HAMBURGER BEITRÄGE
ZUR MATHEMATIK

Heft 259

Analogues of Cayley graphs for topological groups

Bernhard Krön and Rögnvaldur G. Möller

ANALOGUES OF CAYLEY GRAPHS FOR TOPOLOGICAL GROUPS

BERNHARD KRÖN AND RÖGNVALDUR G. MÖLLER

ABSTRACT. We define for a compactly generated totally disconnected locally compact group a graph, called a rough Cayley graph, that is a quasi-isometry invariant of the group. This graph carries information about the group structure in an analogue way as the ordinary Cayley graph for a finitely generated group. With this construction the machinery of geometric group theory can be applied to topological groups. This is illustrated by a study of groups where the rough Cayley graph has more than one end and a study of groups where the rough Cayley graph has polynomial growth.

INTRODUCTION

The concept of a Cayley graph has become a standard part of the toolkit used to investigate and describe groups. It has become particularly important in the study of infinite finitely generated groups, where the Cayley graph and related concepts provide a way to treat the group as a geometric object. When the group is finitely generated it can be shown that various properties of Cayley graphs are the same no matter which finite generating set is used to construct the Cayley graph. This part becomes problematic when we consider groups that are not finitely generated.

The aim of this paper is to present a construction of a graph for compactly generated totally disconnected locally compact groups that can be used in a similar way as the Cayley graph is used in the study of finitely generated groups. This construction allows us to apply the machinery of geometric group theory to compactly generated totally disconnected locally compact groups. We illustrate this by looking at the theory of ends of groups and groups of polynomial growth.

There is an extensive literature on group actions on graphs, see e.g. the survey [36], linking together algebraic properties and properties of the group action. The real novelty of the present approach is that we start just with a group, not with a given action of a group on a graph, making this approach a useful tool in group theory.

In [60], Woess studied the automorphism group of and infinite, connected, locally finite transitive graph as topological groups. A neighbourhood of the identity is given by the pointwise stabilizers of finite sets of vertices. Such automorphism groups are compactly generated, totally disconnected and locally compact. Hence they are a special case of the groups studied in the present paper. The crucial difference is that we start only with a given group instead of starting with a group action on a graph. Our construction will yield a group action on a graph, which we will call a "rough Cayley graph". The rough Cayley graphs will turn out to be unique up to quasi-isometry, see Theorem 2. Hence we are in a similar situation as when studying ordinary Cayley graphs of finitely generated groups: The geometric structure (the graph) is uniquely determined (up to quasi-isometry) by the algebraic structure.

In the first section we present definitions and background material on permutation groups, graphs, topological groups and the interplay of these concepts. The construction is presented in Section 2. We start with a compactly generated totally disconnected locally compact group G, choose a compact open subgroup U and a finite set $\{s_1, s_2, \ldots, s_n\}$ such that $U \cup \{s_1, s_2, \ldots, s_n\}$ generates G, and use these to construct a graph X. The graph X is locally finite and connected and the group G acts transitively on X. This graph will be called a *rough Cayley graph* for G. In Section 2 it is shown that any two such graphs for G are quasi-isometric. If G is finitely generated and U is the trivial group then this construction yields a usual Cayley graph of G. Hence finitely generated Cayley graphs are special cases of rough Cayley graphs.

In Sections 3 and 4 we illustrate the use of the concept of rough Cayley graph. In Section 3 we define the space of ends of compactly generated totally disconnected locally compact groups. We prove an analogue of Stallings' Ends Theorem for groups with infinitely many ends in Section 3.2. The construction of rough ends and the analogue of Stallings' Ends Theorem are related to the work of Abels in [1]. This relationship is discussed in Section 3.6. In Section 3.3 we focus on the concept of *accessibility*. The natural translation of the definition of an accessible group into terms appropriate for our totally disconnected locally compact groups is shown to be equivalent to the graph X being accessible in the sense of [53]. In Section 3.5 we investigate the action of group elements on a rough Cayley graph and how this action can be used to divide G into three disjoint classes (*ellpitic, parabolic, hyperbolic*) resembling the classification of isometries in hyperbolic geometry. In the final part of Sections 3 we relate the concept of *ends of pairs of groups* to the rough ends. As a byproduct we deduce a result due to Dunwoody and Roller [11] (also proved by Niblo [43] and Scott and Swarup [49]).

In Section 4 the growth of the graph X is related to the growth of the topological group G. The outcome is a version of Gromov's theorem on groups of polynomial growth for compactly generated totally disconnected locally compact groups. Many of the results and methods used can be found in the papers by Losert [32] and Woess [60].

The final section is a collection of remarks and comments on the previous sections and the possibilities for further work using rough Cayley graphs.

1. Preliminaries on graphs and groups

1.1. **Permutation groups and graphs.** All the graphs in this paper are undirected except the orbital graphs defined below and the structure trees discussed in Chapter 3. Our graphs are without loops or multiple edges. Thus one can think of a graph X as an ordered pair (VX,EX) where VX is a set and EX is a set of two element subsets of VX. The elements of VX are called *vertices* and the elements of EX are called *edges*. Vertices v and u are said to be *neighbours*, or *adjacent*, if $\{v,u\}$ is an edge in X. A *path* of length n from v to u is a sequence $v = v_0, v_1, \ldots, v_n = u$ of vertices, such that v_i and v_{i+1} are adjacent for $i = 0,1,\ldots,n-1$. A graph is connected if for any two vertices v and u there is a path from v to u in the graph. Let $d(x,y)$ denote the length of a shortest path from a vertex x to a vertex y. If X is connected then d is a metric on VX. When we are dealing with different graphs at the same time we will sometimes write d_X instead of d. Let A be a set of vertices in X. The subgraph *spanned* by A is a graph having A as a vertex set and the edge set is the set of all edges in X such that both end vertices belong to A. We say that A is *connected* if the subgraph spanned by A is connected. The *connected components* (or just *components*) of a graph are the maximal connected sets of vertices.

Let G be a group acting on a set Y. The action is *transitive* if for any two points x,y in Y there is an element $g \in G$ such that $gx = y$. For a point $x \in Y$ the *stabilizer in* G of x is the subgroup

$$G_x = \{g \in G \mid gx = x\}.$$

We define the *pointwise stabilizer* of a set $A \subseteq Y$ as the subgroup

$$G_{(A)} = \{g \in G \mid gx = x \text{ for every } x \in A\}.$$

When $A = \{x,y\}$ then we write $G_{x,y}$ for $G_{(A)}$.

Suppose U is a subgroup of a group G. The group G acts on the set G/U of left cosets of U such that the image of a coset hU under an element $g \in G$ is $(gh)U$. This action is transitive. Conversely, if G acts transitively on some set Y and x is a point in Y then Y can be identified with G/G_x in the following way: For each $y \in Y$ we choose an element $h_y \in G$ such that $h_y x = y$. Then the function $\theta : Y \to G/G_x$, where $y \mapsto h_y G_x$, is bijective. For every $y \in Y$ and every element $g \in G$ we get $\theta(gy) = g\theta(y)$, that is, θ gives an isomorphism of G-actions or, phrased differently, θ is *covariant* with the action of G.

The orbits of the stabilizer G_x are called *suborbits* of G and the orbits of G on Y^2 are called *orbitals*. When G acts transitively on Y there is a simple one-to-one correspondence between the orbits of G_x and the orbitals: the suborbit $G_x y$ corresponds to the orbital $G(x,y)$. A (directed) *orbital graph* $X = (VX,EX)$ is formed by letting the set of vertices VX be equal to Y and letting the set of edges EX be a union of some orbitals. The graphs we get by this construction are directed graphs but in our case we want undirected graphs. Hence we use a similar method to construct a graph X where $VX = Y$ and EX is a union of orbits of G on two element subsets of Y. These graphs can be called *undirected orbital graphs*.

It is easy to see that G acts on an undirected orbital graph as a group of graph automorphisms, because if $g \in G$ and $\{x, y\}$ is an edge in an orbital graph, then $\{gx, gy\}$ is in the same orbit and thus also an edge. When all the suborbits of G are finite and the edge set of an orbital graph is a union of finitely many orbitals then this orbital graph is locally finite (i.e. each vertex in it has only finitely many neighbours).

A *block of imprimitivity* for G is a subset A of Y such that for $g \in G$, either $gA = A$ or $A \cap gA = \emptyset$. The existence of a non-trivial proper block of imprimitivity A (*non-trivial* means that $|A| > 1$ and *proper* means that $A \neq Y$) is equivalent to the existence of a non-trivial proper G-invariant equivalence relation $\sim$ on Y. When G acts transitively on Y, the block A and its translates under G give the $\sim$-classes, and conversely, if $\sim$ is a non-trivial proper G-invariant equivalence relation then each $\sim$-class is a non-trivial proper block of imprimitivity for G. If $\sim$ is a G-invariant equivalence relation on Y then G permutes the $\sim$-classes and thus G acts on the set $Y/\sim$ of equivalence classes.

Finally, we review the definition of a *Cayley graph* of a group. Let G be a group and S a subset of G. The (undirected) Cayley graph $\mathrm{Cay}(G,S)$ of G with respect to S has the set of elements in G as a vertex set and $\{g,h\}$ is an edge if $h = gs$ or $h = gs^{-1}$ for some s in S. The Cayley graph $\mathrm{Cay}(G,S)$ is connected if and only if S generates G. The left regular action of G on itself gives us a transitive action of G as a group of graph automorphisms on $\mathrm{Cay}(G,S)$.

1.2. **Topological groups and the permutation topology.** A topological space is said to be *totally disconnected* if the only connected subsets are single element sets. It is an old result of van Dantzig that a totally disconnected locally compact group always contains a compact open subgroup (see [5] or [21, Theorem 7.7]). A topological group G is *compactly generated* if there is a compact subset that generates G.

Let G be a group acting on a set Y. The action of G on Y can be used to introduce a topology on G. (The survey paper by Woess [60] contains a detailed introduction to this topology.) The topology of a topological group is completely determined by a neighbourhood basis of the identity element. The *permutation topology* on G is defined by choosing as a neighbourhood basis of the identity the family of pointwise stabilizers of finite subsets of Y, i.e. a neighbourhood basis of the identity is given by the family of subgroups

$$\{G_{(F)} \mid F \text{ is a finite subset of } Y\}.$$

Think of Y as having the discrete topology and elements of G as maps $Y \to Y$. Then the permutation topology is equal to the topology of pointwise convergence, and it is also the same as the compact-open topology. A sequence $(g_i)_{i\in\mathbb{N}}$ of elements in G has an element $g \in G$ as a limit if and only if for every $x \in Y$ there is a number N (depending on x) such that $g_n x = gx$ for every $n \geq N$.

Various properties of the action of G on Y are reflected in properties of this topology on G. For instance, the permutation topology on G is Hausdorff if and only if the action of G on Y is faithful (*faithful* means that the only element of G that fixes all the points in Y is the identity). Moreover, G is totally disconnected if and only if the action is faithful.

When G is a permutation group on Y, that is, G acts faithfully on Y, one can think of G as a subgroup of $\mathrm{Sym}(Y)$, the group of all permutations of Y. We say that G is a *closed permutation group* if it is a closed subgroup of $\mathrm{Sym}(Y)$, where $\mathrm{Sym}(Y)$ has the permutation topology.

Let us now turn the tables and assume that G is a topological group and U a compact open subgroup of G. Define $Y = G/U$. Let $x = U \in Y$, that is, x is equal to the coset U. Thus $G_x = U$. Suppose $F = \{y_1, \ldots, y_n\}$ is a finite subset of Y and $g_1, \ldots, g_n$ are elements in G such that $g_i x = y_i$. Then

$$\begin{aligned} G_{(F)} &= G_{y_1} \cap \cdots \cap G_{y_n} = (g_1 G_x g_1^{-1}) \cap \cdots \cap (g_n G_x g_n^{-1}) \\ &= (g_1 U g_1^{-1}) \cap \cdots \cap (g_n U g_n^{-1}). \end{aligned}$$

This implies that in the permutation topology on G with respect to the action of G on $Y = G/U$, all the elements in the neighbourhood basis of the identity are open in the given topology on G. Therefore the permutation topology is contained in the topology on G. The permutation topology can be different from the topology on G. For instance because the permutation topology does not separate points in

$K = \bigcap_{g\in G} gUg^{-1}$, the kernel of the action of G on Y. Note that for every $y \in Y$ the orbit $G_x y = Uy$ is finite. This is so because, if g is an element of G such that $gx = y$ then $U \cap gUg^{-1}$ is an open subgroup of the compact subgroup U and thus

$$|G_x y| = |G_x : (G_x \cap G_y)| = |U : (U \cap gUg^{-1})| < \infty.$$

Therefore, all suborbits in the action of G on Y are finite.

Compactness has a very natural interpretation in the permutation topology as shown in the following lemma. A subset of a topological space is said to be *relatively compact* if it has compact closure. The following lemma was formulated by Woess ([60, Lemma 2]) for automorphism groups of locally finite connected graphs, but it holds in our more general setting as well where it is not assumed that the action of the group is faithful.

Lemma 1. ([60, Lemma 2]) *Let G be a totally disconnected locally compact group and U a compact open subgroup of G. Set $Y = G/U$. A subset A of G is relatively compact in G if and only if the set Ax is finite for every x in Y.*

Furthermore, if A is a subset of G and Ax is finite for some x in Y then Ax is finite for all x in Y.

Turning back to the case when G is a permutation group on Y, we notice that G is closed in the permutation topology of $\mathrm{Sym}(Y)$ if and only if G_x is closed in $\mathrm{Sym}(Y)$. (It is obvious that if G is closed in $\mathrm{Sym}(Y)$ then G_x is also closed. For the reverse implication assume that f is an element in $\mathrm{Sym}(Y)$ that is contained in the closure of G. Since the set $U = \{g \in \mathrm{Sym}(Y) \mid g(x) = f(x)\}$ is an open neighbourhood of f it must contain an element from G. Suppose $g \in G$ such that $g(x) = f(x)$. If V is some open neighbourhood of f in $\mathrm{Sym}(Y)$ then $U \cap V$ is an open neighbourhood of f that intersects gG_x. But the the set gG_x is closed and thus $g \in gG_x \subseteq G$. Hence G is closed.) It is easy to show that if G is a closed permutation group and all the suborbits of G are finite then G_x is compact and G is a totally disconnected locally compact group (see [60, Lemma 1]).

2. ROUGH CAYLEY GRAPHS

Definition 1. *Let G be a topological group. A connected graph X is said to be a* rough Cayley graph *of G if G acts transitively on X and the stabilizers of vertices are compact open subgroups of G.*

When G is a topological group with a compact open subgroup U then one can construct a rough Cayley graph in the following fashion which resembles the construction of the ordinary Cayley graph of a group: Suppose G is a topological group and U a compact open subgroup. For a subset S of G form the ordinary Cayley graph of G with respect to S. Then define X as the graph that has vertex set G/U and two distinct left cosets xU and yU are adjacent if there are elements $g \in xU$ and $h \in yU$ such that g and h are adjacent in Y. The natural action of G on the set of left cosets of U gives an action of G on the graph X.

Conversely, suppose X is a rough Cayley graph of a topological group G. Let U be the stabilizer of a vertex x in X. Find a family $\{g_i\}_{i\in I}$ of elements from G such that $\{g_i(x) \mid i \in I\}$ equals the set of neighbours of x in X. Set $S = U \cup \{g_i(x) \mid i \in I\}$ and define Y as the ordinary Cayley graph of G with respect to S. Then the quotient graph of Y with respect to U (as defined in the previous paragraph) is equal to X.

In Sections 3 and 4 it is shown that when the group G is a compactly generated totally disconnected locally compact group then a rough Cayley graph carries information about the group in much the same way as an ordinary Cayley graph of a finitely generated group does.

Theorem 1. *Let G be a totally disconnected locally compact group. Then G has a connected locally finite rough Cayley graph if and only if G is compactly generated.*

A more detailed version of this theorem (see below) can be found in [39]. There the immediate purpose was to show that the subgroup of FC^--elements in a compactly generated totally disconnected locally compact group is closed. Here the aim is a general investigation of the relationship between the group and the graph we construct.

Theorem 1^+([39, Corollary 1]) *Let G be a totally disconnected compactly generated locally compact group. Then there is a locally finite connected graph X such that:*

(i) *G acts as a group of automorphisms on X and is transitive on VX;*
(ii) *for every vertex v in X the subgroup G_v is compact and open in G;*
(iii) *if* $\mathrm{Aut}(X)$ *is equipped with the permutation topology then the homomorphism $\pi : G \to \mathrm{Aut}(X)$ given by the action of G on X is continuous, the kernel of this homomorphism is compact and the image of π is closed in* $\mathrm{Aut}(X)$.

Conversely, if G acts as a group of automorphisms on a locally finite connected graph X such that G is transitive on the vertex set of X and the stabilizers of the vertices in X are compact and open, then G is compactly generated.

We will give two different constructions of the graph X in the first half of the Theorem 1^+ above. The first one follows the construction used in [39], whereas the second one uses the ordinary Cayley graph of G with respect to some compact generating set and is related to the concept of a *topological graph* from Abels' paper [1] (this relationship will be discussed in Section 3.6). In the following, let G be a compactly generated topological group with a compact generating set S and a compact open subgroup U.

The first construction is based on the following Lemma.

Lemma 2. (Cf. [39, Lemma 2]) *Let G be a compactly generated totally disconnected locally compact group. Let U be a compact open subgroup of G. Then there is a finite set $T = \{h_1, \dots, h_n\}$ such that $H = \langle h_1, \dots, h_n \rangle$ acts transitively on the set of left cosets G/U. Furthermore, every element in G can be written as $h_{i_1} h_{i_2} \cdots h_{i_k} u$ where $u \in U$.*

The set T can be found as follows. The left cosets of U form an open covering of a compact generating set S. Hence there is a finite subcovering consisting, say, of $U, g_1U, \dots, g_kU$. Each double coset Ug_iU is compact and the set $W = U \cup (Ug_1U) \cup \cdots \cup (Ug_kU)$ is thus also compact. Hence it is possible to find finitely many elements $h_1, \dots, h_n$, none of which is contained in U, such that $U \cup h_1U \cup \cdots \cup h_nU = W$. Set $T = \{h_1, \dots, h_n\}$. That $US \subseteq W = TU$ follows from the proof of Lemma 2 in [39].

