Mathematisches Seminar

Hamburger Beiträge
zur Mathematik
Heft 235

Adjacency preserving mappings of 2×2-Hermitian Matrices

Wen-ling Huang

Universität Hamburg

Adjacency preserving mappings of 2×2-Hermitian Matrices

Wen-ling Huang

2000 Mathematics Subject Classification. 15A99, 51D20.

Key words. Geometry of matrices, symmetric matrices, Hermitian matrices, mappings preserving adjacency, distance preserving mappings.

Abstract. We determine all mappings φ from the space of 2×2 Hermitian matrices which take pairs of adjacent matrices to pairs of adjacent matrices and such that the image of φ contains two matrices which are not adjacent.

1 Introduction

L.K. Hua initiated the study of the geometries of matrices [5, 6, 7, 8], namely the geometry of symmetric matrices, the geometry of Hermitian matrices, the geometry of rectangular matrices, and the geometry of alternate matrices. One aim is to characterize the transformation group of the geometry by the geometric invariant *adjacency*. In the fundamental theorems of the geometries, the bijections of the space of matrices which take adjacent matrices to adjacent matrices and vice versa, are determined.

It is a nature and important question posed in [25] whether in the geometry of matrices the bijective homomorphism (adjacency preserving in one direction) is already an isomorphism. This is answered affirmatively in the case of rectangular matrices [18] and in the symmetric and Hermitian case under some assumptions on the involution [15]. Recentally, W.-l. Huang and P. Šemrl determined all the adjacency preserving mappings in the geometry of $n \times n$ complex Hermitian matrices [17].

There is a close relation between the geometry of algebraic homogeneous spaces and the geometry of matrices [3, 12, 24]. In the Grassmann space, it is proved that any bijection which preserves adjacency in one direction preserves adjacency in both directions [11]. In the geometry of symplectic polarities any surjective mapping from the space to itself which preserves adjacency is then bijective and its inverse preserves adjacency [13]. This result has been generalized for projective near polygons which include the dual polar spaces [16]. Furthermore in the paper [14], all adjacency preserving mappings of the geometry of symplectic polarities have been determined under the milder assumption that for any maximal isotropic subspace there is a maximal isotropic subspace such that their images are disjoint.

In this paper we study 2×2 Hermitian matrices over any division ring $\mathcal{D}$, ch$\mathcal{D} \neq 2$, which possesses an involution $^-$ satisfies the assumption that the set of fixed elements F of $\mathcal{D}$ with respect to $^-$ must be field contained in the center of $\mathcal{D}$. This includes the case $F = \mathcal{D}$, i.e. the case of the 2×2 symmetric matrices. The fundamental theorem of the geometries of 2×2 Hermitian matrices over a field has been proved by Z.-X. Wan in [22]under the assumption that F is a proper sub field of $\mathcal{D}$. L.-P. Huang and Z.-X. Wan generalized this result to 2×2 Hermitian matrices over a quaternion division algebra in [26, 27].

We are able to weaken the assumptions in the fundamental theorem of the geometries of 2×2 symmetric and Hermitian matrices. we determine all mappings φ which preserve adjacency such that the image of φ contains at least two matrices which are not adjacent. It turns out that such a mapping need not be surjective if the underlying division ring admits a monomorphism which commutes with the involution and which is not surjective.

This paper is organized as follows. In section 2, we state the fundamental theorem of the geometry of symmetric and Hermitian matrices and our main result Theorem 2. In section 3, the definition of the projective space of Hermitian matrices is given. We then study some properties of this space. In section 4 we prove the main result. In section 5 we discuss applications to special relativity.

The key idea in the proof of our result is to extend the mapping φ to the projective space of Hermitian matrices. We show that this extended mapping preserves adjacency, and then we may apply known results on the projective space. The restriction of the extended mapping to the space of Hermitian matrices is a solution of our problem, whenever it takes Hermitian matrices to Hermitian matrices.

2 The Theorem

Let n be an integer ≥ 2, $\mathcal{D}$ a division ring which possesses an involution $^-$. An $n \times n$ matrix H over $\mathcal{D}$ is called *Hermitian* if $\overline{H}^t = H$. Denote by $\mathcal{H}$ the space of $n \times n$ Hermitian matrices over $\mathcal{D}$. When $^-$ is the identity and $\mathcal{D}$ is a field F, $\mathcal{H}$ is usually denoted by $\mathcal{S}_n(F)$, called the space of $n \times n$ symmetric matrices over F. Let $A, B \in \mathcal{H}$. A, B are said to be adjacent and we write $A \sim B$ if $\operatorname{rank}(A - B) = 1$. The Fundamental Theorem of the geometry of Hermitian matrices (and symmetric matrices) reads as follows.

