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The algebra of metric betweenness I: Subdirect representations, retractions, and equational characterization of weakly median graphs

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THE ALGEBRA OF METRIC BETWEENNESS I: SUBDIRECT REPRESENTATION, RETRACTIONS, AND EQUATIONAL CHARACTERIZATION OF WEAKLY MEDIAN GRAPHS

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Abstract. We bring together algebraic concepts such as equational class and various concepts from graph theory for developing a structure theory for graphs that promotes a deeper analysis of their metric properties. The basic operators are Cartesian multiplication and gated amalgamation or, alternatively, retraction. Specifically, finite weakly median graphs are known to be decomposable (relative to these operators) into smaller pieces that in turn are parts of hyperoctahedra, the pentagonal pyramid, or of certain triangulations of the plane. This decomposition scheme can be interpreted as Birkhoff's subdirect representation in purely algebraic terms. Then the weakly median graphs can be identified with the discrete members of an equational class of ternary algebras satisfying five (independent) axioms on two to four points. This demonstrates that the median algebras featured by Avann and Sholander half a century ago and, more generally, Isbell's isotropic media can be generalized much further, without losing the close ties with graphs.

1. Introduction

Structure theories for graphs that directly allude to algebraic concepts, such as variety and subdirect representation, have been developed rather sporadically. In universal algebra, a variety (alias primitive class) is a class of algebras endowed with any finitary operations that is closed under taking homomorphic images, subalgebras and (direct) products. By Birkhoff's theorem, varieties are exactly the equational classes, i.e. they consist of all algebras of the same type satisfying a (possibly infinite) number of equations [41, 61]. In graph theory, varieties have rather been understood to be classes closed under retracts and products, but then an equational theory is not necessarily in sight. The choice of product is not unique here: it could e.g. either be the strong product or the Cartesian product [45]. In the former case, absolute retracts (of reflexive graphs) come into play [43], whereas for the latter product one is dealing with median

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graphs and their generalizations [14]. Subdirect representation in the algebraic context is an embedding into a product such that the projections onto the factors are surjective. By another theorem of Birkhoff, every algebra admits a subdirect representation by subdirectly irreducible algebras (characterized as the algebras for which the nontrivial congruences do not intersect in the equality relation). There have been attempts to imitate subdirect representation for categories of graphs ([42, 51]), but they seem to be somewhat too general (i.e. with too few subdirectly irreducibles) in order to be useful for specific graph-theoretic questions. There were also approaches to go from graphs to algebras: the so-called graph algebras, introduced in [52], use a (quasi-trivial) partial binary operation on the vertex set of a graph for codifying the edges that is extended to a full operation by adjoining a zero. This has served as a framework for constructing algebras with unusual properties (cf. [32]) rather than for a deeper understanding of graphs.

Some classes of graphs, possessing distinctive features of the geometry of their shortest paths, can be interpreted in algebraic terms quite naturally. Median graphs and their algebras constitute the simplest instance: the ternary operation on the vertex set associates to each triplet u, v, w the unique *median* $x = (uvw)$, i.e. the vertex x lying simultaneously on shortest paths between the three pairs from the triplet; see [1, 2, 53, 54, 55], cf. [9]. The median algebras resulting from this association are all subdirect products of the two-element algebra K_2. Extending the list of subdirectly irreducible algebras in order to encompass all complete graphs then yields quasi-median algebras or isotropic media [14, 46]. What is remarkable in this context is that the purely algebraic avenue leads to objects that can be regarded as particular graphs in the finite case, which admit quite a number of alternative characterizations. Now that quasi-median graphs have been generalized further to weakly median graphs, whereby maintaining a decomposition into simple building stones ("prime graphs") [7], one may wonder whether the algebra goes along with it, too. Somewhat surprisingly, it does - although the list of prime graphs includes quite different types of graphs: induced subgraphs of hyperoctahedra and triangulations of certain plane graphs. It is perhaps no accident that the prime models all have some geometric interpretation.

All graphs $G = (V, E)$ occurring here are undirected, connected, and without loops or multiple edges. The *distance* $d(u, v)$ between two vertices u and v is the length of a shortest (u, v)-path, and the *interval* $I(u, v)$ between u and v consists of all vertices on shortest (u, v)–paths, that is, of all vertices (metrically) *between* u and v:

$$I(u, v) = \{x \in V : d(u, x) + d(x, v) = d(u, v)\}.$$

An induced subgraph of G (or the corresponding vertex set A) is called *convex* if it includes the interval of G between any of its vertices. By the *convex hull* $\mathrm{conv}(W)$ of

W in G we mean the smallest subset of V (or induced subgraph of G) that contains W. An *isometric subgraph* of G is an induced subgraph in which the distances between any two vertices are the same as in G. In particular, convex subgraphs are isometric. A graph G is *weakly modular* [6, 12, 24] if its distance function d satisfies the following conditions:

Triangle condition (T): for any three vertices u, v, w with $1 = d(v, w) < d(u, v) = d(u, w)$ there exists a common neighbor x of v and w such that $d(u, x) = d(u, v) - 1$.

Quadrangle condition (Q): for any four vertices u, v, w, z with $d(v, z) = d(w, z) = 1$ and $2 = d(v, w) \leq d(u, v) = d(u, w) = d(u, z) - 1$, there exists a common neighbor x of v and w such that $d(u, x) = d(u, v) - 1$.

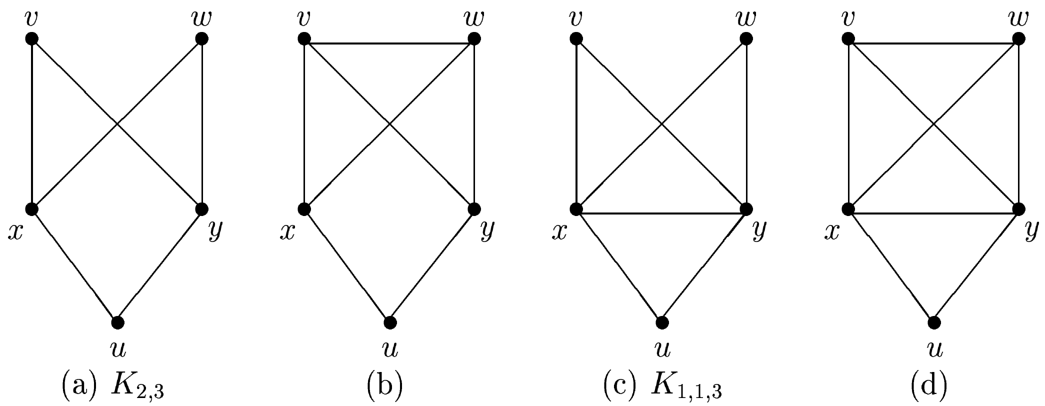


FIGURE 1. Weakly modular graphs that are not weakly median

These two conditions are fulfilled by modular [16], pseudo-modular [11], quasi-median graphs [14, 45], pre-median graphs [21], incidence graphs of dual polar spaces [19], and bridged graphs [36, 56]. Recall that a graph is called *bridged* if it does not contain any isometric cycle of length greater than 3, or alternatively, if the *neighborhood* $N(A) = \{y \in V |\ y \text{ is adjacent to some } x \in A\}$ of every convex set A of G is convex. (T) and (Q) can be merged into a single condition; namely, a graph G is weakly modular if and only if it satisfies

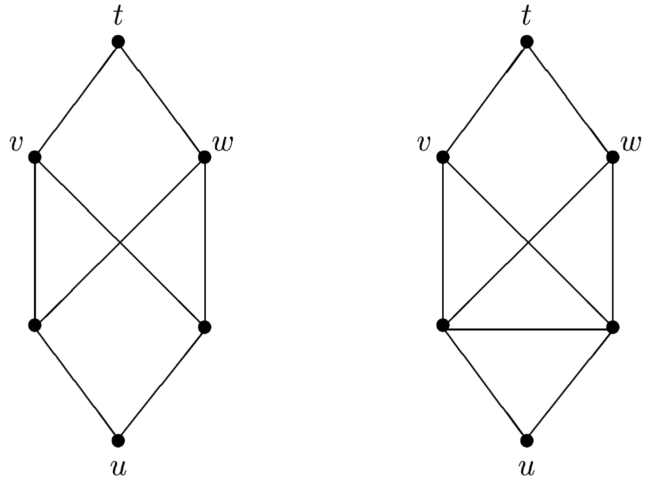
(TQ): for any three vertices u, v, w such that v and w are at distance 2 and have some common neighbor z with $2d(u, z) > d(u, v) + d(u, w)$, there exists a common neighbor x of v and w with $2d(u, x) < d(u, v) + d(u, w)$.

Indeed, (Q) is a trivial consequence of (TQ); and in order to derive (T) from (TQ) proceed by induction on $d(u, v) = d(u, w)$ and apply (TQ) first to the triplet u, v', w

for some neighbor v' of v in $I(u, v)$, which yields a common neighbor w' of v' and w such that $d(u, v') = d(u, w')$. Then apply (TQ) to the triplet v, w, u' where u' is the common neighbor of v' and w' in $I(u, v')$. This establishes (T). Conversely, in a weakly modular graph, (TQ) evidently holds for those triplets u, v, w where $d(u, v) = d(u, w)$, by virtue of (Q). Now, if $d(u, w) = d(u, v) + 1$ and $2d(u, z) > d(u, v) + d(u, w)$ for some common neighbor z of v and w, then employing (T) and (Q) yields vertices y and t such that t is a neighbor of v in $I(u, v)$ and y is a common neighbor of w, z, and t; the required vertex x is a common neighbor of t, v, and w provided by (T).

A *weakly median* graph is a weakly modular graph that does not contain any two distinct vertices with an unconnected triplet of common neighbors; see Fig. 1. Weakly median graphs thus satisfy the stronger variants, (T!), (Q!), and (TQ!), of the above conditions (T),(Q), and (TQ) which additionally require uniqueness of that neighbor x; see Fig. 2 for two (minimal) instances fulfilling (Q) but not (Q!). Indeed, if a weakly modular graph violates (T!) or (Q!), we obtain one of the graphs of Fig. 1 as an induced subgraph; if in some instance of (TQ) there was yet another vertex x' having the same distances to u, v, w as x, then either (Q!) would be violated or v, w, together with some common neighbor t of x and x' in $I(u, x)$ would constitute an unconnected triplet of common neighbors for x and x'. Since, on the other hand, (TQ!) rejects any of the four graphs indicated in Fig. 1, we can therefore state that a graph G is weakly median if and only if it satisfies (TQ!).

An induced subgraph H of a graph G is called *gated* if for every vertex x outside H there exists a vertex x' (the *gate* of x) in H such that each vertex y of H is connected with x by a shortest path passing through the gate x'; cf [34]. G is a *gated amalgam* of two graphs G_1 and G_2 if G_1 and G_2 are (isomorphic to) two intersecting gated subgraphs of G whose union is all of G. In regard to a decomposition scheme involving multiplication and amalgamation, a graph with at least two vertices is said to be *prime* if it is neither a proper weak Cartesian product [45] nor a gated amalgam of smaller graphs. For instance, the only prime median graph is the two-vertex complete graph K_2; see [46, 59]. More generally, the prime quasi-median graphs are exactly the complete graphs [14, 46]; see [45] for more information about quasi-median graphs. In [7], we established that the prime weakly median graphs are precisely (i) the *5-wheel* (a 5-cycle plus a pivot vertex adjacent to all vertices of the cycle), (ii) the *subhyperoctahedra* (induced subgraphs of hyperoctahedra, that is, multipartite graphs of the form $K_{i_1,i_2,i_3,\ldots}$ with $1 \leq i_j \leq 2$) different from the singleton graph K_1, the 3-vertex path $P_2 = K_{1,2}$, and the 4-cycle $C_4 = K_{2,2}$, and (iii) the *two-connected* K_4*- and* $K_{1,1,3}$*-free bridged graphs*, which are exactly the graphs embeddable in the plane such that all inner faces are triangles and all inner vertices have degrees larger than 5.



(a) (b)

FIGURE 2. Graphs fulfilling (Q) but not (Q!)

The paper is organized as follows. The next section describes the intrinsic algebras associated with any graph. Particular interest attaches to those ("apiculate") graphs for which there is a unique intrinsic algebra. Some basic equations can readily be established for the corresponding ternary algebras. In Section 3 we show that successive gated amalgamations lead to the subdirect representation of the resulting associated algebra by subdirectly irreducibles whenever they begin with a class of ("prime") graphs that possess only trivial gated subgraphs (and hence yield simple algebras). If these prime building stones have some additional properties, all fulfilled for prime weakly median graphs, then this subdirect representation can also be interpreted in terms of retracts (and Cartesian products), as is demonstrated in Section 4. Although for infinite weakly median graphs one does not necessarily have a finite decomposition scheme, the subdirectly irreducibles (viz., the prime constituents) can be retrieved as the weak Cartesian factors of the blocks relative to a canonical tolerance (Section 5). Much of this follows from the elegant theory of fiber-complementedness developed by Chastand [21]. Finitely generated weakly median graphs then turn out to be finite (Section 6). In Section 7, the convex hulls of metric triangles in weakly median graphs are determined. In the particular case of prime triangulations, the sides of any metric triangle extend to separating convex paths that partition the planar graph into regions that are relevant for locating point and interval "shadows" involving the corners of the metric triangle. This is a basic tool for verifying a number of equations in weakly median graphs that reflect their geometric structure. The equations considered in Section 8 capture a number of graph properties enjoyed by weakly median graphs.

Several combinations of these equations are then characteristic for this class of graphs, and a number of subclasses (such as the class of quasi-median graphs) can be described by some stronger equations (Section 9). The final section presents the main result (Theorem 4), by which weakly median graphs can be identified with "discrete" ternary algebras satisfying one of three sets of independent equations in four variables.

2. Intrinsic algebras and apiculate graphs

Every graph G with vertex set V can be turned into a ternary algebra, called an *apex algebra* of G [14]: an *apex operation* $(\ldots) : V^3 \to V$ maps any triplets u, v, w and u, w, v to a vertex $x = (uvw) = (uwv) \in I(u,v) \cap I(u,w)$, called a u-apex relative to v and w, such that $I(u,x)$ is maximal with respect to inclusion; see Fig. 3 for an illustration. Consequently,

$$I(u,v) = \{(uvx) : x \in V\} = \{(uxv) : x \in V\},$$

$$I((uvw),v) \cap I((uvw),w) = \{(uvw)\}.$$

The following equations then trivially hold:

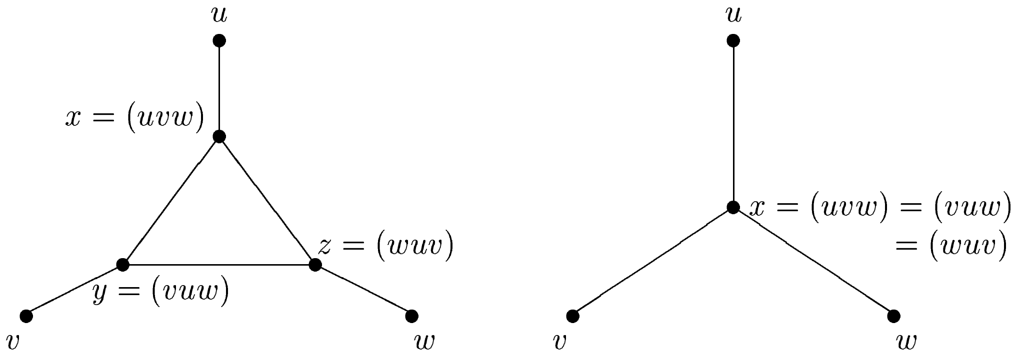
(A1)	$(uvv) = v$	(right majority),
(A1′)	$(uuv) = u$	(left majority),
(A2)	$(uvw) = (uwv)$	(right symmetry),
(A3)	$(vu(uvw)) = (uvw)$	(twisted left absorption),
(A3′)	$(uv(uvw)) = (uvw)$	(left absorption),
(A4)	$((uvw)vw) = (uvw)$	(right absorption),
(A4′)	$((uvw)uv) = (uvw)$	(left-right absorption).

Note that (A1′),(A2), and (A4′) are exactly the axioms 2, 1, and 5, respectively, of Isbell [46, p. 322], whereas his axiom 4b is a consequence of (A2) and (A3). Note that (A3′) follows from (A3) because

$$(uv(uvw)) = (uv(vu(uvw)) = (vu(uvw)) = (uvw).$$

Further, (A1), (A2), and (A3) together imply (A1′):

$$(uuv) = (uvu) = (uv(vuu)) = (vuu) = u.$$

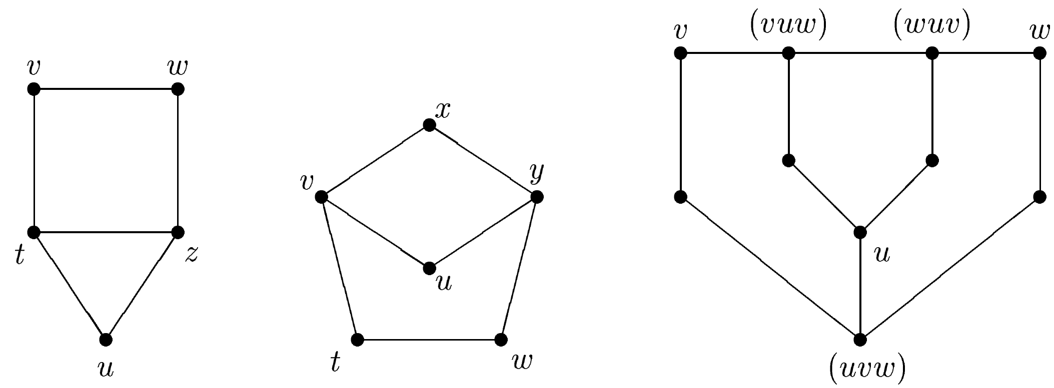


(a) non-symmetric (b) symmetric

FIGURE 3. Apex operation

The inherent non-uniqueness of apices does not permit a canonical choice for (uvw), but at least one could employ a priority rule. Namely, assume that the vertices are enumerated by some ordinal, providing the vertex set with a priority order. Then, if the vertices u, v, w admit a median, define $(uvw) = (uwv) = (vuw) = (vwu) = (wuv) = (wvu)$ as the median of u, v, w having highest priority; else, let $(uvw) = (uwv)$, $(vuw) = (vwu)$, and $(wuv) = (wvu)$ each be a respective apex of highest priority. We will refer to the resulting apex algebra as to a *priority* apex algebra.

When a graph G admits several distinct apex operations, these operations can be iterated to generate further ternary operations in the following way. Let $\nabla_u(v, w)$ be the smallest set S minus $\{v, w\}$ such that S includes v, w, and (uxy) for all $x, y \in S$ and apex operations $(\ldots)$ of G. Then any ternary operation $(\ldots)$ on V such that $(uvw) \in \nabla_u(v, w)$ is called an *intrinsic operation* of G, and the set V together with this operation is an *intrinsic algebra* of G. The interval $I(u, v)$ can be recovered from any intrinsic algebra just as in the case of an apex algebra. Observe that any intrinsic operation satisfies the above equations (A1)-(A4′) except possibly (A4). Trivially, there exists a unique vertex $s \in \nabla_u(v, w)$, referred to as the *imprint* of v, w with respect to u, that is at minimal distance to u, satisfying $s \in I(u, t)$ for all $t \in \nabla_u(v, w)$. For example, the imprint of v, w with respect to u in the graphs of Fig. 1 equals u, whereas an apex operation would select either x or y for this particular triplet u, v, w. The *imprint operation* of G then assigns to each triplet u, v, w the imprint of v, w with respect to u. It fulfills all of the above equations including (A4). This imprint function was introduced by Feder [37, 38] as the appropriate generalization of the imprint function of a quasi-median graph [29, 45] to arbitrary graphs. A different generalization is used in [18] under the same name, which constitutes the "median function" m of Tardif [58]: $m(u, v, w)$ is the gate of u in the smallest gated set containing v and w; see Lemma 1(e) below. Particular interest attaches to the case where imprint and apex operations coincide, that is, when the graph possesses a unique intrinsic algebra. In this case, we say that the graph is *apiculate*; see Fig. 4 for examples. In other words, G is apiculate if and only if for any vertex a the vertex set of G is a meet-semilattice with respect to the base-point order $\preceq_a$ defined by $u \preceq_a v \Leftrightarrow u \in I(a,v)$, that is, $I(a,v) \cap I(a,w) = I(a,(avw))$. Then every principal ideal $I(a,b)$ of the meet-semilattice $(V, \preceq_a)$ is a lattice, where the b-apex relative to $w, x \in I(a,b)$ is their join. Each of these lattices is modular when G is weakly median. Indeed, the quadrangle and triangle conditions imply that $(I(a,b), \preceq_a)$ is lower and upper semimodular, which is equivalent to modularity because of finite length; see [30]. This (semi)lattice condition alone, of course, does not characterize weakly median graphs. For instance, all base-point orders in a geodesic graph (e.g. an odd cycle or the Petersen graph) yield tree semilattices, whence geodesic graphs are apiculate.



(a) house (b) (c)

FIGURE 4. Apiculate graphs

Proposition 1. *A graph G is apiculate if and only if some intrinsic operation of G satisfies one of the equations*

(A5) $(uv(uwx)) = (u(uvw)x)$ (associativity),

(A5′) $(u(uvw)(uv(uwx))) = (uv(uwx))$ (monotonicity).

Proof. In order to verify (A5) for an apiculate graph G, set $a := (uv(uwx))$ and $b := (u(uvw)x)$. Note that $a, b \in I(u,v) \cap I(u,w) \cap I(u,x)$. Since (uvw) and (uwx) are the u-apices relative to v, w and w, x, respectively, we also obtain that

$$a, b \in I(u,(uvw)) \cap I(u,(uwx)).$$

Hence $b \in I(u,v) \cap I(u,(uwx))$. Since a is the u-apex relative to v and (uwx) and the graph G is apiculate, we conclude that $b \in I(u,a)$. Analogously, one can show that $a \in I(u,b)$. Evidently, this implies that $a = b$.

When (A5) is satisfied, then in the particular instance

$$(u(uvw)(u(uvw)x)) = (u(uvw)x)$$

of (A3′) we can replace $(u(uvw)x)$ by $(uv(uwx))$ and thus obtain (A5′). Hence (A5) implies (A5′).