Definition 2. *Let G be a compactly generated totally disconnected locally compact group. A compact open subgroup U together with a finite set T as described in Lemma 2 above is said to form a* good *generating set.*

Construction 1. The graph X is defined such that the vertex set is G/U and the edge set is $G\{v, h_1v\} \cup \cdots \cup G\{v, h_nv\}$, where $G\{v, h_iv\}$ denotes the orbit of the set $\{v, h_iv\}$ under the diagonal action of G. The group G acts transitively as a group of automorphisms on X. It follows from [39, Lemma 1] that the graph X is connected. The orbit of h_iv under G_v is finite since $|G_v : G_{v,h_iv}|$ is finite. Hence the graph X is locally finite.

Construction 2. Form the Cayley graph $\mathrm{Cay}(G,S)$ of G with respect to a compact generating set S. The left regular action of G on itself gives us a transitive action of G on $\mathrm{Cay}(G,S)$. The left cosets of U form the classes of an equivalence relation on the vertices of $\mathrm{Cay}(G,S)$ that is preserved by the action of G. Define X as the quotient graph of $\mathrm{Cay}(G,S)$ with respect to this equivalence relation. The vertices of X are the left cosets of U. Two vertices g_0U and h_0U in X are adjacent if there are elements g in g_0U and h in h_0U such that $h = gs$ or $h = gs^{-1}$ for some s in S. Since S is a generating set for G, the graph $\mathrm{Cay}(G,S)$ is connected and thus the quotient graph X is also connected. A vertex gU in X which is the neighbour of the vertex U must intersect $US \cup US^{-1}$. This latter set is compact because a set that is the product (in the group G) of two compact sets is compact. The set $US \cup US^{-1}$ can thus be covered with finitely many left cosets of U and U is hence only adjacent to finitely many vertices in X. Since X is a transitive graph, it follows that X is locally finite.

Assume that G acts transitively on a locally finite connected graph X such that the stabilizer of a vertex v is a compact open subgroup U of G. (For instance, G could be the automorphism group of a transitive graph X endowed with the permutation topology. We identify the vertex set of X with G/U and then choose a finite set $\{g_1, \dots, g_n\}$ of group elements such that $\{g_1v, \dots, g_nv\}$ is the set of neighbours of v. The graph X is the same as the graph we get in Constructions 1 and 2 using the compact open subgroup U and $T = \{g_1, \dots, g_n\}$. Thus if X is a rough Cayley graph for a compactly generated totally disconnected locally compact group G then there is a compact open subgroup U and a finite set T such that $X = \mathrm{RCay}(G,U,T)$.

Every connected locally finite transitive graph is a rough Cayley graph of its automorphism group. But, not every rough Cayley graph is a Cayley graph of some group, since there are examples of infinite transitive connected locally finite graphs that are not isomorphic to a Cayley graph of some groups, e.g. [7].

Notation. A rough Cayley graph of G constructed above by using a good generating set consisting of a compact open subgroup U and a finite set T is denoted by $\mathrm{RCay}(G,U,T)$.

Remark 1. Constructions such as described above have been widely used. The first instance is in a paper from 1964 by Sabidussi [47] where it is shown how a finite transitive graph can always be described as a quotient of a Cayley graph.

In order for this construction to be truly useful in group theory we would like to show that the choice of the subgroup U and the finite set T has only a limited effect on the properties of $\mathrm{RCay}(G,U,T)$. The concept of quasi-isometry was introduced by Gromov [17] and has been widely used and studied since.

Definition 3. *Two metric spaces (X,d_X) and (Y,d_Y) are said to be* quasi-isometric *if there is a map $\varphi : X \to Y$ and constants $a \geq 1$ and $b \geq 0$ such that for all points x_1 and x_2 in X*

$$a^{-1}d_X(x_1,x_2) - a^{-1}b \leq d_Y(\varphi(x_1),\varphi(x_2)) \leq ad_X(x_1,x_2) + ab,$$

and for all points $y \in Y$ we have

$$d_Y(y,\varphi(X)) \leq b.$$

A map φ between two metric spaces satisfying the above conditions is called a quasi-isometry.

Two connected graphs X and Y are called quasi-isometric if (VX,d_X) and (VY,d_Y) are quasi-isometric. Note that $d_X(x_1,x_2) \geq 1$ for all distinct vertices x_1 and x_2. It is worth noting that in the case of connected graphs this implies that the definition of quasi-isometry can be simplified by replacing the first inequality with

$$a^{-1}d_X(x_1,x_2) - a^{-1}b \leq d_Y(\varphi(x_1),\varphi(x_2)) \leq ad_X(x_1,x_2).$$

Being quasi-isometric is an equivalence relation on the class of metric spaces.

Theorem 2. *Let G be a compactly generated totally disconnected locally compact group. Any two connected locally finite rough Cayley graphs of G are quasi-isometric.*

It is convenient to have an explicit description of a quasi-isometry that is in some sense canonical. Suppose $X_1 = \mathrm{RCay}(G,U_1,T_1)$ and $X_2 = \mathrm{RCay}(G,U_2,T_2)$ are rough Cayley graphs of G (recall that every rough Cayley graph can be presented in this way). The vertex sets of X_1 and X_2 can be identified with G/U_1 and G/U_2, respectively. Let H_1 be a set of representatives of the left cosets of U_1 in G. Define a map $\psi : VX_1 \to VX_2$ such that if $v = hU_1$ in VX_1, where $h \in H_1$, then $\psi(v) = hU_2 \in VX_2$. Using the map ψ we give a more explicit version of Theorem 2. In particular we prove that ψ is *quasi-co-variant* with the actions of G on X_1 and X_2. That is, there is a constant c such that $d_{X_2}(\psi(gv), g\psi(v)) \leq c$ for all $v \in VX_1$.

Theorem 2$^+$ *Let G be a compactly generated totally disconnected locally compact group. Suppose $X_1 = \mathrm{RCay}(G,U_1,T_1)$ and $X_2 = \mathrm{RCay}(G,U_2,T_2)$ are rough Cayley graphs of G. The map ψ defined above is a quasi-isometry and there is a constant c such that for every vertex v in X_1 and every element g in G we have*

$$d_{X_2}(g\psi(v), \psi(gv)) \leq c.$$

Theorem 2$^+$. The proof is split up into three cases depending on the relationship between U_1 and U_2.

Case 1. Assume that $U_1 = U_2$

Under this assumption we can identify the vertex sets of the two graphs and the map ψ defined above becomes the identity map.

Because G only has finitely many orbits on the edges of X_1 there is a constant a such that whenever v and u are neighbours in X_1 then $d_{X_2}(v,u) \leq a$. By choosing a large enough, we may also assume that $d_{X_1}(v,u) \leq a$ for any vertices v and u which are adjacent in X_2. ¿From this it follows that for any pair of vertices v and u we have

$$\frac{1}{a} d_{X_1}(v,u) \leq d_{X_2}(v,u) \leq a d_{X_1}(v,u).$$

Thus X_1 and X_2 are quasi-isometric. In this case, ψ is the identity map. Hence $d_{X_2}(g\psi(v), \psi(gv)) = 0$ and ψ is co-variant with the action of G.

Case 2. Assume that $U_1 \geq U_2$. The left cosets of U_1 give us a G-invariant equivalence relation $\sim$ on the vertices of X_2 (two cosets of U_2 belong to the same $\sim$-class if they are both contained in the same U_1 coset). Since U_2 is an open subgroup of the compact group U_1, we see that $|U_1 : U_2| < \infty$ and the $\sim$-classes are finite. We can identify the vertex set of the quotient graph $X_3 = X_2/\sim$ with the vertex set of X_1. By the first case above, the identity map $\psi_1 : X_1 \to X_3$ is a co-variant quasi-isometry and there is a constant a such that $\frac{1}{a} d_{X_1}(v,u) \leq d_{X_3}(\psi_1(v), \psi_1(u)) \leq a d_{X_1}(v,u)$ for every pair v and u of vertices in X_1. Define a map $\psi_2 : X_3 \to X_2$ such that $\psi_2(hU_1) = hU_2$ where h is in H_1. Let c denote the diameter of the $\sim$-classes in X_2. Then for every pair of vertices v, u in X_3 the following inequalities hold,

$$d_{X_3}(v,u) \leq d_{X_2}(\psi_2(v), \psi_2(u)) \quad \text{and} \quad d_{X_2}(\psi_2(v), \psi_2(u)) \leq (c+1) d_{X_3}(v,u).$$

For every pair of vertices v, u in X_1 this implies,

$$d_{X_3}(\psi_1(u), \psi_1(v)) \leq d_{X_2}(\psi_2\psi_1(u), \psi_2\psi_1(v))$$

and

$$d_{X_2}(\psi_2\psi_1(u), \psi_2\psi_1(v)) \leq (c+1) d_{X_3}(\psi_1(u), \psi_1(v)).$$

Set $\psi = \psi_2 \circ \psi_1$. Then

$$\begin{aligned} \frac{1}{a} d_{X_1}(u,v) &\leq d_{X_3}(\psi_1(u), \psi_1(v)) \leq d_{X_2}(\psi(u), \psi(v)) \\ &\leq (c+1) d_{X_3}(\psi_1(u), \psi_1(v)) \leq a(c+1) d_{X_1}(u,v), \end{aligned}$$

and therefore ψ is a quasi-isometry from X_1 to X_2. Suppose $v = hU_1$ is a vertex in X_1 and g is an element of G. We write $h'U_1 = ghU_1$ where h and h' are in H_1. Then $\psi(gv) = h'U_2$ and $g\psi(v) = ghU_2$. Both $h'U_2$ and ghU_2 belong to the same $\sim$-class and thus $d_{X_2}(g\psi(v), \psi(gv)) \leq c$.

Case 3. Let us now look at the general case. Set $U_3 = U_1 \cap U_2$. Define X_3 as a rough Cayley graph of G with respect to U_3 and some finite set T_3 as described in Construction 1. Define a map $\psi_1 : X_1 \to X_3$ such that if $v = hU_1$, with h in H_1, then $\psi_1(v) = hU_3$. Let H_3 denote a set of coset representatives of U_3. Then define $\psi_2 : X_3 \to X_2$ such that if $v = hU_3$, where $h \in H_3$, then $\psi_2(v) = hU_2$ (note that ψ_2 does not depend on the choice of coset representatives in H_3). The map ψ_2 is a quasi-isometry. It is also worth noting that ψ_2 is co-variant with the action of G. Because both ψ_1 and ψ_2 are quasi-isometries we can conclude that $\psi = \psi_2 \circ \psi_1$ is a quasi-isometry. Since ψ_2 is co-variant with the action of G, we see that $g\psi(v) = g(\psi_2 \circ \psi_1(v)) = \psi_2(g\psi_1(v))$. We know from Case 2 that there is a constant c_1 such that $d_{X_3}(g\psi_1(v), \psi_1(gv)) \leq c_1$ for all vertices v in X_1. Since the map ψ_2 is an quasi-isometry we conclude that there is a constant c such that

$$d_{X_2}(g\psi(v), \psi(gv)) = d_{X_2}(\psi_2(g\psi_1(v)), \psi_2(\psi_1(gv))) \leq c.$$

□

Remark 2. (i) We can also ask about the effect of the particular choice of a set of coset representatives when constructing ψ. Suppose θ and θ' are two quasi-isometries from X_1 to X_2, constructed with respect to different choices of coset representatives. Then $\theta(v)$ and $\theta'(v)$ are vertices in X_2 corresponding to two left U_2 cosets that both intersect the same left U_1 coset. There are only finitely many left U_2 cosets that intersect a given left U_1 coset. Suppose $h_1U_2, \ldots, h_kU_2$ is the collection of all the cosets that intersect U_1 then $gh_1U_2, \ldots, gh_kU_2$ is the collection of all the cosets that intersect gU_1. When we consider the left cosets of U_2 as vertices in X_2 we see that the diameter in X_2 of cosets that intersect a given left coset of U_1 is always the same. Hence there is a constant c such that $d_{X_2}(\theta(v), \theta'(v)) \leq c$ for all vertices v in X_1.

(ii) In the construction of a rough Cayley graph and the proof of Theorem 2^+ we only use the property of compact open subgroups that any two such subgroups U_1 and U_2 are commensurable (i.e. $U_1 \cap U_2$ has finite index in both U_1 and U_2) and that a compact open subgroups is commensurable with its conjugates. Suppose G is a group acting transitively on two locally finite connected graphs X_1 and X_2. Furthermore, assume that for all v_1 in VX_1 and all v_2 in VX_2, the subgroups $U_1 = G_{v_1}$ and $U_2 = G_{v_2}$ are commensurable. Then the same argument as in the proof above shows that the graphs X_1 and X_2 are quasi-isometric.

Theorem 3. *Let G be a totally disconnected locally compact group. Suppose G acts on a connected locally finite graph X such that the stabilizers of vertices are compact open subgroups and G has only finitely many orbits on VX. Then G has a locally finite rough Cayley graph X' which is quasi-isometric to X.*

Proof. Choose some vertex v in X and denote the orbit of v by A. Since G has only finitely many orbits on the vertex set of X there is a number k such that for each vertex u in X there is some vertex in A in distance at most k from u. Construct a new graph Y such that Y has the same vertex set as X and two vertices u and u' are adjacent in Y if and only if $d_X(u,u') \leq 2k+1$. The graph Y is also locally finite and the group G acts on Y. Let X' denote the subgraph of Y spanned by A. Suppose u and u' are some vertices in X'. Since the graph X is connected there is a path $u_0 = u, u_1, \ldots, u_{n-1}, u_n = u'$ in X. For each vertex u_i we can find a vertex v_i in A such that $d_X(v_i,u_i) \leq k$ and

$$d_X(v_i,v_{i+1}) \leq d_X(v_i,u_i) + d_X(u_i,u_{i+1}) + d_X(u_{i+1},v_{i+1}) \leq 2k+1.$$

Either $v_i = v_{i+1}$, or v_i and v_{i+1} are adjacent in Y. From the sequence $u, v_1, \ldots, v_{n-1}, u'$ we can thus get a path in X' from u to u'. Therefore X' is a connected locally finite graph and G acts transitively on X'. It follows from the construction that X' is quasi-isometric to X. □

The following Corollary is proved by using Theorem 3 in combination with the latter part of Theorem 1^+.

Corollary 1. *Let G be a totally disconnected locally compact group. Suppose G acts on a connected locally finite graph X such that the stabilizers of vertices are compact open subgroups and G has only finitely many orbits on VX. Then G is compactly generated.*

Let H be a subgroup of G. The quotient topology on G/H is the finest topology such that the projection from G to G/H is continuous. A subset $\{gH \mid g \in A\}$ of G/H, $A \subset G$, is open, if and only if AH is open in G. A subgroup H of a topological group G is said to be cocompact if the quotient space G/H is compact. Suppose G is a totally disconnected locally compact group acting transitively on a set Ω such that the stabilizers of points in Ω are compact open subgroups of G. It is shown in [38, Lemma 7.5] and [42, Proposition 1] that a subgroup H of G is cocompact if and only if H has only finitely many orbits on Ω. The first part of the corollary below is well known, and is also true without the assumption that the group G is totally disconnected.

Corollary 2. *Let G be a compactly generated totally disconnected locally compact group and H a closed cocompact subgroup of G.*

(i) *The subgroup H is compactly generated.*
(ii) *If Y_H and Y_G be connected locally finite rough Cayley graph of H and G, respectively, then Y_H and Y_G are quasi-isometric.*

Proof. The group H is in its own right a totally disconnected locally compact group. Let X be some rough Cayley graph for G. Then H acts on X with only finitely many orbits and the stabilizers of vertices are compact open subgroups of H. By Theorem 3, H acts transitively on a connected locally finite graph X' such that X' is quasi-isometric to X and the stabilizers in H of vertices in X' are compact open subgroups of H. Theorem 1 says that the group H is compactly generated and the graph X' is a rough Cayley graph of H. If Y_G is some rough Cayley graph of G and Y_H is some rough Cayley graph of H then Y_G is quasi-isometric to X and Y_H is quasi-isometric to X' and hence Y_G and Y_H are quasi-isometric to each other. □

3. Ends of compactly generated groups

3.1. **Preliminaries on ends and structure trees.**

3.1.1. *Ends of graphs.* There are various ways of defining the ends of a graph. The graph theoretical approach is to define the ends as equivalence classes of rays. A *ray* in a graph X is a sequence of distinct vertices $v_0, v_1, \ldots$ such that v_i and v_{i+1} are adjacent for all i. A *line* in X is a two way infinite sequence $\ldots, v_{-1}, v_0, v_1, v_2, \ldots$ of distinct vertices such that v_i and v_{i+1} are adjacent for all i.

Definition 4. ([18]) Let X be a connected graph. Two rays R_1 and R_2 in X are said to be in the same *end* of X if there is a ray R_3 in X which contains infinitely many vertices from both R_1 and R_2.

If X is a tree then two rays are in the same end if and only if their intersection is a ray.

It is easy to check that *being in the same end* is an equivalence relation on the set of rays in X. The equivalence classes are called the *ends* of X and the set of ends is denoted by ΩX.

Another way of phrasing the definition is to say that R_1 and R_2 are in the same end if and only if for every finite set $F \subseteq VX$ there is a path in $VX \setminus F$ from a vertex in R_1 to a vertex in R_2. This in turn leads to yet another reformulation of the definition: two rays R_1 and R_2 are not in the same end if and only if one can find a finite set F of vertices and distinct components C_1 and C_2 of $VX \setminus F$ such that C_1 contains infinitely many vertices of R_1 and C_2 contains infinitely many vertices of R_2. A locally finite connected graph X has more than one end if and only if there is a finite set of vertices F such that $VX \setminus F$ has more than one infinite component.