Theorem 1. *Let $\mathcal{D}$ be a division ring which possesses an involution $^-$ and denote the set of fixed elements of $^-$ in $\mathcal{D}$ by F. If $^-$ is the identity map, assume that $\mathcal{D}$ is a field. If $^-$ is not the identity map, assume that F is a subfield of $\mathcal{D}$ and is contained in the center of $\mathcal{D}$. Let n be an integer ≥ 2. If*

$n = 2$, assume that $\mathcal{D}$ is a field. Then any bijective map φ from $\mathcal{H}$ to itself for which both the map φ and its inverse φ^{-1} preserve the adjacency in $\mathcal{H}$ is of the form

$$X^\varphi = \alpha P X^\sigma \overline{P}^t + H_0 \qquad \text{for all } X \in \mathcal{H}, \tag{1}$$

where $\alpha \in F^* := F \setminus \{0\}$, $P \in \mathrm{GL}_n(\mathcal{D})$, $H_0 \in \mathcal{H}$, and σ is an automorphism of $\mathcal{D}$ which commutes with $^-$, i.e., $\overline{a}^\sigma = \overline{a^\sigma}$ $\forall a \in \mathcal{D}$, unless $n = 3$ and $\mathcal{D} = \mathbb{F}_2$ and $^-$ is the identity map of $\mathbb{F}_2$. In this latter case, there is an extra bijective map ϵ of $\mathcal{S}_3(\mathbb{F}_2)$, and φ might also be the product of a map of the form (1) and ϵ. Conversely, any map of the form (1) or ϵ is bijective, and both the map and its inverse preserve the adjacency.

This theorem was proved by L.K. Hua, Z.-X. Wan et al., cf. [4, 5, 6, 7, 8, 9, 13, 20, 21, 22, 23].

From now on, we consider the case $n = 2$, $\mathcal{D}$ is a division ring with $\mathrm{ch}\mathcal{D} \neq 2$, F is the set of fixed elements of $^-$ in $\mathcal{D}$, and assume F is a subfield of $\mathcal{D}$ with $|F| > 5$ and is contained in the center of $\mathcal{D}$. The symmetric case is also included.

Theorem 2. *Let $\mathcal{D}$ be a division ring, $\mathrm{ch}\mathcal{D} \neq 2$, which possesses an involution $^-$ and assume the set of fixed elements of $^-$ in $\mathcal{D}$, denoted by F, is a subfield of $\mathcal{D}$ and contained in the center of $\mathcal{D}$, with $|F| > 5$. Let $\mathcal{H}_2(\mathcal{D})$ denote the space of 2×2 Hermitian matrices over $\mathcal{D}$. Let $\varphi : \mathcal{H}_2(\mathcal{D}) \to \mathcal{H}_2(\mathcal{D})$ be a mapping which satisfies*

1. *If X, Y are adjacent, then X^φ, Y^φ are adjacent, for all $X, Y \in \mathcal{H}_2(\mathcal{D})$.*
2. *There exist $P \neq Q$ such that $P^\varphi \neq Q^\varphi$ are not adjacent.*

Then

$$X^\varphi = (A + X^\sigma C)^{-1}(B + X^\sigma D) \quad \text{or} \quad X^\varphi = (A + \overline{X}^\sigma C)^{-1}(B + \overline{X}^\sigma D)$$

where A, B, C, D are 2×2 matrices over $\mathcal{D}$ with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \overline{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}^t = \alpha \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \alpha \in F \setminus \{0\}$$

and $\sigma : \mathcal{D} \to \mathcal{D}$ is a monomorphism with $\overline{X^\sigma} = \overline{X}^\sigma$.

Remark 1. If $\sigma : \mathcal{D} \to \mathcal{D}$ is an automorphism, then φ is bijective.

Corollary 1. *When $\mathcal{D} = \mathbb{C}$ is the field of complex numbers, and $-$ is the complex conjugate, then any mapping $\varphi : \mathcal{H}_2(\mathbb{C}) \to \mathcal{H}_2(\mathbb{C})$, which satisfies the conditions 1. and 2. in the Theorem 2, is of the form*

$$\begin{aligned} X^\varphi &= \lambda T X \overline{T}^t + H \quad \text{or} \\ X^\varphi &= \lambda T \overline{X}\, \overline{T}^t + H, \end{aligned}$$

where $\lambda \in \{\pm 1\}$, T is a 2×2 invertible complex matrix and H is a 2×2 Hermitian matrix.

Proof. id and $-$ are the only monomorphisms of $\mathbb{C}$ which commute with $-$, this implies that the matrix $C = 0$ in the Theorem 2. $\square$

Corollary 2. *When $\mathcal{D} = \mathbb{R}$ is the field of real numbers, and $-$ is the identity, then any mapping $\varphi : \mathcal{S}_2(\mathbb{R}) \to \mathcal{S}_2(\mathbb{R})$, which satisfies the conditions 1. and 2. in the Theorem 2, is of the form*

$$X^\varphi = \lambda T X T^t + S$$

where $\lambda \in \{\pm 1\}$, T is a 2×2 real invertible matrix and S is a 2×2 real symmetric matrix.

3 The space $\mathcal{PH}_2(\mathcal{D})$

Let $\mathcal{D}$ be a division ring possessing an involution $^-$, ch$\mathcal{D} \neq 2$. Denote F be the set of fixed elements of $^-$ in $\mathcal{D}$ and assume that F is a subfield of $\mathcal{D}$ and is contained in the center of $\mathcal{D}$. Let

$$K = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is the 2×2 identity matrix.