Finally, if (A5′) holds, then consider any u-apex x relative to v and w. Then $(u(uvw)x) = x$ holds by (A5′), whence $x = (uvw)$ follows and therefore G is apiculate. □

A graph G is a *Pasch graph* [8, 60] if it satisfies the following analogue of the Pasch axiom of elementary geometry: for any five vertices u, v, w, x, y with $x \in I(u, v)$ and $y \in I(u, w)$, the intervals $I(v, y)$ and $I(w, x)$ intersect. This in turn is equivalent to the requirement that for any three vertices u, v, w the (*interval-*)*shadow*

$$I(v, w)/u = \{x \in V : I(u, x) \cap I(v, w) \neq \emptyset\}$$

is convex [26]. Recall that a subset A of V is called *convex* if $I(x, y) \subseteq A$ for all $x, y \in A$. The key feature of Pasch graphs is the following separation property (by which they are actually characterized): each pair of disjoint convex sets can be extended to a pair of complementary convex sets, called *halfspaces*; see [26]. Since weakly median graphs are exactly the weakly modular Pasch graphs [26], all subsequent properties established for Pasch graphs thus hold for weakly median graphs as well. For instance, intervals $I(u, v)$ are convex [60, Proposition 4.15] and, trivially, every (*point-*)*shadow* $v/u = \{v\}/u$ (also called extension of v from u [50]) is convex in a Pasch graph.

Proposition 2. *Every Pasch graph G is apiculate.*

Proof. Pick an arbitrary triplet u, v, w and let x be a vertex in $I(u, v) \cap I(u, w)$ furthest away from u. Suppose by way of contradiction that there exists a vertex $y \in I(u, v) \cap I(u, w)$ outside the interval $I(u, x)$. By the Pasch axiom, the intervals $I(x, w)$ and $I(y, v)$ have a vertex z in common. From the choice of x and y we infer that $z \neq x, y$. Since $x, y \in I(u, v) \cap I(u, w)$ and $z \in I(x, w) \cap I(y, v)$, we conclude that $x, y \in I(z, u)$ and $z \in I(u, v) \cap I(u, w)$, contrary to the choice of x and y. □

By this observation, the weakly modular apiculate graphs are exactly the weakly median graphs (as the four forbidden five-vertex graphs are not apiculate). The Petersen graph shows that an apiculate graph is not necessarily a Pasch graph even when intervals and point-shadows are convex. The latter condition can be turned into an equation, as we see next.

Proposition 3. *An apiculate graph has convex point-shadows exactly when its imprint operation satisfies*

(A6) $(u(uvw)(vwx)) = (uvw)$.

Proof. Assume that (A6) holds. If $v, w \in y/u$ and $x \in I(v, w)$, then

$$(u(uvw)x) = (u(uvw)(vwx)) = (uvw),$$

whence $x \in (uvw)/u \subseteq y/u$ because G is apiculate. Conversely, convexity of $(uvw)/u$ yields $(vwx) \in (uvw)/u$, which is expressed by (A6). □

Three (not necessarily distinct) vertices x, y, z of a graph G are said to form a *metric triangle* xyz if the intervals $I(x, y), I(y, z)$, and $I(z, x)$ pairwise intersect only in the common end vertices. If $d(x, y) = d(y, z) = d(z, x) = k$, then this metric triangle is called *equilateral* of *size* k. A (degenerate) equilateral metric triangle of size 0 is simply a single vertex. We say that a metric triangle xyz is a *quasi-median* of the triplet u, v, w if

$$\begin{aligned} d(u, v) &= d(u, x) + d(x, y) + d(y, v), \\ d(v, w) &= d(v, y) + d(y, z) + d(z, w), \\ d(w, u) &= d(w, z) + d(z, x) + d(x, u). \end{aligned}$$

Note that this definition is more general than the specific notion used in the context of quasi-median graphs [14, 45, 47] in that here quasi-medians are not necessarily equilateral (or of minimum size). Observe that, for every triplet u, v, w, a quasi-median xyz can be constructed in the following way: first select any vertex x from $I(u, v) \cap I(u, w)$ at maximal distance to u, then select a vertex y from $I(v, x) \cap I(v, w)$ at maximal distance to v, and finally select any vertex z from $I(w, x) \cap I(w, y)$ at maximal distance to w. In the case that the quasi-median is degenerate ($x = y = z$), it is a median of the triplet u, v, w.

Proposition 4. *An apiculate graph G has unique quasi-medians, that is, $(uvw), (vuw), (wuv)$ form the quasi-median for any triplet u, v, w of vertices, if and only if*

(A7) $(u(uvw)(vuw)) = (uvw)$,
or equivalently,

(A7′) $(u(vuw)w) = (uvw)$.

Proof. First observe that (A7) and (A7′) are equivalent:

$$\begin{aligned} (u(uvw)(vuw)) &= (u(vuw)(uvw)) && \text{by (A2)} \\ &= (u(u(vuw)v)w) && \text{by (A5)} \\ &= (u(uv(vuw))w) && \text{by (A2)} \\ &= (u(vuw)w) && \text{by (A3).} \end{aligned}$$

As noticed above, one can always construct a quasi-median of u, v, w of the form $(uvw)yz$. Hence, if u, v, w admit a unique quasi-median, then it must be of the form $(uvw)(vuw)(wuv)$. Conversely, if the latter is a quasi-median of u, v, w, then any quasi-median xyz satisfies $x \in I(u, (uvw))$ etc., so that $x = (uvw)$ etc. follows because xyz is a metric triangle. □

Observe that (A6) together with (A2) implies (A7) (simply let $x = u$), that is, an apiculate graph with convex point-shadows has unique quasi-medians. An apiculate graph with unique quasi-medians need not have convex shadows v/t (Fig. 4(b)), and an apiculate graph may have multiple quasi-medians (Fig. 4(c)).

All the properties discussed in this section (apiculate, Pasch, convexity of point-shadows, uniqueness of quasi-medians, etc.) are preserved under Cartesian multiplication (understood as a finitary operation for graphs) and gated amalgamation (for the latter, the proofs are similar to the one for the Pasch property; see [60, Theorem 5.14]).

3. Gated amalgams as subdirect products

For the algebraic framework we will assume throughout (unless stated otherwise) that the graphs under consideration are endowed with their imprint operations. The vertex set V together with the imprint operation $u, v, w \mapsto (uvw)$ then constitutes the *imprint algebra* of the graph $G = (V, E)$. If the graph G has a name or an acronym, its imprint algebra will be referred to by the same name or acronym; thus, the imprint algebra of the triangle K_3 is then the triangle algebra or K_3 algebra, for short. The algebraic terms "subalgebra", "direct product", "homomorphism", "congruence", "subdirect product" etc. that refer to the imprint algebra(s) have the usual meaning [41], and we will briefly speak, for example, of the congruences etc. of the graph G that carries the imprint algebra. The direct product of graphs in this algebraic sense is, of course, traditionally referred to as the Cartesian product. If ψ is a homomorphism from G into another graph, then the congruence on G associated with ψ, called the *kernel* of ψ, is denoted by $\ker\psi = \{(x, y) \in V^2 | \psi x = \psi y\}$. A *tolerance* of G is a reflexive and symmetric binary relation ξ on V compatible with the ternary operation:

$$u\xi x, v\xi y, w\xi z \text{ implies } (uvw)\xi(xyz).$$

A *block* of ξ is any maximal set of pairwise tolerant vertices. The transitive tolerances are then the congruences. By virtue of transitivity and (A2), an equivalence relation θ is a congruence exactly when

$$v\theta w \text{ implies } (vxy)\theta(wxy) \text{ and } (xyv)\theta(xyw).$$

The congruence block containing x is denoted by $[x]$ (usually with a suffix referring to the congruence).

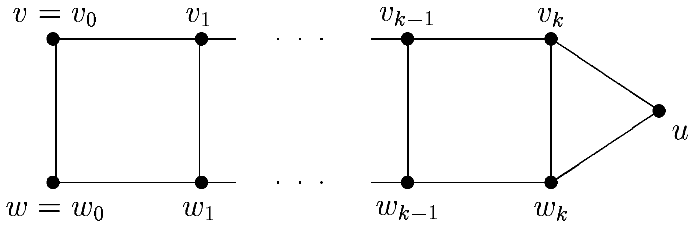


FIGURE 5. k-house

To give an example, consider the k-house ($k \geq 1$) in Fig. 5, generalizing the house. For any tolerance ξ of this graph different from the "all" relation ι, pairs x, y of distinct vertices can be tolerant only if $\{x, y\} \subseteq \{v_0, \ldots, v_k\}$ or $\{x, y\} \subseteq \{w_0, \ldots, w_k\}$. Indeed, $v_i \xi w_i$ for some i implies $v = (vwv_i)\xi(vww_i) = w$, whence $u = (uvw)\xi(uww) = w$, and analogously, $u\xi v$; moreover, $v = (w_k vv)\xi(w_k uv) = w_k$ and, similarly, $w\xi v_k$, yielding $\xi = \iota$ (because tolerance blocks are convex; see Lemma 1(b) below), a contradiction. In the same way, if u is tolerant with v_k or w_k, then v_k and w_k are tolerant, and we are back in the previous case. This proves the claim. It is easy to see that v_i and v_j for some $0 \leq i < j \leq k$ are tolerant exactly when w_i and w_j are tolerant. Therefore the tolerances of the k-house different from ι are in one-to-one correspondence with the tolerances of the path P_k with k edges. Hence the total number of tolerances of the k-house equals the Catalan number $\binom{2k+2}{k+1}/(k+2)$ plus 1; see [4].

The k-house is directly indecomposable (with respect to Cartesian multiplication) but it can be decomposed subdirectly. For each $1 \leq i \leq k$, the k-house G, labelled as in Fig. 5, admits a congruence θ_i with blocks $\{v_0, \ldots, v_{i-1}\}$, $\{v_i, \ldots, v_k\}$, $\{w_0, \ldots, w_{i-1}\}$, $\{w_i, \ldots, w_k\}$, and $\{u\}$. Thus, each homomorphic image G/θ_i constitutes a house. Since $\theta_1, \ldots, \theta_k$ intersect in the equality relation ω, the imprint algebra of G is embedded as a subalgebra in the product of the house algebras $G/\theta_1, \ldots, G/\theta_k$, that is, the k-house is a subdirect product of k houses (by virtue of Birkhoff's theorem; cf. [41]). The house itself is subdirectly irreducible but not simple, as it has a single nontrivial congruence.

Gated sets are readily described in terms of the imprint algebra. We say that a subset A of the vertex set V is an *ideal* if $(VAA) = \{(vab) | v \in V, a, b \in A\} \subseteq A$; note that $A \subseteq (VAA)$ holds trivially by right majority. The smallest ideal $\ll W \gg$ that contains a nonempty subset W of V is generated as follows:

$$W_0 := W \text{ and } W_k := (VW_{k-1}W_{k-1}) \text{ for } k \geq 1, \text{ so that } \ll W \gg := \cup_{k\geq 1} W_k.$$

We employ the following notation in the case of a gated amalgam $G = (V, E)$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ along their (nonempty) intersection $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$. For a gated subgraph $H = (W, F)$ of G, we say that W is a *gated* *set* and the mapping from V to W which assigns to every vertex of G its gate in H is the *gate map* of H (and W). Since $G_1 \cap G_2$ is a gated subgraph of G, both G_1 and G_2 are gated subgraphs of G. The gates of a vertex x of G in G_1, G_2, and $G_1 \cap G_2$ are denoted by x_1, x_2, and $x' = (x_1)_2 = (x_2)_1$, respectively. For $u, v, w \in V$ one has $|\{u, v, w\} \cap V_i| \geq 2$ for some $i = 1, 2$, and therefore $(uvw) = (u_i v_i w_i)$.

Lemma 1. *Let G be a graph.*

(a) *A nonempty subset A of V is gated if and only if it is an ideal of the imprint algebra of G.*

(b) *Every block of a tolerance ξ of G is gated.*

(c) *The gate map ψ_A of a gated set A is characterized as the mapping $\psi : V \to A$ that satisfies any one of the two identities $(uvx) = (uv\psi x)$ and $(xuv) = (\psi xuv)$ for all $u, v \in A$ and $x \in V$.*

(d) *Every tolerance ξ of G is compatible with every gate map ψ_A, that is, $v\xi w$ implies $\psi_A v \xi \psi_A w$ for all $v, w \in V$.*

(e) *Every tolerance ξ of G is compatible with the ternary operation m defined by letting $m(u, v, w)$ be the gate of u in the smallest gated set $\ll v, w \gg$ containing v and w.*

Proof. (a) If A is gated, then any u-apex x relative to $v, w \in A$ necessarily coincides with its gate in A, whence $(uvw) \in \nabla_u(v, w) \subseteq A$, which proves that A is an ideal. Conversely, if A is an ideal, then for $u \in V$ choose $v \in A$ nearest to u. For any $w \in A$ we get $(uvw) \in A \cap I(u, v)$, whence $v = (uvw)$ is the gate of u in A.

(b) If $u, v, w \in V$ and $v\xi w$, then $(uvw)\xi(uvv) = v$. Moreover, for $x \in V$ with $v\xi x$ and $w\xi x$ we obtain $(uvw)\xi(uxx) = x$. This shows that every block of ξ is an ideal and hence gated by (a).

(c) Since $\nabla_u(v, x) \subseteq I(u, v) \subseteq A$ for $u, v \in A$ and $x \in V$, one obtains $\nabla_u(v, \psi_A x) = \nabla_u(v, x)$ and hence $(uvx) = (uv\psi_A x)$. Similarly, as $\nabla_x(u, v) \subseteq A$, we get $\psi_A x \in I(x, (xuv))$ and therefore $(xuv) = (\psi_A xuv)$. Conversely, if $\psi : A \to V$ satisfies at least one of the two identities, then substituting $\psi_A x$ for u and ψx for v yields $\psi_A x = \psi x$ for $x \in V$ in either case because $\psi_A x \in I(x, \psi x)$.

(d) If $v\xi w$, then indeed

$$\psi_A v = (v\ \psi_A v\ \psi_A w)\xi(w\ \psi_A v\ \psi_A w) = \psi_A w.$$

(e) By (a) and in view of the algebraic generation of $\ll v, w \gg$, the gate $m(u, v, w)$ of u can be generated from a finite number of vertices in finitely many steps by employing imprints, so that there is a polynomial function (called algebraic function in [41]) $p_{u,v,w}$ such that

$$p_{u,v,w}(r, s, t) \in \ll s, t \gg \text{ for all } r, s, t \in V \text{ and } p_{u,v,w}(u, v, w) = m(u, v, w).$$

Given six vertices u, v, w, x, y, z with $u\xi x$, $v\xi y$, and $w\xi z$, define another polynomial function q by

$$q(r, s, t) := (u\ p_{u,v,w}(r, s, t)\ p_{x,y,z}(r, s, t)).$$

Then, as tolerances are compatible with all polynomial functions, we infer that $m(u, v, w)\xi m(x, y, z)$ because

$$q(u, v, w) = (u\ p_{u,v,w}(u, v, w)\ p_{x,y,z}(u, v, w)) = (u\ m(u, v, w)\ p_{x,y,z}(u, v, w)) = m(u, v, w)$$

and analogously $q(x, y, z) = m(x, y, z)$. □

The identities in (c) do not entail that a gate map ψ is necessarily a homomorphism. Consider, for instance, the gate map ψ from a 6-cycle to any of its edges. In fact, all edges of a bipartite graph G constitute gated subgraphs (isomorphic to K_2), but the corresponding gate maps are all homomorphisms exactly when G is a median graph. This follows from a more general observation; see Proposition 13 below.

The operation $m : V^3 \to V$ defined in (e) is in fact a *local polynomial function* [33, 44] of the imprint algebra, that is, it can be interpolated by polynomial functions on all finite subsets of V^3. To see this, first note that a finitary meet operation in any meet-semilattice $(V, \preceq_a)$ is a polynomial function. Hence, with the notation in the proof of (e), we can define a polynomial function q_W for every finite subset W of V by letting $q_W(r, s, t)$ be the meet of $p_{u,v,w}(r, s, t)$ for all choices of $u, v, w \in W$ in the meet-semilattice $(V, \preceq_r)$. Then

$$q_W(x, y, z) \preceq_x p_{x,y,z}(x, y, z) = m(x, y, z)$$

and $q_W(x, y, z) \in \ll y, z \gg$ yields $q_W(x, y, z) = m(x, y, z)$ for all $x, y, z \in W$.

Lemma 2. *Let the graph G be the gated amalgam of graphs G_1 and G_2 along $G_1 \cap G_2$ such that either gate map from G_i $(i = 1, 2)$ to $G_1 \cap G_2$ is a homomorphism. Then the gate map ψ of a gated set A of G is a homomorphism if and only if either gate map $\psi_i : V_i \to A \cap V_i$ for $A \cap V_i \neq \emptyset$ is a homomorphism.*

Proof. Necessity is trivial, as ψ_i is the concatenation of ψ and the gate map $x \mapsto x_i$ of G_i, which is also a homomorphism since the gate map $x \mapsto x'$ of $G_1 \cap G_2$ is such: if e.g. more than one of u, v, w belong to $G_2 - G_1$, say the latter two, then

$$(uvw)_1 = (u_2vw)_1 = (u_2vw)' = (u'v'w') = (u_1v_1w_1)$$

because $v_1 = v'$ and $w_1 = w'$.

As to sufficiency, if some V_i includes A, then ψ is the concatenation of the gate map of G_i and ψ_i, whence it is a homomorphism. Otherwise, A is a gated amalgam of $A \cap V_1$ and $A \cap V_2$. If $\{u, v\} \subseteq V_i$ for some $i = 1, 2$, then

$$\psi(uvw) = \psi(uvw_i) = \psi_i(uvw_i) = (\psi_i u \psi_i v \psi_i w_i) = (\psi_i u \psi_i v w_i)$$

$$= (\psi_i u \psi_i v w) = (\psi u \psi v w) = (\psi u \psi v \psi w).$$

The case $\{v, w\} \subseteq V_i$ is settled analogously. □

Lemma 3. *Let G be the gated amalgam of graphs G_1 and G_2. For congruences θ_1 and θ_2 of G_1 and G_2, respectively, that restrict to the same congruence of $G_1 \cap G_2$, the relation $\theta_1 \cup \theta_2$ is is the smallest tolerance of G extending θ_1 and θ_2, and*

$$\theta = \theta_1 \cup \theta_2 \cup \theta_1 \circ \theta_2 \cup \theta_2 \circ \theta_1$$

is the smallest congruence of G extending θ_1 and θ_2.

Proof. To prove the first assertion, let $u(\theta_1 \cup \theta_2)x, v(\theta_1 \cup \theta_2)y, w(\theta_1 \cup \theta_2)z$, and assume that, say, u, v, x, y are from G_1 whereas w, z are from G_2. Then the gate map $'$ of $G_1 \cap G_2$ turns $w\theta_2 z$ into $w'\theta_2 z'$, so that $w'\theta_1 z'$ by hypothesis and hence

$$(uvw) = (uvw')\theta_1(xyz') = (xyz),$$

as required.

Clearly the restriction of θ to G_i equals θ_i $(i = 1, 2)$. Note that $u\theta_1 \circ \theta_2 v$ exactly when $u\theta_1 u'\theta_2 v$, or equivalently, $u\theta_1 v'\theta_2 v$.

As to transitivity of θ, assume $u\theta v\theta w$. Then $u\theta w$ follows trivially if either both u, v or both v, w belong to one of G_1 and G_2. Therefore assume that, say, u, w are in $G_1 - G_2$ and v in $G_2 - G_1$. Then $u\theta_1 v'\theta_1 w$ and hence $u\theta w$.

To prove compatibility with the imprint operation of the amalgam, assume $u\theta w$ with u from G_1, say. Then $u'\theta w'$ by Lemma 1(d) applied to θ_1 and θ_2, since either $u\theta_1 w$ or $u\theta_1 w'\theta_2 w$. If G_1 contains w and at least one of x, y, then

$$(uxy) = (ux_1y_1)\theta_1(wx_1y_1) = (wxy).$$

Else, with w in G_1 but x, y from $G_2 - G_1$, we obtain

$$(uxy) = (u'xy)\theta_2(w'xy) = (wxy).$$

Therefore it only remains to consider the case $u\theta_1 w'\theta_2 w$. By what has been shown, we infer $(uxy)\theta(w'xy)\theta(wxy)$ and hence $(uxy)\theta(wxy)$ by transitivity. Finally, $(xyu)\theta(xyw)$ is established analogously. □

From the first assertion of Lemma 3 we obtain in particular (by letting $\theta_1 = \iota_1$ and $\theta_2 = \iota_2$ be the respective "all" relations) that the two constituents of a gated amalgam are the blocks of a tolerance. This observation suggests a natural generalization of the notion of pairwise gated amalgamation: a graph G is a "tolerance" amalgam of a graph family $(G_i | i \in I)$ if the graphs G_i $(i \in I)$ constitute the blocks of a tolerance of G that covers the edge set of G. We will investigate the finest tolerance of this kind in the case of weakly median graphs; see Section 5 below.

We will apply now the preceding lemma to the situation where A is a gated set in the amalgam G of G_1 and G_2. Let A_1 and A_2 be the sets of gates of A in G_1 and G_2, respectively. Consider the relation $\theta = \theta(A)$ on G to G_i $(i = 1, 2)$ defined by

$$x\theta(A)y \Leftrightarrow (xy\psi_A y) = y \text{ and } (yx\psi_A x) = x \text{ where } \psi_A \text{ is the gate map of } A.$$

Thus, $x\theta y$ means that $x, y, \psi_A y, \psi_A x$ constitute a *metric rectangle,* where $x, \psi_A y \in I(y, \psi_A x)$ and $y, \psi_A x \in I(x, \psi_A y)$, implying $d(x, y) = d(\psi_A x, \psi_A y)$ and $d(x, \psi_A x) = d(y, \psi_A y)$. Then $\theta = \theta_1 \cup \theta_2 \cup \theta_1 \circ \theta_2 \cup \theta_2 \circ \theta_1$ for the restrictions θ_1 and θ_2 of θ to G_1 and G_2, respectively. Indeed, if e.g. $x \in G_1 - G_2$ and $y \in G_2 - G_1$, then for the gate y' of y in $G_1 \cap G_2$ it follows that $x\theta_1 y' \theta_2 y$. Hence, under the hypothesis that θ_1 and θ_2 are congruences of G_1 and G_2, respectively, one infers from Lemma 3 that θ is a congruence of G. In this case, θ is the smallest congruence that has A as one of its blocks.

