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On end degrees and infinite cycles in locally finite graphs

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Abstract

We introduce a natural extension of the vertex degree to ends. For the cycle space $\mathcal{C}(G)$ as proposed by Diestel and Kühn [5, 6], which allows for infinite cycles, we prove that the edge set of a locally finite graph G lies in $\mathcal{C}(G)$ if and only if every vertex and every end has even degree. In the same way we generalise to locally finite graphs the characterisation of the cycles in a finite graph as its 2-regular connected subgraphs.

1 Introduction

The cycle space of a finite graph is the $\mathbb{Z}_2$-vector space consisting of all sums of edge sets of cycles. In its naive extension to infinite graphs, where the elements of the cycle space are finite edge sets, many theorems known for finite graphs become false. Recent studies, see Diestel [2] for an introduction and a survey, strongly suggest that these issues arise because in infinite graphs, certain 'infinite cycles' should be allowed. This idea leads to a definition of the cycle space $\mathcal{C}(G)$ for locally finite graphs G, in which the cycles are those subgraphs of G whose closure in $|G|$, the compactification of G by its ends, is a *circle*, ie. a homeomorphic image of the unit circle. We will introduce $\mathcal{C}(G)$, which has been proposed by Diestel and Kühn [5, 6], formally in the next section.

Using $\mathcal{C}(G)$, almost all properties of the cycle space of a finite graph can be extended to locally finite graphs. Yet, to date there is no generalisation of one of the most basic characterisations of the elements of the cycle space:

Theorem 1. *Let H be a subgraph of a finite graph G. Then $E(H)$ is an element of the cycle space of G if and only if every vertex of G has even degree in H.*

Simple examples show that for infinite graphs it is not sufficient to consider vertex degrees. Consider, for instance, the double ray D. Since $|D|$ is homeomorphic to the unit interval, it does not contain any circles, hence, it follows that $\mathcal{C}(D) = \{\emptyset\}$. Thus $E(D) \notin \mathcal{C}(D)$, even though every vertex of D has degree 2 in D. The problem here seems to arise from the ends rather than the vertices of the considered graph. Diestel and Kühn [6] raised the following problem:

Problem 2. *Characterise the circles and the elements of the cycle space of an infinite graph in purely combinatorial terms, such as vertex degrees and 'degrees of ends'.*

We introduce a concept of end degrees that is a natural extension of vertex degrees; instead of incident edges we count edge-disjoint rays (for degrees in subgraphs it will be necessary to substitute 'rays' with 'arcs'). This notion allows us to solve the first part of Problem 2; we prove a straightforward adaption of the well-known fact that the cycles in a finite graph are exactly its 2-regular connected subgraphs.

Theorem 3. *Let C be a subgraph of a locally finite graph G. Then $\overline{C}$ is a circle if and only if $\overline{C}$ is topologically connected and every vertex or end x of G with $x \in \overline{C}$ has degree two in C.*

Depending on its degree, an end can be assigned a parity, ie. the label 'even' or 'odd'—as long as it has finite degree. Inspired by Laviolette [9], who introduced a concept to measure the parity of vertices of infinite degree, we assign a parity also to ends of infinite degree. A classification of ends into even and odd ends has already been achieved by Nash-Williams [11] for the case of eulerian graphs with only finitely many ends. Our definition coincides with Nash-Williams' in these graphs but covers all locally finite graphs. Moreover, with our definition the following important special case of Problem 2 becomes true, which is the main result of this paper.

Theorem 4. *Let G be a locally finite graph. Then $E(G) \in \mathcal{C}(G)$ if and only if every vertex and every end of G has even degree.*

An extension of this characterisation to arbitrary subgraphs of G would solve Problem 2 completely. We shall offer a conjecture in that respect (see Section 4).

The paper is organised as follows. After some basic definitions in Section 2, we introduce and discuss our end degree concept in Sections 3 and 4. Theorem 4 will be proved in Section 5. In Section 6, we show Theorem 3 and other results, and in the last section, we briefly discuss an alternative end degree.

2 Definitions

The basic terminology we use can be found in Diestel [3]. A 1-way infinite path is called a *ray*, a 2-way infinite path is a *double ray*, and the subrays of a ray or double ray are its *tails*. Two rays in a graph $G = (V, E)$ are *equivalent* if no finite set of vertices separates them; the corresponding equivalence classes of rays are the *ends* of G. We denote the set of these ends by $\Omega = \Omega(G)$. A ray R in an end ω will also be called an ω-ray.

Let us define a topology on G together with its ends; if G is locally finite, then this topology is known as its *Freudenthal compactification*. We begin by endowing G itself (without ends) with the usual topology of a 1-complex. (Thus, every edge is homeomorphic to the real interval $[0, 1]$, and the basic open neighbourhoods of a vertex v are the unions of half-open intervals $[v, z)$, one for every edge e at v with z an inner point of e.) In order to extend this topology to $G \cup \Omega$, we take as a basis of open neighbourhoods of a given end $\omega \in \Omega$ the sets of the form

$$C(S, \omega) \cup \Omega(S, \omega) \cup E'(S, \omega)\,,$$

where $S \subseteq V$ is a finite set of vertices, $C(S, \omega)$ is the (unique) component of $G - S$ in which every ray from ω has a tail, $\Omega(S, \omega)$ is the set of all ends $\omega' \in \Omega$ whose rays have a tail in $C(S, \omega)$, and $E'(S, \omega)$ is any union of half-edges $(z, y]$, one for every S–$C(S, \omega)$ edge $e = xy$ of G, with z an inner point of e and y a vertex of $C(S,\omega)$. Let $|G|$ denote the corresponding topological space on the point set $V \cup \Omega \cup \bigcup E$. We shall freely view G and its subgraphs either as abstract graphs or as subspaces of $|G|$. We note that $|G|$ is a Hausdorff space.

A continuous image of the unit interval $[0,1]$ in $|G|$ is a *topological path*. The images of 0 and 1 are the *endpoints* of the topological path. A homeomorphic image of $[0,1]$ in $|G|$ is called an *arc* in $|G|$. Analogously to ω-rays, let us say that an arc is an ω-arc, if the end ω is one of its endpoints. The following lemma can be found in Hall and Spencer [8, p. 208].

Lemma 5. *Every topological path with distinct endpoints x, y in a Hausdorff space X contains an arc between x and y.*

A set $C \subseteq |G|$ is a *circle* if it is homeomorphic to the unit circle. Then C includes every edge of which it contains an inner point, and the graph consisting of these edges and their endvertices is the *cycle* defined by C. Conversely, it is not hard to show [5] that $C \cap G$ is dense in C, so every circle is the closure in $|G|$ of its cycle and hence defined uniquely by it. Note that every finite cycle in G is also a cycle in this sense, but there can also be infinite cycles; see [2] for examples. The edge set of a cycle is called a *circuit*.

Call a family $(D_i)_{i\in I}$ of subsets of E *thin* if no vertex of G is incident with an edge in D_i for infinitely many i. (Thus in particular, no edge lies in more than finitely many D_i.) Let the *sum* $\sum_{i\in I} D_i$ of this family be the set of all edges that lie in D_i for an odd number of indices i, and let the *cycle space* $\mathcal{C}(G)$ of G be the set of all sums of (thin families of) circuits, finite or infinite. Symmetric difference as addition makes of $\mathcal{C}(G)$ a $\mathbb{Z}_2$-vector space, which coincides with the usual cycle space of G when G is finite. We remark that $\mathcal{C}(G)$ is closed also under taking infinite thin sums [5, 6], which is not obvious from the definitions.

The following theorem will serve as our main tool to decide whether a graph is an element of its own cycle space, namely if and only if it contains no odd cut.

Theorem 6 (Diestel and Kühn [5]). *Let G be a locally finite graph, and let $Z \subseteq E(G)$. Then $Z \in \mathcal{C}(G)$ if and only if $|F \cap Z|$ is even for every finite cut F of G.*

A finite cut F of G *separates* a set $S \subseteq V(G)$ from an end $\omega \in \Omega$ if it meets every ray of ω that starts in S. This is equivalent to that the (unique) component C of $G - F$ with $\omega \in \overline{C}$ is disjoint from S. Similarily, F *separates* two ends ω and ω', if the closure of each component of $G - F$ contains at most one of ω, ω'.