Let $\perp$ be a polarity defined on $\mathcal{D}^4$ by:

$$x^\perp = \left\{ y \mid (y_1, ..., y_4) K (\overline{x}_1, \ldots, \overline{x}_4)^t = 0 \right\}.$$

Let W be a k-dimensional subspace of $\mathcal{D}^4$, $1 \leq k \leq 4$. Let $(w_1, \ldots, w_k)$ be any basis of W. Then we call the $k \times 4$ matrix

$$\begin{pmatrix} w_{11} & \ldots & w_{14} \\ \vdots & & \vdots \\ w_{k_1} & \ldots & w_{k4} \end{pmatrix} \quad \text{where } w_i = (w_{i1}, \ldots, w_{i4})$$

a *matrix representation* of W. We write a matrix representation of two dimensional subspaces W of $\mathcal{D}^4$ in the block form

$$(X \;\; Y),$$

where both X and Y are 2×2 matrices. Then W is totally isotropic if and only if

$$(X \;\; Y) \, K \left(\overline{X} \;\; \overline{Y} \right)^t = 0$$

if and only if

$$X \overline{Y}^t = Y \overline{X}^t. \tag{2}$$

The condition (2) is independent of the choice of the matrix representation. We call $(X \;\; Y)$ a *homogeneous coordinate* of W. If rank $X = 2$, then

$(I\ \ X^{-1}Y)$ is also a homogeneous coordinate of W, and $X^{-1}Y$ is a 2×2 Hermitian matrix. Conversely, for every 2×2 Hermitian matrix H, $(I\ \ H)$ is a homogeneous coordinate of a two-dimensional totally isotropic subspace. We now define a point-line geometry $\mathcal{PH}_2(\mathcal{D})=(\mathcal{P},\mathcal{L})$, where the point set $\mathcal{P}$ consists of all the two-dimensional totally isotropic subspaces W. W is called a *finite point*, if rank $X=2$ for any homogeneous coordinate $(X\ \ Y)$ of W, and W is called *a point at infinity* if rank $X\leq 2$. Then the set of 2×2 Hermitian matrices is the set of finite points of the geometry $\mathcal{PH}_2(\mathcal{D})$.

Definition 1 (see [24], Definition 5.7, 6.12). The *arithmetic distance* between two points W_1 and W_2 is defined by

$$\mathrm{ad}(W_1,W_2)=\dim(W_1+W_2)-2.$$

Definition 2 (see [24], Definition 5.7, 6.12). Two points W_1,W_2 are called *adjacent* if $\mathrm{ad}(W_1,W_2)=1$. A *line* of the geometry $\mathcal{PH}_2(\mathcal{D})$ is defined as a maximal set of pair wise adjacent points. It is a set of two-dimensional totally isotropic subspaces which contain a common one-dimensional totally isotropic subspace.

For any two points $W_1,W_2\in\mathcal{PH}_2(\mathcal{D})$ with homogeneous coordinates $(X_1\ \ Y_1)$, $(X_2\ \ Y_2)$, it is proved in [24], Proposition 5.44, 5.48, 6.43 and 6.47, that

$$\mathrm{ad}(W_1,W_2)=\mathrm{rank}\big(\,(X_1\ \ Y_1)\ K\ \big(\overline{X}_2\ \ \overline{Y}_2\big)^t\,\big)=d(W_1,W_2)$$

where $d(W_1,W_2)$ is the graph theoretic distance between $W_1,W_2\in\mathcal{PH}_2(\mathcal{D})$.

Let H_1,H_2 be two Hermitian matrices over $\mathcal{D}$. Then

$$\mathrm{rank}\big(\,(I\ \ H_1)\ K\ \big(I\ \ \overline{H}_2\big)^t\,\big)=\mathrm{rank}(H_1-H_2)=d(H_1,H_2).$$

We have that the definition of the distance on $\mathcal{PH}_2(\mathcal{D})$, restricted to $\mathcal{H}_2(\mathcal{D})$, coincides with the distance on $\mathcal{H}_2(\mathcal{D})$.

Property 1. Two adjacent points $W_1,W_2\in\mathcal{PH}_2(\mathcal{D})$ determine a unique line

$$\overline{W_1W_2}:=\{W=W^{\perp}\mid W<W_1+W_2\}=\{W=W^{\perp}\mid W>W_1\cap W_2\}.$$

For any pair wise adjacent points $W_1,W_2,W_3\in\mathcal{PH}_2(\mathcal{D})$, it holds $\overline{W_1W_2}=\overline{W_2W_3}=\overline{W_1W_3}$, then there is no "triangle" in $\mathcal{PH}_2(\mathcal{D})$.

Define

$$\infty:=(0\ \ I)\,.$$

Property 2. For any point $W=(X\ \ Y)\in\mathcal{PH}_2(\mathcal{D})$ we have $d(W,\infty)=2$ if, and only if, W is a finite point, since $2=d(W,\infty)=\mathrm{rank}\big(\,(X\ \ Y)\,K\overline{(0\ \ I)}^{t}\big)=\mathrm{rank}X$.

Define $D_1(\infty)$ be the set of points which are adjacent to ∞, and $D_2(\infty)$ be the set of points at distance 2 from (∞), by property 2, $D_2(\infty)$ is the set of finite points.

Property 3. For any point W and a line $l = \overline{W_1W_2}$ with $W \notin l$,

$$P := \big(W \cap (W_1 + W_2)\big) + (W_1 \cap W_2)$$

is a point of l and $d(W, P) = 1$, $d(W, Q) = 2$ for all $Q \in l \setminus \{P\}$.

Property 4. There are two types of lines l.