"Prefiber" has originally been employed by [57] as a synonym for "gated set", thus alluding to the fact that all "fibers" of a Cartesian product, which are the blocks of the kernels of the projections onto the factors, are gated sets. We have seen above that gated sets need not be congruence blocks. Even if they are, they do not necessarily participate in a subdirect representation of the graph. Take the house (Fig. 4(a)) for instance: it has only one nontrivial congruence, viz. the one with blocks $\{v, t\}, \{w, z\}, \{u\}$, so that the house is subdirectly irreducible. The gated set $A = \{u, z, t\}$ (the triangle in the house) is thus not the block of a congruence, but it has the following property, investigated by Chastand [21]: the pre-image of every vertex of A under the gate map ψ_A is a gated set in the given graph. We will refer to such gated sets as *prefibers,* thus replacing the earlier redundant use of the name. Graphs in which all gated sets are prefibers are called *fiber-complemented* in [21]. Every fiber of a direct product is a prefiber in our sense. When we consider arbitrary subdirect products, then the pre-images of single vertices under a canonical projection are trivially blocks of congruences (viz. the kernels of the projection), but they are not necessarily prefibers. Take the gated amalgam of a 6-cycle and a 4-cycle along an edge. This constitutes a subdirect product of C_6 and K_2, but one pre-image (an edge) of a vertex from the factor K_2 is not a prefiber because its gate map partitions the 6-cycle into convex but not gated parts. We suggest to call a gated set A (or the corresponding induced subgraph) a *fiber* if both the kernel $\ker\psi_A$ of the gate map ψ_A of A and the relation $\theta(A)$ are congruences. Thus, a prefiber A is a fiber exactly when there exists a congruence θ that includes $A \times A$ but intersects $\ker\psi_A$ only in the equality relation ω. Indeed, for a fiber, $\theta(A)$ is the desired congruence θ. Conversely, if $x\theta y$ for θ as described and $\psi = \psi_A$ maps x to a and y to b, then

$$\psi(xyb) = (\psi x \psi y \psi b) = (abb) = b = \psi y$$

and $(xyb)\theta(yyb) = y$, whence $(xy\psi y) = (xyb) = y$ because $\ker\psi_A \cap \theta = \omega$. Analogously, $(yx\psi x) = x$, whence $\theta \subseteq \theta(A)$. The converse inclusion is obvious since $\psi x \theta \psi y$ implies $(\psi xxy)\theta(\psi yxy)$.

If A is a fiber, then the mapping $x \mapsto ([x]\ker\psi_A, [x]\theta(A))$ is an embedding of G into the product of $G/\ker\psi_A$ (isomorphic to the subalgebra A) and $G/\theta(A)$. Indeed, if $(xy\psi_A y) = y, (yx\psi_A x) = x$, and $\psi_A x = \psi_A y$, then $x = y$ follows immediately. In this way, every nontrivial fiber leads to a proper subdirect decomposition. If the fiber A is a separating set of G, as is the case when G is amalgamated from G_1 and G_2 along A, then $G/\theta(A)$ is an amalgam of $G_1/\theta_1(A)$ and $G_2/\theta_2(A)$ (where $\theta_i(A)$ is the restriction of $\theta(A)$ to G_i for $i = 1, 2$) along the cut vertex A of $G/\theta(A)$. Hence $[x]\theta(A) \mapsto ([x_1]\theta_1(A), [x_2]\theta_2(A))$ sets up a subdirect representation of $G/\theta(A)$ by $G_1/\theta_1(A)$ and $G_2/\theta_2(A)$.

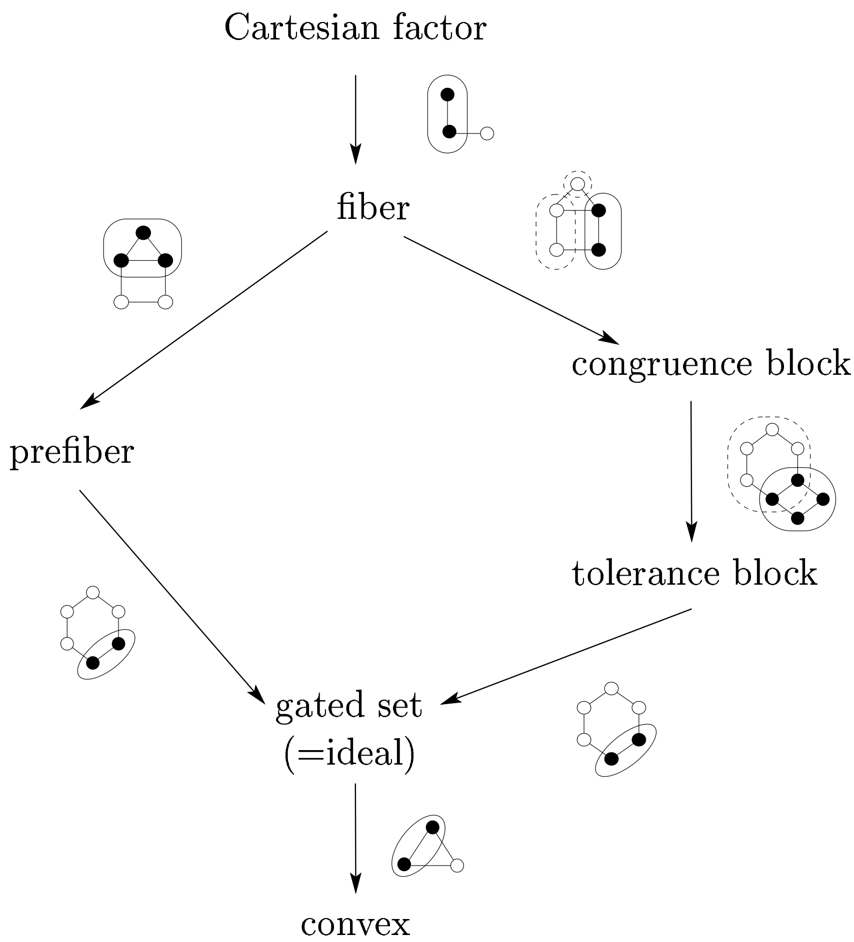


FIGURE 6. Implications between fiber concepts, with counter-examples indicated for the reverse directions

Note that the (finite) intersection of fibers is again a fiber. Trivially, fibers in a Cartesian product are exactly the Cartesian products of fibers in the factors. As to *fiber amalgamation,* that is, gated amalgamation along a common fiber, all fibers of the constituents stay fibers in the amalgam, and the new fibers of the amalgam are all gated amalgams of fibers in the constituents. In particular, when we start off from graphs having no nontrivial gated subgraphs, then Cartesian multiplication and gated amalgamation produces gated subgraphs that are all fibers. We summarize this discussion in the following result.

Theorem 1. *Successive fiber amalgamations from Cartesian products constitute subdirect products. Namely, let $\mathcal{K}$ be a class of graphs having only trivial gated subgraphs, and let $\mathcal{L}$ be the class of all graphs obtained via successive gated amalgamations from Cartesian products of graphs from $\mathcal{K}$. Then every graph from $\mathcal{L}$ is a (finite) subdirect product of graphs from $\mathcal{K}$ (which yield simple algebras). In particular, every finite weakly median graph is the subdirect product of prime weakly median graphs.*

The class $\mathcal{L}$ in Theorem 1 obtained from the class $\mathcal{K}$ of all finite graphs having only trivial gated subgraphs has been studied by Chastand [21]: it coincides with the class of finite fiber-complemented graphs. These graphs can be characterized by a single equation in terms of the ternary operation m, which assigns to each triplet u, v, w the gate of u in $\ll v, w \gg$.

Corollary 1. *For a (finite) graph $G = (V, E)$ the following conditions are equivalent:*

(i) *G is fiber-complemented;*

(ii) *the ternary algebra (V, m) is isomorphic to the imprint algebra of a quasi-median graph;*

(iii) *the ternary algebra (V, m) satisfies Isbell's isotropy law*

$$m(m(u, v, w), x, y) = m(m(u, x, y), m(v, x, y), m(w, x, y)).$$

Proof. (i) implies (ii): By the preceding theorem, G is a subdirect product of graphs having only trivial gated subgraphs ("elementary fiber-complemented graphs" [21]). The ternary operation m on each subdirect factor is therefore the *dual discriminator* [39], that is, $m(u, v, w)$ equals v if $v = w$ and equals u otherwise. Hence m coincides with the imprint operation of the complete graph on the same vertex set. Since every congruence of G is also compatible with the operation m by Lemma 1(e), the algebra (V, m) is a subdirect product of dual discriminator algebras and hence can be interpreted as the imprint algebra of a quasi-median graph [14, 46].

(ii) implies (iii): Quasi-median graphs are known to satisfy isotropy [14, 46].

(iii) implies (i): Suppose that G is not fiber-complemented. Then G includes a gated set A such that the pre-image $\psi_A^{-1}(y)$ of some vertex $y \in A$ is not gated. Thus there exist vertices $v, w \in \psi_A^{-1}(y)$ and a vertex u of G such that $\psi_A u = x \neq y$ and $I(u,v) \cap I(u,w) = \{u\}$. Clearly $u \in \psi^{-1}_{\ll x,y \gg}(x)$ and $v, w \in \psi^{-1}_{\ll x,y \gg}(y)$, whence $m(m(u,v,w),x,y) = m(u,x,y) = x$ but $m(v,x,y) = m(w,x,y) = y$, so that isotropy is violated here. □

Contrasting with the particular situation of the preceding corollary, the algebras (V, m) in general do not fit into the axiomatic framework of imprint algebras. For instance, the house (Fig. 4(a)) violates the twisted left absorption law with respect to the operation m because $m(t, w, m(w, t, v)) = m(t, w, v) = t \neq m(w, t, v)$.

Theorem 1 entails that every member G of that class $\mathcal{L}$ is isomorphic to an isometric subgraph of a Cartesian product of some graphs from the class $\mathcal{K}$. This isometric embedding may then be compared with the so-called *canonical isometric embedding* [40], described as follows. Two edges $e = xy$ and $f = uv$ of a graph $G = (V, E)$ are in Winkler's relation Θ if $d(x,u) + d(y,v) \neq d(x,v) + d(y,u)$. This relation on the edge set E is trivially reflexive and symmetric but not necessarily transitive. Let Θ^* denote the transitive closure of Θ, and let $E_1, \dots, E_k$ be the blocks of Θ^*. Let G_i $(i = 1, \dots, k)$ be the graph having the connected components of the graph $(V, E - E_i)$ as its vertices, with two different components being adjacent when connected by an edge from E_i; alternatively, one can view G_i as the graph resulting from the contraction of all edges in E_i. This contraction induces a natural projection α_i from G onto G_i. Then the map $\alpha : G \to G_1 \Box \dots \Box G_k$ defined by $\alpha v = (\alpha_1 v, \dots, \alpha_k v)$ constitutes an isometric embedding, which is the finest isometric embedding of G into a Cartesian product (whence the name "canonical").

In general, a subdirectly irreducible apiculate graph may still have a nontrivial canonical isometric embedding into a Cartesian product, since the kernels of the projections onto the factors need not be congruences. For example, consider the 8-vertex graph G obtained from a 4-cycle by gluing four triangles along its four edges (so that the edge set of the resulting graph can be partitioned into the edge sets of the four triangles). Then the relation Θ^* on the edge set has two blocks, with either block comprising the edges of a pair of opposite triangles. Thus, the weakly median graph $K_{1,1,2} \Box K_{1,1,2}$ constitutes the Cartesian product for the canonical isometric embedding of G, although G itself has no nontrivial gated subgraphs.

In contrast to this example, all pairs of edges in a two-connected graph G are Θ^* related whenever the graph has an ample supply of isometric odd cycles. In an odd cycle every edge is in relation Θ to its two "opposite" edges, so that Θ^* has only one block. Now, if the cycle space of G has a basis consisting of isometric odd cycles, then Θ^* has a single block; this can be proven by a straightforward induction, similarly as in the proof of [7, Lemma 3]. Such a graph G has only trivial gated subgraphs, since any isometric odd cycle sharing an edge with a gated subgraph H would be included in H, and therefore some cycle C properly intersecting H in an edge (guaranteed by two-connectedness) would not be the modulo 2 sum of isometric odd cycles. Summarizing, we can record the following observation.

Corollary 2. *Let $\mathcal{K}_{odd}$ be the class of graphs comprising K_2 and all two-connected graphs for which the cycle spaces have bases consisting of isometric odd cycles, and let $\mathcal{L}_{odd}$ be the class of all graphs obtained via successive gated amalgamations from Cartesian products of graphs from $\mathcal{K}_{odd}$. Then for every graph $G \in \mathcal{L}_{odd}$ the subdirect representation from Theorem 1 constitutes a canonical isometric embedding. In particular, every finite weakly median graph has a canonical isometric embedding into a Cartesian product of prime weakly median graphs.*

A variant of Corollary 2 has been established in [21, Corollary 6.2] for pre-median graphs G, which generalize weakly median graphs: G is *pre-median* if G is weakly modular such that neither $K_{2,3}$ (Fig. 1(a)) nor the graph of Fig. 1(b) is an induced subgraph.

4. Subdirect products as retracts

A *retraction* φ of a graph $H = (W, F)$ is an idempotent nonexpansive mapping of H into itself, that is, $\varphi^2 = \varphi : W \to W$ with $d(\varphi x, \varphi y) \leq d(x, y)$ for all $x, y \in W$. The induced subgraph of H constituting the image of H under φ is called a *retract* of H. Retracts are isometric subgraphs, but the converse is not true in general: C_6 is an isometric subgraph but is not a retract of the 3-cube $K_2 \square K_2 \square K_2$. The retracts of hypercubes are precisely the median graphs [5], and more generally, the quasi-median graphs are obtained as the retracts of Hamming graphs, viz. (weak) Cartesian products of complete graphs [20, 62].

A retract G of a graph H need not be a subalgebra of H (that is, of the imprint algebra of H); take, for instance, a 3-star $G = K_{1,3}$ in $H = K_{2,3}$. If, however, H is apiculate (so that imprint and apex operations coincide), then the retract G necessarily is a subalgebra of H. Gated amalgams cannot in general be obtained as retracts of Cartesian products of the constituents. The smallest counter-example is given by the gated amalgam of C_5 and K_2 along a vertex: this graph is a subdirect product of C_5 and K_2 but cannot be obtained as a retract of $C_5 \square K_2$.

The retractions from binary products yield the key information for deciding whether an isometric subgraph $G = (V, E)$ of a Cartesian product $H = H_1 \square \ldots \square H_n$ is a retract. According to the elegant result of Feder (Theorem 6.35 of [38]), G is a retract of H exactly when the following two projection criteria are met:

(1) G coincides with the largest induced subgraph of H that has the same images under the projections onto all H_i $(1 \leq i \leq n)$ and $H_i \square H_j$ $(1 \leq i < j \leq n)$ as G; and

(2) each of these images constitutes a retract of the corresponding factor H_i or product $H_i \square H_j$.

Under the additional requirement that G be a subdirect product of $H_1, \ldots, H_n$, condition (1) is automatically fulfilled; moreover, if the factors have no nontrivial gated subgraphs, the images of G in the binary products that remain to be checked for retractions are unions of two fibers. These observations essentially follow from purely algebraic results, the first of which is due to Bergman [17]: A subdirect product V of ternary algebras $V_1, \ldots, V_n$ satisfying the majority law $(aab) = (aba) = (baa) = a$ is uniquely determined by its images under the canonical projections onto $V_i \times V_j$ for $1 \leq i < j \leq n$. Namely, every point $x = (x_1, \ldots, x_n) \in V_1 \times \ldots \times V_n$ for which all coordinate pairs (x_i, x_j) have pre-images in V under the canonical projections must belong to V. The straightforward proof is by induction on n. For $n = 3$, to start with, any three pre-images $(x_1, x_2, c), (x_1, b, x_3), (a, x_2, x_3) \in V$ of points $(x_1, x_2), (x_1, x_3), (x_2, x_3)$ projected from V immediately restore

$$x = (x_1, x_2, x_3) = ((x_1 x_1 a), (x_2 b x_2), (c x_3 x_3)) \in V$$

by means of the majority law. The second result (also formulated here only for ternary algebras) generalizes an observation of Fried and Pixley [39] on dual discriminator algebras.

Lemma 4. *Let a ternary algebra V be a subdirect product of two algebras V_1 and V_2 that satisfy the majority law and have only trivial ideals. Then either the factorization is trivial ($V \cong V_1$ or $V \cong V_2$), or $V = V_1 \times V_2$ is the whole direct product, or V is the union of two ideals (fibers) of the form $\{v_1\} \times V_2$ and $V_1 \times \{v_2\}$.*

Proof. If $(u_1, u_2), (v_1, v_2), (v_1, w_2) \in V$, then $(v_1, (u_2 v_2 w_2)) = ((u_1 v_1 v_1), (u_2 v_2 w_2)) \in V$. Hence, the pre-image $\{v_1\} \times W_2$ of v_1 (for some $\emptyset \neq W_2 \subseteq V_2$) under the (first) projection of V onto V_1 is an ideal of V, whence it is either a singleton or equals V_2 by the hypothesis. The analogous statement holds with respect to the second projection. It is then easy to see that only three cases can occur for the pre-images of single points in V_1 and V_2 : (i) they are all singletons, (ii) they are equal to the associated fibers of $V_1 \times V_2$, and (iii) they are singletons except for two fibers $\{v_1\} \times V_2$ and $V_1 \times \{v_2\}$. This proves the assertion of the lemma. $\square$

Returning to graphs, we thus have the following result that specifies Feder's theorem in the algebraic scenario.

Corollary 3. *Every subdirect product of graphs $H_1, \dots, H_n$ that have no nontrivial gated subgraphs is a retract of $H_1 \Box \dots \Box H_n$ if and only if for all $1 \le i < j \le n$ the gated amalgam of two copies of H_i and H_j along a single common vertex is a retract of $H_i \Box H_j$.*

In general, it is difficult to decide whether the gated amalgam of two graphs along a vertex is a retract of the two graphs: even when the second graph is fixed as K_2, this decision problem is as difficult as the SAT problem and hence NP-complete [38, Lemma 6.32]. As we will show below, these retraction questions are closely related with a combing property of graphs which comes from the geometric theory of groups [35]. Let b be a distinguished vertex ("base point") of a graph G and let k be an integer. Two paths $P(x,b), P(y,b)$ in G connecting two vertices x, y to b are called *k-fellow travelers* if $d(x', y') \le k$ holds for each pair of vertices $x' \in P(x,b), y' \in P(y,b)$ with $d(x,x') = d(y,y')$. A *geodesic k-combing* of G with respect to the base point b comprises shortest paths $P(x,b)$ between b and all vertices x such that $P(u,b)$ and $P(v,b)$ are k-fellow travelers for any edge uv of G. One can select the combing paths so that their union is a spanning tree T_b of G that is rooted at b and preserves the distances from b to all vertices. The neighbor $f(x)$ of x in the unique path of T_b connecting x with the root will be called the *father* of x. A geodesic 1-combing of G with respect to b (also referred to as a mooring in G onto $\{b\}$; see [22]) thus amounts to a tree T_b preserving the distances to the root b such that if u and v are adjacent in G then $f(u)$ and $f(v)$ either coincide or are adjacent in G. The k-houses for $k \ge 1$ admit geodesic 1-combings with respect to all base points. In [23, 28] it is noticed (using [27]) that for bridged graphs every spanning tree returned by Breadth-First-Search (BFS) starting from b provides a geodesic 1-combing. Trivially the same holds for hyperoctahedra and 5-wheels.

For any two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with base points a and b, respectively, we denote by $G_1 \dot{+}_{a,b} G_2$ the gated amalgam of the two fibers of $G_1 \Box G_2$ that share the vertex (a,b), i.e., the subgraph of $G_1 \Box G_2$ induced by $(\{a\} \times V_2) \cup (V_1 \times \{b\})$. The equivalence of the first two conditions in the following lemma is due to [22, Theorem 2.3.1].

Lemma 5. *For graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with at least two vertices the following conditions are equivalent:*

(i) *$G_1 \dot{+}_{a,b} G_2$ is a retract of $G_1 \Box G_2$ for each vertex (a,b) of $G_1 \Box G_2$;*
(ii) *$G_1 \dot{+}_{a,0} K_2$ and $G_2 \dot{+}_{b,0} K_2$ are retracts of $G_1 \Box K_2$ and $G_2 \Box K_2$, respectively, for all vertices a of G_1, b of G_2, and vertex 0 of K_2;*
(iii) *G_1 and G_2 each have geodesic 1-combings with respect to all base points.*

Proof. (i) implies (ii): Take a neighbor c of b in G_2, so that we can regard the edge bc as a copy of K_2. Then the given retraction φ from $G_1 \Box G_2$ to $G_1 \dotplus_{a,b} G_2$ restricts to a retraction from $G_1 \Box K_2$ to $G_1 \dotplus_{a,b} K_2$ because the convex hull of $G_1 \dotplus_{a,b} K_2$ in $G_1 \Box G_2$ equals $G_1 \Box K_2$, which is therefore mapped into itself by φ.

(ii) implies (iii): Let K_2 have the vertices 0 and 1. Since the retraction φ from $G \Box K_2$ to $G_1 \dotplus_{a,0} K_2$ maps $(V_1 - \{a\}) \times \{1\}$ into $V_1 \times \{0\}$ we can define a "father" map $f_1 : V_1 \to V_1$ that preserves or collapses the edges of G_1 via

$$f_1(a) = a \text{ and } \ (f_1(x), 0) = \varphi(x, 1) \text{ for } x \neq a.$$

Then, f_1 maps the base point a to itself and any other vertex x of G_1 to a neighbor $f_1(x)$ of x in $I(a, x)$. Let the spanning tree T_a of G_1 consist of all edges $f_1(x)x$ for $x \in V_1 - \{a\}$. This tree obviously preserves the distances from a to all vertices, and moreover, the paths have the 1-fellow property. Hence the paths in T_a emanating from a provide a geodesic 1-combing of G_1. Analogously, one obtains a geodesic 1-combing of G_2.