If C is a subgraph of G, write $\partial_G C$ or, where no confusion is possible, ∂C, for the cut $E_G(C, G - C)$. Call a connected induced subgraph C of G a *region*, if the cut ∂C is finite. The region C is *even* resp. *odd* if $|\partial C|$ is even resp. odd.

3 End degrees

As ends are equivalence classes of rays, the degree of an end should in some way be related to its rays. Also, the rays may be seen as somewhat analogous to the incident edges of a vertex, whose number is the degree of the vertex. Therefore, as a first try, we might count the maximal number of disjoint rays in an end (sometimes called the *multiplicity* of the end). For this notion of an end degree, however, Theorem 4 fails as the graph G in Figure 1 demonstrates. Since G contains an odd cut, its edge set is not an element of $\mathcal{C}(G)$ by Theorem 6. But each vertex degree is even, and the maximal number of disjoint rays in each end is two.

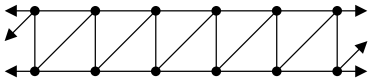

Figure 1: The multiplicity of each end is even, but $E(G) \notin \mathcal{C}(G)$.

Looking more closely we see that although the maximal number of disjoint rays in each end is even, the maximal number of *edge-disjoint* rays is odd, namely three. Thus with this measure instead we would have correctly decided that $E(G) \notin \mathcal{C}(G)$. Formally, let us define the *degree* of an end ω in a graph G as

$$d(\omega) := \sup\{|\mathcal{R}| \,:\, \mathcal{R} \text{ is a set of edge-disjoint } \omega\text{-rays}\} \in \mathbb{N} \cup \{\infty\}.$$

Although we will not use this, we shall prove later on (Lemma 17) that the supremum is attained, ie. if $d(\omega) = \infty$, then there exists an infinite set of edge-disjoint ω-rays. Andreae [1] proves a similar result.

Our degree concept clearly divides the ends of finite degree into even and odd ends, but how are we to deal with ends of infinite degree? We may not simply treat them as odd ends, since the edge set of the infinite grid obviously is an element of its cycle space but the only end of the grid has infinite degree.

On the other hand, classifying all ends of infinite degree as even is not any better: consider the graph G in Figure 2. All vertex degrees are even and both ends have infinite degree, but G has an odd cut (which together with Theorem 6 implies that $E(G) \notin \mathcal{C}(G)$).

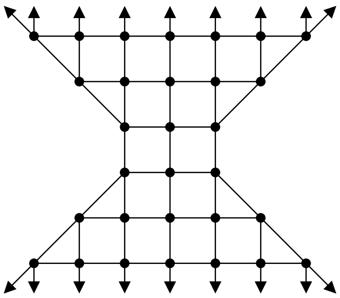

Figure 2: Both ends have infinite degree, but $E(G) \notin \mathcal{C}(G)$.

Consequently, the degree, if infinite, is not sufficiently fine enough to determine the parity of an end. For an adequate refinement we will use the following characterisation of ends with even finite degree.

Lemma 7. *In a locally finite graph G let $\omega \in \Omega(G)$ have finite degree k. Then the following statements are equivalent:*

(i) *k is even;*

(ii) *there is a finite $S \subseteq V(G)$ such that for every finite set $S' \supseteq S$ of vertices the maximal number of edge-disjoint ω-rays starting in S' is even.*

Proof. Consider a set $\mathcal{R}$ of edge-disjoint ω-rays of maximal cardinality $|\mathcal{R}| = k$, and let U be the set of starting vertices of $\mathcal{R}$. Then, for every finite set $S' \supseteq U$, $\mathcal{R}$ has maximal cardinality among all sets of edge-disjoint ω-rays starting in S'. Thus, putting $S := U$, we deduce that (i) implies (ii). Also, (ii) implies (i), which we see by choosing $S' = S \cup U$. □

Observe that as every finite set $S \subseteq V(G)$ gives (essentially) rise to a neighbourhood $\hat{C}(S, \omega)$ of ω, condition (ii) in Lemma 7 can be alternatively formulated using these neighbourhoods, or using regions whose closures contain ω:

(ii′) There is a region A of G with $\omega \in \overline{A}$ such that for every region $B \subseteq A$ of G with $\omega \in \overline{B}$ the maximal number of edge-disjoint rays of ω starting outside B is even.

This motivates the following definition of the parity of an end: an end ω of a locally finite graph is said to be *even* if ω satisfies (ii) of Lemma 7. Otherwise ω is *odd*. Thus, ω is odd if and only if for all finite $S \subseteq V(G)$ there is a finite set $S' \supseteq S$ such that the maximal number of edge-disjoint ω-rays starting in S' is odd. By Lemma 7, an end ω of finite degree is even if and only if $d(\omega)$ is even.

However, it is not possible to extend this notion literally to subgraphs H of G. First of all, we cannot simply measure the degrees of the ends of H (as opposed to those of G). This is not surprising as H is embedded in the space $|G|$. If H is a double ray, for instance, then (viewed as a graph on its own and not as a subgraph) it has two ends, each of which has degree 1. On the other hand, the tails of H may lie in the same end of G, in which case $\overline{H}$ is a circle in $|G|$. Thus the ends contained in $\overline{H}$ should have degree 2 in H, not 1. Therefore, we only consider ends of G (and not of H).

Even taking that into account, the literal extension to subgraphs fails: Consider the bold subgraph of the graph in Figure 3, and let ω be the end of G "to the right". Then, if we count the edge-disjoint ω-rays that lie in H, we find that apart from tail-equivalence there is only one ω-ray. But as $\overline{H}$ is a circle, we would expect the end to have degree 2. In contrast, if we consider the

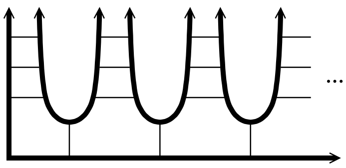

Figure 3: In subgraphs, counting edge-disjoint rays is not enough.

maximal number of edge-disjoint ω-*arcs* in $\overline{H}$ instead of counting edge-disjoint ω-rays in H we obtain the desired degree 2. Counting arcs will indeed turn out to be successful, and the following proposition, which we shall prove in the next section, shows that in G it makes actually no difference whether we count rays or arcs:

Proposition 8. *Let G be a locally finite graph, and let $\omega \in \Omega(G)$. Then for every finite $S \subseteq V(G)$ the maximal number of edge-disjoint ω-rays starting in S equals the maximal number of edge-disjoint ω-arcs starting in S.*

Hence, for a subgraph H of a locally finite graph G, and $\omega \in \Omega(G)$, we define, analogously to the definition of $d(\omega)$ given above, the degree of ω in H as

$$d_H(\omega) := \sup\{|\mathcal{R}| : \mathcal{R} \text{ is a set of edge-disjoint } \omega\text{-arcs in } \overline{H}\} \in \mathbb{N} \cup \{\infty\}.$$

We note that the supremum is attained (see Lemma 17). Further, observe that $d(\omega) = d_G(\omega)$. Indeed, suppose otherwise, ie. $d(\omega) < d_G(\omega)$. So, in particular, $d(\omega)$ is finite. For a set of $d(\omega)+1$ edge-disjoint ω-arcs, let $S \subseteq V(G)$ be a choice of exactly one vertex from each of the arcs. Then, by Proposition 8, there are also $d(\omega) + 1$ edge-disjoint ω-rays starting in S, a contradiction.

The parity of an end in H is defined as follows:

Definition 9. *An end ω of G is* even *in H if there is a finite $S \subseteq V(G)$ such that for every finite $S' \subseteq V(G)$ with $S' \supseteq S$ the maximal number of edge-disjoint ω-arcs in $\overline{H}$ starting in S' is even. Otherwise, ω is* odd *in H.*

Note that by Proposition 8, the definition of parity is consistent with the one given previously. Furthermore, it can be seen similarly as in the proof of Lemma 7 that for an end ω with finite degree in H, ω has even degree in H if and only if $d_H(\omega)$ is even.