1. $\infty \notin l$. Then l contains exactly one point at infinity.
2. $\infty \in l$. Then l contains only points at infinity.

Remark 2. Identify the line $l = \overline{W_1W_2}$ with the one dimensional isotropic subspace $W_1 \cap W_2$, then $\mathcal{PH}_2(\mathcal{D})$ is an antihermitian dual polar space of rank 2, and then a generalized quadrangle (see e.g. [2]), i.e., the following three axioms are satisfied:

(Q1) Two points are contained in at most one line.

(Q2) If l is a line, and $P \notin l$ is a point, then there is a unique point of L which is adjacent to P.

(Q3) No point is adjacent to all others.

The following properties of the space $\mathcal{PH}_2(\mathcal{D})$ are well known in classical generalized quadrangles. However we give some proofs here for reader's convenience.

In $\mathcal{PH}_2(\mathcal{D})$ we have the following closure theorem:

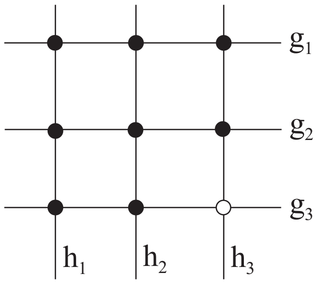
(C) Let g_1, g_2, g_3 and h_1, h_2 be lines with

$$\begin{aligned} g_i \cap g_j &= \emptyset \qquad \forall i, j = 1, 2, 3,\ i \neq j, \\ g_i \cap h_j &\neq \emptyset \qquad \forall i = 1, 2, 3,\ j = 1, 2, \\ h_1 \cap h_2 &= \emptyset. \end{aligned}$$

Then for any line h_3,

$$\begin{aligned} h_i \cap h_3 &= \emptyset \qquad \forall i = 1, 2, \\ g_i \cap h_3 &\neq \emptyset \qquad \forall i = 1, 2, \end{aligned}$$

implies $h_3 \cap g_3 \neq \emptyset$.



Proof. Any line l is a set of two-dimensional totally isotropic subspaces which contain a common one-dimensional totally isotropic subspace. Denote this one-dimensional subspace by $[l]$. For any two distinct lines g, h,

$$g \cap h \neq \emptyset \quad \Leftrightarrow \quad [g] + [h] = g \cap h \quad \Leftrightarrow \quad [g] < [h]^{\perp} \quad \Leftrightarrow \quad [h] < [g]^{\perp}.$$

Since h_1, h_2 intersect g_1, g_2, g_3, we have $[g_i] < [h_j]^{\perp}$, $i = 1,2,3$, $j = 1,2$. This implies $[g_1] + [g_2] + [g_3] < [h_j]^{\perp}$, $j = 1,2$. Then $[g_1] + [g_2] + [g_3]$ is two-dimensional, and $[g_3] < [g_1] + [g_2]$. Since h_3 intersects g_1 and g_2, thus $[g_1] + [g_2] < [h_3]^{\perp}$ and $[g_3] < [g_1] + [g_2] < [h_3]^{\perp}$, which implies $g_3 \cap h_3 \neq \emptyset$. $\square$

Definition 3. A *regulus* is a configuration which consists of two families $\{g_i \mid i \in I\}$, $\{h_i \mid i \in I\}$ of lines with the following properties:

- $g_i \cap g_j = \emptyset$, $h_i \cap h_j = \emptyset$ for all $i, j \in I$, $i \neq j$.
- $g_i \cap h_j \neq \emptyset$ for all $i, j \in I$.
- For all points $X \in g_i$ there exists a unique h_j with $X \in h_j$.
  For all points $Y \in h_i$ there exists a unique g_j with $Y \in g_j$.

Property 5. For any two distinct lines g, h there exists a regulus Σ containing g and h. If $g \cap h = \emptyset$, then there is a unique regulus Σ containing g_1, g_2.

Property 6. Let X, Y be two points contained in a regulus Σ. Let g, h, l be lines in Σ with $X \in g$, $Y \in h$, $g \cap l = \emptyset$, $h \cap l = \emptyset$ and $g \cap h = \emptyset$. Then there are lines $g' \neq g$ and $h' \neq h$ with $g \cap g' = X$, $h \cap h' = Y$ and $l \cap g' = \emptyset$, $l \cap h' = \emptyset$ such that l, g', h' are contained in a regulus Σ' if, and only if, X, Y are adjacent.

Proof. Let X, Y and l, g, g', h, h' and Σ, Σ' be points, lines and reguli as described above. Denote by P the point in l which is adjacent to X. Then the line $\overline{XP}$ is contained in Σ and therefore intersects h. The line $\overline{XP}$ is also contained in Σ' and therefore intersects h'. Since there is no "triangle" in $\mathcal{PH}_2(\mathcal{D})$, we have $Y = h \cap h' \in \overline{XP}$, i.e. X, Y are adjacent.