(iii) implies (i): Let f_1 and f_2 denote the two father maps in the spanning trees T_a and T_b that yield the geodesic 1-combings of G_1 and G_2. We construct a retraction φ from $G_1 \Box G_2$ to $G_1 \dotplus_{a,b} G_2$ as follows. For each vertex (u, v) of $G_1 \Box G_2$ take the smallest $k \geq 0$ such that either $f_1^k(u) = a$ or $f_2^k(v) = b$; then $\varphi(u, v)$ is set to $(f_1^k(u), f_2^k(v))$. In other words, we repeatedly apply the father map pair (f_1, f_2) to (u, v) until we first reach one of the two fibers containing (a, b). By definition, φ is an idempotent map to $G_1 \dotplus_{a,b} G_2$. To show that φ is nonexpansive, that is, φ preserves or collapses edges, assume without loss of generality that vw is an edge of G_2 with $d(b, v) \leq d(b, w)$. Let $\varphi(u, v) = (f_1^k(u), f_2^k(v))$ with k minimal. Since both father maps preserve or collapse edges, the only nontrivial case to check is when $f_1^k(u) \neq a$ and $f_2^k(v) = b \neq f_2^k(w)$. As vw is an edge, it follows that $f_2^{k+1}(w) = b$, whence $\varphi(u, w) = (f_1^{k+1}(u), b)$ is adjacent to $\varphi(u, v)$. □

From this lemma and Lemma 6.32 of [38] we conclude that recognizing graphs which have a geodesic 1-combing is NP-complete. For some classes of graphs the lemma provides us with geodesic 1-combings that are not simply constructed via BFS. For instance, the class of Helly graphs (alias absolute retracts of reflexive graphs [15]) is trivially closed under gated amalgamations along vertices. Therefore the existence of geodesic 1-combings is guaranteed for Helly graphs.

Corollary 4. [22] *Let $\mathcal{K}_\Delta$ be the class comprising all subhyperoctahedra, two-connected bridged graphs, and two-connected Helly graphs, and let $\mathcal{L}_\Delta$ be the class of all graphs obtained via successive gated amalgamations from Cartesian products of graphs from $\mathcal{K}_\Delta$. Then every graph from $\mathcal{L}_\Delta$ is a retract of a Cartesian product of graphs from $\mathcal{K}_\Delta$.*

In particular, every finite weakly median graph is a retract of a Cartesian product of prime weakly median graphs, and vice versa.

5. Subdirect representation of infinite weakly median graphs

In establishing the subdirect representation for finite fiber-complemented graphs (Theorem 1), we have not yet employed the full information provided in [21], which extends to the infinite case as well.

Lemma 6. [21] *Let S be the smallest gated subgraph that contains the edge vw of a fiber-complemented graph G. Then the blocks $W_s = \psi_S^{-1}(s)$ $(s \in S)$ of the kernel of the corresponding gate map ψ_S contain isomorphic gated subgraphs U_s $(s \in S)$ such that their union induces a gated subgraph $H \cong S \square U$ (where U may be any U_s), which together with all blocks W_s $(s \in S)$ covers the edge set of G. If H is not all of G, then G is a gated amalgam of $G-(W_s-U_s)$ and W_s along U_s for some $s \in S$ with $U_s \neq W_s$.*

Proof. The prime gated subgraphs of G are precisely the smallest gated subgraphs generated from single edges of G, by virtue of [21, Lemmas 4.4 and 4.8]. The Cartesian decomposition $H \cong S \square U$ follows from [21, Theorem 5.2], and the amalgamation of $G-(W_s-U_s)$ and W_s is established in the proof of [21, Theorem 5.4]. □

For the particular case of weakly median graphs, the preceding lemma can also be inferred from [7]. Namely, by [7, Lemmas 3 and 5], S either comprises the edge vw or is two-connected and null-homotopic. It is explicitly shown in the proofs of [7, Lemmas 8 and 10] that all U_s are gated and each vertex of H belongs to an isomorphic copy of S which is a transversal for the subgraphs U_s $(s \in S)$. This immediately implies that for any vertex x outside H the gate of x in U_t, where t is the gate of x in S, serves as the gate of x in H.

Lemma 7. *The smallest gated subgraph S that contains the edge vw of a fiber-complemented graph G is a prime subgraph giving rise to a minimal nontrivial tolerance/congruence $\theta = \theta(S)$, which has S as one of its blocks. Then kerψ_S is a congruence, which equals the pseudocomplement θ^* of θ in the tolerance lattice of G, that is, kerψ_S is the largest tolerance intersecting θ in the equality relation. Thus, G/θ^* is isomorphic to S.*

Proof. S is a prime graph according to [21, Lemma 4.8]. In view of the product representation $H \cong S \square U$ described in the previous lemma, S is a block of the canonical congruence $\theta_H(S) = \ker\psi_U|_H$ of the Cartesian product H. We can trivially extend θ_H to the required congruence of G so that $\theta - \theta_H$ is the equality relation on $G - H$. Clearly, $\theta_H(S)$ is a minimal nontrivial congruence, and hence so is θ.

The smallest gated subgraph T generated from any edge xy of the subgraph $W_s = \psi_S^{-1}(s)$, where $s \in S$, either coincides with S (if xy belongs to $U_s \subseteq H$) or is included in W_s such that $|T \cap U_s| \leq 1$ (cf. [21, Lemma 4.9]). Therefore $\theta(T) \subseteq \ker\psi_S$, and hence $\ker\psi_S$ equals the union of all (finitary) relational products of minimal congruences that are contained in $\ker\psi_S$. This qualifies $\ker\psi_S$ as a congruence, which intersects $\theta(S)$ in the equality relation. Since every minimal nontrivial congruence different from $\theta(S)$ is contained in $\ker\psi_S$, it follows that $\ker\psi_S = \theta(S)^*$. □

Algebraically, fiber amalgamation is determined by a tolerance with exactly two blocks that is not transitive (i.e. with two intersecting blocks). Thus, successive fiber amalgamation is manifest in a particular tolerance with several blocks, by virtue of Lemma 3. The smallest tolerance β of this kind will then testify to all possible decompositions via fiber amalgamation, just as in the particular case of median graphs [3]. Its blocks are isomorphic to weak Cartesian products of prime constituents; see the next theorem.

Let I be an infinite index set, and let $F_i = (V_i, E_i)$, $i \in I$, be any graphs with at least two vertices. A *weak Cartesian product* of this family of graphs is any connected component H of the "infinitary Cartesian product" with vertex set $\prod_{i \in I} V_i$ and edges xy for which

$$x_j y_j \in E_j \text{ for some } j \in I \text{ and } x_i = y_i \text{ for all } i \in I - \{j\}.$$

Thus, each graph F_i forms a Cartesian factor of H, and H itself constitutes a directed union of (finitary) Cartesian products. Namely, select a base point b of H and consider the fibers $F_i(b)$ of the infinitary Cartesian product that correspond to F_i and contain b. Then the component $F = F(b)$ that contains b comprises those vertices which differ from b in only finitely many coordinates. F is thus the directed union of the convex hulls of all finite subfamilies of $(F_i(b)|i \in I)$, which constitute (finitary) Cartesian products. Algebraically, the imprint algebra of F is a weak direct product (sensu [41, p. 139]) of the imprint algebras of the family $(I_i|i \in I)$. Bergman's theorem for finitary direct products of ternary algebras (satisfying the majority law) immediately extends to weak direct products by virtue of a trivial induction. Then, in particular, a nonempty subset W of the imprint algebra of F is a subalgebra if and only if its projection on each binary product $V_i \times V_j$ ($i \neq j$ from I) constitutes a subalgebra; moreover, it is uniquely determined by these projections. Therefore the congruences $\theta_i = \theta(F_i)$ ($i \in I$) of H, which are complementary in H to the canonical congruences, permute in pairs, that is, $\theta_i \circ \theta_j = \theta_j \circ \theta_i$ for $i \neq j$. This congruence permutation property is characteristic of the weak Cartesian product H among all subgraphs G of H that constitute subdirect products of the graphs F_i ($i \in I$).

Theorem 2. *Every fiber-complemented graph G is a subdirect product of the prime fiber-complemented graphs G/θ^* associated with the minimal nontrivial congruences θ of G. The smallest tolerance β of G that covers the edge set of G equals the intersection of all tolerances with two intersecting blocks and can also be expressed as the union of all relational products of pairwise commuting minimal congruences. Hence the blocks of β are the maximal gated subgraphs of G isomorphic to weak Cartesian products of prime fiber-complemented graphs.*

Proof. The intersection of all $\theta(S)^*$, with S running through the gated subgraphs generated from single edges, equals the equality relation ω; for otherwise, some minimal nontrivial congruence $\theta(T)$ would be contained in this intersection and hence in its pseudocomplement $\theta(T)^*$, which is absurd. Therefore G is a subdirect product of those graphs S, by Birkhoff't theorem (see [41, §20, Theorem 1]).

First note that for any two tolerances ξ_1 and ξ_2 with $\xi_1 \cap \xi_2 = \omega$ the relation $(\xi_1 \circ \xi_2) \cap (\xi_2 \circ \xi_1)$ is the smallest tolerance containing both ξ_1 and ξ_2 because the imprint algebra satisfies the majority law [31, Lemma 3.8]. Hence, if ξ_1 and ξ_2 commute, this tolerance equals $\xi_1 \circ \xi_2 = \xi_2 \circ \xi_1$. A straightforward induction shows that for any tolerances $\xi_1, \ldots, \xi_n$ that pairwise commute and intersect in ω the relational product $\xi_1 \circ \ldots \circ \xi_n$ is the smallest tolerance containing them since it is easy to see that $\xi_1 \circ \ldots \circ \xi_{n-1} \cap \xi_n = \omega$. Applied to the minimal nontrivial congruences of G, this yields that the requirement to cover all edges forces β to contain all relational products $\theta_1 \circ \ldots \circ \theta_n$ $(n \geq 1)$ of minimal nontrivial congruences θ_i that commute in pairs.

Now, assume that some pair (u, x) of vertices is not a member of any of those relational products. Take a shortest path $u_0 = u, u_1, \ldots, u_{k-1}, u_k = x$ in G (necessarily of length $k \geq 2$) and consider the (not necessarily distinct) gated subgraphs S_i generated from the edges $u_i u_{i+1}$ $(i = 0, \ldots, k-1)$, respectively. For the associated congruences $\theta_i = \theta(S_i)$, choose $i < j$ with $j - i$ minimal such that θ_i and θ_j do not commute. Then

$$u\theta_0 \circ \ldots \circ \theta_{i-1} \circ \theta_{i+1} \circ \ldots \circ \theta_{j-1} v \theta_i w \theta_j u_{j+1} \theta_{j+1} \circ \ldots \circ \theta_k x$$

for some vertices v and w from $I(u, x)$. Since θ_i and θ_j do not commute, $I(v, w)$ and $I(w, u_{j+1})$ generate gated subgraphs T_i and T_j, respectively, such that $\theta(T_i) = \theta(S_i), \theta(T_j) = \theta(S_j)$, and $T_i \cap T_j = \{w\}$. Thus, when T_i plays the role of S and w the role of s in Lemma 6, we infer that G is a gated amalgam of $G - (W_s - U_s)$ and W_s along U_s. Since u_{j+1} is contained in $W_s - U_s$, so is x, whereas u belongs to W_v and hence to $G - W_s$. Therefore (u, x) is not a member of the tolerance having $G - (W_s - U_s)$ and W_s as its two blocks. This proves that β, being sandwiched by the union of all (finitary) relational products of commuting families of minimal nontrivial congruences and the intersection of all tolerances with exactly two blocks that intersect, must coincide with both relations, thus establishing the theorem. □

From Corollary 4 and the preceding theorem we can readily derive the retract theorem of Chastand [22].

Corollary 5. [22] *Let* $(G_i|i \in I)$ *be the family of prime constituents of a fiber-complemented graph* G *such that every* G_i $(i \in I)$ *has a geodesic 1-combing. Then* G *is a retract of a weak Cartesian product* H *of* $(G_i|i \in I)$.

Proof. For each pair $i \neq j$ from I there is a retraction φ_{ij} from H to the pre-image G_{ij} of the canonical projection of G onto $G_i \Box G_j$, by virtue of the hypothesis and Lemma 5. The intersection of all subgraphs G_{ij} of H thus includes G and has the same projection to each $G_i \Box G_j$ as G, whence it must equal G by Bergman's theorem. Since every vertex x of G has finite distance to a fixed vertex b of G, there are only finitely many maps among the retraction maps φ_{ij} $(i \neq j)$ for which the restriction $\varphi_{ij}|_{I(b,x)}$ is not the identity map. Hence we can define the infinitary concatenation φ of the family $(\varphi_{ij}|i \neq j \text{ from } I)$ with respect to some (well-)ordering $\prec$ of $I \times I$ by letting $\varphi x = \varphi_{i_n j_n} \circ \ldots \circ \varphi_{i_1 j_1} x$ where $(i_1, j_1) \prec \ldots \prec (i_n, j_n)$ constitute the index pairs for which $\varphi_{i_\nu j_\nu}$ is not the identity on $I(b, x)$. □

In the infinite case, one is interested in conditions of local finiteness (see [49] for an extensive treatment), which involve finite subsets W of the vertex set V and their convex hulls, for instance.

Corollary 6. *The convex hull of a finite set* F *of vertices in a fiber-complemented graph* G *is the subdirect product of finitely many prime fiber-complemented graphs.*

Proof. For every vertex pair u, x there are at most $d(u, x)$ distinct minimal nontrivial congruences θ such that any shortest path from u to x passes through an edge vw with $v\theta w$. This follows from Lemma 7. Hence for all but finitely many minimal nontrivial congruences θ the vertices in F are all congruent modulo the pseudocomplement θ^*. □

6. Join-hull commutativity and finite generation

The interval between two vertices, and thus their convex hull, in an infinite hyperoctahedron, for example, is infinite. But if such obstructions do not occur, then the convex hull of a finite set in an infinite weakly median graph is generated by finitely many finite intervals, as we will see next. A graph $G = (V, E)$ is called a *Peano graph* if its intervals satisfy the following property:

Peano axiom: for any vertices $u, v, w \in V$, $x \in I(u, v)$, and $y \in I(w, x)$, there exists a vertex $z \in I(v, w)$ such that $y \in I(u, z)$.

One can show that Peano graphs are exactly the graphs in which the convexity is *join-hull commutative* (cf. [60]), that is $\operatorname{conv}(A \cup \{x\}) = \cup_{z \in A} I(x, z)$ holds for every convex set A and vertex x. It was shown in [25] that all weakly median graphs fulfill the Peano axiom. In view of the subdirect representation available now, we can give a somewhat shorter proof for this result.

Proposition 5. [25] *Weakly median graphs are Peano graphs.*

Proof. Since gated amalgamation and Cartesian multiplication preserve the Peano property (cf. [8] and [60, Theorem 5.14]), we actually need to verify the Peano axiom only in the prime case by virtue of Corollary 6. Since all subhyperoctahedra trivially satisfy this property, we can assume that the (prime) weakly median graph G under consideration is C_4- and K_4-free.

Let $x = v_0, v_1, \ldots, v_k = v$ be a shortest path connecting x and v. The Peano property for u, v, w, x, y then follows from that of u, v_i, w, v_{i-1}, y for $i = 1, \ldots, k$. Indeed, this yields a sequence $z_1, \ldots, z_k$ of vertices with $z_{i-1} \in I(u, z_i)$ for $i = 1, \ldots, k$ (where $z_0 = y$), so that $z = z_k$ is the required vertex. Similarly, given a shortest path $x = x_0, x_1, \ldots, x_j = y$, the Peano axiom for u, v, w, x, y is settled once we have shown it for $u, z_{i-1}, w, x_{i-1}, x_i$ by producing the necessary vertex z_i for all $i = 1, \ldots, j$ (where $z_0 = v$). Therefore we can assume that x is adjacent to v and y.

If $d(v, w) > d(x, w)$, then we can choose $z = y$ in order to fulfill the Peano axiom. If $d(v, w) < d(x, w)$, then v, x, y form a triangle because G satisfies (Q) and is C_4-free. Then, by (T), v and y have a common neighbor v' in $I(v, w)$. If $d(u, x) = d(u, y)$, then we can choose $z = v$. Therefore assume $d(u, v) = d(u, y)$. Then $d(u, v') > d(u, v)$ so that $z = v'$ does the job, because otherwise either (T!) is violated (if $d(u, v') < d(u, v)$) or we would have a triangle (formed by v, v', y) equidistant to u, which must have a common neighbor one step closer to u, thus producing a forbidden K_4.

Therefore only the case $d(v, w) = d(x, w)$ remains to be investigated. If y is adjacent to v, then $z = v$ can be chosen. Otherwise, by (T), v and x have a common neighbor v' in $I(v, w)$ different from y. As G is C_4-free, v' and y are adjacent, which by (T) then have a common neighbor v'' in $I(v', w)$. Now, if $d(u, y) = d(u, x)$, then $d(u, v) = d(u, v')$ and thus $z = v'$ is the desired vertex, because otherwise x, y, v' would form a triangle equidistant to u. Hence we can assume $d(u, y) = d(u, v)$. Then x is the median of u, v, y, and therefore v' as a common neighbor of v, y, and x must be at distance $d(u, v)$ from u as well. If $d(u, v'') \leq d(u, v)$, then we get a conflict with (T!) or a forbidden triangle equidistant to u. Therefore $d(u, v'') > d(u, y)$, whence $z = v''$ concludes the proof. □

Lemma 8. *All intervals in a weakly median graph G are finite when G does not contain an induced infinite $K_{1,\ldots,2}$ (a countable complete graph minus one edge).*

Proof. When some infinite intervals occur, take one, $I(u, v)$, for which $d(u, v)$ is as small as possible. If $d(u, v) = 2$, then $I(u, v)$ constitutes a subhyperoctahedron and hence G contains $K_{1,\dots,2}$ as an induced subgraph. Therefore assume $d(u,v) \geq 3$. If the set of neighbors of v in $I(u,v)$ was finite, then $I(u,v)$ would be the union of finitely many finite (sub)intervals. Therefore v has an infinite number of different neighbors $w_0, w_1, w_2, \dots$ at distance $d(u,v) - 1$ to u. By weak modularity w_0 has a common neighbor x_i with each w_i ($i \geq 1$). Since $I(u,w_0)$ contains all x_i ($i \geq 1$) but must be finite by hypothesis, at least one interval $I(x_j, v)$ for some $j \geq 1$ is infinite, contrary to the minimality of $d(u,v) \geq 3$. □

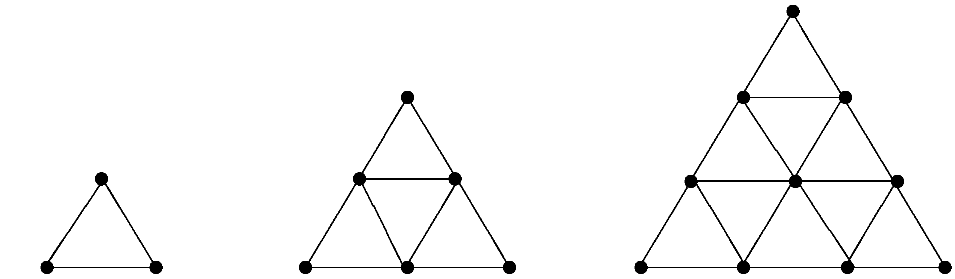
Proposition 6. *The subalgebra S generated by a finite set X in a weakly median graph G is contained in a finite (weakly median) induced subgraph of G that constitutes a subalgebra of the imprint algebra of G.*

Proof. According to Corollary 6, there are only finitely many nontrivial projections of X into the prime factors in a subdirect representation of G. In all factors that are not infinite subhyperoctahedra the convex hulls of the projected vertices from X are finite by Lemma 8. Every finite subset (with at least two vertices) of an infinite subhyperoctahedron can be connected by adding at most one vertex, so that $K_{1,2}, C_4$, or a finite subhyperoctahedron arises. Taking the Cartesian product of the former finite convex hulls and the latter finite graphs results in a finite weakly median graph that is a subalgebra containing X and hence S. □

By this proposition, every equation that holds in the imprint algebras of all finite weakly median graphs is also true for all infinite weakly median graphs.

7. Deltoids

The *triangular grid* is the tessellation of the plane into equilateral triangles of equal (unit) size. The convex hull Δ of a metric triangle xyz of size $k \geq 0$ in the triangular grid either is a single vertex (if $k = 0$) or constitutes an equilateral triangle of size k that is subdivided into unit triangles by lines parallel to its sides. We refer to such a graph as a *k-deltoid* with *corners* x, y, z and *sides* $I(x,y), I(x,z), I(y,z)$ (see Fig. 7 for $k = 1, 2, 3$). The interior Δ° of the deltoid Δ is defined as Δ minus the three sides of Δ; the interior $I^\circ(a,b)$ of a side $I(a,b)$ of Δ (where $a, b \in \{x,y,z\}$) equals $I(a,b) - \{a,b\}$.



(a) $k = 1$ (K_3) (b) $k = 2$ (sun) (c) $k = 3$

FIGURE 7. The first three k-deltoids

Proposition 7. *An apiculate graph G is weakly median if and only if the convex hull* $\mathrm{conv}(u, v, w)$ *of any metric triangle uvw is a Cartesian product of deltoids.*

Proof. By Corollary 6, $\mathrm{conv}(u, v, w)$ is the subdirect product of finitely many prime weakly median graphs. Since metric triangles in gated amalgams must belong to one of the constituents, it follows (by Theorem 2) that $\mathrm{conv}(u, v, w)$ is a Cartesian product of prime graphs. Each of these prime factors is then the convex hull of a metric triangle. Since metric triangles in 5-wheels and induced subgraphs of hyperoctahedra are just triangles and thus 1-deltoids, we can henceforth assume that G is a prime weakly median bridged graph. Note that the metric triangle uvw in G is necessarily equilateral such that all vertices in $I(v, w)$ have the same distance to u, by virtue of weak modularity [12, 24]. If $I(v, w)$ was not a path, then it would include two adjacent vertices x, y equidistant to v because G is bridged. Then, by applying (T) three times (to x, y with respect to u, v, w), we obtain three distinct common neighbors of x and y: two in $I(v, w)$ and one in $I(u, x)$. This necessarily yields either an induced $K_{1,1,3}$ or a K_4, which, however, are forbidden in a prime weakly median bridged graph. Therefore the three intervals $I(u, v), I(v, w), I(w, u)$ are convex paths.