For a set $C \subseteq VX$, we define the (*vertex*) *boundary* ∂C as the set of vertices in $VX \setminus C$ that are adjacent to a vertex in C. The *coboundary* δC is defined as the set of edges that have one end vertex in C and the other one in $VX \setminus C$.

¿From Definition 4 it is evident that if a set of vertices $C \subseteq VX$ with finite boundary contains infinitely many vertices from some ray R then C also contains infinitely many vertices from every ray in the same end as R. Thus it is reasonable to say that C contains the end that R is in. Let ΩC denote the set of ends that are contained in C. If $F \subseteq VX$ is finite and two ends ω and ω' are in different components of $VX \setminus F$ then we say that F *separates* the ends ω and ω'. In this paper we are predominantly concerned with locally finite graphs. In a locally finite graph any two distinct ends can also be separated by removing finitely many edges.

Ends come in two basic sizes: thick and thin. An end ω is said to be *thick* if it contains an infinite set of pairwise disjoint rays, and *thin* otherwise. For an end ω define $m_1(\omega)$ as the supremum of the cardinalities of sets of pairwise disjoint rays in ω. Halin proves in [19] that if ω is thin then $m_1(\omega)$ is finite.

One can also think of the ends as a boundary of the graph. This becomes clearer if we give a topological definition. This definition can be traced back to Freudenthal's thesis in 1931, [12, 13], and the ideas are adapted to locally finite graphs in [14].

Now we add the assumption that X is locally finite. Let $\mathscr{F}$ denote the set of all finite subsets of VX. For $F \in \mathscr{F}$ define $\mathscr{C}_F$ as the set of all infinite components of $VX \setminus F$. If F_1 and F_2 are two elements of $\mathscr{F}$ such that $F_1 \subseteq F_2$ then there is a natural projection $\mathscr{C}_{F_2} \to \mathscr{C}_{F_1}$: a component of $VX \setminus F_2$ being mapped to the component of $VX \setminus F_1$ that contains it. Thus $\{\mathscr{C}_F\}_{F \in \mathscr{F}}$ ordered such that $\mathscr{C}_{F_1} \leq \mathscr{C}_{F_2}$ if $F_1 \subseteq F_2$. Let Ω denote its inverse limit. Now we want to identify Ω and ΩX. An element of Ω can be represented as a family $(C_F)_{F \in \mathscr{F}}$ such that if $F_1 \subseteq F_2$ then $C_{F_2} \subseteq C_{F_1}$. Given an end $\omega \in \Omega X$ it is easy to find the corresponding element in Ω: for $F \in \mathscr{F}$ we let C_F denote the component of $VX \setminus F$ that ω belongs to and then $(C_F)_{F \in \mathscr{F}}$ does the trick. The next step is to show how we find the end corresponding to an element in Ω. Let $(C_F)_{F \in \mathscr{F}}$ be an element in Ω. Take a strictly increasing sequence $F_1 \subset F_2 \subset \ldots$ of finite subsets of VX such that $VX = \bigcup_{i \in \mathbb{N}} F_i$. Then $\{C_{F_i}\}_{i \in \mathbb{N}}$ is a decreasing sequence. First of all it is clear that any two ends in X are separated by some set F_i. Hence there is at most one end ω that belongs to all of the sets C_{F_i}. However, one can find a ray that includes at least one vertex from ∂C_{F_i} for all $i \in \mathbb{N}$. The end that this ray belongs to is contained in all the sets of the sequence $\{C_{F_i}\}_{i \in \mathbb{N}}$. For a finite set F of vertices we find i such that $F \subseteq F_i$. Then $C_{F_i} \subseteq C_F$ and thus $\omega \in C_F$. Hence ω is the only end that is contained in C_F for every $F \in \mathscr{F}$.

The inverse limit construction gives a topology on ΩX. A basis of open sets for this topology is given by sets ΩC where $C \subseteq VX$ and C has finite boundary. If we extend this topology to $VX \cup \Omega X$ then a basis of open sets is given by the sets of the form $C \cup \Omega C$ where C has finite boundary. When X is locally finite then it is easy to see that $VX \cup \Omega X$ is compact with this topology and ΩX is also compact. This topology on $VX \cup \Omega X$ is also compact without the assumption of local finiteness, see [27, Theorem 1], but the set of ends ΩX alone without the set of vertices VX is not compact in general. One can view $VX \cup \Omega X$ as a compactification of VX.

The usefulness of the concept of quasi-isometry introduced in the last section is that various structural properties are preserved under quasi-isometries. One of these properties is the number of ends. Think now of the ends as a compactification of the graph. If ψ is a quasi-isometry between two locally finite graphs X_1 and X_2 then ψ has a unique extension $\overline{\psi}: X_1 \cup \Omega X_1 \to X_2 \cup \Omega X_2$ that is continuous when $X_1 \cup \Omega X_1$ and $X_2 \cup \Omega X_2$ are viewed as topological spaces. The restriction of $\overline{\psi}$ to ΩX_1 is a homeomorphism $\Psi : \Omega X_1 \to \Omega X_2$ (cf. [35, Proposition 1] and [27, Theorem 6]).

Remark 3. It is a common theme in graph theory to study what happens if some vertices are removed from the graph. The end concept defined above is a natural extension of these ideas to infinite graphs. Instead of removing a finite set of vertices we could as well remove a finite set of edges or a set of vertices which is bounded with respect to natural metric of the graph. Then we obtain *edge ends* and *metric ends*, respectively. These concepts are compared in [27]. In locally finite graphs, these different end concepts coincide.

3.1.2. *Automorphisms and ends.* In this section we assume that X is a connected locally finite graph. Local finiteness is not necessary for all the results described but it simplifies the discussion, and our interest is in applications to locally finite rough Cayley graphs.

It is clear from the definition of ΩX that an automorphism of X has a natural action on ΩX. As shown by Halin in his fundamental paper [20], the action on the ends gives vital clues to how the automorphism behaves. The same is also evident from Tits' paper [54], where group actions on infinite trees are studied. Halin shows how automorphisms of X can be divided up into three disjoint classes. For an automorphism g of X one of the following holds:

(i) g leaves invariant some non-empty finite subset of VX;
(ii) g fixes precisely one thick end and does not satisfy (i);
(iii) g fixes precisely two thin ends and does not satisfy (i).

Automorphisms that satisfy (ii) or (iii) are often called *translations*. For a translation g it is possible to find a line in X such that some power of g that acts like a non-trivial translation on that line. If X is a tree then g will act like a translation on the line. Automorphisms that satisfy (i) are called *elliptic*, those that satisfy (ii) are called *parabolic* and those that satisfy (iii) are called *hyperbolic* (or *proper translations*).

It is simple to describe how one finds an invariant line in cases (ii) and (iii). Suppose that g does not satisfy (i). Set $n_0 = \min d(g^k(v), v)$, where k is a non-zero integer and v a vertex in X. Find $k_0 \in \mathbb{N}$ and $v_0 \in VX$ such that $d(g^{k_0}(v_0), v_0) = n_0$. Then take a path P of length n_0 from v_0 to $g^{k_0}(v_0)$ and set $L = \bigcup_{i\in\mathbb{Z}} g^{ik_0}P$. It is obvious that L is an invariant infinite path and that g^{k_0} acts like a translation on L. The interesting thing proved by Halin (see [20, Theorem 7]) is the fact that L is a line, i.e. that L consists of distinct vertices. Say the line L is the sequence $\ldots, v_{-1}, v_0, v_1, v_2, \ldots$. The ends that the rays $v_0, v_{-1}, \ldots$ and $v_0, v_1, \ldots$ belong to are fixed by g, and these are the only ends fixed by g (see [20, Theorem 8]). If g is parabolic then g fixes only one end of X and both rays belong to the same end. In the case where g is hyperbolic the two rays will belong to distinct ends of X. If we suppose that $g^{k_0}(v_0) = v_l$ with $l > 0$ then the end that the ray $v_0, v_1, \ldots$ belongs to is denoted by $\mathscr{D}(g)$. The end $\mathscr{D}(g)$ called the *direction* of g. The ray $v_0, v_{-1}, \ldots$ belongs to the direction $\mathscr{D}(g^{-1})$. Note that if v is a vertex in X and g is either parabolic or hyperbolic then the sequence $g^n(v)$ converges to $\mathscr{D}(g)$ in the topology on $VX \cup \Omega X$ (see [59, Lemma 2.4]).

Now one can ask for the existence of hyperbolic automorphisms in $\mathrm{Aut}(X)$. This question is answered by the following result of Jung [24].

Theorem 4. (Cf. [24, Theorem 1]) *Let X be a connected locally finite transitive graph. Suppose C is an infinite subset of VX with infinite complement and finite boundary. Then there is an element $g \in \mathrm{Aut}(X)$ such that $gC \subsetneq C$ and g is a hyperbolic automorphism.*

Not only does a transitive group of automorphisms of a locally finite graph with more than one end contain hyperbolic elements, they are abundant. Theorem 4 implies that in connected locally finite transitive graphs with more than one end the directions of hyperbolic automorphisms are dense in ΩX, where ΩX has the topology defined towards the end of Section 3.1.1. Pavone has shown that if in addition there is no end fixed by the action of $\mathrm{Aut}(X)$ then the directions of hyperbolic elements are bilaterally dense in ΩX, i.e. if U_1 and U_2 are disjoint open sets in ΩX then there is some hyperbolic automorphism g of X such that $\mathscr{D}(g^{-1}) \in U_1$ and $\mathscr{D}(g) \in U_2$ (see [45, Theorem 5]).

¿From Theorem 4 it can be deduced that if X has more than two ends then there are no isolated points in ΩX. As a consequence we obtain Theorem 5 below which was proved by Hopf in [22] in the context of Freudenthal's ends of locally compact connected spaces, see [12, 13]. For the case of vertex ends in non-locally finite graphs see [20, Corollary 15], and for the case of metric ends see [29, Corollary 3.15] and [30, Theorem 4].

Theorem 5. *An infinite connected locally finite transitive graph has either* 1 *or* 2 *ends, or ΩX is a Cantor set.*

3.1.3. *Structure trees.* The fundamental results behind the theory of structure trees are from the book by Dicks and Dunwoody [6, Chapter II], but the connections and uses of this theory to study infinite graphs and group actions on such graphs are developed in [34] and [53]. The survey paper [36] gives an overview of this technique and the main results on group actions on graphs with infinitely many ends.

For a subset $e \subseteq VX$ we set $e^* = VX \setminus e$. A *cut* (or more precisely *an edge cut*) is a subset $e \subseteq VX$ such that the coboundary δe of e is finite. A cut e is said to be *tight* if both e and e^* are connected subsets of X. Define $\mathscr{B}_n X$ to be the Boolean ring generated by all cuts c such that $|\delta c| \leq n$. We also define $\mathscr{B}X$ as the Boolean ring generated by all the cuts. All the elements in $\mathscr{B}X$ are cuts.

A set E of cuts is said to be a *nested* if for each choice of $e, f \in E$ one of the intersections

$$e \cap f, \quad e \cap f^*, \quad e^* \cap f, \quad \text{or} \quad e^* \cap f^*$$

is empty, (i.e. $e \subseteq f^*$, $e \subseteq f$, $e^* \subseteq f^*$ or $e^* \subseteq f$). A nested set E is a *tree set* if for all elements $e, f \in E$ such that $e \subseteq f$ there are only finitely many elements $g \in E$ such that $e \subseteq g \subseteq f$. A tree set E is called *undirected* if whenever $e \in E$ then $e^* \in E$. A tight cut e is called a *Dunwoody-cut* (or a *D-cut*) if $E = Ge = \{ge \mid g \in G\}$ is a tree set. Furthermore, we say that a tree set is tight if every element is a tight cut. In their book [6] Dicks and Dunwoody prove the following remarkable theorem.

Theorem 6. ([6, Theorem II.2.20]) *Let X be a connected graph and $G \leq \mathrm{Aut}(X)$. Then there is a chain of G-invariant undirected tree sets $E_1 \subseteq E_2 \subseteq \dots$ in $\mathscr{B}X$ such that all elements in E_n are tight and E_n generates $\mathscr{B}_n X$ for all n.*

¿From a tight undirected G-invariant tree set E in $\mathscr{B}X$ we can build a directed tree $\overrightarrow{T} = \overrightarrow{T}(E)$. This construction is first described in [8, Theorem 2.1]. It is also treated in [6, Section II.1], [28], [36] and [53]. The reader is referred to those references for more details and proofs.

For elements $e, f \in E$ we define $f \Subset e$ if $f \subsetneq e$ and if $f \subseteq g \subseteq e$ then $e = g$ or $f = g$. Define a relation $\sim$ on E such that $e \sim f$ if $e = f$ or $f^* \Subset e$. In the proof of [8, Theorem 2.1] it is shown that $\sim$ is an equivalence relation. The vertex set of $\overrightarrow{T}$ is the set of $\sim$-classes. There is a one-to-one correspondence between the edges of $\overrightarrow{T}$ and the elements of E: An element $e \in E$ corresponds to a directed edge $\vec{e}$ from the $\sim$-class of e^* to the $\sim$-class of e. Hence we may consider elements of E as edges of $\overrightarrow{T}$. We have already defined an inversion * on the set E and we can also define an inversion * on $E\overrightarrow{T}$ so that if $\vec{e} = (\alpha, \beta)$ is an edge in $E\overrightarrow{T}$ then $\vec{e}^* = (\beta, \alpha)$. We see that if $e \in E$ and $\vec{e} = (\alpha, \beta)$ of $\overrightarrow{T}$ then e^* corresponds to the edge $\vec{e}^* = (\beta, \alpha)$. Furthermore, $e, f \in E$ correspond to edges $\vec{e} = (\alpha, \beta)$ and $\vec{f} = (\beta, \gamma)$ in $\overrightarrow{T}$ if and only if $f \Subset e$. Define a partial ordering on $E\overrightarrow{T}$ such that $\vec{e} \geq \vec{f}$ if there is a path of distinct vertices $\alpha_0, \alpha_1, \dots, \alpha_{n-1}, \alpha_n$ in T such that $\vec{e} = (\alpha_0, \alpha_1)$ and $\vec{f} = (\alpha_{n-1}, \alpha_n)$. The tree set E is also partially ordered with respect to inclusion.

From the construction it is clear that the two partial ordered sets E and $E\overrightarrow{T}$ are order isomorphic. Let T denote the undirected graph which has the same vertex set as $\overrightarrow{T}$ and $\{\alpha,\beta\}$ is an edge if and only if (α,β) is an edge in $\overrightarrow{T}$. The directed graph $\overrightarrow{T}$ is a tree in the sense that the undirected graph T is a tree. The tree $\overrightarrow{T}=\overrightarrow{T}(E)$ is called a *structure tree* of X. Note that structure trees are directed graphs.

Our group G acts on the tree set E and the equivalence relation $\sim$ is invariant under this action. Hence G acts both on $\overrightarrow{T}$ and T. The space of ends $\Omega\overrightarrow{T}$ of the directed graph $\overrightarrow{T}$ is defined as being the same as the space of ends of the undirected graph T.

Next we define maps $\varphi : VX \to V\overrightarrow{T}$ and $\Phi : \Omega X \to VT\cup\Omega\overrightarrow{T}$. Think of an element e in E as a directed edge $\vec{e}=(\alpha,\beta)$ in the tree $\overrightarrow{T}$. We say that $\vec{e}$ *points towards* a vertex γ in $\overrightarrow{T}$ if γ is in the same component of $\overrightarrow{T}\setminus\{\vec{e},\vec{e}^*\}$ as β. That is, a path from α to γ must contain β. Let v be a vertex in X. We locate $\varphi(v)$ by the property that all the elements in E that contain v should point towards $\varphi(v)$, when they are viewed as edges of $\overrightarrow{T}$. Suppose that e and f are elements in E and $\vec{e}=(\alpha,\beta)$ and $\vec{f}=(\beta,\gamma)$ are the corresponding edges in $\overrightarrow{T}$. If $\varphi(v)=\beta$ then $v\in e$ but $v\notin f$. In fact, e is minimal in E subject to containing v and f is maximal subject to not containing v. From this we can observe that for $v,u\in VX$ it follows that $\varphi(v)\neq\varphi(u)$ if and only if there is an element $e\in E$ such that e contains precisely one of the vertices v and u (i.e. v and u belong to different components of X when the edges in δe have been removed). The group G acts on both VX and $V\overrightarrow{T}$, and the map φ commutes with these actions.

An element $e\in E$ thought of as a directed edge $\vec{e}=(\alpha,\beta)$ in the tree $\overrightarrow{T}$, *points towards* an end ω in $\overrightarrow{T}$ if ω is in the same component of $\overrightarrow{T}\setminus\{\vec{e},\vec{e}^*\}$ as β. The map Φ is defined in a similar way as φ: those elements in $E\overrightarrow{T}$ that contain an end ω of X should point towards $\Phi(\omega)$ when considered as edges in $\overrightarrow{T}$. For an end ω we may get an infinite sequence of decreasing cuts in E, all of which contain ω. This sequence defines a ray R in $\overrightarrow{T}$ and we define $\Phi(\omega)$ as the end of $\overrightarrow{T}$ that R belongs to. If there is no such sequence then there is a vertex α in $V\overrightarrow{T}$ such that any cut in E that points to α contains ω. In this case we set $\Phi(\omega)=\alpha$. The vertices of $\overrightarrow{T}$ that are in the image of Φ are recognizable as those vertices that have infinite degree or whose pre-image under φ is infinite, [34, Lemma 4]. If $\Phi(\omega)$ is a vertex $\alpha\in V\overrightarrow{T}$ then we say that the end ω *lives inside* the vertex α. As for vertices of X, we see that two ends of X have distinct images under Φ if there is some element $e\in E$ such that δe separates the two ends. Take a tree set E that generates $\mathscr{B}_nX$. Then, if some two ends can be separated by a set containing n or fewer edges then there will be an element $e\in E$ such that δe separates the ends (i.e. the ends belong to different components of X when the edges in δe have been removed). Again it is clear that the map Φ commutes with the actions of G on ΩX and $V\overrightarrow{T}\cup\Omega T$.