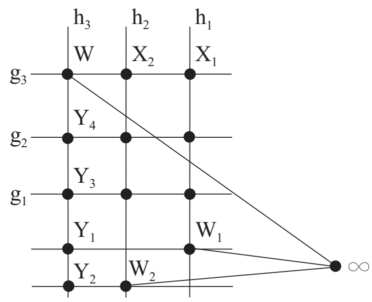
Now let X, Y and l, g, h and Σ be points, lines and a regulus as above, and let $X \neq Y$ be adjacent. Choose $P \in l$ such that X, P are not adjacent, then P, Y are not adjacent. Let g' be any line with $g' \neq g$, $g' \ni X$, $g' \cap l = \emptyset$. Then there is a uniquely determined point $Q \in g'$ adjacent to P. Since P, Y are not adjacent, there is a uniquely determined point $R \in \overline{PQ}$ adjacent to Y. Define $h' := \overline{YR}$. $\square$

Property 7. There are two types of reguli Σ:

1. If $\infty \in \Sigma$, then there are two lines $g, h \in \Sigma$ with $g \cap h = \infty$. The points of $g \cup h$ are the infinite points of Σ, and the other points of Σ are the finite points of Σ.

2. If $\infty \notin \Sigma$, then any line $l \in \Sigma$ contains exactly one point at infinity.

Property 8. Let $W \neq \infty$ be a point at infinity and let g_3 and h_3 be two distinct lines which intersect in W and which do not contain ∞. Then there are distinct lines g_1, g_2, h_1, h_2 such that g_i, h_j intersect in finite points for all $i, j = 1, 2, 3$ with $(i, j) \neq (3, 3)$.



Proof. Let X_1, X_2 be two distinct finite points of g_3. Choose any line $h_1 \neq g_3$ which contains X_1, then $h_1 \cap h_3 = \emptyset$. Choose a line $h_2 \ni X_2$ such that h_1, h_2, h_3, g_3 are contained in a regulus Σ. Let W_i be the unique infinite points satisfying $W_i \in h_i$, $i = 1, 2$. Denote by Y_i the point in h_3 which is adjacent to W_i, $i = 1, 2$. Since $|F| > 5$, then we have more than four finite points on h_3. Choose finite points $Y_3, Y_4 \in h_3 \setminus \{Y_1, Y_2, W\}$ and define g_1, g_2 $\neq h_3$ to be the unique lines in the regulus Σ which contain Y_3, Y_4, respectively. Then we have that $g_i \cap h_j$, $i, j \in \{1, 2\}$, are finite points. $\square$

Lemma 1. *Let φ be an injective map from $\mathcal{PH}_2(\mathcal{D})$ to itself, which preserves the distance. Then there are a monomorphism $\sigma : \mathcal{D} \to \mathcal{D}$ with $\overline{\sigma} = {}^{-\sigma}$ and a matrix $T \in GL_4(\mathcal{D})$ with $TK\overline{T}^t = \alpha K$, where $\alpha = \overline{\alpha}$, $\alpha \neq 0$, such that*

$$W^\varphi = W^\sigma T \quad \text{or} \quad W^\varphi = \overline{W}^\sigma T.$$

Proof. Since φ preserves the distance, φ is an isomorphism between $(\mathcal{P}, \mathcal{L}) = \mathcal{PH}_2(\mathcal{D})$ and $\varphi(\mathcal{PH}_2(\mathcal{D}))) = (\varphi(\mathcal{P}), \varphi(\mathcal{L}))$. Here the set of point of $\varphi(\mathcal{PH}_2(\mathcal{D}))$ is

$$\varphi(\mathcal{P}) := \{\varphi(P) \mid P \in \mathcal{PH}_2(\mathcal{D})\}$$

and the set of lines of $\varphi(\mathcal{PH}_2(\mathcal{D}))$ is

$$\varphi(\mathcal{L}) := \{\varphi(l) \mid l \in \mathcal{L}\},$$

where $\varphi(l) \in \mathcal{L}$ is the unique line which contains $\{\varphi(P) \mid P \in l\}$. Identify the elements of $\mathcal{L}$ with the one dimensional isotropic subspace (see remark 2). The dual of $\mathcal{PH}_2(\mathcal{D})$, $\mathcal{PH}_2(\mathcal{D})^* := (\mathcal{L}, \mathcal{P})$ is an antihermitian polar space over $PG(3, \mathcal{D})$ with respect to the polarity $\perp$. Then also $\varphi(\mathcal{PH}_2(\mathcal{D})^*)$ is an antihermitian polar space and there exists a subdivision ring E of $\mathcal{D}$ and an involution $\tilde{}$ on E, such that $\varphi(\mathcal{PH}_2(\mathcal{D})^*)$ is up to an isometry $\psi : \mathcal{PH}_2(\mathcal{D})^* \to \mathcal{PH}_2(\mathcal{D})^*$ the antihermitian polar space over $PG(3, E)$ associated with the polarity

$$x^{\perp'} = \left\{ y \mid (y_1, ..., y_4) K (\tilde{x}_1, \ldots, \tilde{x}_4)^t = 0 \right\}.$$