A complete solution of Problem 2 requires an analogon of Theorem 4 for subgraphs H of G. The forward direction of such an analogon can be proved easily with the same methods as used for Theorem 4. Furthermore, if G has only countably many ends the problem is not overly difficult (Proposition 25). In view of this, and in view of Theorems 3 and 4, and two more results in Section 6, which demonstrate that the end degrees behave in many aspects similar to vertex degrees in finite graphs, we offer the following conjecture:

Conjecture 10. *Let H be a subgraph of a locally finite graph G. Then $E(H) \in \mathcal{C}(G)$ if and only if every vertex and every end has even degree in H.*

4 A cut criterion for the end degree

In this section we prove Proposition 8. The other result of this section is Corollary 19, which yields a criterion for the parity of an end in terms of cut cardinalities. Let us start with a simple lemma that shows how we can construct a topological path by piecing together infinitely many arcs.

Lemma 11. *Let G be a locally finite graph, and let for $n \in \mathbb{N}$, $\phi_n : [0,1] \to |G|$ be a homeomorphism such that if $A_n := \phi_n([0,1])$ it holds that:*

(i) *$A_n \cap A_m \subseteq V(G) \cup \Omega(G)$ for $n \neq m$; and*

(ii) *$\phi_n(1) = \phi_{n+1}(0)$ for all n.*

Then there is an $x \in |G|$ such that $\bigcup_{n=1}^{\infty} A_n \cup \{x\}$ is a topological path from $\phi_1(0)$ to x.

Proof. Instead of the ϕ_n let us consider compositions with suitable homeomorphisms $\phi'_n : [1 - 2^{-(n-1)}, 1 - 2^{-n}] \to A_n$. Together the ϕ'_n define, by (ii), a continuous function $\phi' : [0,1) \to |G|$. As $|G|$ is compact, the sequence $\phi_1(0) = \phi'(1/2), \phi_2(0) = \phi'(3/4), \ldots$ has an accumulation point x. We claim that $\phi : [0,1] \to |G|$ defined by $\phi(s) := \phi'(s)$ for $s \in [0,1)$ and by $\phi(1) := x$ is continuous.

Let a neighbourhood V of x be given, and note that because of (i) and (ii), none of the $\phi_n(0)$ is an inner point of an edge, and thus x is an end. Then there is a basic open neighbourhood $\hat{C}(S,x) \subseteq V$ that contains all but finitely many of the $\phi_n(0)$. By (i), only finitely many of the A_n meet the finite cut $\partial C(S,x)$. So, there is an N such that $A_n \subseteq \hat{C}(S,x)$ for $n \geq N$. Consequently, $\phi^{-1}(V)$ contains the open set $(1-2^{-N}, 1]$, and thus is a neighbourhood of 1 in $[0,1]$. $\square$

Menger's theorem applied to the line graph implies that between any two finite edge sets E_1, E_2 in a graph there are as many edge-disjoint E_1–E_2 paths as the minimal number of edges needed in order to separate E_1 and E_2. The following lemma generalises this result to arcs. Let us say that an arc A is an E_1–E_2 arc if it has exactly one edge in E_1, exactly one in E_2, and these are incident with an endpoint of A.

Lemma 12. *Let H be a subgraph of a locally finite graph G. Let $E_1, E_2 \subseteq E(H)$ be finite. Then the maximal number of edge-disjoint E_1–E_2 arcs in $\overline{H} \subseteq |G|$ equals the minimum k such that there is a finite set $X \subseteq E(G)$ separating E_1 from E_2 in G with $k = |X \cap E(H)|$.*

We need two lemmas for the proof, the first of which is a standard tool in infinite graph theory:

Lemma 13 (König's Infinity Lemma). *Let $W_1, W_2, \ldots$ be an infinite sequence of disjoint non-empty finite sets, and let H be a graph on their union, such that for $n \geq 2$ every vertex in W_n has a neighbour in W_{n-1}. Then H contains a ray $v_1 v_2 \ldots$ with $v_n \in W_n$ for all n.*

Theorem 14 (Diestel and Kühn [6]). *Every element of the cycle space of a graph is a disjoint union of circuits.*

Proof of Lemma 12. Let S be a finite vertex set such that $E_1 \cup E_2 \subseteq E(G[S])$, let $v_1, v_2, \ldots$ be an enumeration of $V(G)$, and put $G_n := G[S \cup \{v_1, \ldots, v_n\}]$ for $n \in \mathbb{N}$. Let $\mathcal{L}_n$ be the set of all sets M satisfying

(i) M is a set of pairwise edge-disjoint subgraphs of H;

(ii) for each $L \in M$ there is an E_1–E_2 path P with $P \cap G_n = L$; and

(iii) $|M| \geq k$.

Let us show that $\mathcal{L}_n$ is non-empty for each n. Contract each component of $G - G_n$ to a vertex (keeping parallel edges but deleting loops), and denote the resulting finite graph by $\tilde{G}_n$. Let $\tilde{H}_n$ be the subgraph of $\tilde{G}_n$ that consists of the edges in $E(H) \cap E(\tilde{G}_n)$ together with the incident vertices. Now, let $\tilde{M}$ be a set of edge-disjoint E_1–E_2 paths in $\tilde{H}_n$ of maximal cardinality. By Menger's theorem applied to the line graph, there is a finite set $\tilde{X} \subseteq E(\tilde{H})$ of cardinality $|\tilde{M}|$ that separates E_1 from E_2 in $\tilde{H}_n$. Then $X := \tilde{X} \cup (E(\tilde{G}_n) \setminus E(\tilde{H}_n))$ separates

E_1 from E_2 in $\tilde{G}_n$, and $|X \cap E(H)| = |\tilde{M}|$. Next, observe that X is also an E_1–E_2 separator in G, implying $|\tilde{M}| = |X \cap E(H)| \geq k$. Put $M := \{\tilde{L} \cap G_n : \tilde{L} \in \tilde{M}\}$, and note that this choice satifies (i) and (iii). Furthermore, by replacing each vertex of $\tilde{G}_n - G$ that $\tilde{L}$ meets with a path through the respective component of $G - G_n$, we easily find for the corresponding $L \in M$ a path such that (ii) is satisfied. Thus, $\mathcal{L}_n \neq \emptyset$.

Define a graph on the vertex set $\bigcup_{n=1}^{\infty} \mathcal{L}_n$ with edges $M_n M_{n+1}$ for $M_n \in \mathcal{L}_n$ and $M_{n+1} \in \mathcal{L}_{n+1}$ if $M_n = \{L \cap G_n : L \in M_{n+1}\}$. As every $M \in \mathcal{L}_{n+1}$ has a neighbour in $\mathcal{L}_n$ we may apply Lemma 13, which yields a ray $M_1 M_2 \ldots$ with $M_n \in \mathcal{L}_n$ for all $n \geq 1$. For each $L_1 \in M_1$ there is a sequence $L_1, L_2, \ldots$ with $L_n \in M_n$ and $L_n = L_{n+1} \cap G_n$. More precisely, there are, by (iii), k such sequences so that their respective unions $L^1 = \bigcup L_n^1 \ldots, L^k = \bigcup L_n^k$ are mutually edge-disjoint. Furthermore, $L^i \subseteq \overline{H}$, and $L^i \cap G_n = L_n^i \in M_n$ for $i = 1, \ldots, k$.

We claim that each of the L^i contains an E_1–E_2 arc. Indeed, consider an $i \in \{1, \ldots, k\}$, and let v and w be the endvertices of the path P for which $P \cap G_1 = L_1^i$. Introduce a new vertex z to G, and link it to v and w. Denote by Q the resulting v–w path vzw. We want to show that $E(L^i \cup Q)$ is an element of the cycle space of $G' := G \cup Q$. To this end, let F be a finite cut of G', and choose n large enough so that $F \subseteq E(G_n \cup Q)$. Denote by P_n the v–w path with $P_n \cap G_n = L_n^i$, which exists by (ii). Then

$$\begin{aligned} F \cap E(L^i \cup Q) &= F \cap (E(G_n \cap L^i) \cup Q) \\ &= F \cap (E(G_n \cap L_n^i) \cup Q) = F \cap E(P_n \cup Q). \end{aligned}$$

The last intersection is an even set as $E(P_n \cup Q)$ is a circuit in G', and hence, $E(L^i \cup Q)$ an element of the cycle space of G', by Theorem 6. Thus, by Theorem 14 there is a circle D with $Q \subseteq D \subseteq L^i \cup Q$. Hence, $A^i := D \setminus (Q \setminus \{v, w\})$ is an arc from v to w. Therefore, there are k edge-disjoint E_1–E_2 arcs $A^1, \ldots A^k$ in $\overline{H}$.