The following Lemma describes the relationship between the action of G on X and the action of G on $\overrightarrow{T}$.

Lemma 3. ([34, Corollary 1]) *Let X be a connected locally finite graph and $\overrightarrow{T}=\overrightarrow{T}(E)$ some structure tree of X, where E is a tight undirected tree set.*

(i) *If $g\in\mathrm{Aut}(X)$ acts like a translation on $\overrightarrow{T}$ then g acts like a translation on X and g is hyperbolic.*
(ii) *If $g\in\mathrm{Aut}(X)$ is a translation (a parabolic or hyperbolic automorphism of X) then either g acts as a translation on $\overrightarrow{T}$ or there is a unique vertex of $\overrightarrow{T}$ fixed by g and that vertex has infinite degree.*
(iii) *If $g\in\mathrm{Aut}(X)$ is hyperbolic then there is a tight undirected tree set E_g such that g acts as a translation on $\overrightarrow{T}(E_g)$.*

3.1.4. *Ends of groups.* The number of ends of a finitely generated group G is defined as the number of ends of a Cayley graph of G with respect to some finite generating set. As will be explained in the next section, the choice of a finite generating set will not affect the outcome.

¿From Theorem 5 it follows that a Cayley graph of a finitely generated group either has no ends, one end, two ends or infinitely many ends. The structure of groups with more than one end is described in the following theorems. The first one, which is a conjunction of results of Hopf [22, Satz 5] and Wall [56, Lemma 4.1], gives a clear description of the groups that have precisely two ends.

Theorem 7. *Let G be a finitely generated group. Then the following are equivalent:*

(i) *G has precisely two ends;*
(ii) *G has an infinite cyclic subgroup of finite index;*
(iii) *G has a finite normal subgroup N such that G/N is either isomorphic to the infinite cyclic group or to the infinite dihedral group.*

Definition 5. *A group G is said to* split *over a subgroup H if G can be decomposed into a non-trivial amalgamated free product $A *_H B$ of subgroups A and B (non-trivial means that $H \neq A$ and $H \neq B$), or if G is an HNN-extension $A *_H x$, where x denotes the stable letter.*

Theorem 8. (Stalling's Ends Theorem, [52]) *Suppose G is a finitely generated group with more than one end. Then G splits over a finite subgroup.*

This theorem can be deduced from the general theory of structure trees (described in the previous section) with the aid of Bass-Serre theory of groups acting on trees (see [50]). In Bass-Serre theory it is usually assumed that a group G acts on a tree T *without inversion*, meaning that no element in the group transposes a pair of adjacent vertices. ¿From the Bass-Serre theory of groups acting on trees we need the following.

Theorem 9. (Cf. [50, Theorem 6]) *Suppose G is a group acting without inversion on a tree T such that G has just a single orbit on the edges of T. Suppose $\{u,v\}$ is an edge in T. If G has two orbits on the vertices of T then $G = G_u *_{G_{u,v}} G_v$. If G has just one orbit on the vertices of T then G can be written as a HNN-extension $G_u *_{G_{u,v}} x$.*

The condition that the group acts without inversion is not a serious restriction, because by replacing T with its barycentric subdivision (adding a new vertex at the "middle" of each edge) we are sure to get an action without inversion.

Suppose G acts transitively on a locally finite connected graph with infinitely many ends. We find a Dunwoody-cut e of X and define $\overrightarrow{T}$ as the structure tree of X with respect to the tree set $E = Ge \cup Ge^*$. The group G acts on $\overrightarrow{T}$ and also on the undirected tree T, but we can not be sure that the action is without inversion. Suppose e corresponds to an edge $\{\alpha,\beta\}$ in T. If G acts on T without inversion then $G = G_\alpha *_{G_e} G_\beta$, or G is an HNN-extension $G_\alpha *_{G_e} x$. If G acts with inversion, i.e. there is an element in g such that $g\alpha = \beta$ and $g\beta = \alpha$, then $G = G_\alpha *_{G_e} H$ where H is the setwise stabilizer of the set $\{\alpha,\beta\}$. This says that G splits over a group $G_{\delta e}$ where e is some Dunwoody-cut in G and $G_{\delta e}$ denotes the subgroup of all elements in G that leave the set δe invariant. Note that if e is a cut then the set A_e of vertices in e that are adjacent to vertices in e^* is the set $\partial(e^*)$. The setwise stabilizer H_1 of e in G is equal to the intersection of the setwise stabilizers of ∂e and $\partial(e^*)$. Let H_2 denote the stabilizer of a vertex in X. Both ∂e and $\partial(e^*)$ are finite so H_1 is commensurable with H_2, meaning that both indices $|H_1 : H_1 \cap H_2|$ and $|H_2 : H_1 \cap H_2|$ are finite.

With further reference in mind we state the outcome of the considerations above.

Corollary 3. *Let G be a group acting transitively on a locally finite connected graph X with more than one end. Then G splits over a subgroup commensurable with the stabilizer in G of a vertex in X.*

3.2. **Stallings' Ends Theorem for rough ends.** The theorem from Section 2 that any two rough Cayley graphs of a compactly generated totally disconnected locally compact group are quasi-isometric and the result from Section 3.1.1 that locally finite graphs which are quasi-isometric have homeomorphic end spaces allow us to use rough Cayley graphs to define ends for compactly generated totally disconnected locally compact groups.

Definition 6. *The space of* rough ends *of a compactly generated totally disconnected locally compact group G is the end space of a rough Cayley graph of G.*

The following corollary to Theorem 3 links together the rough ends of a compactly generated totally disconnected locally compact group G and the rough ends of a closed cocompact subgroup.

Corollary 4. *Let G be a compactly generated totally disconnected locally compact group and H a closed cocompact subgroup. Then the spaces of rough ends of G and H are homeomorphic. In particular H has the same number of rough ends as G.*

Proof. By Corollary 2 a rough Cayley graph of G is quasi-isometric to a rough Cayley graph of H. Hence they must have homeomorphic spaces of rough ends. □

Let X_1 and X_2 be rough Cayley graphs for G. Suppose $\psi : X_1 \to X_2$ is a quasi-isometry like in Theorem 2^+. Let $\overline{\psi}$ and Ψ be as described at the end of Section 3.1.1. Because $\overline{\psi}$ is continuous, we see that if a sequence of vertices v_i in X_1 converges to an end ω in ΩX_1 then the sequence $\overline{\psi}(v_i)$ converges to the end $\overline{\psi}(\omega) = \Psi(\omega)$ in ΩX_2. The action of G on X_1 induces an action of G on ΩX_1. Hence gv_i converges to $g\omega$. Similarly, the sequence $g\psi(v_i)$ in X_2 must converge to the end $g\Psi(\omega)$ in ΩX_2 and the sequence $\psi(gv_i)$ must converge to the end $\Psi(g\omega)$. Because there is a constant c such that $d_{X_2}(g\psi(v_i), \psi(gv_i)) \leq c$ for all i, we can conclude that the sequences $g\psi(v_i)$ and $\psi(gv_i)$ converge to the same end of X_2. Thus $g\Psi(\omega) = \Psi(g\omega)$ and the map Ψ is covariant with the action of G. In this context one can also note that different choices of coset representatives when constructing the map ψ do not affect the map Ψ. These considerations are so fundamental in what follows that we state the results as a theorem.

Theorem 10. *Let X_1 and X_2 be rough Cayley graphs for some compactly generated totally disconnected locally compact group. Let $\psi : X_1 \to X_2$ be a quasi-isometry as in Theorem 2^+. There is a unique extension $\overline{\psi} : VX_1 \cup \Omega X_1 \to VX_2 \cup \Omega X_2$ of ψ whose restriction Ψ to ΩX_1 is a homeomorphism $\Omega X_1 \to \Omega X_2$ which is covariant with the actions of G on ΩX_1 and ΩX_2, i.e. $\Psi(g\omega) = g\Psi(\omega)$ for all $\omega \in \Omega X_1$ and $g \in G$.*

Now we turn our attention to group theoretic properties related to rough ends. Our aim is to show that a compactly generated totally disconnected locally compact group with more than one rough end splits over some compact open subgroup, and thus to derive an analogue of Stallings' Ends Theorem.

Let G act transitively on a locally finite connected graph X with infinitely many ends. Then Dunwoody's theory of structure trees yields an action on a directed tree with infinitely many ends which has at most two orbits on the edges, and consequently at most two orbits on the vertices. We will show how we also can go the other way.

If we start with an action of G on a tree T such that the stabilizers of edges are compact open subgroups of G and G has only finitely many orbits on the edges then we will show how T can be used to construct a tree set of Dunwoody-cuts of some rough Cayley graph. In our discussion it is sometimes convenient to think of T as a directed graph $\overrightarrow{T}$. This is purely a formal device to ease the presentation. The vertex sets of T and $\overrightarrow{T}$ are the same and each undirected edge $\{u,v\}$ in T is represented by two directed edges (u,v) and (v,u) in $\overrightarrow{T}$. In the following we will be discussing a tree set E, the set of edges $E\overrightarrow{T}$ of the directed tree $\overrightarrow{T}$ and ET the set of edges of T. These sets are related and we will typically use e and f to denote elements of E, for the corresponding elements of $E\overrightarrow{T}$ will be denoted with $\vec{e}$ and $\vec{f}$ and $\overline{e}$ and $\overline{f}$ for the corresponding elements of ET.

Let u be a fixed vertex in T and let E_u denote the set of edges in T with u as an end vertex. The set $U_{\overline{e},\overline{f}}$ of elements of G that map a given edge $\overline{e}$ of T to a given edge $\overline{f}$ of E_u is open because the edge stabilizers are open. The set of elements of G that map a given edge $\overline{e}$ of T to a an edge of E_u is thus open because it is the union of the open sets $U_{\overline{e},\overline{f}}$ for $f \in E_u$. Suppose $\{u,v\}$ and $\{u,w\}$ are distinct edges in T. Both

$$V_1 = \{g \in G \mid g\{u,v\} \in E_u\} \quad \text{and} \quad V_2 = \{g \in G \mid g\{u,w\} \in E_u\}$$

are open subsets of G. Their intersection is an open set and $G_u = V_1 \cap V_2$. Hence G_u is an open subgroup of G. (This conclusion is obviously also true if there is only one edge in T with u as an end vertex.) Because G_u is an open subgroup it is also a closed subgroup of G.

The stabilizer V of an edge $\{u,v\}$ in T is a compact open subgroup of G. Because G_u is both open and closed, the subgroup $G_u \cap V$ is open and compact and is equal to the stabilizer of the edge (u,v) in $\overrightarrow{T}$. Let U denote some compact open subgroup of G_u. Define X as $\mathrm{RCay}(G,U,S)$ where U together with

$S = \{s_1,\dots,s_n\}$ form a good generating set. Let $\vec{e} = (v,w)$ be an edge in $\overrightarrow{T}$. Remove the edge $\{v,w\}$ from T and the tree T splits into two subtrees $T_{\vec{e}}$ and $T_{(\vec{e})*}$, where $T_{\vec{e}}$ contains w and $T_{\vec{e}^*}$ contains v. Define

$$c_{\vec{e}} = \{gU \mid gu \in VT_{\vec{e}}\} \subset G/U = VX.$$

Note that $c_{(\vec{e})^*} = VX \setminus c_{\vec{e}} = (c_{\vec{e}})^*$. The assumption that $U \subseteq G_u$ guarantees that if $h \in gU$ then $hu = gu$.

Theorem 11. *Let G be a compactly generated totally disconnected locally compact group. Suppose G acts on a tree T such that the stabilizers of the edges are compact open subgroups of G and G has only finitely many orbits on the edges of T. Suppose $\overrightarrow{T}, U, S, X$ and u are as above.*

Then, for an edge $\vec{e}$ in $\overrightarrow{T}$ the set $c_{\vec{e}}$ is a cut in X and it is possible to choose S such that $c_{\vec{e}}$ is connected. The set $E = \{c_{\vec{e}} \mid e \in E\overrightarrow{T}\}$ is a tree set. If the map $E\overrightarrow{T} \to E$, $\vec{e} \mapsto c_{\vec{e}}$ is bijective and the sets $c_{\vec{e}}$ are connected then $\overrightarrow{T}$ is isomorphic to the structure tree $\overrightarrow{T}(E)$. If T has an edge $\overline{f}$ such that both components of $T \setminus \{\overline{f}\}$ contain infinitely many vertices from the orbit of the vertex u in T then X has more than one end.

Proof. Our first task is to prove that $c_{\vec{e}}$ is indeed a cut of X. This means we have to show that $\delta c_{\vec{e}}$ is finite. By Lemma 1, each orbit of $G_{\vec{e}}$, the stabilizer of the edge $\vec{e} = (v,w)$ in $\overrightarrow{T}$, on the edges of T is finite (note that T need not be locally finite) and thus every orbit $G_{\vec{e}}$ on the vertex set of T is also finite. We split the proof of the finiteness of $\delta c_{\vec{e}}$ up into two parts.

Define a graph Y such that the vertex set of Y is the same as the vertex set of T but the edge set of Y is

$$EY = G\{u,s_1u\} \cup \dots \cup G\{u,s_nu\}.$$

The group G acts on Y as a group of automorphisms. The distance in T between the end vertices of edges in Y is bounded, because G has only finitely many orbits on the edges of Y. We show that there are only finitely many edges in Y with one end vertex in $VT_{\vec{e}}$ and the other in $VT_{\vec{e}^*}$. Suppose there are infinitely many edges in Y with one end vertex in $VT_{\vec{e}^*}$ and the other in $VT_{\vec{e}}$. This would allow us to find infinitely many edges $\{v_i,w_i\}$ in EY with the following three properties:

(i) $v_i \in VT_{\vec{e}^*}$ and $w_i \in VT_{\vec{e}}$ for all i,
(ii) all these edges are in the same G-orbit, and
(iii) $d_T(v,v_i) = d_T(v,v_j)$ and $d_T(w,w_i) = d_T(w,w_j)$ for all i and j.

The third item follows from the fact that if $a = \min\{d_T(u,s_iu)\}$ then $d_T(v_i,w_i) \le a$ for all i and since $v_i \in T_{(\vec{e})*}$ and $w_i \in T_{\vec{e}}$ there are only finitely many possibilities for $d_T(v,v_i)$ and $d_T(w,w_i)$. An element in G that maps an edge $\{v_i,w_i\}$ to an edge $\{v_j,w_j\}$ must fix the edge $\{v,w\}$ in T. This leads to a contradiction since the stabilizer of the edge $\{v,w\}$ in G is compact and thus has only finite orbits on the edges of T, and therefore the orbits on the vertices of T are also finite. Whence there are only finitely many edges in Y with one end vertex in $VT_{\vec{e}^*}$ and the other in $VT_{\vec{e}}$.

Let Y' denote the subgraph spanned by the orbit of u under G. Both end vertices of an edge in Y always belong to VY'. Define a map $\theta : VX \to VY'$ such that $\theta(hU) = hu$. It follows from the assumption that $U \le G_u$ that θ is well defined. The fibers of θ form a G invariant equivalence relation $\sim$ on VX and the map θ is covariant with the action of G on these equivalence classes. From the way X and the edge set of Y are defined, it is clear that an edge in X is either mapped to an edge of Y' or both end vertices are mapped to the same vertex of Y'. Suppose there were infinitely many edges in X with one end vertex in $c_{\vec{e}}$ and the other in $c_{\vec{e}^*}$. These will be mapped to edges in Y' that have one end vertex in $VT_{\vec{e}}$ and the other in $VT_{\vec{e}^*}$. There are only finitely many such edges. Infinitely many of the edges in X with one end vertex in $c_{\vec{e}}$ and the other in $c_{\vec{e}^*}$ would then be mapped to the same edge e' in Y, and, because G has only finitely many orbits on the edges of X, infinitely many of these would belong to the same G-orbit. If f is an edge in X and f and gf are both mapped by θ to the same edge e' in Y then $g \in G_{e'}$. Hence the group $G_{e'}$ would have infinite orbits on the edges of X. But the stabilizer of the edge e' is a compact subgroup of G and would therefore have finite orbits on the edges of X. We have reached a contradiction and conclude that there are only finitely many edges in X with one end vertex in $c_{\vec{e}}$ and the other one in $c_{\vec{e}^*}$.

Let $\vec{e}$ now be some edge in $\overrightarrow{T}$. It is not clear that $c_{\vec{e}}$ is a connected subset of X, but it is clear that $c_{\vec{e}}$ has only finitely many connected components, because $\delta c_{\vec{e}}$ is finite. Choose one vertex from each component of $c_{\vec{e}}$ to get a collection of vertices $v_1,\ldots,v_m$. Now replace the graph X by the graph one gets by adding the sets $G\{v_i,v_j\}$ for $i\neq j$ to the set of edges in X. (One can also think of this in terms of adding to the set S group elements t_{ij} such that $\{U,t_{ij}U\}$ is in the same G-orbit as $\{v_i,v_j\}$.) Note that this new graph is locally finite and the set $c_{\vec{e}}$ is connected and so are also all the cuts in the G-orbit of $c_{\vec{e}}$. The group G has only finitely many orbits on the edges of T and thus only finitely many orbits on the cuts $c_{\vec{e}}$ one gets from T. Therefore one only needs to repeat the above construction finitely many times for $c_{\vec{e}_1},\ldots,c_{\vec{e}_k}$ where $\vec{e}_1,\ldots,\vec{e}_k$ are representatives for the orbits of G on the edges of $\overrightarrow{T}$ to get a locally finite connected graph X such that $c_{\vec{e}}$ is a connected cut for every edge $\vec{e}$ in $\overrightarrow{T}$.