Since the isotropic subspaces of $PG(3, E)$ with respect to $\perp'$ are also isotropic subspaces of $PG(3, \mathcal{D})$ with respect to $\perp$,

$$\left(I \;\middle|\; \begin{matrix} 0 & x \\ -\tilde{x} & 0 \end{matrix} \right)$$

is in $\mathcal{PH}_2(E) \subset \mathcal{PH}_2(\mathcal{D})$. Then hermitian matrix

$$\begin{pmatrix} 0 & x \\ -\tilde{x} & 0 \end{pmatrix}$$

over $(E, \tilde{})$ is hermitian over $(\mathcal{D}, \bar{})$ for all $x \in E$, and $\tilde{} = \bar{}|E$. By [19, Theorem 8.6], $\psi \circ \varphi$ is induced by a semi-linear bijection from $PG(3, \mathcal{D}) \to PG(3, E)$, $(W^{\varphi})^{\psi} = W^{\sigma'} T'$, or $(W^{\varphi})^{\psi} = \overline{W}^{\sigma'} T'$, where $T' \in GL_4(E)$ satisfies $T' K \overline{T'}^t = \alpha' K$, $\alpha' = \overline{\alpha'}$, $\alpha' \neq 0$, and σ' is an isomorphism from $\mathcal{D}$ to E with $\overline{\sigma'} = {}^{-\sigma'}$. Compose the above transformations with the inverse of ψ(use [19, Theorem 8.6] again) , we obtain the lemma. $\square$

4 Proof of Theorem 2

Let $\varphi : \mathcal{H}_2(\mathcal{D}) \to \mathcal{H}_2(\mathcal{D})$ with

$$X, Y \text{ are adjacent} \quad \Rightarrow \quad X^{\varphi}, Y^{\varphi} \text{ are adjacent}$$

for all $X, Y \in \mathcal{H}_2(\mathcal{D})$. Let there exist $P \neq Q$ such that $P^{\varphi} \neq Q^{\varphi}$ are not adjacent.

Step 1. We have $d(P, X) = d(P^{\varphi}, X^{\varphi})$ for all X in $\mathcal{H}_2(\mathcal{D})$.

Proof. In the case $d(P, X) = 0, 1$ we have $d(P, X) = d(P^{\varphi}, X^{\varphi})$, since φ preserves adjacency. Now let $d(P, X) = 2$. In [15], Lemma 2.5 and Lemma 2.6, it states for $|F|$ is infinity, there exists a sequence of consecutively adjacent points $Q = Q_0, ..., Q_k = X$, $Q_i \in \mathcal{H}_2(\mathcal{D})$ and a sequence of points $R_1, ..., R_k$, $R_i \in \mathcal{H}_2(\mathcal{D})$ such that $d(P, Q_i) = 2$ for all $i = 0, 1, ..., k$, $d(P, R_i) = 1$ for all $i = 1, ..., k$ and Q_i, R_{i+1}, Q_{i+1} are adjacent for all $i = 0, ..., k-1$. The proof remains valid if $|F| > 5$. Therefore such sequences $(Q_i), (R_i)$ described above exist in our case. Since $d(P, Q_0) = 2 = d(P^\varphi, {Q_0}^\varphi)$, $d(P, R_1) = 1 = d(P^\varphi, {R_1}^\varphi)$, and Q_0, R_1, Q_1 are adjacent, we have $d(P^\varphi, {Q_1}^\varphi) = 2$. Analogously, $d(P^\varphi, {Q_2}^\varphi) = 2$,..., $d(P^\varphi, {Q_k}^\varphi) = 2$, i.e. $d(P^\varphi, X^\varphi) = 2$. □

Step 2. φ is injective, and $d(X, Y) = d(X^\varphi, Y^\varphi)$ for all X, Y in $\mathcal{H}_2(\mathcal{D})$.

Proof. We consider the case $X, Y \in \mathcal{H}_2(\mathcal{D})$ with $d(X, Y) = 2$. Let Z be a point in $\mathcal{H}_2(\mathcal{D})$ with $d(P, Z) = 2 = d(Z, X)$. Then by step 1. we have

$$d(P^\varphi, Z^\varphi) = 2 = d(Z^\varphi, X^\varphi) = d(X^\varphi, Y^\varphi).$$

The injectivity of φ is then obvious. □

In the following, we will identify $\mathcal{H}_2(\mathcal{D})$ and the finite points of $\mathcal{PH}_2(\mathcal{D})$. For any line g of $\mathcal{PH}_2(\mathcal{D})$, $\infty \notin g$, g contains a unique point at infinite and the other points of g are finite points. Define g^φ be the maximal set of adjacent points of $\mathcal{PH}_2(\mathcal{D})$ containing the image of all finite points of g under φ, then g^φ is a line of $\mathcal{PH}_2(\mathcal{D})$ and for all $\infty \notin h \neq g$, $g^\varphi \neq h^\varphi$, by step 2.

Step 3. Let g, h be two lines in $\mathcal{PH}_2(\mathcal{D})$, $\infty \notin g, h$. Then $g \cap h = \emptyset$, if and only if, $g^\varphi \cap h^\varphi = \emptyset$.

Proof. If $g \cap h = \emptyset$, then there exist two finite points $X \in g$ and $Y \in h$ such that X and Y are adjacent. If $g \cap h \neq \emptyset$, then for any $X \in g$ and $Y \in h$, $X, Y \neq g \cap h$, we have that X and Y are not adjacent. Thus by step 2, $g \cap h = \emptyset$ implies $g^\varphi \cap h^\varphi = \emptyset$. Now let $g \cap h \neq \emptyset$. If $g \cap h$ is a finite point, then $g^\varphi \cap h^\varphi$ is a finite point. If $g \cap h =: W$ is a point at infinity, we choose lines $g_1, g_2, g_3 := g, h_1, h_2, h_3 := h$ as described in Property 4. Let X_{ij} be the intersection of g_i and h_j for all $i, j = 1, 2, 3$ with $(i, j) \neq (3, 3)$. X_{ij} are finite points, so X_{ij}^φ are defined. By step 2. and the closure theorem (C) we have that the lines $\overline{X_{13}^\varphi X_{23}^\varphi} = g^\varphi$ and $\overline{X_{31}^\varphi X_{32}^\varphi} = h^\varphi$ intersect in a point. □

By step 3. we can extend φ to $\mathcal{PH}_2(\mathcal{D}) \setminus \{\infty\}$. Let $W \neq \infty$ be a point at infinity. Choose any two lines g, h with $W = g \cap h$ and $\infty \notin g, h$. Define $W^\varphi := g^\varphi \cap h^\varphi$. W^φ is well-defined since there is no triangle in $\mathcal{PH}_2(\mathcal{D})$.

