal{K}$. For every two vertices $u, v \in G_n$ that both send infinitely many edges to C there is a finite cut $F_{u,v} \subseteq E(C)$ separating $N(u) \cap V(C)$ from $N(v) \cap V(C)$ in C, because of (3). Let F_C be the union of all such cuts $F_{u,v}$. Note that F_C is finite, as there are only finitely many pairs u, v. Therefore, $C - F_C$ has only finitely many components, and each of these sends only finitely many edges to $G - G_n$. Let $\mathcal{K}'$ be the set of all the components of some $C - F_C$, for all $C \in \mathcal{K}$. Observe that $\mathcal{K}'$ is again finite, that for every $D \in \mathcal{K}'$ there is at most one vertex u_D that sends infinitely many edges to D, and that this vertex u_D (if it exists) lies in G_n.

In G, contract every $D \in \mathcal{K}'$ to a vertex v_D, keeping parallel edges but deleting loops. If two vertices of the resulting graph are joined by infinitely many edges, then these are u_D and v_D for some $D \in \mathcal{K}'$. In a second step, we now contract all these edges $u_D v_D$, again keeping parallel edges. We obtain a finite multigraph G'_n with $V(G_n) \subseteq V(G'_n)$ and $E(G_n) \subseteq E(G'_n)$; note in particular that the edge set of G'_n is finite, despite parallel edges. By subdividing edges we may assume that G'_n is a simple graph. Since in all these contractions only loops were deleted, every cut in G'_n is also a cut in G. Hence, p induces an admissible even map p'_n on G'_n. By assumption there is a faithful cycle cover of (G'_n, p'_n). Every cycle in that cover can be extended to a finite cycle in G. The family of these cycles then satisfies (i) and (ii), thus proving $W_n \neq \emptyset$.

The rest of the proof is again analogous to that of Theorem 1.4 as that every element of $\mathcal{C}(\tilde{G})$ is an edge-disjoint union of cycles [6]. □

Using the same techniques as above, we can also extend Theorem 4.3:

Theorem 5.4. *Let G be a graph that satisfies (3) and does not contain the Petersen graph as a minor, and let $p : E(G) \to \mathbb{N}$ be any admissible map. Then (G, p) has a faithful cycle cover.*

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