Let E denote the set of all the cuts of X which we get in this way by removing the edges of $\overrightarrow{T}$. That E is a tree set follows from the construction, because the ordering of E by inclusion mirrors the ordering of $E\overrightarrow{T}$ given by the tree, precisely as in the relationship between a tree set of cuts and a structure tree. If this map is bijective then it gives an isomorphism of the tree $\overrightarrow{T}$ and the structure tree $\overrightarrow{T}(E)$. The last statement in the Theorem follows trivially because if $\{v,w\}$ is such an edge and $\vec{e}=(v,w)$ then both $c_{\vec{e}}$ and the complement of $c_{\vec{e}}$ contain infinitely many vertices and hence X must have more than one end. □

Remark 4. (i) If G is a finitely generated group then the theorem above says that every action of G on a tree with finite edge stabilizers corresponds to a tree set of Dunwoody-cuts of some ordinary Cayley graph for a finite generating set.

(ii) It is possible that two different edges in $\overrightarrow{T}$ give the same cut of X. Suppose that $\vec{e}=(v,w)$ and $\vec{e}'=(v',w')$ are edges in $\overrightarrow{T}$. The cuts $c_{\vec{e}}$ and $c_{\vec{e}'}$ are equal if and only if the trees $T_{\vec{e}}$ and $T_{\vec{e}'}$ contain precisely the same vertices from the orbit of u under G. It is possible to "prune" the tree T to get a new tree that G acts on where this situation does not arise. If $\vec{e}=(v,w)$ is an edge in $\overrightarrow{T}$ and $T_{\vec{e}}$ contains no vertex from the orbit Gu then we delete $T_{\vec{e}}$ together with the edge $\{v,w\}$ from T. When all subtrees of this form have been deleted we are left with a new tree T' containing the orbit of u under G. Note that the action of G on T restricts to an action on T'. It still may happen that there are edges $\overline{e}$ and $\overline{e}'$ in T such that corresponding directed edges in $\overrightarrow{T}'$ give rise to the same cut in E in X, but that must be because $T\setminus\{\overline{e},\overline{e}'\}$ has a component not including any vertices from Gu. Because of the pruning already done we can conclude that all the vertices in this component have degree 2. Hence there is a path $v_0,v_1,\ldots,v_{k-1},v_k$ in the tree T' that starts with the edge $\overline{e}$ and ends with the edge $\overline{e}'$ and all vertices in this path, with the possible exceptions of v_0 and v_k, have degree 2. Throw out the vertices $v_1,\ldots,v_{k-1}$ and put in a single edge $\{v_0,v_k\}$. When all such instances have been treated then we are left with a tree T'' that G acts on and the action on this tree gives rise to precisely the same cuts of X as the action on the original tree T but now the map from the edges of $\overrightarrow{T}''$ to the set of cuts is bijective. Hence T'' is isomorphic to the structure tree.

Theorem 1 says that a totally disconnected locally compact group acting transitively on a connected locally finite graph with compact open stabilizers of vertices is compactly generated. We use this result to prove that the stabilizer of a vertex in a structure tree (as in Theorem 11) is compactly generated. To do so we use the following construction from [53, Section 7]. Let X be a graph and $\overrightarrow{T}$ a structure tree of X with respect to some tree set consisting of tight cuts (i.e. each cut is a connected subset of VX). Suppose G is a group acting transitively on X. (Note that in [53, Section 7] the group action considered is the action of the full automorphism group of X, but all the arguments hold true for any transitive group action on X.) Given a vertex α in $\overrightarrow{T}$ we want to produce a connected subgraph X_α of X such that G_α acts with finitely many orbits on X_α.

For an element $e\in E$ (corresponding to an edge $\vec{e}$ of $\overrightarrow{T}$) and a natural number q we define $R_q(e)$ as the subgraph of X spanned by the vertices in e (where we think of e as a cut of X) that are at distance less or equal to q from the vertex boundary ∂e of e in X. Using the property that e is connected we can clearly choose q so large that the following condition (†) is satisfied:

(†) If $P_1,\ldots,P_r$ are pairwise edge-disjoint paths in $e\cup\partial e$ between vertices in ∂e, and all other vertices in these paths are contained in e then $R_q(e)\cup\partial e$ contains pairwise edge-disjoint paths $P_1',\ldots,P_r'$

such that P_i' and P_i have the same end vertices for all $i = 1, \dots, r$ and all vertices in the paths P_i' apart from the end vertices are in $R_q(e)$.

Because G acts with only finitely many orbits on $E\overrightarrow{T}$, we can find a number q such that $R_q(e)$ satisfies (†) for all $\vec{e} \in E\overrightarrow{T}$. Let X_α be the subgraph of X induced by the union of the set $\varphi^{-1}(\alpha)$ and the vertices in $R_q(e)$ for all $e \in E\overrightarrow{T}$ of the form (α, β).

Because G has only finitely many orbits on $E\overrightarrow{T}$, we know that G_α (the stabilizer of α in G) has only finitely many orbits on the set of edges in $\overrightarrow{T}$ with initial vertex α. Also note that the group G_α acts transitively on the vertices in $\varphi^{-1}(\alpha)$. The subgraph X_α of X is invariant under G_α and, by the above, G_α acts on X_α with only finitely many orbits. The ends of X that are mapped to α by the structure map are in a natural correspondence to the ends of X_α. It is instructive to go through the argument that shows this. Suppose R is a ray in an end ω that is mapped to α. Suppose $\vec{e} = (\alpha, \beta)$ is an edge in $\overrightarrow{T}$ and e is the corresponding cut in X. Then the end ω does not lie in e, so e will at most contain finitely many disjoint finite subpaths from R. The end vertices of these finite subpaths are all in ∂e. Now we use property (†) to replace each of these finite paths with a path in X_α that has the same end vertices. The resulting 1-way infinite path may have repeated vertices but because the graph is locally finite this 1-way infinite path will contain a ray R' which is clearly also in the end ω. Using (†) one can also show that two rays in X_α that belong to the same end of X must also belong to the same end of X_α. Hence the end of X_α that R' belongs to does not depend on the choice of the ray R in ω and we have the promised correspondence between ends of X mapped by the structure map to α and ends of X_α.

Theorem 12. *Suppose G is a compactly generated totally disconnected locally compact group and X is some rough Cayley graph of G. If $\overrightarrow{T}$ is a structure tree of X then the stabilizers of edges in $\overrightarrow{T}$ are compact open subgroups of G and the stabilizers of vertices in $\overrightarrow{T}$ are compactly generated subgroups of G that are both closed and open in G.*

Proof. An edge in $\overrightarrow{T}$ corresponds to a Dunwoody-cut e of X. A group element stabilizing an edge in T must stabilize (setwise) the coboundary of a Dunwoody-cut. The boundary δe is a finite set of edges and the stabilizer of each edge of X is a compact open subgroup of G. Hence we see that the stabilizer of an edge in $\overrightarrow{T}$ is a compact open subgroup of G.

Let us now look at the stabilizer of a vertex α in $\overrightarrow{T}$. Suppose that $\vec{e}$ is an edge in $\overrightarrow{T}$ that has α as one end vertex. Since G_α contains the compact open subgroup $G_{\vec{e}}$ we conclude that G_α is a closed open subgroup of G. The group G_α acts with finitely many orbits on the locally finite connected graph X_α with compact open stabilizers of the vertices. By Corollary 1, we can conclude that G_α is compactly generated. □

Theorem 13. *Let G be a compactly generated totally disconnected locally compact group.*

The group G splits over some compact open subgroup if and only if it has more than one rough end.

More precisely, if G has more than one rough end then $G = A *_C B$ or $G = A *_C x$ where the subgroups A and B are compactly generated and open, and C is a compact open subgroup.

Proof. If some rough Cayley graph of G has more than one end then we get an action of G on a tree such that the stabilizers of edges are compact open subgroups of G and the stabilizers of vertices are closed open subgroups of G that are compactly generated (by Theorem 12). Now we can refer to the Bass-Serre theory of groups acting on trees to get the result.

To prove the latter part of the theorem we use the Bass-Serre theory to get an action of G on a tree T with just one orbit on the edges and such that the stabilizers of edges are conjugates of C. First suppose that G can be written as an HNN-extension $G = A *_C x$. Then G acts on a tree with just one orbit on the vertices and clearly $\overrightarrow{T}$ is isomorphic to the structure tree we get from Theorem 11 and the result now follows from Theorem 12. Now suppose that $G = A *_C B$. Then G has two orbits on the vertices of the Bass-Serre tree T. If $\overrightarrow{T}$ is isomorphic to the structure tree we get from Theorem 11 then there is nothing more to do. But it could happen that T is not isomorphic to the structure tree. This will only happen when the vertices in one of the orbits on T have degree 2. Let us assume that B is the stabilizer of vertex of degree 2 in T. Then B is compact. When we prune the structure tree by removing the vertices of degree 2 we get a tree T' such that T is the barycentric subdivision of T' and $\overrightarrow{T'}$ is isomorphic to the structure tree we get from T. The stabilizers of vertices in this structure tree are conjugates of A which is therefore compactly generated. □

Remark 5. Abels, [1, Struktursatz 5.7 and Korollar 5.8], proves much the same result but his approach is very different. The methods used in this paper are compared with Abels' methods in Section 3.6.

Remark 6. In this context it is worth noting a result of Morris and Nicholas, [41], that a locally compact group that can be expressed as a nontrivial free product must be discrete.

Example 1. The group $\mathrm{SL}_2(\mathbb{Q}_p)$ is a free product with amalgamation of two copies of $\mathrm{SL}_2(\mathbb{Z}_p)$. Hence $\mathrm{SL}_2(\mathbb{Q}_p)$ has infinitely many rough ends.

Theorem 13 in conjunction with Corollary 4 give the following.

Corollary 5. *Let G be a compactly generated totally disconnected locally compact group and H a closed cocompact subgroup. Then G splits over a compact open subgroup if and only if H splits over a compact open subgroup.*

Theorem 14. (Cf. [40, Proposition 2.3]) *Let G be a compactly generated totally disconnected locally compact group. Suppose that the space of rough ends has precisely two points. Then G has a compact open normal subgroup N such that G/N is either isomorphic to $\mathbb{Z}$ or D_∞, the infinite dihedral group.*

Proof. Let X be a rough Cayley graph of G. Then X is a locally finite graph with two ends and G acts transitively on X. By [40, Proposition 2.3] there is a normal subgroup N with finite orbits (and thus compact closure) such that G/N is either equal to $\mathbb{Z}$ or D_∞. To complete the proof we have to show that N is open. The condition on G/N implies that if A_1 and A_2 are some two distinct orbits of N then the subgroup of G that stabilizes both A_1 and A_2 setwise is N. Since A_1 and A_2 are both finite we conclude that N is open. □

3.3. **Accessibility.** The usefulness of rough Cayley graphs can be demonstrated further by considering the concept of *accessibility*.

Definition 7. *A finitely generated group is said to be* accessible *if it has an action on a tree T such that:*

(i) *the number of orbits of G on the edges of T is finite;*
(ii) *the stabilizers of edges in T are finite subgroups of G;*
(iii) *every stabilizer of a vertex in T is a finitely generated subgroup of G and has at most one end.*

We only need to change the above definition slightly to fit into our framework.

Definition 8. *A compactly generated totally disconnected locally compact group is said to be* accessible *if it has an action on a tree T such that:*

(i) *the number of orbits of G on the edges of T is finite;*
(ii) *the stabilizers of edges in T are compact open subgroups of G;*
(iii) *every stabilizer of a vertex in T is a compactly generated subgroup of G and has at most one rough end.*

Accessibility also has a graph theoretical aspect.

Definition 9. ([53, p. 249]) *Let X be a connected locally finite graph. If there is a number k such that any two distinct ends can be separated by removing k or fewer edges from X then the graph X is said to be* accessible.

As pointed out in [44, Theorem 0.4], the property of a locally finite connected transitive graphs being accessible is preserved by quasi-isometries.

It was an open question for a long time if every finitely generated group is accessible. Dunwoody proved in [9] that every finitely presented group is accessible, but in [10] he constructed a finitely generated group that is not accessible. Thomassen and Woess prove in [53, Theorem 1.1] that a finitely generated group is accessible if an only if its Cayley graphs are accessible. They also proof that a locally finite connected transitive graph X is accessible if and only if there is a number n such that $\mathscr{B}_n X = \mathscr{B} X$ (see [53, Theorem 7.6], for notation see Theorem 6 above). With reference to Theorem 6 this implies that X is accessible if and only if there is a tree set $E_n \subseteq \mathscr{B}_n X$ that generates $\mathscr{B} X$.

The following is an analogue of the result of Thomassen and Woess.

Theorem 15. *Let G be a compactly generated totally disconnected locally compact group. Then G is accessible if and only if every rough Cayley graph of G is accessible.*

Proof. First we suppose that a rough Cayley graph X of G is accessible. Then we can find a structure tree $\overrightarrow{T} = \overrightarrow{T}(E)$ of X such that for any vertex α of $\overrightarrow{T}$ the graph X_α has at most one end, [53, Theorem 7.6 and Proposition 7.7]. This means that $\Phi^{-1}(\alpha)$ contains at most one end. The rough Cayley graphs of G_α are quasi-isometric to the one ended graph X_α (see Theorem 3). Hence the subgroup G_α has at most one rough end. The rest of the conditions in Definition 8 follow from Theorem 12.

Assume now that the group G is accessible. Let T be a tree that G acts on such that the conditions in Definition 8 are satisfied. By Theorem 11 we can view $\overrightarrow{T}$ as a structure tree of some rough Cayley graph X of G with respect to some tree set E (we may assume that $\overrightarrow{T}$ has been pruned, see Remark (ii) following the proof of Theorem 11) since the pruning process does not affect the properties listed in Definition 8. If the graph X is not accessible then there is a vertex α of $\overrightarrow{T}$ such that the graph X_α has more than one end. The compactly generated subgroup G_α acts on X_α with only finitely many orbits. By Theorem 3 any rough Cayley graph of G_α is quasi-isometric to X_α. The rough Cayley graphs for G_α will consequently have more than one end contradicting the assumptions on the action of G on $\overrightarrow{T}$. Hence X is accessible. Accessibility of locally finite transitive graphs is invariant under quasi-isometries. It follows that all rough Cayley graphs of G are accessible. □

3.4. **Co-compact free subgroups.** Let G be a finitely generated group and X a Cayley graph of G with respect to some finite generating set. It is a well known result, often attributed to Gromov but also found in a slightly different form in Woess' paper [58], that X is quasi-isometric to a tree if and only if G has a finitely generated free subgroup with finite index. For *graphs of groups* the reader is referred to Serre's book [50].

Theorem 16. *Let G be a compactly generated totally disconnected group with a co-compact finitely generated free subgroup.*

(i) *Some (hence, every) rough Cayley graph of G is quasi-isometric to some tree if and only if G has an expression as a fundamental group of a finite graph of groups such that all the vertex and edge groups are compact open subgroups of G.*
(ii) *Assume also that the group G is unimodular. Then some (hence, every) rough Cayley graph of G is quasi-isometric to some tree if and only if G has a finitely generated free subgroup that is cocompact and discrete.*

Proof. (i) Assume first that X is a rough Cayley graph of G and that X is quasi-isometric to some tree T. It follows from [58, Proposition 2] or from [30, Theorem 6] that a locally finite graph that is quasi-isometric to a tree has no thick ends. From [37] it is known that an inaccessible graph has uncountably many thick ends. Thus our graph X is accessible and there is a tree set $E_n \subseteq \mathscr{B}_n X$ which generates $\mathscr{B} X$. ¿From [53, Theorem 7.3] and [34, Lemma 4] one concludes that the structure tree $\overrightarrow{T} = \overrightarrow{T}(E)$ is locally finite. Suppose that e is an edge in X that is contained δf for some cut f in X and assume that α is an end vertex of the corresponding edge $\vec{f}$ in $\overrightarrow{T}$. The orbit of e under G_α is contained in the union of the coboundaries of the edges in $\overrightarrow{T}$ that have α as an end vertex. By assumption, there are only finitely many such edges in $\overrightarrow{T}$ and we conclude that the orbit $G_\alpha e$ is finite. Thus G_α is relatively compact. The stabilizer in G of a vertex v in $\overrightarrow{T}$ is a closed and open subgroup by Theorem 12. Hence G_α is a compact open subgroup of G. The action of G on the structure tree $\overrightarrow{T}$ gives an expression of G as a fundamental group of a graph of groups such that both the edge and vertex groups in this graph of groups are compact open subgroups of G.

Conversely, assume that G has an expression as a fundamental group of a graph of groups where all the vertex and edge groups are compact open subgroups of G. This gives us an action of G on a locally finite tree such that all the stabilizers of vertices are compact open subgroups of G and G has only finitely many orbits on the vertex set of the tree. By Theorem 3 the graph T is quasi-isometric to some rough Cayley graph of G.

(ii) Suppose first that G is unimodular with a rough Cayley graph that is quasi-isometric to some tree. By the above we get an action of G on a locally finite tree $\overrightarrow{T}$ such that G has only finitely many orbits on the vertices of $\overrightarrow{T}$ and the stabilizers of vertices are compact open subgroups of G. It follows from [2, Section 4] that G contains a discrete finitely generated free subgroup F acting with finitely many orbits on $\overrightarrow{T}$. Hence F is cocompact in G.