Since there is a finite set $X \subseteq E(G)$ separating E_1 from E_2 in G such that $k = |X \cap E(H)|$, there cannot be more than $k+1$ arcs in $\overline{H}$ connecting E_1 and E_2, as each of them meets X. □

Corollary 15. *Let G be a locally finite graph, let H be a subgraph, and let $C_1 \supseteq C_2$ be regions of G. Then the maximal number of edge-disjoint $\partial C_1 \cap E(H)$–$\partial C_2 \cap E(H)$ arcs in $\overline{H}$ equals the minimum k such that there is a region D with $C_1 \supseteq D \supseteq C_2$ and $|\partial D \cap E(H)| = k$.*

Proof. By Lemma 12, the maximal number of edge-disjoint $\partial C_1 \cap E(H)$–$\partial C_2 \cap E(H)$ arcs in $\overline{H}$ equals the minimal $k' \in \mathbb{N}$ such that there is a finite $X \subseteq E(G)$ with $|X \cap E(H)| = k'$ which separates $\partial C_1 \cap E(H)$ from $\partial C_2 \cap E(H)$ in G. Any such X with $|X|$ minimal gives rise to a region D as above, hence $k = k'$. □

We obtain a Mengerian criterion:

Lemma 16. *Let G be a locally finite graph, let H be a subgraph, let $\omega \in \Omega(G)$, and let $S \subseteq V(G)$ be finite. Then the maximal number of edge-disjoint ω-arcs in $\overline{H}$ starting in S equals the minimum $|F \cap E(H)|$ over all cuts F separating S from ω.*

Proof. First, choose a region $C_1 \subseteq C_0 := C(S,\omega)$ with $\omega \in \overline{C_1}$ such that $|\partial C_1 \cap E(H)|$ is minimal among all regions $C \subseteq C_0$ with $\omega \in \overline{C}$. To prove the assertion it suffices to find a set of edge-disjoint ω-arcs in $\overline{H}$ starting in S, where each of the arcs uses exactly one edge from $\partial C_1 \cap E(H)$.

Next, observe that the closure of one of the components C of $C_1 - N(G-C_1)$ contains ω. Choose C_2 such that $|\partial C_2 \cap E(H)|$ is minimal among all regions $C \subseteq C_1$ satisfying $C \cup N(C) \subseteq C_1$ and $\omega \in \overline{C}$. Continuing in this manner, we obtain regions C_i with $\omega \in \overline{C_i}$ that satisfy for $i \geq 1$:

$$C_{i+1} \cup N(C_{i+1}) \subseteq C_i; \textit{ and} \tag{1}$$

$$|\partial C_i \cap E(H)| \leq |\partial D \cap E(H)| \textit{ for every region } D \textit{ with } C_i \supseteq D \supseteq C_{i+1}. \tag{2}$$

We claim that

there exists a set $\mathcal{A}$ of edge-disjoint ω-arcs in $\overline{H}$ starting outside C_1 such that $|\partial C_1 \cap E(H)| = |\mathcal{A}|$. (3)

Indeed, because of (2), Corollary 15 yields for every $i \geq 1$ a set $\mathcal{P}_i$ of edge-disjoint $(\partial C_i \cap E(H))$–$(\partial C_{i+1} \cap E(H))$ arcs in $\overline{H}$, with $|\mathcal{P}_i| = |\partial C_i \cap E(H)|$. Hence, for any edge $e \in \partial C_1 \cap E(H)$ there is an $A_1 \in \mathcal{P}_1$ that starts in e. The arc A_1 ends in an edge $e' \in \partial C_2 \cap E(H)$, and thus there exists an arc $A_2 \in \mathcal{P}_2$ such that $A_1 \cap A_2 = e'$. In this manner we find a sequence $A_1, A_2, \ldots$ so that A_i and A_{i+1} overlap in exactly one edge of $\partial C_{i+1} \cap E(H)$. As each $A_i \in \mathcal{P}_i$ is an $(\partial C_i \cap E(H))$– $(\partial C_{i+1} \cap E(H))$ arc, A_i and A_j are disjoint for $|i-j| > 1$. Thus, by deleting the last vertex and all inner points of the last edge in each A_i, we obtain a sequence of edge-disjoint arcs to which we may apply Lemma 11. This yields an $x \in |G|$ together with an arc $A_e \subseteq \bigcup_{i=1}^{\infty} A_i \cup \{x\}$ in $\overline{H}$ that starts in e and ends in x, which uses exactly one edge in each $\partial C_i \cap E(H)$.

Suppose $x \neq \omega$. Then, there exists a finite set $T \subseteq V(G)$ that separates ω and x. By (1), there is an $N \in \mathbb{N}$ such that $T \cap V(C_N) = \emptyset$. Thus, C_N is completely contained in one component of $G - T$, and only one of ω, x lies in $\overline{C_N}$, a contradiction. So, we obtain for each e an ω-arc A_e, and all these arcs are edge-disjoint since for each i the arcs in $\mathcal{P}_i$ are.

Finally, using Corollary 15 we lengthen the A_e in order to obtain a set of edge-disjoint ω-arcs which start in S (this is possible by the minimal choice of C_1). Note that each of these arcs indeed uses exactly one edge from $\partial C_1 \cap E(H)$. □

Similarly we can show that for a given subgraph H of G and an end $\omega \in \Omega(G)$ there indeed exists a maximal set of edge-disjoint ω-arcs in $\overline{H}$.

Lemma 17. *Let G be a locally finite graph, let H be a subgraph, and let $\omega \in \Omega(G)$ such that $d_H(\omega) = \infty$. Then there is an infinite set of edge-disjoint ω-arcs in $\overline{H}$.*

Proof. As in the proof of Lemma 16 we define a sequence of regions C_i satisfying (1) and (2). Again, Corollary 15 yields for each $i \in \mathbb{N}$ a set of edge-disjoint $(\partial C_{i-1} \cap E(H))$–$(\partial C_i \cap E(H))$ arcs, which we piece together with the help of Lemma 11 to obtain a set of edge-disjoint ω-arcs in $\overline{H}$ (whose union contains all edges of all of the $\partial C_i \cap E(H)$). Note that this set is indeed infinite, since $d_H(\omega) = \infty$, and hence by Lemma 16, we may assume $\partial C_{i-1} \cap E(H) < \partial C_i \cap E(H)$ for all $i \in \mathbb{N}$. □

We finally prove Proposition 8, which we restate:

Proposition 8. *Let G be a locally finite graph, and let ω be an end of G. Then for every finite set $S \subseteq V(G)$ the maximal number of edge-disjoint ω-rays starting in S equals the maximal number of edge-disjoint ω-arcs starting in S.*

Proof. By Lemma 16, the maximal number of edge-disjoint ω-arcs starting in S equals the minimal cardinality of a finite cut that separates S from ω. This minimal cardinality, on the other hand, equals the maximal number of edge-disjoint ω-rays starting in S: there clearly cannot be more ω-rays, and conversely, using Menger's theorem applied to the line graph, we can piece together the rays we need along minimal separating cuts, in a similar fashion as in Lemma 16. □

Two further results that follow immediately from Lemma 16 are characterisations of end degrees in terms of cut cardinalities:

Corollary 18. *Let G be a locally finite graph, let H be a subgraph, and let $\omega \in \Omega(G)$. Then $d_H(\omega) = k \in \mathbb{N}$ if and only if k is the smallest integer such that every finite $S \subseteq V(G)$ can be separated from ω with a finite cut that shares exactly k edges with $E(H)$.*

Corollary 19. *Let G be a locally finite graph, let H be a subgraph, and let $\omega \in \Omega(G)$. Then ω has even degree in H if and only if there is a finite $S \subseteq V(G)$ such that for every finite $S' \subseteq V(G)$ with $S' \supseteq S$ it holds: if $F \subseteq E(G)$ is a finite cut separating S' and ω with $|F \cap E(H)|$ minimal, then $|F \cap E(H)|$ is even.*

5 Proof of Theorem 4

The forward direction follows from Theorem 6, which ensures that every finite cut of G is even, and thus together with Corollary 19 implies the assertion.