To complete the proof, we proceed by induction on the size $k \geq 2$ of uvw. Let $v = x_0, x_1, \ldots, x_{k-1}, x_k = w$ be the convex path constituting $I(v, w)$. Applying (T) to each pair x_{i-1}, x_i with respect to u, we obtain vertices $y_1, \ldots, y_k$ at distance $k - 1$ to u such that y_i is adjacent to x_{i-1}, x_i for $i = 1, \ldots, k$. Since $I(v, w)$ is a convex path and G is bridged, the vertices $y_1, \ldots, y_k$ are different and induce a convex path with $d(y_1, y_k) = k - 1$. Hence uy_1y_k is a metric triangle of size $k - 1$. By the induction hypothesis, $\mathrm{conv}(u, y_1, y_k)$ is a $(k - 1)$-deltoid. This together with $I(u, v)$ induces a k-deltoid Δ. For $k = 2$, suppose by way of contradiction that $I(u, x_1)$ contains a third common neighbor z of u and x_1 besides y_1 and y_2. Then z must be adjacent to y_1 and y_2 because G is bridged, thus producing a forbidden K_4. Hence assume $k \geq 3$. Let t be the neighbor of u in $I(u, v)$. Then, by exchanging the roles of u and v, we obtain another convex $(k-1)$-deltoid with corners v, t, and x_{k-1}. This is necessarily induced by the convex path $I(v, x_{k-1})$ together with the $(k-2)$-deltoid with corners y_1, t, and y_{k-1}. An analogous statement can be made when u and w are interchanged. We conclude that Δ is the union of three convex $(k-1)$-deltoids (each containing exactly one of u, v, w). Therefore each pair of vertices at distance 2 in Δ belongs to a convex $(k-1)$-deltoid, so that by [7, Lemma 1] Δ is convex in this case. This establishes the "only if" part of the proposition.

To establish the converse, observe that when the convex hull of any metric triangle (uvw) in G is a Cartesian product of deltoids then all vertices of $I(v, w)$ are equidistant to u. This property entails that G is weakly modular by [24, Theorem 2]. Since G is apiculate, G is weakly median. □

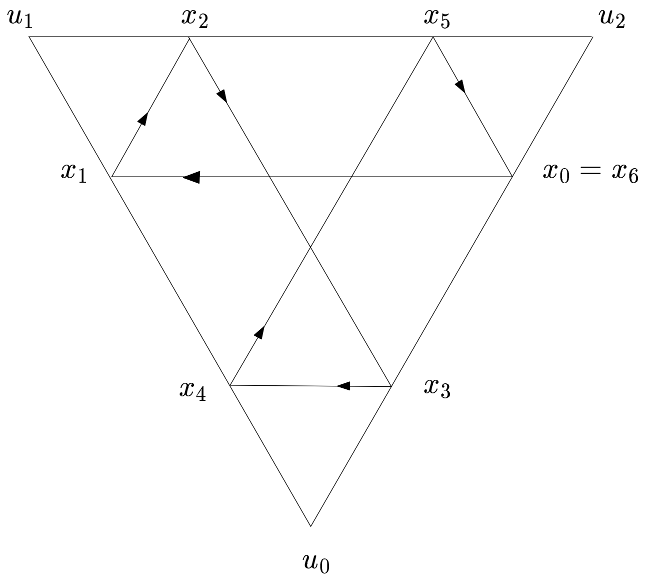


FIGURE 8. The billiard law in a deltoid

As subalgebras, deltoids are generated by their three corners plus one neighbor of a corner. To see this, consider a k-deltoid Δ $(k \geq 2)$ with corners u_0, u_1, and u_2. Then for any vertex $x_0 \in I(u_0, u_2)$ with $d(x_0, u_2) = j \leq k/2$, say, one can iteratively define the "billiard sequence"

$$x_{i+1} = (u_{i+1} u_i x_i) \text{ for } 0 \leq i \leq 5$$

where the indices of u are read modulo 3. Then

$$d(x_1, u_1) = d(x_2, u_1) = d(x_3, u_0) = d(x_4, u_0) = d(x_5, u_2) = d(x_6, u_2) = j,$$

whence $x_6 = x_0$. We refer to this property as to the *billiard law;* see Fig. 8. For $j = 1$, in particular, Δ is covered by the three $(k-1)$-deltoids with corner sets $\{u_0, x_0, x_1\}, \{u_1, x_4, x_5\}$, and $\{u_2, x_2, x_3\}$, respectively. In each of these $(k-1)$-deltoids, some vertex from $\{x_0, \ldots, x_5\}$ is adjacent to a corner and thus can start a billiard sequence within this $(k-1)$-deltoid. Thus a trivial induction shows that all vertices of Δ are generated by u_0, u_1, u_2, and x_0. Since deltoids can be arbitrarily large, the distance between two generators of a 4-generated weakly median graph G cannot be bounded from above, quite in contrast to 4-generated quasi-median graphs, which have no more than 868 vertices [10, 48]. Let us call a ternary algebra a *weakly median algebra* if it satisfies all equations true for finite weakly median graphs. The free 3-generated weakly median algebra coincides with the "free taut medium on 3 generators" [46, p. 331] and is represented by the familiar 6-vertex graph of Fig. 3(a). In view of Proposition 7 and the 4-generation of deltoids we conclude that the free 4-generated weakly median algebra is infinite and is not the imprint algebra of a graph (because it has infinite bounded chains).

According to [7], prime weakly median bridged finite graphs $G = (V, E)$ are exactly represented by the plane triangulations in which all inner vertices have degree larger than 5. The neighborhood $N(x) = \{y \in V | \; y \text{ is adjacent to } x\}$ of every inner vertex induces a cycle, whereas the vertex incident with the external face have a path neighborhood. By a *line* of G (thus embedded in the plane) we mean the vertex set of a convex path whose end vertices both belong to the external face of G. Every convex path extends to a line. To show this, let L be a convex path $u = v_0, v_1, \ldots, v_k = v$ $(k \geq 1)$ such that the end vertex v is an inner vertex of the plane graph G. Since the degree of v is then at least 6, there exists a vertex w adjacent to v but not to v_{k-1} such that v is the only common neighbor of v_{k-1} and w. Hence, by [7, Lemma 1], L plus w induces a convex path. This shows that there is an ample supply of lines because every edge can be extended to a line. A line is either separating, that is, its removal disconnects G or it is part of the boundary of the external face. The *border* L of a halfspace A of G comprises the vertices $x \in A$ for which the neighborhood $N(x)$ intersects the complementary halfspace $A' = V - A$. Then $L' = N(L) \cap A'$ is the border of A'. By a *zipper* we mean the square of a path P of length at least 2; the *square* P^2 of a graph P has the same vertex set as P where two vertices are adjacent exactly when they are at distance 1 or 2 in P.

Lemma 9. *Let $G = (V, E)$ be a finite two-connected K_4- and $K_{1,1,3}$-free bridged graph. For a halfspace A of G and its complement $A' = V - A$, the borders L and L' of A and A', respectively, constitute separating lines such that L and L' together induce a zipper.*

Proof. We first claim that any two adjacent vertices u and v of L have a common neighbor in L'. Indeed, if some neighbors $u', v' \in L'$ of u and v, respectively, are adjacent, then one of them is a common neighbor of u and v because G is bridged; else, if $d(u', v') \geq 2$, then u' and v' have a common neighbor $z \in A'$ adjacent to both u and v because A' is convex and G is bridged.

Since neighborhoods of convex sets are convex in a bridged graph, we infer that the borders $L' = N(A) \cap A'$ and L are convex. For $u, v, w \in L$ with $u, w \in N(v)$, there exist $x, y \in L'$ with $x \in N(\{u, v\})$ and $y \in N(\{v, w\})$, by what has been shown above. Necessarily, u and w are non-adjacent, whereas x and y are distinct adjacent vertices, since L and L' are convex and both C_4 and K_4 are forbidden subgraphs. If v had a third neighbor in L, then a vertex $z \in N(\{t, v\}) \cap A'$ together with x, y, and v would form a forbidden K_4. This shows that L induces a convex path. L evidently separates $A - L$ from A' unless $L = A$. In the latter case, L is a boundary line because the neighborhood of every vertex of L must be a path.

Let P denote the bipartite subgraph of G comprising the edges between L and L' (that is, each having one end vertex in L and the other in L'). By what has been shown, $L \cup L'$ is the vertex set of P, every vertex of P has degree 2 except for two vertices (that are end vertices of L or L') which have degree 1, and P is connected. Since G is K_4-free and L and L' induce convex paths, we conclude that P is a path, whence $L \cup L'$ induces a zipper given by P^2. □

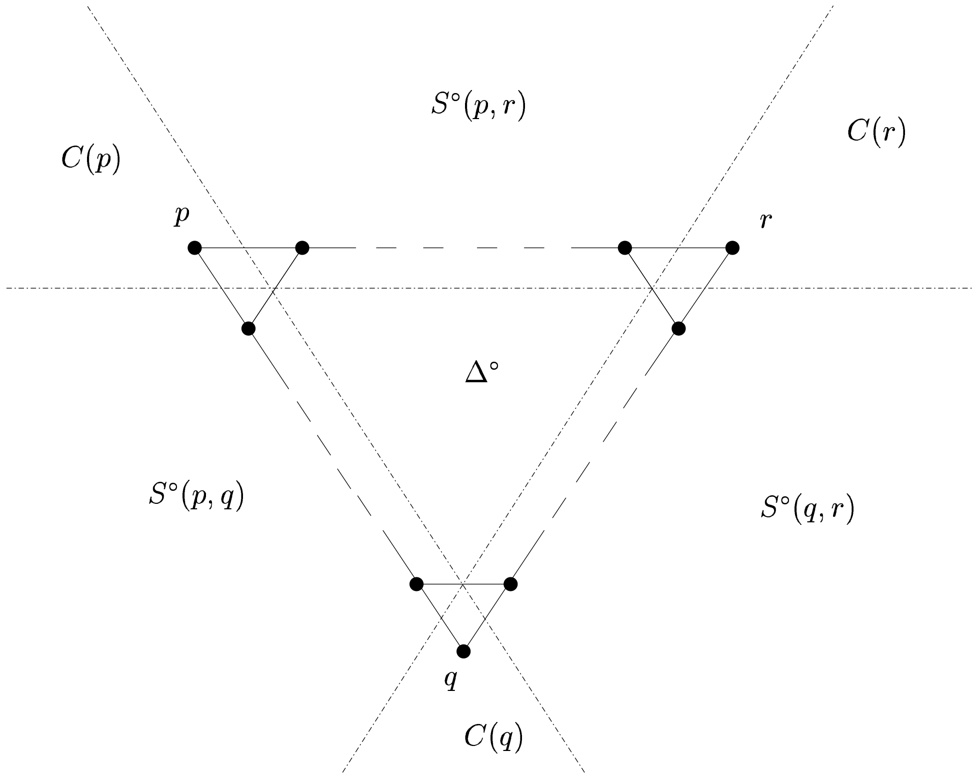


FIGURE 9. Convex partition with respect to a metric triangle pqr

Proposition 8. *Let pqr be a metric triangle of size $k \geq 1$ in a finite two-connected K_4- and $K_{1,1,3}$-free bridged graph $G = (V, E)$, which gives rise to a k-deltoid $\Delta = \mathrm{conv}(p, q, r)$.*

(a) *The sides $I(p, q), I(p, r)$, and $I(q, r)$ of Δ extend to border lines $L(p, q), L(p, r)$, and $L(q, r)$ of halfspaces $H(p, q), H(p, r)$, and $H(q, r)$, respectively, such that*

$$\Delta \cap H(p, q) = I(p, q), \;\; \Delta \cap H(p, r) = I(p, r), \;\; \Delta \cap H(q, r) = I(q, r).$$

The three border lines pairwise intersect only in one corner of Δ each and induce a partition of the vertex set V into the following seven convex sets: the $(k-2)$-deltoid Δ° (for $k \geq 2$), the (nonempty) cones

$$C(p) = H(p, q) \cap H(p, r), \;\; C(q) = H(p, q) \cap H(q, r), \;\; C(r) = H(p, r) \cap H(q, r),$$

and the open sectors

$$S^\circ(p, q) = H(p, q) - (C(p) \cup C(q)),$$
$$S^\circ(p, r) = H(p, r) - (C(p) \cup C(r)),$$
$$S^\circ(q, r) = H(q, r) - (C(q) \cup C(r));$$

see Fig. 9. Moreover, the union of any open sector with one of its two neighboring cones is convex. Each of the corresponding closed sectors

$$S(p,q) = H(p,q) - [(C(p) - L(p,r)) \cup (C(q) - L(q,r))],$$

$$S(p,r) = H(p,r) - [(C(p) - L(p,q)) \cup (C(r) - L(q,r))],$$

$$S(q,r) = H(q,r) - [(C(q) - L(p,q)) \cup (C(r) - L(p,r))]$$

is also convex as well as its union with Δ or with any neighboring cone.

(b) *The three border lines pairwise recombined at the corners of Δ yield altogether six new shortest paths, such as $(L(p,r) \cap C(p)) \cup (L(p,q) - C(p))$. Moreover, the following statements hold for every vertex $u \in C(p)$:*

$$C(p) = p/q \cap p/r,$$

$$\Delta \cup S(q,r) \subseteq p/u,$$

$$S(q,r) \subseteq I(q,r)/u \subseteq I(q,r)/p = H(q,r),$$

$$S^\circ(q,r) = I^\circ(q,r)/u \text{ if } k \geq 2.$$

(c) *Any two lines L_1 and L_2 of G that pass through $I(p,q) - \{p\}$ and $I(p,r) - \{p\}$ are disjoint whenever $L_1 \cap \Delta$ and $L_2 \cap \Delta$ are different.*

Proof. (a): Let q' and r' be the (adjacent) neighbors of p on the paths $I(p,q)$ and $I(p,r)$, respectively. Then, by virtue of [7, Lemma 12], there are unique halfspaces $H(p,q'), H(p,r')$, and $H(q',r')$ that intersect the triangle $\{p,q',r'\}$ exactly in $\{p,q'\}, \{p,r'\}$, and $\{q',r'\}$, respectively. $H(p,q')$ contains q but not the neighbor q'' of q in $I(q,r)$ because $r' \in I(p,q'')$. Therefore $H(p,q')$ is the unique halfspace $H(p,q)$ that includes $I(p,q)$ and is disjoint from the $(k-1)$-deltoid $\Delta - I(p,q)$. Analogously, $H(p,r')$ is the required halfspace $H(p,r)$. The assertion for $H(q,r)$ is settled by symmetry. If $H(q,r)$ and $V - H(q',r')$ intersect in a vertex x, then x is closer to p than to q' and closer to q than to the neighbor of q in $I(p,q)$. Hence the path $I(p,q)$ must include a subpath s_1, s_2, s_3 with $2d(x,s_2) > d(x,s_1) + d(x,s_3)$. This, however, conflicts with $I(p,q)$ being convex because G is bridged and weakly modular. Therefore $H(q,r) \subseteq H(q',r')$ holds.

The border lines $L(p,q), L(p,r)$, and $L(q,r)$ of $H(p,q), H(p,r)$, and $H(q,r)$, respectively, intersect Δ in the corresponding sides of Δ. The lines $L(p,r)$ and $L(q,r)$ meet along a convex path within the cone $C(r)$. Suppose that $L(p,r) \cap L(q,r) \neq \{r\}$. Then this intersection contains some neighbor w of r in $C(r)$. Let $s \in I(p,r)$ and $t \in I(q,r)$ denote the two neighbors of r in Δ. Since $L(q,r)$ is the border of $H(q,r)$, the vertices r and w have a common neighbor v in $V - H(q,r)$ by Lemma 9. Moreover, s and v must be adjacent vertices on the border of $V - H(q,r)$. Then, however, $v \in I(s,w) \subseteq L(p,r)$ contradicts the fact that $L(p,r)$ is a line.

If the intersection of $H(p, q'), H(p, r')$, and $H(q', r')$ had some vertex z in common, then z must be at equal distance to p, q', and r'. Then, as G is weakly modular and C_4-free, we would obtain a common neighbor of p, q', and r' (one step closer to z), thus yielding a forbidden K_4. We conclude that, in particular,

$$H(p, q) \cap H(p, r) \cap H(q, r) = \emptyset,$$

whence the three cones and open sectors are pairwise disjoint. Each open sector is convex because it is the intersection of three halfspaces; for instance,

$$S^\circ(p, q) = H(p, q) - ((H(p, r) \cup H(q, r)).$$

Then, for instance, $S^\circ(p, q) \cup C(q) = H(p, q) - H(p, r)$ is convex as well. In order to show that the cones and open sectors together with Δ° form a partition of V, it remains to verify that

$$\Delta^\circ = V - [H(p, q) \cup H(p, r) \cup H(q, r)].$$

Clearly Δ° is disjoint from the halfspaces $H(p, q), H(p, r)$, and $H(q, r)$. On the other hand, it follows from [7, p. 708] that any vertex $x \notin \Delta$ is located outside the region bounded by the sides of Δ in the canonical representation of G in the plane. Therefore, if $\Delta^\circ \neq \emptyset$, any shortest path connecting x with a vertex of Δ° intersects one of the sides of Δ, say $I(p, q)$, whence x does not lie in the halfspace $V - H(p, q)$. Finally, if $\Delta^\circ = \emptyset$, then the vertices p, q, and r are pairwise adjacent, and x cannot be equidistant from the corners of Δ, say $d(x, p) < d(x, r)$, whence x belongs to $H(p, q)$. This proves the desired equality for Δ°.

From the definition of $S(p, q)$ and $S^\circ(p, q)$ we infer that

$$S(p, q) = S^\circ(p, q) \cup (H(p, q) \cap L(p, r)) \cup (H(p, q) \cap L(q, r)).$$

Taking the union with $\Delta = (\Delta^\circ \cup I^\circ(p, q)) \cup (I(p, r) - \{r\}) \cup (I(q, r) - \{r\}) \cup \{r\}$ this yields

$$\begin{aligned} S(p, q) \cup \Delta &= (S^\circ(p, q) \cup \Delta^\circ) \cup (L(p, r) - H(q, r)) \cup (L(q, r) - H(p, r)) \cup \{r\} \\ &= [(V - H(p, r)) \cap (V - H(q, r))] \cup [L(p, r) - H(q, r)] \\ &\quad \cup [(L(q, r) - H(p, r)] \cup [L(p, r) \cap L(q, r)] \\ &= [(V - H(p, r)) \cup L(p, r)] \cap [(V - H(q, r)) \cup L(q, r)]. \end{aligned}$$

Therefore $S(p, q) \cup \Delta$ is convex because the border of any halfspace together with the complementary halfspace constitute a convex set. Moreover, as $S(p, q) = (S(p, q) \cup \Delta) \cap H(p, q)$, we conclude that $S(p, q)$ is convex. $S(p, q) \cup C(q)$, for instance, is also convex because it is the intersection of $H(p, q)$ and the neighborhood of $V - H(p, r)$. This completes the proof of (a).

(b): Suppose by way of contradiction that $(L(p,r) \cap C(p)) \cup (L(p,q) - C(p))$ is not a shortest path. Then we can select two non-adjacent vertices x and y such that the interval $I(x,y)$ intersects this path exactly in x and y. Since $L(p,r) \cap C(p)$ and $L(p,q) - C(p) = L(p,q) \cap (V - H(p,r))$ are convex paths by part (a), we can assume that $x \in L(p,r) \cap C(p) - \{p\}$ and $y \in L(p,q) - C(p)$. Then, as $I(p,x) \subseteq L(p,r)$ and $I(p,y) \subseteq L(p,q)$, the vertices x, y, and p form a metric triangle. Consequently, the neighbors of p in $I(p,x)$ and $I(p,y)$ are adjacent. Since the neighbors of p in $I(p,r)$ and $I(p,y)$ are also adjacent, this contradicts the convexity of $L(p,r)$, thus establishing the first assertion in (b).

If $u \in p/q \cap p/r$, that is, $p \in I(q,u) \cap I(r,u)$, then $u \in H(p,r) \cap H(p,q) = C(p)$, because $q \in V - H(p,r)$ and $r \in V - H(p,q)$. Conversely, let $u \in C(p)$. Any shortest path P from u to r intersects either the path $L(p,r) \cap C(p)$ or the path $L(p,q) \cap C(p)$ in some vertex x. Then $p \in I(x,r)$ because the subpath from x to r on $L(p,r)$ or $(L(p,q) \cap C(p)) \cup (L(p,r) - C(p))$, respectively, is a shortest path. Therefore, in either case $u \in p/r$. Analogously, we obtain $u \in p/q$.

For $u \in C(p) = p/q \cap p/r$, we immediately get $p, q, r \in p/u$ and hence $\Delta \subseteq p/u$ by convexity of shadows. Then it follows that $I(q,r)/u$ is contained in both p/u and $I(q,r)/p$. The halfspace $H(q,r)$ necessarily includes $I(q,r)/p$. As to the converse, consider a shortest path P from $z \in H(q,r)$ to p. If P intersects $I^\circ(q,r)$, then certainly z belongs to $I(q,r)/p$. Otherwise, P meets either $L(q,r) \cap H(p,r)$ or $L(q,r) \cap H(p,q)$, say, the former. Since $I(p,r) \cup (L(q,r) \cap H(p,r))$ is a shortest path that joins p with a vertex from P and passes through r, we infer that $z \in I(q,r)/p$. A similar argument, applied to $z \in S(q,r)$ and $u \in C(p)$, shows that $z \in I(q,r)/u$ because $L(p,q) \cap H(q,r)$ and $L(p,r) \cap H(q,r)$ are contained in p/u, whence $S(q,r) \subseteq I(q,r)/u$.

To prove the equality claimed for $S^\circ(q,r)$, let q' and r' be the neighbors of q and r in $I(q,r)$. Then, as $I^\circ(q,r)/u$ does not contain q and r, we infer from what has been shown that $I^\circ(q,r)/u$ is included in $H(q,r) - [C(q) \cup C(r)] = S^\circ(q,r)$. To show the converse, recall that $S^\circ(q,r) \subseteq I(q,r)/u$. Assume that some shortest path connects u with a vertex $x \in S^\circ(q,r)$ that passes through q, say. Since $q \in H(p,q)$ but $x \in V - H(p,q)$, the interval $I(q,x)$ contains either q' or a common neighbor y of q and q' on the border of $V - H(p,q)$, by Lemma 9. In either case we get $q' \in I(u,x)$, as required.