Now suppose that G has a cocompact discrete finitely generated free subgroup F. The free group F has a (rough) Cayley graph Y that is a tree. Since F is cocompact we can refer to Theorem 3 and conclude that if X is a rough Cayley graph of G then X is quasi-isometric to the tree Y. □

Corollary 6. *Let G be a compactly generated totally disconnected locally compact group. If G has a cocompact finitely generated free discrete subgroup then G splits over some compact open subgroup and G can be written as $G = A *_C B$ or $G = A *_C x$ where A, B and C are compact open subgroup of G.*

Proof. The free subgroup has a tree as a Cayley graph and, by Corollary 2(ii), a rough Cayley graph for G will be quasi-isometric to this tree. The group G is unimodular since it has a discrete cocompact subgroup and now the result follows from the latter half of Theorem 16. □

Example 2. Before it was mentioned that $\mathrm{SL}_2(\mathbb{Q}_p)$ is a free product with amalgamation of two copies of $\mathrm{SL}_2(\mathbb{Z}_p)$ which are compact open subgroups. Hence the rough Cayley graphs of $\mathrm{SL}_2(\mathbb{Q}_p)$ are all quasi-isometric to trees and, because $\mathrm{SL}_2(\mathbb{Q}_p)$ is unimodular, then $\mathrm{SL}_2(\mathbb{Q}_p)$ has a cocompact discrete finitely generated free subgroup.

3.5. **Types of automorphisms of graphs.** The automorphisms of a connected graph can be split up into three classes: elliptic, parabolic and hyperbolic (see Section 3.1.2). In this section we explore this classification further, considering the case of a compactly generated totally disconnected locally compact group acting on a rough Cayley graph.

Theorem 17. *Let G be a compactly generated totally disconnected locally compact group. Whether an element $g \in G$ acts on a rough Cayley graph X as an elliptic, parabolic or hyperbolic element does not depend on the choice of the rough Cayley graph X.*

Proof. Lemma 1 says that g acts on a rough Cayley graph as an elliptic automorphism if and only if g is a periodic element of the topological group G, i.e. the cyclic subgroup generated by g is relatively compact in G.

Suppose that X_1 and X_2 are two rough Cayley graphs for G. By Theorem 10 above, there is a homeomorphism $\Psi : \Omega X_1 \to \Omega X_2$ that is covariant with the action of G. If g acts like a parabolic automorphism on X_1 then g is not periodic and g fixes precisely one end of X_1. Since the homeomorphism Ψ is covariant with the action of G on ΩX we see that g fixes precisely one end of X_2 and acts as a parabolic automorphism on X_2.

That g acts like a hyperbolic automorphism on X_2 if g acts like a hyperbolic automorphism on X_1 is proved in the same way. □

This theorem allows us to speak of the elements of G as elliptic, parabolic or hyperbolic without any reference to the action of G on a particular rough Cayley graph. It is also possible to describe the properties of being elliptic, parabolic or hyperbolic in more group theoretic terms.

Theorem 18. *Let G be a compactly generated totally disconnected locally compact group with infinitely many ends.*

(i) *An element g in G is elliptic if and only if g is a periodic element of G.*

(ii) *An element in g in G is parabolic if and only if g is not elliptic and whenever G acts on a tree such that the stabilizers of the edges in T are compact open subgroups of G then g fixes a vertex.*
(iii) *An element g in G is hyperbolic if and only if G has an action on a tree such that G acts transitively on the set of neighbours of any vertex, the edge stabilizers are compact open subgroups and g acts as a hyperbolic element.*

This can also be phrased in terms of graphs of groups and splittings of groups. The second item says that when G is written as a graph of groups with compact open edge groups then a parabolic element will always belong to a conjugate of a vertex group. The third item says that G splits over a compact open subgroup C such that $G = A *_C B$ or $G = A *_C x$ and g is not contained in a conjugate of A or B.

of Theorem 18. Item (i) follows immediately from Lemma 1.

We will first look at item (iii). Suppose g is hyperbolic. Let X be some rough Cayley graph of G. Then g fixes two ends of X. Let e be a Dunwoody-cut of X that separates these two ends. Note that the stabilizer of e is a compact open subgroup C of G. Let E denote the tree set $E = Ge \cup Ge^*$. The action of G on $\overrightarrow{T} = \overrightarrow{T}(E)$ gives us the desired action on a tree (see Lemma 3).

Suppose now that G acts on a tree T as described in (iii). We may clearly assume that T has no vertices of degree 2. Let C denote the stabilizer in G of some edge in T. The condition that the stabilizer in G of a vertex acts transitively on all adjacent vertices implies that G acts transitively on the edges of T. Suppose $\vec{e}$ is an edge in $\overrightarrow{T}$ that separates the two ends of T fixed by g. The edge $\vec{e}$ corresponds to a Dunwoody-cut f of some rough Cayley graph X and the related structure tree $\overrightarrow{T}(Gf \cup Gf^*)$ is isomorphic to $\overrightarrow{T}$ (the conditions on T imply that there is no need for pruning as described in Remark (ii) following the proof of Theorem 11). Hence g acts like a hyperbolic automorphism on X and g is hyperbolic.

Part (ii) follows directly from (iii) and Theorem 11. □

These ideas about types of automorphisms can also be linked to the concept of accessibility.

Corollary 7. *Let G be a compactly generated totally disconnected locally compact group. Then G is accessible if and only if G has an action on a tree such that the stabilizers of edges are compact open subgroups of G and all the hyperbolic elements in G act as hyperbolic automorphisms on T.*

Proof. Let X be a rough Cayley graph of G. If G is accessible then the graph X is accessible. Thus there is a number n and a tree set E_n of Dunwoody-cuts of X that generates the Boolean ring $\mathscr{B}X$ and $\mathscr{B}X = \mathscr{B}_n X$. Let $\overrightarrow{T} = \overrightarrow{T}(E)$ be the structure tree. By Theorem 12, the stabilizers of edges in $\overrightarrow{T}$ are compact open subgroups of G and stabilizers of vertices are compactly generated. If $g \in G$ is a hyperbolic element then there is a cut in E that separates the two ends that g fixes and we see that g acts like a hyperbolic automorphism on $\overrightarrow{T}$ (see the proof of [34, Corollary 1 (iii)]).

Conversely, suppose we have an action of G on a tree T satisfying the conditions above. We may assume that the tree has been "pruned" (see the remark following the proof of Theorem 11), because if the conditions in the corollary were satisfied before "pruning" then they will also be satisfied after pruning. ¿From the action of G on T we can get a tree set of cuts of some rough Cayley graph X (see Theorem 11). If α is a vertex in T then G_α acts on the graph X_α with only finitely many orbits. The graph X_α is quasi-isometric to a rough Cayley graph of G_α. If X_α had more than one end then it would follow from Theorem 4 (see also [53, Lemma 8.3]) that G_α would contain an element that acted like a hyperbolic automorphism on X_α and this element would also act like a hyperbolic element on X. The assumed absence of hyperbolic elements from G_α ensures thus that X_α has only one end. Hence X is accessible. □

Corollary 8. *Let G be a compactly generated totally disconnected locally compact group. Then G is accessible if and only if G can be expressed as a fundamental group of a finite graph of groups such that all the edge groups are compact open, all vertex groups are compactly generated and no hyperbolic element of G is contained in a vertex group.*

Proof. This Corollary is a translation of Corollary 7 into terms involving graphs of groups. □

In Section 3.1.2 the direction of a parabolic or hyperbolic automorphism of a graph X is defined. The set of *directions of the graph* X, denoted $\mathscr{D}(X)$, is the set

$$\{\mathscr{D}(g) \mid g \in \mathrm{Aut}(X) \text{ acts as a parabolic or hyperbolic automorphism on } X\}.$$

Suppose that X_1 and X_2 are two rough Cayley graphs for some compactly generated totally disconnected locally compact group G. If ω is the direction of g when g acts on X_1 then $\Psi(\omega)$ (for the definition of Ψ see Theorem 10) is the direction of g when g acts on X_2. Hence, the restriction of Ψ to $\mathscr{D}(X_1)$ is a homeomorphism $\mathscr{D}(X_1) \to \mathscr{D}(X_2)$ (see Theorem 10). Also note that if g and h have the same direction when acting on X_1 they will also have the same direction when acting on X_2. Thus we are justified in talking about the directions of an element in G without referring to a particular action of G on a rough Cayley graph.

Remark 7. The case of a finitely generated group acting on a Cayley graph is a special case of the results above.

Theorem 19. *Let G be a compactly generated totally disconnected locally compact group with more than one rough end. Suppose g and h are hyperbolic elements of G. Then $\{\mathscr{D}(g), \mathscr{D}(g^{-1})\} = \{\mathscr{D}(h), \mathscr{D}(h^{-1})\}$ if and only if the closure of the group $\langle g, h\rangle$ in G has precisely two rough ends.*

Proof. Define H as the closure in G of the group $\langle g, h\rangle$. Assume $\{\mathscr{D}(g), \mathscr{D}(g^{-1})\} = \{\mathscr{D}(h), \mathscr{D}(h^{-1})\}$. We may assume that $\mathscr{D}(g) = \mathscr{D}(h)$ and $\mathscr{D}(g^{-1}) = \mathscr{D}(h^{-1})$. Let X be some rough Cayley graph of G. Let e be a Dunwoody cut separating the two ends that g fixes. The group G acts on the structure tree $\overrightarrow{T} = \overrightarrow{T}(Ge \cup Ge^*)$ and g and h act hyperbolically on $\overrightarrow{T}$. Since g and h have the same fixed ends in X, they also have the same fixed ends in $\overrightarrow{T}$. Hence there is a line L in $\overrightarrow{T}$ such that both g and h act like translations on L, fixing both ends of L. The stabilizer in H of a vertex α on L must be open and compact in H since the stabilizer of a vertex must fix the whole line and stabilizers of edges are open and compact in H. Thus the line L is a rough Cayley graph of H and H has only two rough ends.

Now suppose that H has exactly two rough ends. Let $X = \mathrm{RCay}(G, U, S)$ be a rough Cayley graph of G and assume that g and h are contained in S. Let v denote the vertex U in X. The subgraph Y of X that is spanned by the orbit Hv is connected (because $\{v, gv\}$ and $\{v, hv\}$ are edges in X). The graph Y is a rough Cayley graph of H and thus, by assumption, has two ends. These ends are the directions of g and g^{-1} as well as the directions of h and h^{-1}, and therefore $\{\mathscr{D}(g), \mathscr{D}(g^{-1})\} = \{\mathscr{D}(h), \mathscr{D}(h^{-1})\}$. □

Example 3. Looking at the above theorem one is led to ask about the relationship between the number of ends of the finitely generated subgroup $\langle g, h\rangle$ of G and the number of rough ends of the closure of $\langle g, h\rangle$ in G. The following example shows that $\langle g, h\rangle$ can have just one end whilst its closure can have more than one rough end.

Define a graph X such that X consists of two disjoint lines $\ldots, x_{-1}, x_0, x_1, \ldots$ and $\ldots, y_{-1}, y_0, y_1, \ldots$ with additional edges $\{x_i, y_{i+1}\}$ and $\{y_i, x_{i+1}\}$ for all i. The graph X has two ends. The automorphism group of X has a subgroup G of index 2 that fixes both the ends of X. Let g be an automorphism of X such that $g(x_i) = x_{i+1}$ and $g(y_i) = y_{i+1}$ for all i and let f be an automorphism that transposes x_0 and y_0 but fixes all other vertices. Set $h = gf$. The group $\langle g, h\rangle$ can translate along the lines and also interchange x_i and y_i for finitely many values of i (the group $\langle g, h\rangle$ is the restricted wreath product of $\mathbb{Z}_2$ with $\mathbb{Z}$ and has appeared in the literature recently as *the lamplighter group*) and it has only one end. The closure of $\langle g, h\rangle$ in G is equal to G (it is the unrestricted wreath product of $\mathbb{Z}_2$ with $\mathbb{Z}$) where one can translate, and also interchange x_i and y_i on any set of values for i. The closure of $\langle g, h\rangle$ is G and is a compactly generated locally compact group with two rough ends.

Corollary 9. *Let G be a finitely generated group with more than one end. Suppose g and h are hyperbolic elements of G. Then $\{\mathscr{D}(g), \mathscr{D}(g^{-1})\} = \{\mathscr{D}(h), \mathscr{D}(h^{-1})\}$ if and only if the group $\langle g, h\rangle$ has precisely two ends.*

Proof. The same argument as in the proof of Theorem 19 can be used here, indeed if we view G as a having the discrete topology this is just a special case of Theorem 19. □

Suppose now that X is an infinite connected locally finite graph. Jung and Watkins [26, Lemma 5.2 and Theorem 5.13] prove that if there is some automorphism of X that has only finitely many orbits then X has precisely two ends. For the following corollary, recall that if a totally disconnected locally compact group G acts on a set Ω with open and compact point stabilizers then a subgroup H of G is cocompact if and only if H has only finitely many orbits on Ω, see the remarks before Corollary 2.

Corollary 10. *Let G be a compactly generated totally disconnected locally compact group with more than one rough end. Suppose g and h are hyperbolic elements of G. Then* $\{\mathscr{D}(g),\mathscr{D}(g^{-1})\} = \{\mathscr{D}(h),\mathscr{D}(h^{-1})\}$ *if and only if* $\langle h\rangle$ *is cocompact in the closure of* $\langle g,h\rangle$ *in G.*

The following is well know, e. g., see a more general result in the same vein in [25, Theorem 2.5].

Corollary 11. *Let G be a compactly generated totally disconnected locally compact group with infinitely many ends. Let X be a rough Cayley graph of G. Suppose g and h are hyperbolic elements of G such that there is no end of X fixed by both g and h. Then there are integers n and m such that* $\langle g^n,h^m\rangle$ *is a free group.*

Let us now look at the special case of a finitely generated group acting on its Cayley graph. The next result answers a question of Pavone, [46, p. 69].

Theorem 20. *Let G be a finitely generated group and S a finite generating set. Define X as the Cayley graph of G with respect to the generating set S. Suppose g and h are elements of G that act on X as hyperbolic automorphisms. If* $\mathscr{D}(g)=\mathscr{D}(h)$ *then* $\mathscr{D}(g^{-1})=\mathscr{D}(h^{-1})$*.*

Proof. Suppose $\mathscr{D}(g)=\mathscr{D}(h)$, but $\mathscr{D}(g^{-1})\neq\mathscr{D}(h^{-1})$. We may assume that both g and h belong to the generating set S, because adding them to S will not change the assumptions that $\mathscr{D}(g)=\mathscr{D}(h)$ and $\mathscr{D}(g^{-1})\neq\mathscr{D}(h^{-1})$. Set $H=\langle g,h\rangle$. The end $\mathscr{D}(g)$ is fixed by H.

Let X' be the Cayley graph of H with respect to the generating set $S'=\{g,h\}$. The graph X' can be regarded as a subgraph of X. Let F be a finite set of vertices of X such that the distinct ends $\mathscr{D}(g)$, $\mathscr{D}(g^{-1})$ and $\mathscr{D}(h^{-1})$ belong to distinct components of $VX\setminus F$. Each of these components will contain infinitely many vertices from X'. There are integers n_1,n_2,n_3 such that all the elements $g^{n_1},g^{n_1+1},\ldots$ are contained in the component containing $\mathscr{D}(g)$, all the elements $g^{n_2},g^{n_2-1},\ldots$ are contained in the component containing $\mathscr{D}(g^{-1})$ and all the elements $h^{n_3},h^{n_3-1},\ldots$ are contained in the component containing $\mathscr{D}(h^{-1})$. Hence $VX'\setminus F$ has at least three infinite components in X'. Thus the group H has infinitely many ends. But a group with infinitely many ends acting on its Cayley graph can not fix an end (see, [51, Proposition 2 and Corollary 5]).

The assumption that $\mathscr{D}(g^{-1})\neq\mathscr{D}(h^{-1})$ leads to contradiction and we conclude that $\mathscr{D}(g^{-1})=\mathscr{D}(h^{-1})$. □

In his paper Pavone proves a similar result when G is a finitely generated word hyperbolic group with infinite boundary, see [46, Theorem 3]. The following corollary is an analogue to [46, Corollary 4].

Corollary 12. *Let G be a finitely generated group with infinitely many ends. Set* $\mathscr{D}(G)=\{\mathscr{D}(g)\mid g\in G,\ g \text{ is not elliptic.}\}$*. The map* $f:\mathscr{D}(G)\to\mathscr{D}(G)$ *such that* $f(\mathscr{D}(g))=\mathscr{D}(g^{-1})$ *is well defined and discontinuous at every point of* $\mathscr{D}(g)$*.*

Proof. That f is well defined follows from Theorem 20. We have to show that f is discontinuous in every point. Let X be a finitely generated Cayley graph of G. A basis for the topology of ΩX (the end space of G) consists of all sets of the form ΩC where C is a cut of X. Sets of the type $\Omega C\cap\mathscr{D}(G)$ form a basis of the subspace topology on $\mathscr{D}(G)$. Because X has infinitely many ends, there are disjoint base elements U, V and W such that $\omega\in U$ and $f(\omega)\in V$. By the bilateral denseness of the directions of G (see discussion at the end of Section 3.1.2), there is a hyperbolic element h in G such that $\mathscr{D}(h)\in U$ and $\mathscr{D}(h^{-1})\in W$. Then $f(\mathscr{D}(h))=\mathscr{D}(h^{-1})\in W$ and $f(\mathscr{D}(h))\notin V$. Whence f is discontinuous at ω. □

Example 4. Theorem 20 does not generalize to compactly generated totally disconnected locally compact groups. Let T denote the regular tree of degree 3. Set $G=\mathrm{Aut}(T)$. Then G with the permutation group topology is a compactly generated totally disconnected locally compact group and T is a rough Cayley graph of G. Let G_ω be the subgroup of G that fixes a given end ω of T. This subgroup is closed in G and is thus a totally disconnected locally compact group and T is also a rough Cayley graph of G_ω. Since G_ω acts transitively on T and the stabilizers of vertices are compact open subgroups of G_ω, we see from the latter part of Theorem 1 that G_ω is compactly generated. Suppose now that g is an hyperbolic element in G_ω. The only ends fixed by g are $\mathscr{D}(g)$ and $\mathscr{D}(g^{-1})$ so ω must be equal to either $\mathscr{D}(g)$ or $\mathscr{D}(g^{-1})$. Say $\omega = \mathscr{D}(g^{-1})$. The group G_ω fixes only the end ω. So G_ω contains an element f such that $f(\mathscr{D}(g)) \neq \mathscr{D}(g)$. Thus if $h = fgf^{-1}$ then $\mathscr{D}(h^{-1}) = \mathscr{D}(g^{-1})$ but $\mathscr{D}(h) \neq \mathscr{D}(g)$.