For the backward direction suppose that $E(G) \notin \mathcal{C}(G)$. Observe that we may assume G to be connected, which means in particular that G is countable. We shall find a sequence $C_1 \supseteq C_2 \supseteq \ldots$ of regions of G that satisfy

(i) $\partial_G C_n$ is an odd cut, for $n \geq 1$;

(ii) $C_n \cup N(C_n) \subseteq C_{n-1}$, for $n \geq 2$; and

(iii) if D is a region of G with $C_{n-1} \supseteq D \supseteq C_n$ then $|\partial_G D| \geq |\partial_G C_{n-1}|$, for $n \geq 2$,

Then G has an odd end, contradicting the assumption, as desired. Indeed, by piecing together paths in the C_n, we see that there is a ray R which has a tail in every C_n. Let ω be the end with $R \in \omega$, and consider any finite $S \subseteq V(G)$. Choose I large enough such that $C_I \subseteq C(S, \omega)$, which is possible by (ii). So, every cut that separates $S' := N(C_I)$ from ω has cardinality at least $|\partial_G C_I|$, by (iii). Thus by (i) and Corollary 19, ω has odd degree.

For our construction, we need a further condition for $n \geq 1$. Let us call a region C of a graph H a *k-region* if $|\partial_H C| = k$.

(iv) for every k-region $D \subseteq C_n$ of G with $k < |\partial_G C_n|$ there is an $\ell \in \mathbb{N}$ and even regions $K_1, \ldots, K_\ell \subseteq C_n$ such that $|\partial_G K_i| \leq k$ for all i, and $V(D) \subseteq \bigcup_{i=1}^{\ell} V(K_i)$.

This condition, of course, is trivially satisfied if k is even.

As $E(G) \notin \mathcal{C}(G)$, Theorem 6 ensures the existence of odd regions in G. Choose any odd region C_1 such that $\partial_G C_1$ has minimal cardinality. This choice satisfies (i) and (iv), and that is all we required for $n = 1$.

Now, suppose the C_i to be defined for $i \leq n$. In order to find a suitable C_{n+1}, we shall contract certain even k-regions D of G (contained in C_n) for which $k < |\partial_G C_n|$. In the resulting minor, which has only big cuts, we will choose a small odd cut, which in G induces the desired region C_{n+1}.

We will construct this minor in several steps. More precisely, for each even integer $m < |\partial_G C_n|$ we define a minor G^m of $G =: G^0$, which will have the properties:

(a) G^m is obtained from G^{m-2} by contracting disjoint infinite m-regions K of G^{m-2} with $E(K) \subseteq E(C_n)$ for $m \geq 2$; and

(b) $|E(D) \cap E(G^m)| < \infty$ for every k-region D of G with $D \subseteq C_n$ and $k \leq m$.

Observe that, by (a) and as all vertices of G are even, all vertices of G^m have even degree too. We claim that (b) together with (iv) implies for $m < |\partial_G C_n| - 2$:

(c) every k-region D of G^m with $E(D) \subseteq E(C_n)$ and $k \leq m+1$ is finite.

Indeed, consider a k-region D of G^m with $E(D) \subseteq E(C_n)$ and $k \leq m+1$. By uncontracting, we obtain from D a region D' of G with $\partial_G D' = \partial_{G^m} D$ and $E(D) \subseteq E(D')$. Since by assumption $m < |\partial_G C_n| - 2$, we get that $k < |\partial_G C_n|$. Then (iv) implies that there is a finite set $\mathcal{K}$ of regions $K \subseteq C_n$ such that their union contains all vertices of D' (and thus also all but finitely many edges of D'). Each $K \in \mathcal{K}$ is an ℓ-region with even $\ell \leq k = m+1$. As $m+1$ is odd we get $\ell \leq m$, and hence by (b), that $E(K) \cap E(G^m)$ is finite. Thus $|E(D') \cap E(G^m)| < \infty$, and hence D is finite. This establishes (c).

As G is connected, $G^0 = G$ obviously satisfies (b), which is all we required for $m = 0$. So, assume $m \geq 2$, and G^i to be constructed for all even $i < m$. We define a sequence $(L_j)_{j \in \mathbb{N}}$ of (not necessarily induced) subgraphs of G^{m-2}; by contracting the components of their union L we obtain G^m.

Consider an enumeration $R_1, R_2, \ldots$ of all infinite m-regions of G^{m-2} with $E(R_i) \subseteq E(C_n)$ (such an enumeration is possible since $E(G^{m-2}) \subseteq E(G)$ is countable). Put $L_1 := R_1$, and let for $j > 1$,

$$L_j := L_{j-1} \cup R_j \ \ \textit{if} \ \ \partial_{G^{m-2}} R_j \cap E(L_{j-1}) = \emptyset \tag{4}$$

and $L_j := L_{j-1}$ otherwise. Note that in the former case each component of L_{j-1} is either contained in R_j or disjoint from R_j. Thus, by induction on j, every component K of L_j is an infinite m-region of G^{m-2}.

Put $L := \bigcup_{j \in \mathbb{N}} L_j$, and consider a component K of L. Certainly, K is an infinite induced subgraph in G^{m-2} with $E(K) \subseteq E(C_n)$. We claim that $k := |\partial_{G^{m-2}} K| = m$. Clearly, $k \leq m$ as otherwise there would already be a component $K' \subseteq K$ of some L_j with $|\partial_{G^{m-2}} K'| > m$, which is impossible. On the other hand, $k \geq m$, by (c) for $m-2$; thus $k = m$, as desired. We now obtain G^m from G^{m-2} by contracting the components of L to one vertex each (keeping multiple edges but deleting loops). Obviously, G^m satisfies (a).

Before we show the validity of (b), let us prove for all even $i < m < |\partial_G C_n|$ that it holds that:

($*$) for every region $D \subseteq C_n$ of G with $\partial_G D \subseteq E(G^i)$ there is a (possibly empty) induced subgraph $D' \subseteq C_n$ that satisfies

(I) there are finitely many regions $K_1, \dots, K_\ell$ of G each of which contracts to a vertex of degree $\leq i+2$ in G^{i+2} such that $V(D) \setminus V(D') \subseteq \bigcup_{j=1}^{\ell} V(K_j)$;

(II) if there is a region C of G with $D \subseteq C$ and $\partial_G C \subseteq E(G^{i+2})$, then also $D' \subseteq C$;

(III) $|\partial_G D'| \leq |\partial_G D|$; and

(IV) $\partial_G D' \subseteq E(G^{i+2})$.

Observe that D' has at most $|\partial_G D'|$ components (since G is connected). Each of these is a region of G with properties (II)–(IV).

Let us now show ($*$). Given D as above, choose an induced subgraph $\tilde{D} \subseteq C_n$ such that $|\partial_G \tilde{D} \setminus E(G^{i+2})|$ is minimal among all induced subgraphs that satisfy (I), (II), (III) and $\partial_G \tilde{D} \subseteq E(G^i)$ (which is possible as D itself has these four properties). If $|\partial_G \tilde{D} \setminus E(G^{i+2})| = 0$, we may put $D' := \tilde{D}$, so suppose otherwise. Then, by (a), there is an $(i+2)$-region K of G^i with $E(K) \subseteq E(C_n)$, which is contracted to a vertex in G^{i+2} and for which holds that $\partial_G \tilde{D} \cap E(K) \neq \emptyset$. Denote by $\tilde{D}^i$ the image of $\tilde{D}$ in G^i, ie. the induced subgraph of G^i with $\partial_{G^i} \tilde{D}^i = \partial_G \tilde{D}$ and $E(\tilde{D}) \cap E(G^i) = E(\tilde{D}^i)$.