(c): Any line L of G that contains q and some vertex of $I(p,r) - \{p\}$ necessarily intersects Δ in the side $I(q,r)$. Then L stays within $H(q,r)$ because it cannot intersect $C(p)$, or $S^\circ(p,q)$, or $S^\circ(p,r)$, by the first equality of part (b) and last statement of (b), respectively. A line L' of G passing through the open sides $I^\circ(p,q)$ and $I^\circ(p,r)$ of Δ cannot contain any of the three corners of Δ because it is a convex path. Hence L' is disjoint from the three cones, by the first inclusion in statement (b). Hence, if L' contained a vertex from $S^\circ(q,r)$, then L' would pass through $I^\circ(q,r)$, which is impossible. Therefore $L' \subseteq V - H(q,r)$, namely, L' is included in the union of $S^\circ(p,q), S^\circ(p,r)$, and $\Delta - I(q,r)$. Clearly, $L' \cap \Delta$ is a line of Δ that is uniquely determined by its end point in $I^\circ(p,q)$ as well as by its other end point (in $I^\circ(p,r)$). Assume that $x_i \in I^\circ(p,q)$ and $y_i \in I^\circ(p,r)$ are the end points of $L_i \cap \Delta$ $(i = 1,2)$. If $L_1 \cap \Delta$ and $L_2 \cap \Delta$ are different, then they constitute the sides of two distinct subdeltoids $\Delta_1 = \mathrm{conv}(p, x_1, y_1)$ and $\Delta_2 = \mathrm{conv}(p, x_2, y_2)$ of Δ, where, say, $\Delta_1 \subset \Delta_2$. When we apply the preceding observations to Δ_2 instead of Δ and let L_1 and L_2 play the roles of L and L', we can conclude that $L_1 \subseteq V - H(x_2, y_2)$ and $L_2 \subseteq H(x_2, y_2)$, so that L_1 and L_2 do not meet. □

Lemma 10. *Let pqr be a metric triangle of size $k \geq 1$ in a finite two-connected K_4- and $K_{1,1,3}$-free bridged graph $G = (V, E)$.*

(a) *If $v \in C(q)$ and $w \in C(r)$, then $(pvw) = p$. In particular, if $u \in C(p)$ and $p \in I(u,v) \cap I(u,w)$, then $(uvw) = p$.*
(b) *If $v \in C(q)$ and $y \in S(p,r)$, then $I(v,y) \cap I(p,r) = I((pry),(rpy))$ and $(vry) = (qr(rpy))$.*
(c) *If $u \in C(p)$ and $v, w \in C(r)$, then $(uvw) \in S^\circ(p,r) \cup C(r)$.*
(d) *If y is a vertex of $H(p,r)$, then $(ypq) = (rp(qpy))$.*

Proof. We denote the convex hull of $\{p,q,r\}$ by Δ.

(a): The first statement follows from

$$I(p,v) \cap I(p,w) \subseteq (S(p,q) \cup C(q)) \cap (S(p,r) \cup C(r)) = \{p\},$$

by Proposition 8(a).

(b): From the inclusion $S(p,r) \subseteq I(p,r)/v$ established in Proposition 8(b) we obtain $I(v,y) \cap I(p,r) \neq \emptyset$. Since $I(p,r) \subseteq q/v$, all vertices of $I(p,r)$ have the same distance to v. On the other hand, letting $p' = (pry)$ and $r' = (rpy)$, the vertex y is equidistant to all vertices of the path $I(p',r')$ and has a larger distance to the remaining vertices of $I(p,r)$. Therefore $I(v,y) \cap I(p,r) = I(p',r')$, as required. Note that, as a consequence, $I(q,y) \cap \Delta = \mathrm{conv}(q,p',r')$.

By what has just been shown, there exists a shortest path from v to y via q, $q' = (qrr')$, and r'. Therefore q' belongs to $I(v,r) \cap I(v,y)$. Then, as

$$\begin{aligned} I(q',r) \cap I(q',y) &\subseteq I(q',r) \cap \Delta \cap I(q,y) \\ &= I(q',r) \cap \mathrm{conv}(q,p',r') = \{q'\}, \end{aligned}$$

we obtain $q' = (vry)$.

(c): Suppose the assertion is false. Then without loss of generality we may assume that uvw is a metric triangle (of size ≥ 1). The sides $I(u,v)$ and $I(u,w)$ of the deltoid conv(u, v, w) intersect the line $L(q, r) \cap H(p, r)$ of $H(p, r)$ in two distinct vertices y and z such that, say, $y \in I(r, z)$. Further, $I(u, w)$ and $I(u, v)$ meet the line $L(p, q) \cap H(p, q)$ of $H(p, q)$ in unique vertices x and x', respectively. Then conv(x, q, z) is a deltoid, which intersects the convex paths $I(u, w)$ and $I(u, v)$ in the different paths $I(x, z)$ and $I(x', y)$. From Proposition 8(c) we infer that any two lines of G extending $I(u, w)$ and $I(u, v)$, respectively, must be disjoint, which is absurd.

(d): If $y \in C(p)$, then $(rp(qpy)) = (rpp) = p = (ypq)$, as required. If $y \in C(r)$, then $(ypq) = r = (rpq) = (rp(qpy))$ by part (a). Finally, if y belongs to the open sector $S^\circ(p, r)$, then, by the first assertion of (b), (pry) is the vertex of the intersection $I(q, y) \cap I(p, r)$ closest to p, whence $(ypq) = (pry)$. On the other hand, $(qpy) = (qp(pry))$ by the second assertion of (b), and therefore $(rp(qpy)) = (rp(qp(pry)) = (pry)$, completing the proof. □

Corollary 7. *Let pqr be a triangle in a finite two-connected, K_4- and sun-free weakly median bridged graph G. If $x \in C(p)$ and $y \in C(q)$, then $I(x, y)$ contains both p and q.*

Proof. $I(r, x) \cap I(r, y) = \{r\}$ by Lemma 10(a). Since $r \notin I(x, y)$ and G is sun-free, the triplet r, x, y has a (unique) quasi-median of size 1. Then, as $p \in I(r, x)$ and $q \in I(r, y)$, this quasi-median is the triangle rpq, as required. □

8. Equations in four variables

In order to characterize weakly median graphs among apiculate graphs algebraically, we have to translate the interval conditions defining weak modularity into equations in terms of the imprint operation. Two series of equations then come into play, the first of which (see Lemma 13 below) is implied by axiom 4a of Isbell [46]. Since we focus on equations in at most four variables, we will first have a look at all 4-element algebras realized within apiculate graphs. Note that up to isomorphism there are only two different 3-element subalgebras of the imprint algebras of graphs, viz. the imprint algebras of the path P_2 of length 2 and the triangle K_3.

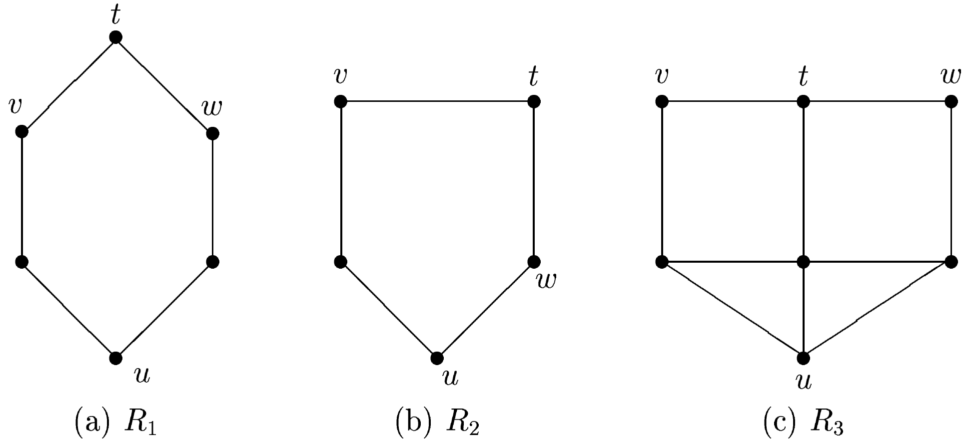


FIGURE 10. 4-element subalgebras $\{t, u, v, w\}$ of imprint algebras

Lemma 11. *The following list describes (up to isomorphism) all 4-element subalgebras $R = \{t, u, v, w\}$ of the imprint algebras of apiculate graphs, relative to the number n of triangle subalgebras of R:*

($n = 0$) *the imprint algebras of the path P_3, the star $K_{1,3}$, and the cycle C_4;*

($n = 1$) *the imprint algebra of the triangle with an edge attached, and the subalgebra R_1 (Fig. 10(a)) of the C_6 algebra;*

($n = 2$) *the $K_{1,1,2}$ algebra, and the subalgebra R_2 (Fig. 10(b)) of the C_5 algebra;*

($n = 3$) *the subalgebra R_3 (Fig. 10(c)) of the imprint algebra of the amalgam of two house algebras along a convex 2-path;*

($n = 4$) *the K_4 algebra.*

Proof. If the algebra $R = \{t, u, v, w\}$ can be realized as a graph, that is, R is the imprint algebra of a 4-vertex graph, then we have one of the six graphs described in the lemma. Henceforth let $R_n = \{t, u, v, w\}$ be a (proper) subalgebra of the imprint algebra of some apiculate graph G_n such that R_n is different from the preceding six algebras and R_n harbors exactly n distinct triangle subalgebras (which thus constitute metric triangles in G_n). Then R_n must be a *quasi-trivial algebra*, that is, $(xyz) \in \{x, y, z\}$ for all $x, y, z \in R_n$. Necessarily, $1 \leq n \leq 3$ holds, and we may assume that uvw is a metric triangle in G_n. Hence t together with any two from u, v, w forms either the K_3 algebra or the P_2 algebra.

Case $n = 1$: Then, as $\{t, u, v, w\}$ is not a P_3 algebra, at least one of the three P_2 algebras $\{t, u, v\}, \{t, u, w\}$, and $\{t, v, w\}$ has t in between the other two vertices, say

$t \in I(v, w)$. Consequently, v and w must belong to $I(t, u)$, whence R_1 can be realized within the 6-cycle G_1 as indicated in Fig. 10(a).

Case $n = 2$: Let $\{t, u, v\}$ be the second triangle subalgebra of R_2. If $I(t, w)$ contains v, then it also includes u, whence R_2 would be the imprint algebra of K_4 minus an edge, contrary to the hypothesis. Therefore neither u nor v belongs to $I(t, w)$, whence $t \in I(v, w)$ or $w \in I(v, t)$, say, the former holds. Then $w \in I(t, u)$ follows, showing that R_2 can be realized within the 5-cycle algebra G_2 of Fig. 10(b) as claimed.

Case $n = 3$: We can assume that $\{t, u, v\}$ and $\{t, u, w\}$ are triangle subalgebras and that $\{t, v, w\}$ is a path algebra with $t \in I(v, w)$. Then, evidently, R_3 can be represented as shown in Fig. 10(c). □

The algebras R_1, R_2, and the partial algebra R_2^* defined next emerge when the triangle and quadrangle conditions are violated. For instance, the imprint algebras of all cycles of length at least 7 harbor both R_1 and R_2 subalgebras. The partial algebra R_2^* equals R_2 (Fig. 10(b)) except that (vuw) and (vwu) are not specified and thus left undetermined. Observe that R_2 does not occur in a house algebra, but R_2^* is shared by the C_5 and house algebras.

Lemma 12. *Let G be a graph.*

(a) *If G violates* (Q), *then every intrinsic algebra of G contains a R_1 subalgebra.*
(b) *If G violates* (Q!), *then the imprint algebra of G contains a R_1 subalgebra.*
(c) *If G violates* (T), *then every intrinsic algebra of G contains a R_2^* partial algebra. Moreover, when G is apiculate, this partial algebra extends to either a R_2 subalgebra or a house subalgebra of the imprint algebra of G.*

Proof. Consider an instance u, v, w, z that violates condition (Q). We can assume $I(u, v) \cap I(u, w) = \{u\}$, so that uvw is a metric triangle. Then $t = z$ satisfies the additional three betweenness properties required for R_1. Therefore $\{t, u, v, w\}$ constitutes the algebra R_1 with respect to any intrinsic operation of G. If, instead, (Q) is fulfilled for this quartet but with more than one possible choice for x, then there are at least two v-apices relative to u and w, so that the imprint of u and w with respect to v equals v and, analogously, the imprint of u and v with respect to w equals w. We may assume that $(uvw) = u$, so that R_1 arises.

Now consider a triplet u, v, w as described in the triangle condition (T). Then we can assume that uvw is a metric triangle. If this triplet does not admit the desired vertex x, then any neighbor t of w in $I(u, w)$ is at distance 2 to v. Hence $\{t, u, v, w\}$ with (vtu) and (vut) unspecified constitutes a copy of the partial algebra R_2^*, where the roles of t and w are interchanged with respect to Fig. 10(b). □

Lemma 13. *Let G be a graph.*

(a) *If some intrinsic algebra of G satisfies one of the equations*

(A8) $((uwx)(vwx)u) = (uwx)$,

(A8′) $(((uwx)(vwx)u)wx) = ((uwx)(vwx)u)$,

then G is weakly modular.

(b) *If some intrinsic algebra of G satisfies the equation*

(A9) $((uwx)((uvx)wx)u) = (uwx)$,

then G satisfies (Q)*; moreover,* (Q!) *holds provided that the intrinsic operation is the imprint operation.*

(c) *If some intrinsic algebra of G satisfies the equation*

(A10) $((((uvw)vx)wx)(vwx)((uvw)vx)) = (((uvw)vx)wx)$,

then G satisfies (T).

Proof. First observe that (A9) and (A10) are particular instances of (A8).

Suppose that R_1 is a subalgebra of some intrinsic algebra of G, where now u, x, v, w play the roles of t, u, v, w in Fig. 10(a). Then

$$(uwx) = w, \ (uvx) = v = (vwx), \text{ and } (wvu) = u,$$

whence (A8′) and (A9) are violated.

Finally suppose that the partial algebra R_2^* is found within some intrinsic algebra of G, where now x, v, w, u play the roles of t, u, v, w in Fig. 10(b). Then

$$(uwx) = x, \ (vwx) = v, \text{ and } (xvu) = u = (uvw),$$

whence (A8′) and (A10) are violated.

Summarizing, this shows that under the hypothesis of (a) R_1 and R_2^* are forbidden, whereas for (b) R_1 alone and for (c) R_2^* alone are forbidden. Hence Lemma 12 completes the proof. □

One can derive (A8′) from (A8) by means of (A4): denote the two sides of (A8) by y (left) and z (right), respectively, then $(ywx) = (zwx) = z = y$. Recall that imprint and apex algebras always satisfy (A4). Obviously, when a graph G has diameter 2 (such as the graphs of Fig. 1), (A4) is fulfilled by all intrinsic operations of G.

In order to verify (A8) for an intrinsic algebra of a given graph, it suffices to check only those quartets u, v, w, x of distinct vertices for which $(uwx) \neq u$ and $(vwx) \neq w, x$. In the case of (A9) we can additionally assume that

$$v = (uvx) \notin I(u, (uwx)) \cup I((uwx), x),$$

provided that the intrinsic operation is an apex operation. Hence, in particular, $v = (uvx)$ and (uwx) then do not lie on a common shortest path between u and x. This immediately proves claim (b) in the following example.

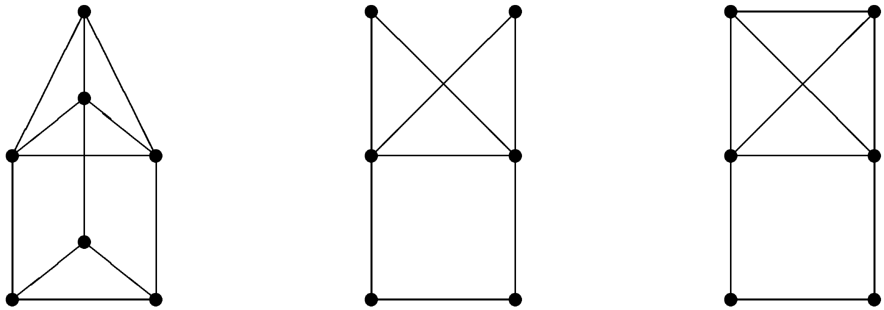
Example 1. (a) (A8) and hence (A8′) are satisfied by the imprint algebras of the graphs of Fig. 1(a,b) and by all intrinsic algebras of the graphs of Fig. 1(c,d).
(b) The imprint algebra of a geodesic graph (such as C_5) fulfills (A9).
(c) The C_6 algebra satisfies (A10).

To establish claim (a), first note that the C_4 and $K_{1,1,2}$ subalgebras of the imprint algebras of the graphs of Fig. 1(a,b) satisfy (A8). Therefore we can assume that the distinct vertices u, v, w, x (where $(uwx) \neq u$ and $(vwx) \neq w, x$) are not covered by either subgraph. $K_{2,3}$ then does not accomodate such a quartet. In the second graph of Fig. 1, the only choices yield $\{v, w, x\}$ as a triangle with u adjacent to exactly one of w and x but non-adjacent to v, so that (A8) is evidently satisfied here.

As for the second assertion in (a), observe that the $K_{1,1,2}$ and K_4 subalgebras satisfy (A8). Hence we can assume (in addition to the above premises) that exactly one of the central vertices of the graph of Fig. 1(c) or Fig. 1(d) is from u, v, w, x. If u serves as a central vertex, then we obtain $((uwx)(vwx)u) = ((uwx)vu) = (uwx)$ because (uwx) is either u or a central vertex. In $K_{1,1,3}$, neither u, nor w, nor x could play the role of a central vertex under the premises. But in the other graph $\{v, w, x\}$ could form a triangle with either the vertex w or x being central, which then equals both sides of (A8).

As for (b), the vertices (uwx) and (uvx) lie on the unique shortest path between u and x. If $(uwx) \in I((uvx), x)$, then also $(uwx) \in I((uvx), w)$ holds, yielding $((uvx)wx) = (uwx)$ because geodesic graphs are apiculate. Consequently, the left-hand side of (A9) is also (uwx). Finally, if $(uvx) \in I((uwx), x)$, then $((uvx)wx) \in I((uwx), x)$ from which we infer that (uwx) is between $((uvx)wx)$ and u, whence the left-hand side of (A9) is again (uwx).

As to (c), if $(uvw) \in \{v, w\}$ or $(vwx) \in \{w, x\}$, then (A10) clearly holds. Therefore we can assume $(uvw) = u$ and $(vwx) = v$ in order to verify (A10) in C_6. Then only the case $(uvx) = u$ remains to be checked for (A10), but this cannot be reconciled with $(vwx) = v$ in the 6-cycle.



(a) (b) (c)

FIGURE 11. Imprint algebras enjoying (A9) and (A11)

The preceding example shows that (A8) does not imply (A5), and that (T) is not a consequence of (A9) plus (Q), and (Q) not a consequence of (A10) plus (T).

Proposition 9. *The imprint algebra of an apiculate graph G of diameter 2 satisfies* (A9) *if and only if G does not contain an induced subgraph isomorphic to the graph of Fig.* 4(b) *or its companion that has the additional chord tu. In particular, the imprint algebras of the house and the graphs of Fig.* 11 *satisfy* (A9).

Proof. The two forbidden graphs (see Fig. 4(b) where tu is now a potential chord) are apiculate and of diameter 2 but evidently their imprint algebras violate (A9): the left-hand side becomes u whereas the right-hand side is y.

As to the converse, let the quartet u, v, w, x violates (A9). Then, by the diameter constraint, we can assume that $d(u,x) = 2$, and, moreover, $v = (vwx)$ and $y = (uwx)$ are two distinct common neighbors of u and x. Since (yvu) is then the left-hand side of (A9) and y equals the right-hand side, these two vertices are different by hypothesis, and therefore u, v, y, x induce a 4-cycle in G. The vertex w must be a neighbor of y but cannot be adjacent to u, x, or v (because the graphs of Fig. 1(a,b) are forbidden). Hence there exists a common neighbor t of v and w in G. In order to avoid an induced subgraph from Fig. 1, the vertex t is nonadjacent to y and adjacent to at most one of u, x. □

The second suite of equations that we will later employ in the algebraic characterization rejects R_2 and R_3 subalgebras.

Lemma 14. *If some intrinsic algebra of a graph G fulfills any one of the three equations*

(A11) $((wux)(uvw)v) = (w(uvw)(vu(wux)))$,

(A11′) $((wux)(uvw)v) = (w(uvw)(v(uvw)(wux)))$,

(A11″) $((wux)(uvw)(vuw)) = ((wuv)(uvw)((vuw)(uvw)(wux)))$, *then it cannot contain a R_2 or a R_3 subalgebra. If some apex algebra of G fulfills* (A11) *or* (A11$'$)*, then G is apiculate.*

Proof. Suppose that some intrinsic algebra of G includes a R_2 subalgebra $\{u, v, w, x\}$ such that w, u, v, x play the roles of t, u, v, w in Fig. 10(b). Then the left-hand sides of (A11) and its two variants all equal $(xuv) = x$, whereas the right-hand sides are equal to $(wuv) = w$. Hence all three equations are violated.

To see that G is apiculate whenever (A11) or (A11$'$) holds for some apex operation, consider any vertex $x \in I(u,v) \cap I(u,w)$ such that $I(x,v) \cap I(x,w) = \{x\}$. We wish to show that x and (uvw) coincide. We may assume that $I(v,x) \cap I(v,(uvw)) = \{v\}$. Denote the left-hand side of (A11) and (A11$'$) by $p = p'$, and the corresponding right-hand sides by q and q'. Then

$$p = p' = (x(uvw)v),$$
$$q = (w(uvw)x),$$
$$q' = (w(uvw)v).$$

If (A11) holds, then $p = q \in I(v,x) \cap I(w,x) = \{x\}$, whence $x \in I((uvw), w)$ and consequently, $x = (uvw)$ since (uvw) is a u-apex relative to v and w. If instead (A11$'$) holds, then

$$p = q' \in I(v,x) \cap I((uvw), w) \subseteq I(u,v) \cap I((uvw), w) = \{(uvw)\},$$

whence $(uvw) \in I(v,x)$ and consequently, $x = (uvw)$. This shows that under either hypothesis the graph G is apiculate. □

Proposition 10. *For Pasch graphs the equations* (A11),(A11$'$)*, and* (A11$''$) *are all equivalent.*

Proof. From Proposition 2 we know that a Pasch graph is apiculate. We may assume that $x = (wux)$. Put $t = (vux)$, $t' = (v(uvw)x)$, and $t'' = ((vuw)(uvw)x)$, so that the sides of the three equations become

$$p = p' = (x(uvw)v),$$
$$p'' = (x(uvw)(vuw)),$$
$$q = (w(uvw)t),$$
$$q' = (w(uvw)t'),$$
$$q'' = ((wuv)(uvw)t'').$$

The shadow $(vuw)/v$ trivially contains u, w, and (vuw). Since point-shadows are convex, we have $(uvw), (wuv), x \in I(u,w) \subseteq (vuw)/v$. Therefore $(vuw)/v$ includes $p = p', p'', q, q', q'', t''$, as well as t, t' (because the graph is apiculate), whence $p =$

$p'' \in I(x,(vuw))$ and $t' = t'' \in I((vuw),(uvw))$. Then $t,(uvw) \in I(u,t')$. Further, $t,(vuw),(uvw) \in I(u,v) \subseteq (wuv)/w$. Because this point-shadow then contains t' as well, we infer that it also harbors q and q', whence $q, q' = q'' \in I((uvw),(wuv))$. This, en passant, establishes the equivalence of (A11$'$) and (A11$''$). Now, since $t \in I(u,t')$ and $u, t' \in q'/w$, the point-shadow q'/w contains t. Hence $q' \in I(w,t) \cap I(w,(uvw))$, and as a consequence

$$q \in I(q',(uvw)) \cap I(q',t).$$

The (convex) point-shadow p/x contains the point t' because $t' \in I((uvw),v) \subseteq p/x$. Since $t' \in I(v,t) \subseteq I(v,x)$, we can summarize this information by

$$p, t \in I(t',x) \subseteq I(v,x).$$

Now assume that $p = q$ holds. Because $q \in I((uvw),(wuv)) \subseteq I((uvw),w)$, we have $(uvw) \in I(u,p) = I(u,q)$. Then, as $t \in I(u,t')$, the intervals $I(p,t)$ and $I(t',(uvw))$ have a vertex z in common by the Pasch axiom applied to $u, t', p, t, (uvw)$. Then, as p and t belong to the (convex) interval $I(t',x)$, we infer $z \in I(t',x) \cap I(t',(uvw))$. Since t' is the v-apex relative to x and (uvw), we conclude that $z = t'$. Hence $t' \in I(t,p) = I(t,q) \subseteq I(t,w)$, and consequently, $p \in I(t',w)$. Therefore $p = q \in I(w,t') \cap I(w,(uvw)) = I(w,q')$. On the other hand, we know that $p = q \in I(q',(uvw))$. This entails $q' = q$.