Step 4. If $X, Y \in \mathcal{PH}_2(\mathcal{D}) \setminus \{\infty\}$ are adjacent, then X^φ, Y^φ are adjacent.

Proof. Let Σ be a regulus which contains X, Y, l, g, h, g', h' be lines not containing ∞ as in property 6. Then $l^\varphi, g^\varphi, h^\varphi$ are contained in a regulus, and $l^\varphi, g'^\varphi, h'^\varphi$ are contained in a regulus. So $X^\varphi = g^\varphi \cap g'^\varphi$ and $Y^\varphi = h^\varphi \cap h'^\varphi$ are adjacent by property 6. □

After step 4, we are able to extend the definition of g^φ to any line g of $\mathcal{PH}_2(\mathcal{D})$ as the maximal set of adjacent points of $\mathcal{PH}_2(\mathcal{D})$ containing the image of all points of $g \setminus \{\infty\}$ under φ.

Step 5. If $X, Y \in \mathcal{PH}_2(\mathcal{D}) \setminus \{\infty\}$ are not adjacent, then X^φ, Y^φ are not adjacent.

Proof. We only check the case that X is a point at infinity. Choose any line g with $X \in g$ and $\infty \notin g$. There is a unique point $Z \in g$ adjacent to Y. Since X, Y are not adjacent and $\infty \notin g$, Z is a finite point. Let h be the line contains Y and Z, then $\infty \notin h$ and $g \cap h \neq \emptyset$. Thus by step 3, $g^\varphi \cap h^\varphi \neq \emptyset$ and $g^\varphi \neq h^\varphi$. This implies that X^φ, Y^φ are not adjacent. $\square$

Step 6. Choose any lines g, h with $g \cap h = \infty$. Extend g, h to a regulus Σ. Then g^φ, h^φ are contained in a regulus, and $|g^\varphi \cap h^\varphi| = 1$. $\infty^\varphi := g^\varphi \cap h^\varphi$ is well-defined.
Step 7. We have extended φ to the whole $\mathcal{PH}_2(\mathcal{D})$. From the construction of φ we have that $d(X, Y) = d(X^\varphi, Y^\varphi)$ for all $X, Y \in \mathcal{PH}_2(\mathcal{D})$. By Lemma 1 we have

$$W^\varphi = W^\sigma \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{or} \quad W^\varphi = \overline{W}^\sigma \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $\sigma : \mathcal{D} \to \mathcal{D}$ is an monomorphism with $\overline{X^\sigma} = \overline{X}^\sigma$, and A, B, C, D are 2×2 matrices over $\mathcal{D}$ with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} K \overline{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}^t = \alpha K, \qquad \alpha \in F \setminus \{0\}.$$

We look at the restriction of φ to $\mathcal{H}_2(\mathcal{D})$. In the first case we have

$$\begin{aligned} \varphi\big((I \;\; X)\big) &= (I \;\; X)^\sigma \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (I \;\; X^\sigma) \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ &= (A + X^\sigma C \;\; B + X^\sigma D) \in \mathcal{H}_2(\mathcal{D}). \end{aligned}$$

So $A + X^\sigma C$ is invertible for all $X \in \mathcal{H}_2(\mathcal{D})$, and

$$X^\varphi = (A + X^\sigma C)^{-1}(B + X^\sigma D).$$

In the second case we obtain

$$X^\varphi = (A + \overline{X}^\sigma C)^{-1}(B + \overline{X}^\sigma D).$$

5 Application to special relativity

The space of complex 2×2 Hermitian matrices can be identified with four-dimensional Lorentz-Minkowski spacetime, see e.g. [10, Chapter 5.3]. The Hermitian matrix X corresponding to the point $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ is

$$X = \begin{pmatrix} x_0 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 - x_1 \end{pmatrix}.$$

Two Hermitian matrices are adjacent if, and only if, the corresponding points $x = (x_0, x_1, x_2, x_3)$ and $y = (y_0, y_1, y_2, y_3)$ are distinct and have Lorentz-Minkowski distance 0,

$$x \neq y \quad \text{and} \quad (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 - (x_0 - y_0)^2 = 0, \tag{3}$$

i.e., when they are joined by a lightlike line. Thus, if a mapping $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ preserves the relation (3), then either the image of φ is contained in a lightlike line, or, by Corollary 1, φ is the product of a Lorentz transformation and a dilatation. This result generalizes Alexandrov's theorem [1] in the four-dimensional case, where all bijections φ are determined such that φ and φ^{-1} preserve Lorentz-Minkowski distance 0.