Suppose that one of $|E_{G^i}(K \cap \tilde{D}^i, \tilde{D}^i \setminus K)|$, $|E_{G^i}(K \setminus \tilde{D}^i, G^i - (\tilde{D}^i \cup K))|$ is smaller than or equal to $|E_{G^i}(K \cap \tilde{D}^i, K \setminus \tilde{D}^i)|$. Then, putting either $\hat{D}^i = \tilde{D}^i \setminus K$ or $\hat{D}^i = G^i[\tilde{D}^i \cup K]$ we get

$$|\partial_{G^i} \hat{D}^i| \leq |\partial_{G^i} \tilde{D}^i| = |\partial_G \tilde{D}|. \tag{5}$$

Observe that $\partial_{G^i} \hat{D}^i \cap E(K) = \emptyset$, and denote by $\hat{D}$ the induced subgraph of G that we obtain from $\hat{D}^i$ by uncontracting. Then $\partial_G \hat{D} = \partial_G \hat{D}^i$ has fewer edges outside $E(G^{i+2})$ than $\partial_G \tilde{D}$. We claim that this contradicts the minimal choice of $\tilde{D}$. Indeed, $\partial_G \hat{D} \subseteq E(G^i)$, and also (I) and (III) hold for $\hat{D}$: the latter by (5), and for the former observe that each K_i either still contracts to a vertex of degree $\leq i+2$ in G^{i+2} or is contained in a region which does so. Adding K to these regions, we obtain the K_i as desired for (I).

To see that $\hat{D}$ satisfies (II), consider a region $C \supseteq D$ with $\partial_G C \subseteq E(G^{i+2})$. Observe that $\tilde{D} \subseteq C$ because $\tilde{D}$ satisfies (II). Now, since $\partial_G C \cap E(K) = \emptyset$, either $E(K) \subseteq E(C)$ or $E(K) \subseteq E(G - C)$ because K is connected. The latter case is impossible, as $\partial_G \tilde{D} \cap E(K) \neq \emptyset$. Hence, $E(K) \subseteq E(C)$, and thus, as $\hat{D}^i \subseteq G^i[\tilde{D}^i \cup K]$, we get $\hat{D} \subseteq C$, as desired. Note also that $\hat{D} \subseteq C_n$, as $\partial_G C_n \subseteq E(G^{i+2})$ by (a).

We may therefore assume that

$$|E_{G^i}(K \cap \tilde{D}^i, K \setminus \tilde{D}^i)| < |E_{G^i}(K \cap \tilde{D}^i, \tilde{D}^i \setminus K)|, |E_{G^i}(K \setminus \tilde{D}^i, G^i - (\tilde{D}^i \cup K))|,$$

and thus $|\partial_{G^i}(K \cap \tilde{D}^i)|, |\partial_{G^i}(K \setminus \tilde{D}^i)| < |\partial_{G^i} K|$. As K is infinite and as $K \cap \tilde{D}^i$ and $K \setminus \tilde{D}^i$ have only finitely many components (since G is connected), one of these components, say K', is infinite. Now, K' is a region of G^i with $\partial_{G^i} K' \subseteq \partial_{G^i}(K \cap \tilde{D}^i)$ or $\partial_{G^i} K' \subseteq \partial_{G^i}(K \setminus \tilde{D}^i)$. In both cases, $|\partial_{G^i} K'| < |\partial_{G^i} K| = i+2$. Because $E(K') \subseteq E(K) \subseteq E(C_n)$ and $i \leq m-2 < |\partial_G C_n| - 2$, this contradicts (c). We have thus shown ($*$).

Let us prove that G^m also satisfies (b). For this, consider a region $D \subseteq C_n$ of G with $|\partial_G D| \leq m$, and suppose that $E(D) \cap E(G^m)$ is infinite. Assume D to be chosen among all such regions such that i is maximal with $\partial_G D \subseteq E(G^i)$. Now, if $i < m$, then $(*)$ yields a subgraph D'. By (I), all but finitely many of the edges in $E(D) \cap E(G^m)$ lie in $E(D')$. Since D' has only finitely many components, there is one, C say, such that $E(C) \cap E(G^m)$ is infinite. Since, by (III), $|\partial_G C| \leq |\partial_G D|$, and since, by (IV), $\partial_G C \subseteq E(G^{i+2})$, we obtain a contradiction to the choice of D. Thus, we may assume that $i = m$, ie. $\partial_G D \subseteq E(G^m)$.

Therefore, by performing the according contractions we obtain from D an infinite region $\tilde{D}$ of G^{m-2} such that $\partial_{G^{m-2}} \tilde{D} = \partial_G D$ and $E(\tilde{D}) \subseteq E(C_n)$. Because of (c) for $m-2$ and because of $|\partial_G D| \leq m$, we get $|\partial_{G^{m-2}} \tilde{D}| = m$. Hence, the region $\tilde{D}$ appears in the enumeration $R_1, R_2, \ldots$ used in the construction of G^m, ie. there is a j with $\tilde{D} = R_j$. Since $\partial_{G^{m-2}} R_j \subseteq E(G^m)$ it follows from (4) that $E(R_j) \subseteq E(L_j) \subseteq E(L)$. Thus, in G^m all edges of $\tilde{D} = R_j$ are contracted, and consequently, $E(D) \cap E(G^m) = E(R_j) \cap E(G^m) = \emptyset$, a contradiction to $|E(D) \cap E(G^m)| = \infty$.

Having constructed G^m for all $m \leq M := |\partial_G C_n| - 1$, we finally find the region C_{n+1}. Observe that, by (a), $\partial_G C_n$ is a cut of G^M, and that the cut F of G^M that consists of those edges in $E(G^M) \cap E(C_n)$ that in G^M are adjacent to $\partial_G C_n$ has odd cardinality (because by (a), all vertices of G^M are even). Thus, since F is also a cut of G, there exists a region C of G with $\partial_G C \subseteq E(G^M)$ that satisfies (i) and (ii) for $n+1$.

We claim that

$$\text{\textit{any region C of G that for $n+1$ satisfies,} (i), (ii) \textit{and} $\partial_G C \subseteq E(G^M)$ \textit{also satisfies} (iii).} \tag{6}$$

Indeed, consider a k-region D of G with $C_n \supseteq D \supseteq C$. Then $E(C) \cap E(G^M)$ is infinite, by (i). (For this, contract all edges in $E(G^M) \setminus (E(C) \cup \partial_G C)$, and recall that a finite graph always has an even number of odd vertices.) From (b) for $m = M$ it follows that $k \geq M + 1 = |\partial_G C_n|$, as desired for (iii).

Now, choose C_{n+1} such that $\partial_G C_{n+1}$ has minimal (odd) cardinality among all regions satisfying (i), (ii) and (iii) for $n+1$.

To see (iv) for $n+1$, consider a k-region $D \subseteq C_{n+1}$ with $k < |\partial_G C_{n+1}|$. If $k \leq M$, then we can apply (iv) for n, so suppose $k > M$. Furthermore, we may assume that k is odd, as otherwise we can choose $\ell := 1$ and $K_1 := D$. Then D satisfies (i) and (ii) for $n+1$, and we should have chosen D as C_{n+1}, if not (iii) and thus, by (6), also $\partial_G D \subseteq E(G^M)$ fails for D. Repeated use of $(*)$, where we apply $(*)$ in each step to every component of the subgraph D' obtained in the previous step, yields a subgraph D^* of C_n such that each of its components $K_1, K_2, \ldots, K_\ell$ has properties (II)–(IV) for $m = M$. In particular, (II) implies for $1 \leq i \leq \ell$ that $K_i \subseteq C_{n+1}$. Let $\{K_{\ell+1}, \ldots, K_L\}$ be the set of all regions that arose as one of the K_i in one of the applications of $(*)$. Then $V(D) \subseteq \bigcup_{i=1}^{L} V(K_i)$, by (I).

By (III), $|\partial_G K_i| \leq k$ for $i = 1, \ldots, \ell$. Now, if there is an $j \in \{1, \ldots, \ell\}$ such that $|\partial_G K_j|$ is odd, then $K_j \subseteq C_{n+1}$ satisfies (i), (ii), and, by (IV) and (6), also (iii) for $n+1$, contradicting the choice of C_{n+1}. So, $|\partial_G K_i|$ is even and $\leq k$ for $i = 1, \ldots, L$ (for $i > \ell$ this follows from (I) and $k > M$). As $\partial_G C_{n+1} \subseteq E(G^M)$, and thus $\partial_G C_{n+1} \cap \bigcup_{i=\ell+1}^{L} E(K_i) = \emptyset$, and as $K_1, \ldots, K_\ell \subseteq C_{n+1}$, each of the

$K_1, \dots, K_L$ either lies completely in C_{n+1} or is disjoint from it. Together with $D \subseteq C_{n+1}$ this implies that $V(D) \subseteq \bigcup_{K \in \mathcal{K}} V(K)$ for $\mathcal{K} := \{K_i : K_i \subseteq C_{n+1}$ and $1 \leq i \leq L\}$, which proves (iv) for $n+1$. This completes the proof of the theorem.