Finally assume that $p = q'$ holds. Then $q \in I(q',t) = I(p,t) \subseteq I(t',x) \subseteq I(x,v)$ by convexity of intervals. Since $q \in I(q',(uvw))$ and $q' = p = p'' \in I(x,(uvw))$, we infer $q \in I(x,(uvw)) \cap I(x,v) = I(x,p) = I(x,q')$ by convexity of $I(x,(uvw))$, whence $q' \in I(q,(uvw))$, so that $q = q'$ follows. □

From the proof we infer that (A11$'$) and (A11$''$) are equivalent for apiculate graphs having convex point shadows. Pasch graphs in general need not satisfy any of the equations (A8),(A9),(A11),(A11$'$), and (A11$''$) as the 5-cycle C_5 shows. In order to verify (A11$''$) for some Pasch graph (or more generally, a graph with unique quasi-medians), one may assume that

$$v = (vuw) \neq (uvw) \text{ and } x = (wux) \notin (uvw)/v \cup I(w,(wuv))$$

because $(uvw)(vuw)(wuv)$ is then the quasi-median of u, v, w. A quartet u, v, w, x of vertices with these properties does not exist in C_6, for instance. For the house, only one quartet (up to automorphism) is feasible, viz. u, v, w, and $x = z$ as labelled in Fig. 4(a); in this case, $(xuv) = x = (wut) = (wu(vux))$, as required for (A11$''$). All feasible quartets in the graphs of Fig. 11 are already included in some convex prism or house. We summarize and extend these observations in the following example.

Example 2. (a) The imprint algebras of the graphs of Fig. 1 satisfy (A11) and its variants. An apex algebra of any of these graphs satisfies (A11″) exactly when it is a priority apex algebra.
(b) The C_6 algebra, the house algebra, and the imprint algebra of the graphs of Fig. 11 satisfy (A11) and its variants.

As for (a), we may assume $(uvw) \neq (wuv)$ and $x = (wux) \neq (uvw)$. Then the first assertion in (a) is quite evident. (A11″) can potentially be violated in an apex algebra of $K_{1,1,3}$ only when u, v, w are the three vertices of degree 2. Let y and z denote the two central vertices of $K_{1,1,3}$. Then we may assume that x is either u, w, or y. If the apex operation has the priority property, then either side of (A11″) equals the vertex from $\{y, z\}$ that has higher priority. If the priority property does not hold, then we may assume $(vuw) = y$ but $(uvw) = (wuv) = z$, yielding z on the right-hand side of (A11″) and x on the other.

Another type of equations describes the key features of metric triangles more directly, as expressed by the billiard law in deltoids. The resulting equations in the following lemma are a bit lengthy but rather easy to handle.

Lemma 15. (a) *If some intrinsic algebra of a graph G satisfies*

(A12) $\quad (uw(wv(vu(uw(wv(vu(uwx))))))) = ((uvw)(wuv)(uwx)),$

or the weaker equation

(A12′) $\quad (u'w'(w'v'(v'u'(u'w'(w'v'(v'u'x')))))) = x'$

$$\text{where } u' = (uvw), v' = (vwu'), w' = (wu'v'), \text{ and } x' = (u'w'x),$$

then G is weakly modular.
(b) *If some apex algebra of G satisfies* (A12), *then G is weakly median.*
(c) *If the imprint algebra of G satisfies* (A12′), *then G satisfies* (Q!) *and* (T!).

Proof. (a): Recall that u', v', w' as defined in (A12′) form a quasi-median of u, v, w. Then clearly (A12) implies (A12′).

Suppose that R_1 is a subalgebra of some intrinsic algebra of G, where now x, u, v, w play the roles of t, v, u, w in Fig. 10(a). Then $u' = u, v' = v, w' = w$, and $x' = x$, whence the right-hand side of (A12′) equals x, whereas the left-hand side is u. Therefore (A12′) implies (Q) in view of Lemma 12(a).

Now consider a triplet u, v, w as described in the triangle condition (T) but suppose that uvw is a metric triangle. Let x be a neighbor of w in $I(u, w)$. Again, $u' = u, v' = v, w' = w$, and $x' = x$, whence the right-hand side of (A12′) is x. If $(vux) = v$, then the left-hand side of (A12′) is equal to w. Otherwise $(vux) = y$ is adjacent to v and x but not adjacent to w. Then $(wvy) = v$, and we conclude that the left-hand side of (A12′) equals w, so that (A12′) is violated in either case.

(b): The graph G is weakly modular by what has just been shown. Suppose that G contains some unconnected triplet u, v, w having at least two common neighbors.

Case 1: u, v, and w are pairwise nonadjacent. If $|\{(uvw), (vuw), (wuv)\}| \leq 2$, then, say, $y = (uvw) = (wuv)$ is a common neighbor of u, v, and w. Let x be another common neighbor. Then the left-hand side of (A12) is x and the right-hand side equals $(yyx) = y$, yielding a contradiction. So, $y = (uvw), z = (wuv)$, and $x = (vuw)$ are all different such that, say, $x \notin I(y, z)$. Then the right-hand side (yzx) of (A12) is different from the vertex x, which is the left-hand side, again yielding a contradiction.

Case 2: u and v are adjacent. Then w is non-adjacent to u and v, whence $y = (wuv)$ is some common neighbor of u, v, and w. Pick another common neighbor x. Then $(uw(wv(vu(uwx)))) = (uwv) = u$ and further $(uw(wv(vuu))) = (uwy) = y$, but $((uvw)(wuv)(uwx)) = (uyx) = u$, contradicting (A12).

(c): Since a R_1 subalgebra cannot occur under (A12′), G satisfies (A12′) by Lemma 12(b). If the weakly modular graph G violates (T!), then we obtain one of the graphs of Fig. 1(b,d) as an induced subgraph. For u, v, w, x as is indicated in that figure we obtain u as the left-hand side of (A12′) but x as the right-hand side. □

Example 3. (a) Any apex algebra of a pseudo-modular graph satisfies (A12′).
(b) The imprint algebras of $K_{2,3}$ and $K_{1,1,3}$ satisfy (A12).

Statement (a) is obvious because all metric triangles of a pseudo-modular graph have size at most 1 by definition. As for (b), when u, v, and w induce a path or triangle in $K_{2,3}$ or $K_{1,1,3}$, these vertices together with (uwx) are included in some C_4 or $K_{1,1,2}$ subalgebra, which evidently satisfies (A12). Otherwise, u, v, and w are the vertices of degree 2, so that $(uvw) = u, (vuw) = v$, and $(wuv) = w$. Then both sides of (A12) equal (uwx), no matter whether (uwx) is u, or w, or a common neighbor of u, v, and w.

9. Equational characterization of weakly median graphs

The equations (A8)-(A12) and their variants constitute a sufficiently rich pool from which various characterizations of weakly median graphs (as well as subclasses) can spring.

Theorem 3. *The following statements are equivalent for a graph G :*

(i) *G is weakly median;*
(ii) *some intrinsic algebra of G satisfies* (A5), (A9), *and* (A10);

(iii) *some intrinsic algebra of* G *satisfies* (A5) *and* (A8);
(iv) *some intrinsic algebra of* G *satisfies* (A5) *and* (A12);
(v) *some apex algebra of* G *satisfies* (A12);
(vi) *some apex algebra of* G *satisfies* (A8) *and* (A11).

In conditions (ii),(iii),(iv), *and* (vi), *the equations* (A5),(A8), (A11), *and* (A12) *may each be substituted by the corresponding variants* (A5′),(A8′), (A11′), *and* (A12′).

Proof. If any variant of one of the conditions (ii)-(vi) is satisfied, then G is weakly modular by Lemmas 13 and 15. If, in addition, (A5) or (A5′) is satisfied, then G is apiculate by Proposition 1 and hence is weakly median. Condition (v) implies that G is weakly median by Lemma 15. Under condition (vi) or its variants, G is apiculate by Lemma 14.

Conversely, we need to show that the imprint algebra of a weakly median graph G satisfies all of the equations listed in the theorem. We already know that (A5) and (A5′) are satisfied. Since (A12) implies (A12′) and (A8) implies (A9) and (A10), it then remains to verify the three equations (A8),(A11″), and (A12), by virtue of Proposition 10. In view of Theorem 1 and Proposition 6, we may assume that G is prime and finite. Four vertices in a complete graph, or a hyperoctahedron, or a 5-wheel either induce a decomposable graph (C_4 or $K_{1,2}$) or a complete graph K_n ($1 \le n \le 4$), or are included in a fan (see Fig. 12(a) below).

The case when $G \cong K_n$ is readily checked. As to (A8), we may assume that $w \neq x$, so that $(uwx) = u$ is the resulting vertex on either side of (A8). As to (A11″), if $|\{u, v, w\}| \le 2$, then (A11″) trivially holds. Else, both sides of (A11″) equal u for $u = x$ and equal w for $u \neq x$. As to (A12), we already know that (A12) holds when u, v, and w form a triangle. If two of u, v, and w are equal, then this vertex is returned by either side of (A12).

Now, assume that G is a finite two-connected K_4- and $K_{1,1,3}$-free bridged graph. To establish (A8), we may stipulate that u, v, w, x are different such that $(uwx) \neq u$ and $(vwx) = v$. Suppose that (A8) is violated: then $((uwx)vu)) \neq (uwx)$. Pick any neighbor u' of (uwx) in $I((uwx), ((uwx)vu)) \subseteq I((uwx), u)$. Then $(u'wx) = (uwx)$ and $((u'wx)vu') = u'$, so that we may substitute u by u'. Recall [7, Lemma 12] that there are exactly two different halfspaces H and H' that contain $(u'wx)$ but not u'. Hence they both include w and x but do not contain v. Since $(wvx), (xvw) \in I(w, x)$, we conclude that $(wvx), (xvw) \in H \cap H'$. Hence the border lines L and L' of H and H' intersect the paths $I(v, (wvx)) - \{v\}$ and $I(v, (xvw)) - \{v\}$ of the deltoid Δ with corners $v, (wvx)$, and (xvw). By Proposition 8(c), L and L' must coincide because they share the vertex $(u'wx)$. The neighbors u' and $(u'wx)$ participate in some triangle with third vertex y because of two-connectivity. Then y is in the symmetric difference of H

and H', so that y belongs to one of L and L' but not to the other. This contradicts $L = L'$.

To prove (A11″), set $p = (uvw), q = (vuw), r = (wuv)$, and $y = (wux)$. Then Lemma 10(d) applies because $u \in C(p), w \in C(r)$ by Proposition 8(b), and therefore $y = (wux) \in I(u, w) \subseteq H(p, r)$.

Finally, we will establish (A12). Let again pqr denote the quasi-median of u, v, w but now put $y = (uwx)$. Then $y \in I(u, w) \subseteq H(p, r)$. Since $I(u, w)$ is contained in the convex shadows q/v and $I(p, r)/v \subseteq I(p, r)/q$, we conclude that $q \in I(v, y)$ and $\emptyset \neq I(q, y) \cap I(p, r) \subseteq I(v, y) \cap I(p, r)$. We distinguish three cases in regard to the position of y in the halfspace $H(p, r) = C(p) \cup S^{\circ}(p, r) \cup C(r)$.

Case 1: $y \in C(r)$. The right-hand side of (A12) is the vertex $(pry) = r$. To compute the left-hand side, we first employ Lemma 10(a) to obtain $(vuy) = q$ (because $q \in I(v, u) \cap I(v, y)$). Further, we get $(wvq) = q, (uwq) = p, (vup) = p, (wvp) = r$, and $(uwr) = r$ because pqr is the quasi-median of the triplet u, v, w. Therefore the left-hand side of (A12) equals r as well.

Case 2: $y \in p/v \subseteq p/q$, that is, either $y \in C(p)$ or $y \in S^{\circ}(p, r) \cap p/v$. Then the right-hand side of (A12) is the vertex $(pry) = p$, which is clear if $y \in C(p)$ and is a consequence of the first statement in Lemma 10(b) otherwise. In the computation of the left-hand side, we first obtain the vertex $z = (vuy) \in I(p, u) \cap I(p, y) \subseteq C(p)$. Since $r \in I(w, v) \cap I(w, z)$, it follows from Lemma 10(a) that $(wvz) = r$. Then, as $(uwr) = r, (vur) = q, (wvq) = q$, and $(uwq) = p$, we eventually see that both sides of (A12) yield the same vertex.

Case 3: $y \in S^{\circ}(p, r) - p/v$. Then the vertex $p' = (pry)$, which constitutes the right-hand side of (A12), as well as the vertex $q' = (qpp') = (qp(pry))$ are different from p such that $(vpy) = q'$. Then u, v, and y belong to different cones with respect to the deltoid with corners p, q', and p', by virtue of the first equality in Proposition 8(b). Since $q' \in I(v, u) \cap I(v, y)$, we obtain $(vuy) = q'$ by Lemma 10(a) applied to the latter deltoid. Necessarily, $p' \neq r$ and hence $q' \neq q$ because $y \in I^{\circ}(p, r)/v$ by Proposition 8(b). Note that $x_0 = p'$ and $x_1 = q'$ constitute the first two vertices in the billiard sequence relative to the deltoid Δ with corners $u_0 = p, u_1 = q$, and $u_2 = r$. Moving on, we obtain $x_2 = (rqq')$ and so forth, until we eventually reach $x_6 = x_0 = p'$. In the preceding computation we can actually substitute p, q, r by u, v, w, as we will see next. Since $\Delta \subseteq r/w$ by Proposition 8(b), w belongs to the cone of x_2 with respect to the deltoid with corners x_2, q, and $x_1 = q'$, whence $x_2 \in I(w, x_1)$. Then, as rqp is the quasi-median of w, v, u, we have $x_2 \in I(w, q)$ as well, so that Lemma 10(a) applied to this deltoid yields $(wvx_1) = x_2 = (rqx_1)$. In an analogous fashion we then obtain

$(uwx_2) = x_3, (vux_3) = x_4, (wvx_4) = x_5$, and finally $(uwx_5) = x_0 = p'$, so that (A12) is verified. This completes the proof of the theorem. □

None of the conditions (ii)–(vi) can be weakened in a straightforward way. Equation (A5) is indispensable in (ii)–(iv), as can be seen with Examples 1(a) and 3(b). In (ii), both (A9) and (A10) are needed in view of Example 1(c) and Proposition 9. Equation (A12) cannot be replaced by (A12′) in condition (v); see Example 3(a). In (vi), (A8) cannot be weakened to (A9) or (A10) by Proposition 9, Examples 1(c) and 2(b), and (A11) cannot be substituted by (A11″) in view of Example 2(a). From Examples 3(b), 1(a), and 2(a) we deduce that "apex" in (v) or (vi) could not be replaced by "intrinsic".

It is now easy to specify nested subclasses of the class of weakly median graphs by adding stronger equations. Such equations will reject certain prime weakly median graphs as constituents. In some cases, there is a smallest rejected graph that can serve as a forbidden induced subgraph. For example, a weakly median graph for which all constituents are prime pseudo-median graphs is sun-free, i.e., it does not contain the sun (Fig. 7(b)) as an induced subgraph, and vice versa. Recall that the *pseudo-median* graphs are exactly the weakly median graphs in which all quasi-medians have size at most 1 [13]. Now, if one forbids the fan (see Fig. 12(a) below) instead, this excludes the 5-wheel and all two-connected K_4- and $K_{1,1,3}$-free bridged graphs as building stones, so that the prime graphs left are all included in hyperoctahedra. Finally, if the kite (K_4 minus one edge) is forbidden, then the prime constituents are complete graphs, generating all quasi-median graphs.

Proposition 11. *The following statements are equivalent for a graph G:*

(i) *G is weakly median and sun-free;*

(ii) *G is weakly median and its imprint algebra satisfies the equation*

(A13) $((uvw)(wuv)(uwx)) = ((uvw)(wuv)x)$;

(iii) *some apex algebra of G satisfies the equation*

(A14) $(uw(wv(vu(uw(wv(vu(uwx))))))) = ((uvw)(wuv)x)$.

Proof. (i) implies (ii): First, let G be a complete graph. If $|\{u, v, w\}| \leq 2$, then (A13) trivially holds. Else, (A13) becomes an instance of (A3′). Now assume that G is a finite K_4- and sun-free weakly median bridged graph. Let $p = (uvw), q = (vuw), r = (wuv)$, and $y = (uwx)$. If $p = q = r$, then both sides of (A13) are equal to this value. So, assume that p, q, and r form a triangle. Note that the vertex x belongs to the open sector $S^\circ(p, q)$ if and only if $d(x, p) = d(x, q) < d(x, r)$; analogous relationships hold for the other two sectors. Therefore, if $x \in C(r) \cup S^\circ(q, r)$, then $r \in I(p, x)$, and the right-hand side of (A13) is $(prx) = r$. If $x \in C(p) \cup S^\circ(p, q)$, then $p \in I(r, x)$, whence $(prx) = p$. Finally, if $x \in C(q) \cup S^\circ(p, r)$, then $d(x, p) = d(x, r)$ and again

$(prx) = p$ because p and r are adjacent. The left-hand side of (A13) is the vertex (pry). If $x \in C(r) \cup S^{\circ}(q,r)$, then $r \in I(u,x) \cap I(u,w)$ by Corollary 7 and Lemma 10(b). Hence $y = (uwx) \in I(r,w) \subseteq C(r)$, yielding $(pry) = r$. If $x \in C(p) \cup S^{\circ}(p,q)$, then $y = (uwx) \in I(u,x) \cap I(u,w) \subseteq H(p,q) \cap H(p,r) = C(p)$, and therefore $(pry) = p$. If $x \in C(q)$, then $p \in I(u,x) \cap I(u,w)$ by Corollary 7, and therefore $y = (uwx) = p$ by Lemma 10(a), whence $(pry) = p$. Finally, if $x \in S^{\circ}(p,r)$, then $y \in I(u,w) \subseteq C(p) \cup S^{\circ}(p,r)$ by Proposition 8(a). If $y \in C(p)$, then $p \in I(r,y)$, so that $(pry) = p$; otherwise, if $y \in S^{\circ}(p,r)$, then y is equidistant from p and r, yielding $(pry) = p$. We conclude that in each case the two sides of (A13) yield the same vertex.

(ii) implies (iii): Since G is weakly median, equation (A12) is satisfied. Notice that the right-hand side of (A12) equals the left-hand side of (A13), the left-hand sides of (A12) and (A14) coincide as well as the right-hand sides of (A13) and (A14), thus showing that (A14) holds.

(iii) implies (i): The instances of (A12) considered in the proof of Lemma 15 for inferring that G is weakly median all stipulate that $(uwx) = x$, so that (A12) and (A14) coincide in those cases. Therefore G is weakly median and its apex algebra is the imprint algebra. Now, suppose by way of contradiction that G contains a sun with corners u, v, and x. Denote the common neighbor of v and x by w, the common neighbor of u and v by p, and the common neighbor of u and x by y. Then the right-hand side of (A14) is equal to $(pwx) = w$, whereas for the left-hand side we successively compute

$$(uwx) = y,\ (vuy) = p,\ (wvp) = w,\ (uww) = w,$$

$$(vuw) = v,\ (wvv) = v, \quad \text{and } (uwv) = p \neq w,$$

thus violating (A14). □

Every sun contains an induced fan (Fig. 12(a)), while a fan includes $K_{1,1,2}$. The $K_{1,1,2}$-free weakly median graphs are exactly the quasi-median graphs, which have only complete graphs as prime constituents. For the larger class of fan-free weakly median graphs, all prime members are included in hyperoctahedra.

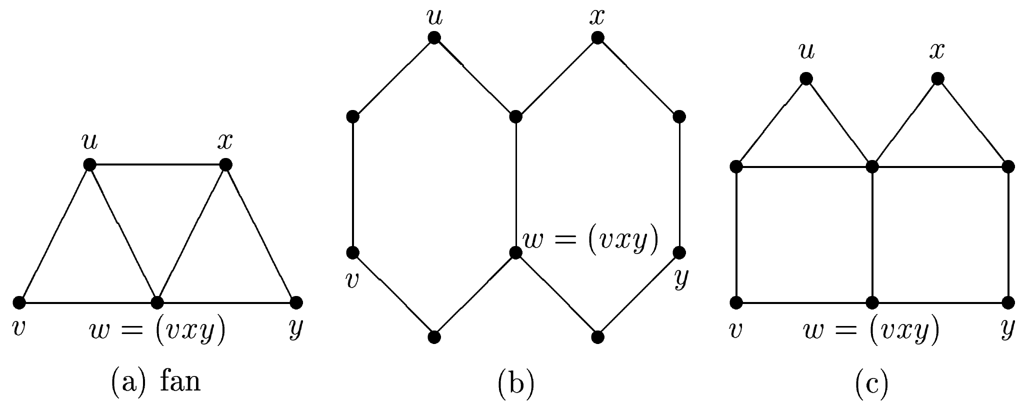


FIGURE 12. Graphs violating (A15)

Lemma 16. *An intrinsic algebra of a graph G satisfies the equation*

(A15) $((uv(vxy))(vxy)(xyv)) = (vxy)$

whenever G has unique quasi-medians and for any four vertices u, v, x, y either $(vxy) = (xyv)$ or $(uv(vxy)) = (vu(vxy))$ holds. In particular, cycle algebras, the house algebra, and the imprint algebra of the Petersen graph satisfy (A15).