Remark 8. Pavone mentions that his [46, Theorem 3] could also be deduced from the theory of convergence groups originating from [15]. Bowditch [3] has shown that if G acts on a locally finite connected graph X with infinitely many ends and the stabilizers of edges in X are all finite then the action on the ends of X is a convergence action. Theorem 20 now follows from [15, Corollary 6.9]. The situation concerning general group actions on infinite end spaces of graphs is different as can be seen from Example 4.

3.6. **Specker compactifications.** Stalling's Ends Theorem says that whether or not a finitely generated group splits over a finite subgroup depends on the number of ends of a Cayley graph with respect to some finite set of generators. Much work has been done on extending and generalizing Stalling's theorem. In this section and the next one, we will look at some of this work and how the concepts introduced relate to the present work. First we look at Abels' construction of Specker compactifications of compactly generated locally compact groups, see Sections 2, 3 and 5 in [1]. Below, Abels' construction is described in graph theoretic terms. We will show that if G is a compactly generated totally disconnected locally compact group then the ideal points in his compactification can be identified with the rough ends of G. Abels uses his construction to derive an analogue of Stallings' Ends Theorem which is roughly the same as our Theorem 13.

Definition 10. *A Specker compactification of a topological group G is a compact space $\hat{G}$ containing G such that*

(i) *G is dense in $\hat{G}$,*
(ii) *$\Omega = \hat{G} \setminus G$ is totally disconnected,*
(iii) *the right regular action of G on itself extends to a trivial action on Ω (i.e. extends to the identity on Ω).*
(iv) *the left regular action of G on itself extends to an action by homeomorphisms on Ω.*

Note that G is open and therefore Ω is closed in the compact space $\hat{G}$. Hence the "boundary" Ω is compact. Abels defines a *topological graph* as a connected graph X with the additional structure that the vertex set is a locally compact topological space and with the property that if K is a compact set of vertices then the vertex boundary ∂K is relatively compact (i.e. has compact closure). Consider connected components when a relatively compact set of vertices is removed from X. The "boundary" Ω of X can now be constructed by using inverse limits in much the same way as in the construction of the ordinary ends of a graph described in Section 3.1.1. The only difference is that the word "finite" is replaced with "relatively compact".

The definition of Ω could also be described by considering equivalence classes of rays. Define a ray R in X to be *properly non-compact* if the intersection of R with any compact subset is finite. If R is a properly non-compact ray and K is a relatively compact set of vertices in X, then we see that only one component of $VX \setminus K$ contains infinitely many vertices of R. Define two properly non-compact rays R_1 and R_2 to be equivalent if whenever K is a relatively compact set of vertices then the same component of $VX \setminus K$ contains infinitely many vertices from both R_1 and R_2. The points in the boundary Ω can now be defined as the equivalence classes of properly non-compact rays. If C is a set of vertices in X which has a relatively compact vertex boundary then we say that an equivalence class of rays belongs to C if C contains infinitely many vertices from some (equivalently, any) ray in the equivalence class. If ∂C is relatively compact then define $\overline{C}$ as the union of C with all the equivalence classes of properly non-compact rays that belong to C. A basis for the open neighbourhoods of a point ξ in the boundary consists of all sets $\overline{C}$ where C is an open set of vertices with relatively compact vertex boundary such that ξ is an element of $\overline{C} \setminus C$.

Let G be a compactly generated totally disconnected locally compact group with infinitely many rough ends. Suppose U is a compact open subgroup of G and that U together with a finite set T forms a good generating set. Let X be the ordinary (undirected) Cayley graph of G with respect to the generating set $U \cup T$. The graph X is an example of a topological graph as described above. The subgraphs of X induced by the left cosets of U are all complete graphs. The rough Cayley graph $X' = \mathrm{RCay}(G,U,T)$ is a quotient graph of X where we contract each left coset of U to a single vertex. Let $\pi : X \to X'$ be the quotient map. The image under π of a properly non-compact ray R in X is not necessarily a ray, but since R is properly non-compact this image will be infinite and since X' is locally finite it will contain some ray R'. Two rays in the image of R will clearly belong to the same end of X'. We can also conclude that two properly non-compact rays R_1 and R_2 in X are in the same equivalence class if and only if the rays R'_1 and R'_2 in X' belong to the same rough end. If we start with a ray in X' then one can find a properly non-compact ray in X that projects onto our given ray in X'. Hence there is a one-to-one correspondence between the rough ends of G and the ideal points in the Specker compactification that Abels defines. Abels deduces an analogue of Stalling's Ends Theorem (see [1, Struktursatz 5.7 and Korollar 5.8]) using methods derived directly from Stalling's proof. Abels result on splittings of groups where the Specker-compactification has infinitely many ideal points is roughly equivalent to Theorem 13.

A priori it seems that Abels' treatment of groups whose Specker-compactification has infinitely many ideal points is more general in scope than our treatment of groups with infinitely many rough ends. But Abels proves that if a compactly generated locally compact group has a Specker-compactification with more than two points then the group must contain a compact open subgroup (see [1, Section 5]). The results in this paper are stated for compactly generated totally disconnected locally compact groups, but it is clear that instead of assuming that the group is totally disconnected it is enough to assume that the group contains a compact open subgroup.

Remark 9. In [27] the first author discusses *metric ends* of graphs. A metric ray is a ray whose infinite subsets are all unbounded. Two metric rays are equivalent if they cannot be separated by a bounded set of vertices. That is, two metric rays are equivalent if whenever we remove a bounded set of vertices then all but finitely many vertices of the two metric rays will always lie in the same component. Metric ends (or proper metric ends in [27]) are the corresponding equivalence classes of metric ends. For locally finite graphs the metric ends are just the same as the ordinary ends. Abels proves, [1, Item 2.3], that in a topological graph a set of vertices is relatively compact if and only if it has finite diameter. From this it follows that the ideal points in Abels' compactification can be identified with the metric ends of the topological graph. The quotient map $\pi : X \to X'$ discussed above is a quasi-isometry and extends to a homeomorphism between the spaces of metric ends of X and X', see [27, Theorem 6].

3.7. **Ends of pairs of groups.** The number of ends of a finitely generated group determines whether or not the group splits over a finite subgroup. Suppose a subgroup C of G is given. We seek a way to define the number of ends of G "relative" to the subgroup C. The aim would then be to show that if this number of ends is greater than 1 then G splits over C or over some subgroup closely related to C. Before discussing two different definitions of the number of ends of G "relative" to C and how these concepts relate to our rough ends, we need some preliminary discussion.

Let G be a finitely generated group and X some Cayley graph of G with respect to some finite generating set. The number of ends of G can be defined as the supremum of the number of infinite connected components when a finite set of vertices is removed from the graph.

Definition 11. *Let X be a Cayley graph of a (finitely generated) group G with respect to some (finite) generating set S. Suppose C is a subgroup of G.*

(i) *Define ${}_CX$ as the quotient graph of X with respect to the right cosets of C (the quotient with respect to the left regular action of C on the vertex set of X). Let ${}_C\pi$ denote the quotient map ${}_C\pi : X \to {}_CX$.*
(ii) *Define X_C as the quotient graph of X with respect to the left cosets of C (the quotient with respect to the right regular action of C on the vertex set of X). Let π_C denote the quotient map $\pi_C : X \to X_C$.*

Remark 10. Note that that X_C is equal to $\mathrm{RCay}(G,C,S\setminus C)$.

The two graphs ${}_CX$ and X_C can be very different. The group G will act transitively on X_C by left multiplication as a group of automorphisms, but the graph ${}_CX$ does not necessarily have a transitive group of automorphisms (see Example 5).

Definition 12. *Let G be a group and C a subgroup of G. A subset of G is called* right-C-finite *if it can be covered with finitely many right cosets of C. A set which is not right-C-finite is called* right-C-infinite. Left-C-finite *and* left-C-infinite *sets are defined in the same way using left cosets of C.*

Below there are two definitions of the number of ends of G relative to a subgroup C that have been discussed in the literature. The notion of *ends of pairs of groups* appeared first in papers by Houghton [23] and Scott [48]. A variant of the idea of ends relative to a subgroup was introduced by Kropholler and Roller in [31]. Here we use the name *coends* for Krophollers and Rollers concept and use the following reformulation, due to Bowditch [4], as a definition.

Definition 13. *Let G be a finitely generated group and C a subgroup of G. Define X as a Cayley graph of G with respect to some finite generating set.*

(i) (Cf. [48, Lemma 1.1]) *The number of ends of the pair G and C, denoted by $e(G,C)$, is defined as the number of ends of the graph ${}_CX$.*
(ii) *The number of* coends*, denoted $\tilde{e}(G,C)$, of C is defined as the maximum number of right-C-infinite components of X when a right-C-finite set of vertices in X is removed.*

It is simple to show that $e(G,C)$ and $\tilde{e}(G,C)$ do not depend on the choice of generators used to construct X. In this section we will consider these concepts from a graph theoretical viewpoint and describe how these concepts relate to the work of the present paper. For a broader discussion and a comparison of these concepts the reader is referred to the survey paper by Wall [57].

Theorem 21. *Let G be a compactly generated totally disconnected locally compact group and U a compact open subgroup of G. Define X as the ordinary Cayley graph of G with respect to some compact generating set. The graphs ${}_UX$ and X_U are both connected and locally finite.*

Proof. It is obvious that the graphs ${}_UX$ and X_U are both connected. Let K denote the compact generating set used to construct X. Suppose that A is a set of vertices in X. The vertex boundary of A in X is contained in the set AK. If A is relatively compact then the set AK is also relatively compact and can be covered with finitely many left (right) cosets of U. Thus we see that if A is a left (right) coset of U then the set of neighbours of A in ${}_UX$ (x_U) is finite. When we form the quotient of X by the right (left) cosets of U we get a locally finite graph. □

Instead of using the topology on G we can impose a group theoretic condition on G and C that allows us to get the same result.

Definition 14. *Two subgroups H and K in G are said to be* commensurable *if $H\cap K$ has finite index in both H and K. The* commensurator *of H is the subgroup of those elements $g\in G$ for which gHg^{-1} is commensurable with H.*

When U is a compact open subgroup of a topological group G then every conjugate of U is commensurable with U.

Theorem 22. *Let G be a finitely generated group and $X=\mathrm{Cay}(G,S)$ a Cayley graph of G with respect to some finite generating set S. Suppose C is a subgroup of G.*

(i) *The graph ${}_CX$ is locally finite.*
(ii) *The commensurator of C is G if and only if each left coset of C is contained in a union of finitely many right cosets and vice versa.*
(iii) *The commensurator of C is G if and only if X_C is locally finite.*

Proof. (i) The vertices of ${}_CX$ are the orbits of C when C acts on the Cayley graph X from the left. Each element g in the vertex set of X has neighbours in only finitely many orbits $Ch_1,\dots,Ch_k$ of C. The neighbours of all the vertices in the orbit Cg will be contained in the union of $Ch_1,\dots,Ch_k$. Hence the degree of a vertex Cg in ${}_CX$ is at most equal to the degree of the vertex g in X. Therefore the graph ${}_CX$ is locally finite.

(ii) Assume the commensurator of C is G. Let us consider a left coset gC of C. The group $C\cap gCg^{-1}$ has finite index in gCg^{-1}. Thus gCg^{-1} can be covered with finitely many right cosets of C, i.e. $gCg^{-1}\subseteq Ch_1\cup\dots\cup Ch_n$. Multiplying on the right with g we get $gC\subseteq Ch_1g\cup\dots\cup Ch_ng$. The prove that each right coset can be covered with finitely many left cosets is similar.

Assume now that each left coset of C is contained in a union of finitely many right cosets and vice versa. Let g be an element in G. Find elements $h_1,\dots,h_n$ in G such that $g^{-1}C\subseteq Cg^{-1}h_1\cup\dots\cup Cg^{-1}h_n$. Then $C\subseteq(gCg^{-1})h_1\cup\dots\cup(gCg^{-1})h_n$ implying that the group $C\cap gCg^{-1}$ has finite index in C. The proof that $C\cap gCg^{-1}$ has finite index in gCg^{-1} is similar.

(iii) Suppose the commensurator of C is G. Let $S=\{s_1,\dots,s_n\}$ denote the set of generators used to construct X. Since the graph X_C is transitive we only need to consider the degree of the vertex in X_C represented by the left coset C. All the vertices in X which are adjacent to some element in C are contained in the right cosets $Cs_1,\dots,Cs_n$. Since each right coset can be covered with finitely many left cosets, we see that the degree of the vertex in X_C representing the right coset C is finite. Hence X_C is locally finite.

Suppose now that X_C is locally finite. The size of the orbit of the vertex in X_C representing the coset gC under C is equal to the index $|C:C\cap gCg^{-1}|$. But the subgroup C fixes the vertex representing C in X_C and the vertices in the orbit of gC all have the same distance from C. Since the graph X_C is locally finite, there are only finitely many vertices in any given distance from C and thus the orbit of gC is finite. Therefore the index $|C:C\cap gCg^{-1}|$ is finite. Letting the vertices C and gC in X_C and the subgroups C and gCg^{-1} change roles, one shows that the index $|gCg^{-1}:C\cap gCg^{-1}|$ is also finite. Hence C and gCg^{-1} are commensurable and we conclude that the commensurator of C is the whole group G. □

¿From Theorem 2^+ and Corollary 3 we can piece together the following:

Theorem 23. *Let G be a finitely generated group. Suppose that C is a subgroup of G such that the commensurator of C is the whole of G. If X and X' are Cayley graphs of G for some finite generating sets then the graphs X_C and X'_C are quasi-isometric. If X is a Cayley graph with respect to some finite generating set and X_C has more than one end then G splits over a subgroup commensurable with C.*

One of the things that makes the concept of ends of pairs of groups more difficult than ordinary ends, or the rough ends, is that the group G does not have a natural action on ${}_CX$. For instance, because of the transitive action of G, the Cayley graph X has either 0, 1, 2 or infinitely many ends, but $e(G,C)$ can take any given integer value, see [48, p. 186].

Example 5. Let G be the infinite dihedral group and C some two element subgroup of G. Let X be some Cayley graph of G with respect to some finite set of generators. The graph ${}_CX$ has only one end and thus $e(G,C)=1$. The graph X_C has two ends. The subsets of G which are right-C-finite are just the finite sets. The number of coends is thus just the same as the number of ordinary ends, i.e. $\tilde{e}(G,C)=2$.

The example above sets the tone for the comparison between the number of ends of ${}_CX$ and X_C and the number of coends of C. First we will have a look at the case when G is a compactly generated totally disconnected group, U a compact open subgroup of G and X a rough Cayley graph of G. The analogue of coends of U would be defined by looking at right-U-infinite components of X when a right-U-finite set of vertices in X is removed from X. But the right-U-finite sets are just the relatively compact subsets of G, and the right-U-infinite sets are just the sets that are not relatively compact. We are thus back to the concepts discussed in the previous section on Specker compactifications.

Theorem 24. *Let G be a compactly generated totally disconnected locally compact group and U a compact open subgroup of G. Define X as the ordinary Cayley graph of G with respect to a compact* *generating set S that includes a generating set for U. Then the graphs X_U and ${}_UX$ are both locally finite and the graph X_U has at least as many ends as the graph ${}_UX$.*

Proof. The graph X is an example of a topological graph as defined in Section 3.6. Let N be the set of vertices in VX which are adjacent to a given right coset of U. The set N is relatively compact, because right cosets of U are compact and X is a topological graph. Hence N can be covered with finitely many right cosets of U. This implies that ${}_CX$ is locally finite. The proof that X_C is locally finite is identical.

The number of ends of a locally finite graph can be defined as the supremum of the number of infinite components when a finite set of vertices is removed. Suppose that we get n infinite components when we remove a finite set F of vertices from ${}_UX$. The pre-image ${}_UF$ under ${}_U\pi$ of F is a finite union of right cosets of U and therefore a compact subset. Note that because ${}_U\pi$ maps a connected set of X to a connected set of X_U, we see that the graph $VX \setminus {}_UF$ has at least n non-compact connected components.

Now consider the graph X_U. Let F_U be a union of finitely many left cosets of U (vertices of X_U) that includes ${}_UF$. Because ${}_UF \subseteq F_U$, we see that $VX \setminus F_U$ has at least as many non-compact components as $VX \setminus {}_UF$. The map π_U maps the set F_U to a finite set F' of vertices of X_U. When regarded as an induced subgraph of X, each left coset of U is a connected graph. Each left coset of U not contained in F_U thus intersects only one component of $VX \setminus F_U$. Thus π_U maps $VX \setminus F_U$ to $VX_U \setminus F'$ and the number of infinite components of $VX_U \setminus F'$ is equal to the number of components of $VX \setminus F_U$. Therefore $VX_U \setminus F'$ has at least as many infinite components as $V_UX \setminus {}_UF$, i.e. at least n components. Hence the graph X_U has at least as many ends as the graph ${}_UX$. □

Theorem 25. *Let G be a finitely generated group and C a subgroup of G.*

(i) ([31, Lemma 2.5]) $\tilde{e}(G,C) \geq e(G,C)$.

(ii) *Suppose that the commensurator of C is G and C is finitely generated. Let X denote some Cayley graph of G with respect to a finite generating set that includes a generating set for C. Then the graph X_C has at least the same number of ends of the graph ${}_CX$.*

This result can be proved by using precisely the same methods as used to prove Theorem 24.

The following result was first noted and proved by Dunwoody and Roller [11, p. 30], but has also emerged in papers by Niblo [43, cf. Theorem B] and Scott and Swarup [49, Theorem 3.12].