6 Properties of the end degree

As an indication that the end degree indeed behaves as expected of a degree, we extend three basic properties of the vertex degree in finite graphs to end degrees in locally finite graphs. At the end of this section, however, we present two examples where end degrees differ from vertex degrees.

The number of odd vertices in a finite graph is always even. We prove the following easy analogon.

Proposition 20. *Let G be a locally finite graph. Then the number of odd vertices and ends in G is even or infinite.*

Proof. Suppose that the set $\mathcal{O}$ of odd vertices and ends has odd cardinality. Observe that there is a finite set $S \subseteq V(G)$ that contains all vertices of $\mathcal{O}$ and separates the ends in $\mathcal{O}$ pairwisely. By Corollary 19, there is for each end $\omega \in \mathcal{O}$ an odd region $A_\omega \subseteq G - S$ with $\omega \in \overline{A_\omega}$. Observe that the A_ω are pairwise disjoint. So, contracting each A_ω to a vertex a_ω we arrive at a graph G' that has an odd number of odd vertices, and in which all end degrees are even. Now, consider two copies of G' and add all edges vv', where v is an odd vertex of G' and v' its copy. The resulting graph has an odd cut, but no odd vertices or ends, a contradiction to Theorems 4 and 6. $\square$

Dirac [7] proved that if a finite graph has minimum vertex degree $k \geq 2$ then it contains a circuit of length $k+1$. This becomes false for infinite graphs: an easy counterexample is the k-regular infinite tree. But the tree ceases to be a counterexample if a minimum degree is also imposed on the ends, and indeed, then Dirac's result extends to locally finite graphs:

Theorem 21. *Let G be a locally finite graph, and let $H \subseteq G$ be a subgraph so that every vertex and every end $x \in \overline{H}$ has degree at least $k \geq 2$ in H. Then there is a circuit $C \subseteq E(H)$ of G with length $\geq k+1$.*

First, note that the theorem is best possible, even for infinite graphs. Indeed, consider disjoint copies $G_1, G_2, \dots$ of K^{k+1}. Identify a vertex in G_1 with a vertex in G_2. Then identify a different vertex in G_2 with a vertex of G_3 and so on. In the resulting graph the minimum vertex degree is k, and it is easy to see that the single end has degree k too, but there is no circuit of length greater than $k+1$. See Figure 4 for an example with $k = 2$.

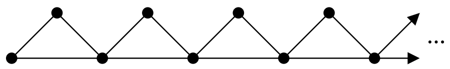

Figure 4: Theorem 21 is best possible for $k = 2$

Next, let us remark that the long circuit provided by the theorem may be infinite, and indeed the result becomes false if we require finite circuits. To see this, consider a k-regular tree H with root r. Let G be the graph obtained by adding an edge between any two vertices which the same distance to r. Then G has a single end, which has infinite degree in H, but H does not contain any finite circuits.

Proof of Theorem 21. First, observe that if H has an infinite block then H contains two disjoint rays that are equivalent in H (and thus also in G). By linking these by a path in H we obtain a double ray whose edge set is an infinite circuit of G.

Therefore, we may assume that every block of H is finite. Next, suppose that there is a block B of H that contains at most one vertex v with $d_B(v) < k$. Pick a longest path in B. One of the endvertices has at least k neighbours on that path, and hence there is a finite circuit of length $\geq k+1$ in B.

So, every block B of H is finite and contains at least two vertices of H with degree $< k$ in B, which then are cutvertices of H. Now, replace every block B of H by a tree $T \subseteq B$ whose leaves are exactly the cutvertices of H incident with B. Then every vertex of the resulting forest $H' \subseteq H$ has degree ≥ 2 as every block contains two cutvertices.

Assume that $E(H')$ does not contain infinite circuits, and let $v_1, v_2, \ldots$ be an enumeration of $V(H')$. We will inductively construct for $n \in \mathbb{N}$ homeomorphisms $\phi_n : [0,1] \to \overline{H'} \subseteq |G|$. Choosing b_0 as any vertex in H' and putting $A_n := \phi_n([0,1])$, we require that for $n \geq 1$ both $a_n := \phi_n(0)$ and $b_n := \phi_n(1)$ are vertices, and satisfy:

(i) $a_n = b_{n-1}$ for $n \geq 2$;

(ii) $A_m \cap A_n = \emptyset$ for $1 \leq m \leq n-2$ and $A_{n-1} \cap A_n = \{b_{n-1}\}$;

(iii) there is a cutvertex v incident with two blocks B, B' of H such that $d_B(v) < k$ and such that A_n contains two edges incident with v, one in $E(B)$ and the other in $E(B')$ (let us call any arc with that property *deficient*); and

(iv) if there is a topological path in $\overline{H'}$ from b_{n-1} to v_n that is edge-disjoint from $B_{n-1} := \bigcup_{i=1}^{n-1} A_i$, then $v_n \in A_n$.

Note that for $n \geq 1$, B_n is a topological path.

In order to construct ϕ_n, assume $\phi_1, \ldots, \phi_{n-1}$ to be defined already. First, suppose there is a topological path as required by (iv). By Lemma 5, either b_{n-1} and v_n are the endpoints of an arc A that is edge-disjoint from B_{n-1}, or $b_{n-1} = v_n$, in which case we put $A := \{v_n\}$. We claim that $A \cap B_{n-1} = \{b_{n-1}\}$. Indeed, otherwise let v be the vertex with $b_{n-1}v \subseteq A$. Then $A \cup B_{n-1}$ contains a topological path from v to b_{n-1} that avoids all inner points of $b_{n-1}v$, and hence, by Lemma 5, also a b_{n-1}–v arc A'. Thus, $A' \cup b_{n-1}v \subseteq A \cup B_{n-1} \subseteq \overline{H'}$ is a circle, contradicting our assumption.

We now lengthen A so that it also satisfies (iii). Because every vertex has degree ≥ 2 in H', and because $\overline{H'}$ does not contain any circles, v_n has a neighbour in $H' \setminus A \cup B_{n-1}$. Continuing in this way, we obtain a v_n–B path in H' that meets $A \cup B_{n-1}$ only in v_n, where B is a block of H which is adjacent to the block that contains v_n. As $B \cap \overline{H'}$ is connected and as $\overline{H'}$ does not contain any circles, B is disjoint from $A \cup B_{n-1}$. So, since B has a cutvertex b with

$d_B(b) < k$, there is a deficient path $P \subseteq H'$ that starts in v_n and is otherwise disjoint from $A \cup B_{n-1}$. Thus, we easily find a homeomorphism $\phi_n : [0,1] \to A \cup P$ which satisfies (i)–(iv).

So suppose there is no topological path as in (iv). Again we find a deficient path $P \subseteq H'$ starting in b_{n-1} which is disjoint from $B_{n-1} \setminus \{b_{n-1}\}$, and the respective homeomorphism $\phi_n : [0,1] \to P$ has properties (i)–(iv).

This process yields a set of arcs A_n, to which we apply Lemma 11. We obtain an $x \in |G|$, which is necessarily an end, such that $A^* := \bigcup_{n=1}^{\infty} A_n \cup \{x\}$ is a topological path from b_0 to x.

The end x has degree k in H, and hence there are k edge-disjoint arcs $R_1, \ldots, R_k \subseteq \overline{H}$ that start in x. Each of the R_i meets $A^* \setminus \{x\}$ in every neighbourhood of x. Indeed, suppose there is a neighbourhood U of x and an index j such that $R_j \cap U$ is disjoint from $A^* \setminus \{x\}$. Since R_j is continuous, there is a subarc of R_j which starts in x and is completely contained in U. Pick a vertex v_m on this subarc, and denote by R the subarc of R_j between x and v_m. Then $\bigcup_{n=m-1}^{\infty} A_n \cup R$ clearly is a topological path from b_{m-1} to v_m which is edge-disjoint from B_{m-1}, a contradiction to (iv) as $v_m \notin A^* \supseteq A_m$.