Proof. The equation $((vu(vxy))(vxy)(xyv)) = (vxy)$ is equivalent to (A7) because setting $u = v$ returns (A7), and conversely, $(vu(vxy)) \in I(v,(vxy))$ and $(vxy) \in I(v,(xyv))$ hold in a graph with unique quasi-medians by Proposition 4. This proves the first assertion of the lemma. Note that any cycle algebra and the Petersen algebra fulfill the stronger property that (vxy) equals either v or (xyv). If in the house algebra, $(vxy) \notin \{v,(xyv)\}$, then $\{v,(vxy)\}$ constitutes a gated edge of the house, whence $(uv(vxy)) = (vu(vxy))$ for any vertex u. □

The two graphs in Fig. 12(b),(c) are gated amalgams of two smaller graphs that both satisfy (A15). In either case, however, the composite graph violates (A15). This shows that gated amalgamation, even along congruence blocks, need not preserve equations.

Proposition 12. *For a graph $G = (V,E)$ the following conditions are equivalent:*

(i) *G is a fan-free weakly median graph;*
(ii) *every maximal induced subhyperoctahedron in G is a prefiber;*
(iii) *the imprint algebra of G satisfies* (A15).

Proof. (i) implies (ii): By [7, Lemma 3], a maximal induced subhyperoctahedron S is gated whenever it includes an induced 4-wheel. If S has no induced 4-wheel, then $S = I(u,v)$, where the common neighbors of u and v are pairwise adjacent. If S is not gated, then by [7, Lemma 1], there is a vertex x outside S adjacent to two vertices

$y, z \in S$. If y, z are different from u, v, we obtain one of the graphs of Fig.1(c,d). Otherwise, a fan occurs.

(ii) implies (i): The graphs of Fig. 1(b,c,d) and the fan contain an induced subgraph $K_{1,1,2}$ such that the fifth vertex u has at least two neighbors in this $K_{1,1,2}$, which therefore cannot be extended to a gated subhyperoctahedron since those four graphs are never included (as induced subgraphs) in subhyperoctahedra. If $K_{2,3}$ occurs as an induced subgraph of G, then any gated subhyperoctahedron S that contains two adjacent vertices w and x of this $K_{2,3}$ cannot include any further vertex of this subgraph. Then, however, either the pre-image $\psi_S^{-1}(w)$ or $\psi_S(x)$ is not convex, contradicting that S is a prefiber.

To establish the triangle condition (T), let u, v, w be a triplet as described in (T). Pick a gated subhyperoctahedron S that contains the two adjacent vertices v and w. Necessarily, the gate x of u in S is a common neighbor of v and w in $I(u, v)$, as required in (T). Finally, as for the quadrangle condition (Q), let u, v, w, z be a quartet as described in (Q). Assuming that $I(u, v) \cap I(u, w) = \{u\}$, we need to show that $d(u, z) = 2$. Take a subhyperoctahedron S that is a prefiber of G and contains the two adjacent vertices w and z. If v belongs to S, then $d(u, z) = 2$ is fulfilled. Therefore assume that v is outside S, whence z is the gate of v in S. The gate t of u in S either equals w or is a neighbor of w in $I(u, w)$. Then the vertex $x = (vtu) \in \ll t, u \gg \subseteq \psi_S^{-1}(t)$ is different from v. Since $d(v, t) = 2 + d(w, t)$, we infer that $d(v, x) = d(z, t)$ and $d(t, x) = 1$, yielding $x \in I(u, v) \cap I(u, w) = \{u\}$. If $t = w$, then $d(u, z) = 2$. Otherwise, $t \in N(w)$ and $d(u, z) = 3$. Consequently, $S = \ll t, z \gg$ is a 2-connected subhyperoctahedron. Now, by interchanging the roles of v and w, we may assume that $\ll s, z \gg$ is a 2-connected subhyperoctahedron for some common neighbor s of v and $u = x$. If $\ll s, z \gg$ and $\ll t, z \gg$ had a vertex $y \neq z$ in common, then it would follow $\ll s, z \gg = \ll y, z \gg = \ll t, z \gg$, a contradiction. Therefore $\ll s, z \gg \cap \ll t, z \gg = \{z\}$ and thus $z \in I(s, t)$, which however is in conflict with $d(s, t) \leq 2$. We conclude that (Q) is satisfied.

(i) implies (iii): If G includes an induced fan, with its vertices labelled as in Fig. 12(a), then

$$((uv(vxy))(vxy)(xyv)) = (uwx) = u \neq w = (vxy),$$

so that (A15) is violated.

(iii) implies (i): Conversely, four vertices in a prime constituent of a fan-free weakly median graph G either induce $K_{1,1,2}$ or are included in a K_4 subgraph or a decomposable subgraph (C_4 or $K_{1,2}$). Clearly, K_4 and $K_{1,1,2}$ meet the condition in Lemma 16 that is sufficient for (A15). □

A stronger version of equation (A8), viz. (A16) below (alias axiom 4a of Isbell [46]), then characterizes the quasi-median graphs. These graphs can be defined as weakly modular graphs without induced $K_{2,3}$ and $K_{1,1,2}$ [14, 47], or alternatively, as weakly median graphs having bipartite intervals.

Proposition 13. *For a graph $G = (V, E)$ the following conditions are equivalent:*

(i) *G is a quasi-median graph;*
(ii) *every maximal complete subgraph of G is a prefiber;*
(iii) *some intrinsic algebra of G satisfies one of the following (equivalent) equations*

(A16) $(u(uwx)(vwx)) = (uwx)$,
(A16') $((vwx)u(uwx)) = (uwx)$.

Proof. (i) implies (ii): By [14, Theorem 1, (iii)⇒(iv)], every maximal complete subgraph of G is gated. Its gate map is a homomorphism by Lemma 7 since quasi-median graphs are fiber-complemented [21].

(i) implies (iii): This follows from [14, Theorem 3].

(ii) implies (i): Clearly $K_{1,1,2}$ is a forbidden induced subgraph, whence every maximal subhyperoctahedron must be a complete graph. From Proposition 12 we then know that G is weakly median.

(iii) implies (i): The equations (A16) and (A16′) are equivalent because either one expresses that $(uwx) \in I(u, (vwx))$ for all $u, v, w, x \in V$. So, assume that (A16) holds for an intrinsic algebra. Then (A8) holds as well. To show that G is apiculate, let v be a vertex from $I(u, w) \cap I(u, x)$ such that $I(u, v)$ is maximal with respect to inclusion. Then $(vwx) = v$ and hence $(uwx) \in I(u, v)$ by (A16), that is, $v = (uwx)$ is the u-apex relative to w and x. Suppose by way of contradiction that some interval $I(u, w)$ is not bipartite, that is, it contains two adjacent vertices v and x equidistant to u. It follows that

$$(u(uwx)(vwx)) = (uxv) \neq x = (uwx),$$

thus violating (A16). Hence G is a quasi-median graph. □

10. Axiomatics of discrete weakly median algebras

So far, we have derived the ternary algebras from specific graphs. Now, with the pool of equations at hand, we are able to reverse the association: starting from a "discrete" ternary algebra fulfilling certain equations as axioms one can recover the ternary algebra as the imprint algebra of an apiculate graph in which the intervals $I(u, v)$ are exactly the sets of elements x satisfying $(uvx) = x$. We say that a ternary algebra satisfying the axioms (A1), (A2), and (A3) is *discrete* if it does not contain an infinite bounded chain as a subalgebra; by a *bounded chain* we mean the median algebra associated with a linear order having a least element as well as a largest element. In particular, a finite chain is the imprint algebra of a path. Trivially, every intrinsic algebra of a (not necessarily finite) graph is discrete because any chain with bounds u and v is included in the interval $I(u,v)$ and hence has at most $d(u,v)+1$ elements.

In an abstract setting, an *interval space* $(V,\circ)$ [60] is a set V together with a binary set-valued operator $\circ$ that assigns to each pair of points a nonempty subset of V (called segment or interval) such that

$$u,v \in u\circ v = v\circ u \text{ and } u\circ u=\{u\}.$$

$(V,\circ)$ is *geometric* [60] if in addition

$$w\in u\circ x \text{ and } v\in u\circ w \text{ imply } v\in u\circ x \text{ and } w\in v\circ x.$$

To any ternary algebra on a set V satisfying the axioms (A1), (A2), (A3) (and hence (A1′) and (A3′) as well) one associates an interval space $(V,\circ)$: by virtue of (A3′) one can define

$$u\circ v=\{(uvx): x\in V\}=\{x\in V: (uvx)=x\},$$

so that $u,v\in u\circ v=v\circ u$ and $u\circ u=\{u\}$ follows from (A1),(A1′),(A2), and (A3).

Lemma 17. *Let* $(V,\circ)$ *be the interval space of a ternary algebra* $\mathcal{A}=(V,(\ldots))$ *satisfying* (A1), (A2), *and* (A3).

(a) $w\in u\circ x$ *and* $v\in u\circ w$ *imply* $v\in u\circ x$ *if and only if* $\mathcal{A}$ *satisfies*

(A17) $\quad (uv(uwx))=(ux(uv(uwx))).$

In particular, (A17) *holds whenever* (A5) *does.*

(b) $w\in u\circ x$ *and* $v\in u\circ w$ *imply* $w\in v\circ x$ *if and only if* $\mathcal{A}$ *satisfies*

(A18) $\quad (uwx)=((uv(uwx))(uwx)x).$

In particular, (A18) *holds whenever either* (A11) *or* (A16) *holds.*

(c) (A18) *implies* (A4′).

Proof. (a) If $w\in u\circ x$ and $v\in u\circ w$ for $u,x\in V$, then

$$v=(uvw)=(uv(uwx))=(ux(uv(uwx)))=(uxv)$$

by (A17) and (A2), whence $v\in u\circ x$. Conversely, $(uwx)\in u\circ x$ and $(uv(uwx))\in u\circ(uwx)$ imply $(uv(uwx))\in u\circ x$, that is, (A17) holds, by the first part of the geometricity condition and (A2).

If (A5) is satisfied, then $(uv(uwx))=(ux(uvw))$ by (A2), and therefore (A17) follows from (A3′).

(b) If $w \in u \circ x$ and $v \in u \circ w$ for $u, x \in V$, then

$$w = (uwx) = ((uv(uwx))(uwx)x) = ((uvw)wx) = (vwx)$$

by (A18) and (A2). Conversely, $(uwx) \in u \circ x$ and $(uv(uwx)) \in u \circ (uwx)$ imply $(uwx) \in (uv(uwx)) \circ x$, that is, (A18) holds, by the second part of the geometricity condition and (A2).

If (A11) holds, then we infer from $w \in u \circ x$ and $v \in u \circ w$ that

$$(vwx) = ((wuv)(uxw)x) = (w(uxw)(xu(wuv))) = (ww(xu(wuv))) = w.$$

Alternatively, if (A16) holds, we derive

$$(xwv) = (x(xuw)(vuw)) = (xuw) = w.$$

In either case, $w \in v \circ x$ is true.

(c) From (A18) we infer that $w \in u \circ x$ and $x \in u \circ w$ imply $w \in x \circ x = x$. Further, (A18) yields $(uvw) \in ((uvw)uv) \circ u$ because $(uvw) \in v \circ u$ and $((uvw)uv) \in v \circ (uvw)$. On the other hand, $((uvw)uv) \in (uvw) \circ u$ holds trivially. Hence $(uvw) = ((uvw)uv)$, that is, (A4′) holds. □

Discreteness carries over from a ternary algebra $\mathcal{A}$ to its corresponding interval space. To a discrete interval space $(V, \circ)$ one associates a graph $G = (V, E)$ by letting $uv \in E$ if and only if $u \circ v = \{u, v\}$. Recall that $(V, \circ)$ is called a *graphic interval space* [6, 60] if $u \circ v = I(u, v)$ holds for any $u, v \in V$, where I is the interval function of the graph G. The edges uv of G are retrieved from $\mathcal{A}$ by the condition $(uvw) \in \{u, v\}$ for all $w \in V$. In [6] we established that a geometric interval space is graphic whenever it satisfies the following *triangle condition:* for any three points u, v, w in V with

$$u \circ v \cap u \circ w = \{u\}, u \circ v \cap v \circ w = \{v\}, u \circ w \cap v \circ w = \{w\},$$

the intervals $u \circ v, u \circ w, v \circ w$ are edges of the underlying graph whenever at least one of them is an edge.

A subalgebra of an intrinsic algebra of a graph G is typically disconnected (taken as a subgraph) in G but may very well yield a graphic interval space in its own right. For instance, every metric triangle in G constitutes a subalgebra isomorphic to the imprint algebra of K_3.

Theorem 4. *The following statements are equivalent for a discrete ternary algebra* $\mathcal{A} = (V, (\ldots))$:

(i) $\mathcal{A}$ *is the imprint algebra of a weakly median graph* $G = (V, E)$;
(ii) $\mathcal{A}$ *satisfies the equations* (A1), (A2), (A5), (A8), *and* (A11);
(iii) $\mathcal{A}$ *satisfies the equations* (A1), (A2), (A5), (A8′), *and* (A11);
(iv) $\mathcal{A}$ *satisfies the equations* (A1), (A2), (A3), (A5), (A12′), *and* (A18).

In particular, any subalgebra of the imprint algebra of a weakly median graph is itself the imprint algebra of some weakly median graph. None of the axioms in (ii)-(iv) *are redundant.*

Proof. From Theorem 3 and Lemma 17 we know that the imprint algebra of a graph satisfies the equations listed in (ii) or (iii), respectively, if and only if the graph is weakly median. Therefore it remains to establish that a ternary algebra $\mathcal{A}$ satisfying one of the three sets of equations is the imprint algebra of a graph. First notice that (A1), (A2), (A5), and (A11) imply (A3). For $u, v, w, x \in V$,

$$\begin{aligned}(wu(uwv)) &= (w(uwv)(vu(wuu))) && \text{by (A1),(A2)}\\ &= ((wuu)(uvw)v) && \text{by (A11),(A2)}\\ &= (uv(uvw)) && \text{by (A1),(A2)}\\ &= (u(uvv)w) && \text{by (A5)}\\ &= (uvw) = (uwv) && \text{by (A1),(A2).}\end{aligned}$$

Hence (A3) holds. From Lemma 17 we conclude that in each case the interval space $(V, \circ)$ of $\mathcal{A}$ is geometric. Suppose by way of contradiction that the interval space $(V, \circ)$ violates the triangle condition for interval spaces. Then there exist adjacent vertices u, v and a vertex w in the graph G associated with $(V, \circ)$ such that $u \circ w \cap v \circ w = \{w\}$, but $u \circ w \neq \{u, w\}$. Then $(wuv) = w$, $(uvw) = u$, and $(vuw) = v$. Let x be a neighbor of u in $u \circ w$, i.e., a vertex such that $(uwx) = x$ and $u \circ x = \{u, x\}$. First assume that $\mathcal{A}$ satisfies the equations from (ii) or (iii). If $(xuv) = u$, then we get a contradiction to axiom (A8) and (A8$'$), respectively, because

$$((xuv)(wuv)x) = (uwx) = x \neq u = (xuv).$$

Therefore $u \notin x \circ v$. Since u is adjacent in G to both x and v, we conclude that $(xuv) = x$ and $(vux) = v$. Employing this in

$$((wux)(uvw)v) = (xuv) = x,$$

$$(w(uvw)(vu(wux))) = (wu(vux)) = (wuv) = w,$$

we see that (A11) is violated, giving a contradiction. Next assume that $\mathcal{A}$ satisfies the equations from (iii). Then u', v', w', and x' derived from u, v, w, and x as in (A12$'$) coincide with u, v, w, and x, respectively. Hence $(v'u'x')$ equals v or u. Consequently, the left-hand side of (A12$'$) is either u or w and hence cannot equal x, which is in conflict with (A12$'$). We conclude that the triangle condition is satisfied when (ii) or (iii) holds. Therefore $(V, \circ)$ is a graphic interval space. Moreover, the given ternary algebra is the imprint algebra of the underlying graph G of $(V, \circ)$. Indeed, let x be a vertex of $I(u,v) \cap I(u,w)$ such that $(uvw) \in I(u,x)$ and $I(u,x)$ is maximal with respect to the inclusion. Then we obtain $(uv(uwx)) = (uvx) = x$ and $(u(uvw)x) = (uvw)$, so that $x = (uvw)$ follows from (A5).

Finally, we will demonstrate that each axiom system in the theorem is irredundant.

Ad (A1): The constant ternary operation on $\{0,1\}$, defined by $(uvw) = 0$, satisfies all the equations listed in Theorem 4 except for (A1).

Ad (A2): The second and third ternary projections in $\{0,1\}$, defined by $(uvw) = v$ and $(uvw) = w$, respectively, satisfy (A1) and (A5) but violate (A2); moreover, (A8), (A8′), and (A11) are fulfilled by the second projection, whereas (A3), (A12′), and (A18) are fulfilled by the third projection.

Ad (A3): Take the integers $\mathbb{Z}_3 = \{0,1,2\}$ modulo 3, and define (uvw) as $u+1$ if $\{u,v,w\} = \{0,1,2\}$ and otherwise (when $|\{u,v,w\}| \leq 2$) via the majority rule. Then (A1), (A1′), and (A2) trivially hold. Consequently, (A3′) is fulfilled whenever $|\{u,v,w\}| \leq 2$. Since $(uvu{+}1)$ equals $u{+}1 = (uvw)$ for $\{u,v,w\} = \{0,1,2\}$, we infer that (A3′) is always true. In contrast, (A3) is violated because $(102) = 2$ but $(01(102)) = (012) = 1$. (A5) and (A18) easily follow from the valid equations (A1), (A1′), (A2), and (A3′) because at least two of the four variables u, v, w, x must be equal. In the case of (A12′), we infer that u', v', w', and x' all equal (uvw), so that this equation evidently holds.

Ad (A5): Consider the subalgebra $R_4 = \{u,v,w,x\}$ of the imprint algebra of $K_{1,1,3}$ as labelled in Fig. 1(c). Then (A1)-(A4′) trivially hold but (A5) is violated (see Proposition 1). Further, (A8) and (A8′), (A11), and (A12′) are satisfied according to Examples 1(a), 2(a), and 3(b), respectively. (A18) holds because R_4 is a subalgebra of an imprint algebra.

Ad (A8) and (A8′): The imprint algebra of the house satisfies (A11) as well as (A1)-(A5) but violates (A8) and (A8′); see Example 2(b) and Lemma 13.

Ad (A11): The algebra $R_3 = \{t,u,v,w\}$ (see Fig. 10(c)) is a subalgebra of the imprint algebra of an apiculate graph and hence satisfies (A1)-(A5) but violates (A11) by Lemma 14. Since (A8) obviously holds whenever two of the four variables are equal or $(uwx) = u$, only one (up to symmetry) instance of (A8) needs explicit checking for R_3 :

$$((vwt)(uwt)v) = (tuv) = t = (vwt),$$

as required. Hence (A8′) holds as well.

Ad (A12′): The C_5 and house algebras trivially satisfy (A1)-(A5) and (A18) but violate (A12′) by Lemma 15.

Ad (A18): Modify the chain algebra of the linear order $0 < 1 < 2 < 3$ by turning $\{0, 1, 2\}$ into a K_3 algebra, that is:

$$(ijk) = \begin{cases} i & \text{if } \{i, j, k\} = \{0, 1, 2\}, \\ (i \wedge j) \vee (i \wedge k) \vee (j \wedge k) & \text{otherwise} \end{cases}$$

Then (A1)-(A3′) clearly hold. To check (A5) and (A12′), we may assume that $\{u, v, w, x\} = \{0, 1, 2, 3\}$. Then (A12′) readily follows because the only nontrivial case is when $\{u, v, w\} = \{0, 1, 2\}$ and $x = 3$. Indeed, in this case, $x' \in \{u, w\}$ and therefore $\{u', v', w', x'\} \in \{0, 1, 2\}$. As for (A5), we distinguish three cases. If $u = 3$, then all brackets $(\ldots)$ on both sides of (A5) are computed in the chain algebra as no bracket can contain the triplet $0, 1, 2$. If $w = 3$, then $(uwx) \in \{u, x\}$ and $(uvw) \in \{u, v\}$, so that either side of (A5) yields u because $(uvx) = u$. Finally, if 3 is one of v or x, say the latter, then $(uwx) \in \{u, w\}$ and $(uvw) = u$, so that again both sides of (A5) equal u. This concludes the proof of Theorem 4. □

Corollary 8. *The following statements are equivalent for a discrete ternary algebra* $\mathcal{A} = (V, (\ldots))$:

(i) $\mathcal{A}$ *is the imprint algebra of a quasi-median graph;*
(ii) $\mathcal{A}$ *satisfies the equations* (A1), (A2), (A5), *and* (A16′)*;*
(iii) $\mathcal{A}$ *satisfies the equations* (A1), (A2), (A3), (A5), *and* (A16).

Proof. (i) implies (ii) by Proposition 13.

(ii) implies (iii): Using (A2), one derives (A1′) from (A16′) and (A2) by setting $u = v$ and $w = x$ in (A16′). Setting $v = w$ in (A16′) yields (A3) by virtue of (A1′). Then (A16) and (A16′) are equivalent because (A3) and (A2) hold.

(iii) implies (i): From Lemma 17 we infer that the algebra $\mathcal{A}$ satisfies (A8) (since (A16) and (A4′) hold) and that its interval space $(V, \circ)$ is geometric. We can therefore proceed as in the proof of the preceding theorem. In establishing the triangle condition, we can replace the argument involving (A11) by one using (A16) instead: for the triplet $u, v, w = (wvu)$ and the vertex $x = (vux) \in I(u, w)$ we get

$$w = (wvx) = (w(vux)(wux)) = (wux) = x,$$

a contradiction.

As to independence of axioms, note that (A16) and (A16′) are satisfied by the constant ternary operation and the third projection of $\{0, 1\}$ as well as by the imprint operation of R_4. Moreover, the 3-element algebra (defined above) that rejects (A3) satisfies (A16). This finishes the proof. □

Further axiomatic characterizations of the imprint algebras of quasi-median graphs can be found in [14, 46].

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