Similarly, the space of 2×2 Hermitian matrices over the quaternions can be identified with six-dimensional Lorentz-Minkowski spacetime. The Hermitian matrix X corresponding to the point $(x_0, \ldots, x_5) \in \mathbb{R}^6$ is

$$X = \begin{pmatrix} x_0 + x_1 & x_2 + ix_3 + jx_4 + kx_5 \\ x_2 - ix_3 - jx_4 - kx_5 & x_0 - x_1 \end{pmatrix},$$

and we obtain that, if a mapping $\varphi : \mathbb{R}^6 \to \mathbb{R}^6$ preserves the relation

$$x \neq y \quad \text{and} \quad (x_1 - y_1)^2 + \ldots + (x_5 - y_5)^2 - (x_0 - y_0)^2 = 0$$

for all $x = (x_0, \ldots, x_5)$, $y = (y_0, \ldots, y_5) \in \mathbb{R}^6$, then either the image of φ is contained in a lightlike line, or φ is the product of a Lorentz transformation and a dilatation.

Acknowledgments

The author would like to thank H. van Maldeghem and P. Šemrl for valuable discussions.

References

[1] A. D. Alexandrov. Seminar report. *Uspekhi Mat. Nauk*, 37(3):187, 1950.

[2] F. Buekenhout, editor. *Handbook of Incidence Geometry*. Elsevier Science B.V., 1995.

[3] W.-L. Chow. On the geometry of algebraic homogeneous spaces. *Ann. Math.*, 50(1):32–67, 1949.

[4] J.-f. Gao, Z.-X. Wan, R.-q. Fenq, and D.-j. Wang. Geometry of symmetric matrices and its applications III. *Algebra Colloquium*, 3:135–146, 1996.

[5] L.K. Hua. Geometries of matrices I. Generalizations of van Staudt's theorem. *Trans. Amer. Math. Soc.*, 57:441–481, 1945.

[6] L.K. Hua. Geometries of matrices I_1. Arithmetical construction. *Trans. Amer. Math. Soc.*, 57:482–490, 1945.

[7] L.K. Hua. Geometries of matrices II. Study of involutions in the geometry of symmetric matrices. *Trans. Amer. Math. Soc.*, 61:193–228, 1947.

[8] L.K. Hua. Geometries of matrices III. Fundamental theorems in the geometries of symmetric matrices. *Trans. Amer. Math. Soc.*, 61:229–255, 1947.

[9] L.K. Hua. Geometries of symmetric matrices over any field with characteristic other than two. *Ann. Math.*, 50:8–31, 1949.

[10] L.K. Hua. *Starting with the Unit Circle*. Springer-Verlag, New York, Heidelberg, Berlin, 1981.

[11] W.-l. Huang. Adjacency preserving mappings of Grassmann spaces. *Abh. Math. Sem. Univ. Hamburg*, 68:65–77, 1998.

[12] W.-l. Huang. On the fundamental theorems of the geometries of symmetric matrices. *Geom. Dedicata*, 78:315–325, 1999.

[13] W.-l. Huang. Adjacency preserving mappings of invariant subspaces of a null system. *Proc. Amer. Math. Soc.*, pages 2451–2455, 2000.

[14] W.-l. Huang. Characterization of the Transformation Group of the space of a Null System. *Results Math.*, 40:226–232, 2001.

[15] W.-l. Huang, R. Höfer, and Z.-X. Wan. Adjacency preserving mappings of symmetric and hermitian matrices. *Aequationes Math.*, 67:132–139, 2004.

[16] W.-l. Huang and A.E. Schroth. Adjacency preserving mappings. *Adv. Geom.*, 3:53–59, 2003.

[17] W.-l. Huang and P. Šemrl. Adjacency preserving mappings of hermitian matrices. *Canad. J. Math.*, 2005. To appear.

[18] W.-l. Huang and Z.-X. Wan. Adjacency preserving mappings of Rectangular matrices. *Beitr. Algebra Geom.*, 45(2):435–446, 2004.

[19] J. Tits. *Buildings of Spherical Type and Finite BN-Pairs*, volume 386 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[20] Z.-X. Wan. Geometry of symmetric matrices and its applications I. *Algebra Colloquium*, 1:97–120, 1994.

[21] Z.-X. Wan. Geometry of symmetric matrices and its applications II. *Algebra Colloquium*, 1:201–224, 1994.

[22] Z.-X. Wan. Geometry of Hamiltonian matrices and its applications I. *Algebra Colloquium*, 2:167–192, 1995.

[23] Z.-X. Wan. Geometry of Hamiltonian matrices and its applications II. *Algebra Colloquium*, 3:107–116, 1996.

[24] Z.-X. Wan. *Geometry of matrices*. World Scientific, Singapore, 1996.

[25] Z.-X. Wan. Geometry of matrices revisited. In K.-P. Shum, E. Taft, and Z.-X. Wan, editors, *An International Congress*, pages 477–486. ICAC '97, Springer, Hong Kong, 1999.

[26] Z.-X. Wan and L.-P. Huang. Geometry of 2×2 herimitian matrices. *Science in China (Series A)*, 45:1025–1037, 2002.

[27] Z.-X. Wan and L.-P. Huang. Geometry of 2×2 Hermitian Matrices II. *Multilinear Algebra*, pages XXX–XXX, 2004.