Let $\phi : [0,1] \to A^*$ be a continuous function with range A^* and $\phi(1) = x$. Choose an $s \in [0,1)$ such that each of the R_i hits A^* in a $\phi(r_i)$ with $r_i < s$. Because of (iii), we may assume that $v := \phi(s)$ is a cutvertex incident with two blocks B, B' of H such that $d_B(v) < k$ and such that A^* contains two edges incident with v, one in $E(B)$ and the other in $E(B')$. Not all of the k arcs R_i can go through the cut $F := E_H(v, B - v)$ of H, which has cardinality $d_B(v) < k$; so assume R_j does not contain any edge of F. Let uw be the (unique) edge in $E(A) \cap F$, and assume $\phi^{-1}(u) \leq \phi^{-1}(w)$. Then $(A \cup R_j) \setminus uw \cup \{u, w\}$ contains a topological path from w to u (simply run from w to x along A, then from x to $\phi(r_j)$ along R_j and finally from $\phi(r_j)$ to u along A). Therefore, there is also an arc $R \subseteq (A \cup R_j) \setminus uw \cup \{u, w\}$ with endpoints u and w, by Lemma 5. Consequently, $R \cup uw \subseteq \overline{H}$ is a circle. Since $E(R)$ is disjoint from F and every B–$(B' - v)$ path in H has to go through F, $|E(R)|$ is infinite. Thus, $E(R \cup uw) \subseteq E(H)$ is an infinite circuit, as desired. □

In a finite graph the cycles are exactly the connected 2-regular subgraphs. We extend this characterisation to locally finite graphs.

Theorem 3. *Let C be a subgraph of a locally finite graph G. Then $\overline{C}$ is a circle if and only if $\overline{C}$ is topologically connected and every vertex or end x of G with $x \in \overline{C}$ has degree two in C.*

We need a lemma:

Theorem 22 (Diestel and Kühn [4]). *When a graph G is locally finite, every closed connected subset of $|G|$ is path-connected.*

Proof of Theorem 3. If $\overline{C}$ is a circle, then it is clearly topologically connected and every vertex and every end $x \in \overline{C}$ has degree two in C.

For the converse direction, Theorem 21 implies that there is a circle $D \subseteq \overline{C}$. Suppose there exists a point $z \in \overline{C} \setminus D$. Theorem 22 yields an arc $A \subseteq \overline{C}$ that starts at z and ends in D. As both A and D are closed, A has a first point in D, ie. a point x such that the subarc A' of A between z and x meets D only in x. Thus, there are three edge-disjoint arcs in C with common endpoint x, two in D and the arc A'. So, x is either a vertex or an end and has degree at least 3 in C, a contradiction. Thus, $\overline{C} = D$. $\square$

Let us now turn to two areas in which end degrees differ in their behaviour from vertex degrees in finite graphs.

For a subgraph H of a graph G, deleting $E(H)$ reduces the degree of a vertex $v \in V(G)$ by its degree in H, ie. $d_G(v) = d_H(v) + d_{G-E(H)}(v)$. Although for an end ω it clearly holds that $d_G(\omega) \geq d_H(\omega) + d_{G-E(H)}(\omega)$, equality is in general not ensured. Consider the $4 \times \infty$-grid, which has a single end. As depicted in Figure 5, the removal of (the edge set of) a ray R leads to a decrease of the end degree from 4 to any of $3, 2, 1$ or 0, depending on how R is chosen. Similarly, deleting a circuit can lead to an odd decrease in the degree.

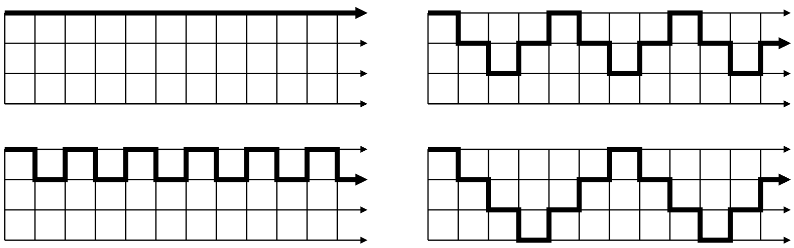

Figure 5: Removal of a ray lets the end degree decrease by 1 or more

The second area where the end degree differs in its behaviour concerns extremal results. A classical theorem by Mader [10], for instance, states that high average degree forces a finite graph to contain a large complete minor. This, however, fails for locally finite graphs even if every end has high degree. Figure 6

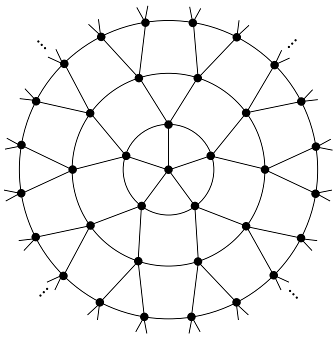

Figure 6: High degree in all vertices and in the single end but planar

indicates how for every $k \geq 5$ a planar k-regular graph with a single end of infinite degree can be constructed. Being planar, such a graph can never contain even a K^5 as a minor.

7 Weakly even ends

Finally, let us briefly discuss an alternative degree concept, which arises from the observation that (ii) of Lemma 7 is equivalent to:

(iii) for every finite $S \subseteq V(G)$ there is a finite set $S' \supseteq S$ of vertices such that the maximal number of edge-disjoint ω-rays starting in S' is even.

Lemma 7 (ii) was our main motivation for our definition of an even end. In the same vein, (iii) leads to the following alternative definition of parity, which differs only in that the quantifiers are exchanged:

Definition 23. *Let H be a subgraph of a locally finite graph G. Call ω* weakly even *in H if for every finite $S \subseteq V(G)$ there is a finite set $S' \supseteq S$ of vertices such that the maximal number of edge-disjoint ω-arcs in $\overline{H}$ starting in S' is even. Otherwise, ω is* strongly odd *in H.*

Observe that an even end is weakly even, and that a strongly odd end is odd. For ends of finite degree the two parity concepts are equivalent; this can be seen in a similar way as the equivalence of (ii) and (iii). For ends of infinite degree, however, this need not be true: consider a ray $v_1v_2\dots$, and replace each edge v_iv_{i+1} by i (subdivided) parallel edges. The obtained graph has a single end, which is both odd and weakly even.

This construction only works because there are odd vertices present. But could an odd end exist in a graph that has all vertices even and all ends weakly even? Or, on the contrary:

Problem 24. *Does Theorem 4 remain true if we substitute "even ends" by "weakly even ends"?*

We have been unable to settle the problem. However, we can answer both this question and Conjecture 10 positively for locally finite graphs with only countably many ends:

Proposition 25. *Let G be a locally finite graph with only countably many ends, and let H a subgraph. Then $E(H) \in \mathcal{C}(G)$ if and only if every vertex has even degree in H and if every end has weakly even degree in H.*

Proof. The forward direction follows immediately from Theorem 6 and Corollary 19. For the backward direction, suppose $E(H) \notin \mathcal{C}(G)$, which by Theorem 6 means that G has a finite cut F with $|F \cap E(H)|$ odd. Let $\omega_1, \omega_2, \dots$ be an enumeration of $\Omega(G)$. We successively define a sequence $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \dots$ of finite sets of disjoint regions $A \subseteq G - F$ of G with $|\partial A \cap E(H)|$ even and such that for each ω_i with $i \leq n$ there is an $A \in \mathcal{A}_n$ with $\omega_i \in \overline{A}$. Put $\mathcal{A}_0 := \emptyset$. In order to define the set $\mathcal{A}_n$ first check whether there is an $A \in \mathcal{A}_{n-1}$ such that $\omega_n \in \overline{A}$, in which case we put $\mathcal{A}_n := \mathcal{A}_{n-1}$. Otherwise consider the (finite) set S of all neighbours of each $A \in \mathcal{A}_{n-1}$ and of the endvertices of the edges in F. As ω_n is weakly even, Lemma 16 yields a region $B \subseteq G - F$ with $\omega_n \in \overline{B}$ and $A \cap B = \emptyset$ for all $A \in \mathcal{A}_{n-1}$. Put $\mathcal{A}_n := \mathcal{A}_{n-1} \cup \{B\}$. Finally, contracting all the disjoint regions $A \in \bigcup_{n=1}^{\infty} \mathcal{A}_n$ to a vertex each yields a finite graph with all vertex degrees even in H that has a cut F with $|F \cap E(H)|$ odd, a contradiction. $\square$

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