Theorem 26. *Let G be a finitely generated group and C a finitely generated subgroup of G. If $e(G,C) > 1$ and the commensurator of C is the whole group G then G splits over a subgroup commensurable with C.*

Proof. Let X be a Cayley graph of G with respect to some finite generating set S of G and choose S such that it includes a generating set for C. We conclude from Theorem 22 and Theorem 25 that the graphs ${}_CX$ and X_C are connected and locally finite, and that the number of ends of X_C is at least equal to the number of ends of ${}_CX$. Since $e(G,C) > 1$, we know that ${}_CX$ has more than one end and hence X_C also has more than one end. The group G acts transitively as a group of graph automorphisms on the locally finite connected graph X_C. The conclusion now follows from Corollary 3. □

4. POLYNOMIAL GROWTH

Let X be a connected graph. For a vertex v and an integer $n \geq 1$, define $B(v,n) = \{u \in VX \mid d(v,u) \leq n\}$. If there are constants c and d such that $|B(v,n)| \leq cn^d$ for all positive integers n then we say that the graph X has *polynomial growth*. This property does not depend on the choice of the vertex v. A finitely generated group is said to have polynomial growth if it has a Cayley graph with polynomial growth. Note that having a polynomial growth is invariant under quasi-isometries and thus the choice of a finite generating set used to construct the Cayley graph is immaterial. Finitely generated groups with polynomial growth were characterized in a famous theorem by Gromov.

Theorem 27. ([16]) *Let G be a finitely generated group with polynomial growth. Then G has a nilpotent subgroup N of finite index.*

The converse, that a finitely generated nilpotent group has polynomial growth, had been shown earlier by Wolf [61]. A group having a nilpotent subgroup of finite index is often said to be *almost nilpotent*. Gromov's theorem was applied to graphs by Trofimov.

Theorem 28. ([55, Theorem 2]) *Suppose X is a connected locally finite graph with polynomial growth and G is a group that acts transitively on X. Then there is a G-invariant equivalence relation σ on the vertex set of X such that the equivalence classes of σ are finite and if K denotes the kernel of the action of G on the equivalence classes then G/K is a finitely generated almost nilpotent group and the stabilizers in G/K of σ-classes are finite.*

It should be noted that Trofimov proves an even stronger result [55, Theorem 1], since he shows that it is possible to find an equivalence relation σ as described in Theorem 28 such that the stabilizer of a vertex in $\mathrm{Aut}(X/\sigma)$ is finite.

The concept of polynomial growth can also be defined for topological groups.

Definition 15. *Let G be a locally compact group generated by a compact neighbourhood V of the identity. Set $V^n = \{g_1g_2\cdots g_n \mid g_i \in V\}$. Let μ denote a Haar measure on G. If there are constants c and d such that $\mu(V^n) \leq cn^d$ for all positive integers n then we say that G has* polynomial growth.

Gromov's theorem has been applied to topological groups by Losert in [32] and [33]. Woess [60] used Losert's results from [32] to give a short proof of Theorem 28.

Theorem 29. *Let G be a compactly generated totally disconnected locally compact group and X some rough Cayley graph of G. Then X has polynomial growth if and only if G has polynomial growth (in the sense of Definition 15).*

We need the following reformulation of a Lemma from [60].

Lemma 4. ([60, Lemma 3]) *Let G be a compactly generated totally disconnected locally compact group. Suppose X is some rough Cayley graph of G. Fix a vertex v_0 in X and define $W = \{g \in G \mid d(v_0, gv_0) \leq 1\}$. Then W is a compact open neighbourhood of the identity and W generates G. Furthermore, g is in $W^n = \{g_1g_2\cdots g_n \mid g_i \in W\}$ if and only if $d(v_0, gv_0) \leq n$.*

Proof. Since X is connected it is easy to see that W generates G. ¿From the definition of W we see that if $g \in W$ then $d(v_0, gv_0) \leq 1$. Assume that if $g \in W^n$ then $d(v_0, gv_0) \leq n$. Note that $W^n \subseteq W^{n+1}$. Suppose $d(v_0, gv_0) = n+1$. Let u be some neighbour of gv_0 such that $d(v_0, u) = n$. Choose $h \in G$ such that $hv_0 = u$. By the induction hypothesis, h is in W^n. Now $d(h^{-1}gv_0, v_0) = d(gv_0, hv_0) = d(gv_0, u) = 1$. Hence $h^{-1}g \in W$ and $g \in hW \subseteq W^{n+1}$. Conversely, it is clear that if $g \in W^n$ then $d(v_0, gv_0) \leq n$. □

of Theorem 29. Assume that the group G has polynomial growth. Define X as a rough Cayley graph with respect to some compact open subgroup U and some finite set T. Let μ be a left invariant Haar measure normalized such that $\mu(U) = 1$. Set v_0 as the vertex in X such that $G_{v_0} = U$. Define W as above. By assumption, there are constants C and d such that $\mu(W^n) \leq Cn^d$. By Lemma 4,

$$W^n = \bigcup_{g_i(v_0) \in B(v_0, n)} g_i G_{v_0}, \tag{4.1}$$

and we see that $\mu(W^n) = |B(v_0, n)|$. Whence $|B(v_0, n)| \leq Cn^d$.

The second half of the proof follows the proof of [60, Theorem 1]. Suppose the graph X has polynomial growth. Fix a vertex v_0 and suppose that C and d are constants such that $|B(v_0, n)| \leq Cn^d$. Let μ be some left invariant Haar measure normalized such that $\mu(G_{v_0}) = 1$. Let W be as in the Lemma above. Hence, by equation (1) above,

$$\mu(W^n) = |B(v_0, n)|\, \mu(G_{v_0}) = |B(v_0, n)| \leq Cn^d.$$

□

Combining the above result with Trofimov's theorem we get the following analogue of Gromov's theorem.

Theorem 30. *Let G be a compactly generated totally disconnected locally compact group. Then G has polynomial growth if and only if G has a normal compact open subgroup K such that G/K is a finitely generated almost nilpotent group.*

Proof. Let X be some rough Cayley graph of G.

If the group G has polynomial growth then, by Theorem 29, the graph X has also polynomial growth. The result now follows directly from Theorem 28 stated above.

Suppose now that G has a normal compact open subgroup K such that G/K is a finitely generated almost nilpotent group. Let S be a finite set of group elements such that $K \cup S$ is a good generating set for G. The rough Cayley graph $\mathrm{RCay}(G,K,S)$ is isomorphic to the Cayley graph $\mathrm{Cay}(G/K,S)$. Since G/K is almost nilpotent, this graph has polynomial growth. Therefore G has polynomial growth and the statement follows from Theorem 27. □

5. COMMENTARY

1. Using rough Cayley graphs one can define a compactly generated totally disconnected locally compact group to be *hyperbolic* if its rough Cayley graphs are hyperbolic in the sense of Gromov. (Note that being hyperbolic is a quasi-isometry invariant.) The results on quasi-isometries between rough Cayley graphs allow us to define the hyperbolic boundary and the group G has a natural action on the boundary.

2. The assumption on our groups being totally disconnected can be relaxed: everywhere in the paper the condition of being totally disconnected can be replaced by the condition that the group contains a compact open subgroup.

One could also put the results in a different setting by starting with a group G and a subgroup U such that the commensurator of U is the whole group G and G can be generated by the union of finitely many cosets of U.

3. A finitely generated group with the discrete topology is an example of a compactly generated totally disconnected locally compact group. Thus our results also hold for finitely generated groups and their Cayley graphs.

4. Instead of considering the rough Cayley graph $Y = \mathrm{RCay}(G,U,T)$ we could study the normal Cayley graph $X = \mathrm{Cay}(G,U \cup T)$ which is then non-locally finite. Let u and v be vertices of Y. That is, u and v are left cosets of U. Let $x \in u$ and $y \in v$ be vertices of X. Then $d_Y(u,v) \leq d_X(x,y)$. Let $u = w_0, w_1, \ldots, v = w_n$ be a path in Y of length n. Then there is a sequence $x = r_0, s_0, r_1, s_1, \ldots, r_n, y = s_n$ of vertices in X (i.e., elements of G) such that r_i and s_i are elements of the left coset w_i and such that $s_i^{-1} r_{i+1}$ is in T. This sequence spans a path in X whose length is less or equal $2n+1$. It follows that $d_Y(u,v) \leq d_X(x,y) \leq 2d_Y(u,v)+1$. This implies that a set of vertices in X is bounded if and only if its projection to Y is bounded. In other words, since Y is locally finite, a set of vertices in X is bounded if and only if its projection to Y is finite. This implies that the metric end space of X (see the remark at the end of Section 3.6) is isomorphic to the end space of Y. It can happen that two left cosets of U can be connected with infinitely many edges in X. Hence the vertex ends and the edge ends of X do in general not correspond to the ends of Y.

For rough Cayley graphs, the property of the group being compactly generated is crucial in order to obtain an end space of the group as an end space of a locally finite graph. One aim of further research could be to drop the condition of being compactly generated and then apply the theory of metric ends in this context.

REFERENCES

[1] Abels, H.: Specker-Kompactifizierungen von lokal kompakten topologischen Gruppen, Math. Z. **135**, 325–361 (1974)

[2] Bass, H.,Kulkarni, R.: Uniform tree lattices, J. Amer. Math. Soc. **3**, 843–902 (1990)

[3] Bowditch, B.H.: Groups acting on Cantor sets and the end structure of graphs, Pacific J. Math. **207**, 31–60 (2002)

[4] Bowditch, B.H.: Splittings of finitely generated groups over two-ended subgroups, Trans. Amer. Math. Soc. **354**, 1049–1078 (2002)

[5] van Dantzig, D.: Zur topologischen Algebra III. Brouwersche und Cantorsche Gruppen, Compos. Math. **3**, 408–426 (1936)

[6] Dicks, W., Dunwoody, M.J.: Groups acting on graphs, Cambridge University Press, Cambridge 1989

[7] Diestel, R., Leader, I.: A conjecture concerning a limit of non-Cayley graphs, J. Algebraic Combin. **14**, 17–25 (2001)

[8] Dunwoody, M.J.: Accessibility and groups of cohomological dimension one, Proc. London Math. Soc. (3) **38**, 193–215 (1979)

[9] Dunwoody, M.J.: The accessibility of finitely presented groups, Invent. Math. **81**, 449–457 (1985)
[10] Dunwoody, M.J.: An inaccessible group. In: Niblo, G.A., Roller, M.A. (d.) The Proceedings of Geometric Group Theory 1991, 75–78, L.M.S. Lecture Notes Series 181, Cambridge University Press 1993
[11] Dunwoody, M.J.,Roller, M.A.: Splitting groups over polycyclic-by-finite subgroups, Bull. London Math. Soc. **25**, 29–36 (1993)
[12] Freudenthal, H.: Über die Enden topologischer Räume und Gruppen, Dissertation Berlin 1931 (Available from `http://www.mathematik.uni-bielefeld.`.)
[13] Freudenthal, H.: Über die Enden topologischer Räume und Gruppen. Math. Z. **33**, 692–713 (1932)
[14] Freudenthal, H.: Über die Enden diskreter Räume und Gruppen, Comment. Math. Helv., **17**, 1–38 (1945)
[15] Gehring, F.W., Martin, G.J.: Discrete quasiconformal groups I, Proc. London Math. Soc. (3) **55**, 331–358 (1987)
[16] Gromov, M.: Groups of polynomial growth and expanding maps, Publ. Math. IHES, **53**, 53–78 (1981)
[17] Gromov, M.: Infinite groups as geometric objects, Proceedings of the International Congress of Mathematicians, Vol. 1. 385–392. PWN, Warsaw, 1984.
[18] Halin, R.: Über unendliche Wege in Graphen, Math. Ann. **157**, 125–137 (1964)
[19] Halin, R.: Über die Maximalzahl fremder unendlicher Wege in Graphen, Math. Nachr. **30**, 119–127 (1965)
[20] Halin, R.: Automorphisms and Endomorphisms of Infinite Locally Finite Graphs, Abh. Math. Sem. Univ. Hamburg **39**, 251–283 (1973)
[21] Hewitt, E., Ross, K.A.: *Abstract harmonic analysis, Volume I*, Springer, Berlin-Göttingen-Heidelberg 1963.
[22] Hopf, H.: Enden offener Räume und unendliche diskontinuierlich Gruppen, Comment. Math. Helv. **16**, 81–100 (1944)
[23] Houghton, C.H.: Ends of locally compact groups and their coset spaces, J. Austral. Math. Soc. **17**, 274–284 (1974)
[24] Jung, H.A.: A note on fragments of infinite graphs, Combinatorica **1**, 285–288 (1981)
[25] Jung, H.A.: On finite fixed sets in infinite graphs, Discrete Math. **131**, 115–125 (1994)
[26] Jung, H.A., Watkins, M.E.: Fragments and automorphisms of infinite graphs, Europ. J. Combinatorics **5**, 149–162 (1984)
[27] Krön, B.: End compactifications in non-locally-finite graphs, Math. Proc. Camb. Phil. Soc. **131**, 427–443 (2001)
[28] Krön, B.: Quasi-isometries between non-locally-finite graphs and structure trees, Abh. Math. Sem. Univ. Hamburg **71**, 161–180 (2001)
[29] Krön, B., Möller, R.G.: Metric ends, fibers and automorphisms of graphs. To appear in Math. Nach.
[30] Krön, B., Möller, R.G.: Quasi-isometries between graphs and trees. Preprint, 2005.
[31] Kropholler, P.H., Roller, M.A.: Relative ends and duality groups, J. Pure Appl. Algebra **61**, 197–210 (1989)
[32] Losert, V.: On the structure of groups with polynomial growth, Math. Z. **195**, 109–117 (1987)
[33] Losert, V.: On the structure of groups with polynomial growth. II, J. London Math. Soc. (2) **63**, 640–654 (2001)
[34] Möller, R.G.: Ends of graphs, Math. Proc. Camb. Phil. Soc. **111**, 255–266 (1992)
[35] Möller, R.G.: Ends of graphs II, Math. Proc. Camb. Phil. Soc. **111**, 455–460 (1992)
[36] Möller, R.G.: Groups acting on locally finite graphs, a survey of the infinitely ended case: In: Campbell, C.M., Hurley, T.C., Robertson, E.F., Tobin, S.J., Ward, J.J. (ed.) Groups'93 Galway/St Andrews, Vol. 2 L.M.S. Lecture Notes Series 212, 426–456. Cambridge University Press 1995
[37] Möller, R.G.: Accessibility and ends of graphs, J. Combin. Theory Ser. B **66**, 303–309 (1996)
[38] Möller, R.G.: Structure theory of totally disconnected locally compact groups via graphs and permutations, Canadian J. Math. **54**, 795–827 (2002)
[39] Möller, R.G.: FC^--elements in totally disconnected groups and automorphisms of infinite graphs, Math. Scand. **92**, 261–268 (2003)
[40] Möller, R.G., Seifter, N.: Digraphical regular representations of infinite finitely generated groups, European J. Combin. **19**, 597–602 (1998)
[41] Morris, S.A., Nicholas, P.: Locally compact topologies on an algebraic free product of groups, J. Algebra **38**, 393–397 (1976)
[42] Nebbia, C.: Minimally almost periodic totally disconnected groups, Proc. Amer. Math. Soc. **128**, 347–351 (2000)
[43] Niblo, G.A.: The singularity obstruction for group splittings, Topology Appl. **119**, 17–31 (2002)
[44] Papasoglu, P., Whyte, K.: Quasi-isometries between groups with infinitely many ends, Comment. Math. Helv. **77**, 133–144 (2002)
[45] Pavone, M.: Bilateral denseness of the hyperbolic limit points of groups acting on metric spaces, Abh. Math. Sem. Univ. Hamburg **67**, 123–135 (1997)
[46] Pavone, M.: On the hyperbolic limit points of groups acting on hyperbolic spaces, Rend. Circ. Mat. Palermo (2), **47**, 49–70 (1998)
[47] Sabidussi, G.: Vertex-transitive graphs, Monatsh. Math. **68**, 426–438 (1964)
[48] Scott, G.P.: Ends of pairs of groups, J. Pure Appl. Algebra **11**, 179–198 (1977).
[49] Scott, G.P., and Swarup, G.A.: Splittings of groups and intersection numbers, Geom. Topol. **4**, 179–218 (2000)
[50] Serre, J.-P.: Trees, Springer 1980.
[51] Soardi, P. and Woess, W., Amenability, unimodularity, and the spectral radius of random walks on infinite graphs, Math. Z., **205** (1990), 471–486.
[52] Stallings, J.R.: On torsion free groups with infinitely many ends, Ann. of Math. **88**, 312–334 (1968)
[53] Thomassen, C., Woess, W.: Vertex–transitive graphs and accessibility, J. Combin. Theory Ser. B **58**, 248–268 (1993)

[54] Tits, J.: Sur le groupe des automorphismes d'un arbre. In: Essays on topology and related topics. (Mémoires dédiés a G.de Rham), 188–211. Springer (1970)

[55] Trofimov, V.I.: Graphs with polynomial growth, Math USSR Sb. **51**, 405–417 (1985)

[56] Wall, C.T.C.: Poincaré complexes I, Ann. of Math. (2) **86**, 213–245 (1967)

[57] Wall, C.T.C.: The geometry of abstract groups and their splittings, Rev. Mat. Complut. **16**, 5–10 (2003)

[58] Woess, W.: Graphs and groups with tree-like properties, J. Combin. Theory Ser. B **47**, 361–371 (1989)

[59] Woess, W.: Boundaries of random walks on graphs and groups with infinitely many ends, Isr. J. Math. **68**, 271–301 (1989)

[60] Woess, W.: Topological groups and infinite graphs. In Diestel, R. (ed.) Directions in Infinite Graph Theory and Combinatorics, Topics in Discrete Math. 3, North Holland, Amsterdam 1992 (Also in Discrete Math. **95**, 373–384 (1991))

[61] Wolf, J.A.: Growth of finitely generated solvable groups and curvature of Riemannian manifolds, J. Differential Geometry **2**, 421–446 (1968)