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Abstract

We prove that every finite 4-connected graph G has at least $\frac{1}{34} \cdot (|E(G)| - 2|V(G)|)$ many contractible edges.

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1 Introduction

All graphs considered here are supposed to be finite, simple, and undirected. For terminology not defined here we refer to [1] or [2].

An edge $e = xy$ in a k-connected graph G is called *k-contractible* if the graph G/e obtained from G identifying x, y and simplifying the result is k-connected. It is easy to see that every edge of a connected graph is 1-contractible, and it is a well known fact that every vertex of a 2-connected graph nonisomorphic to K_3 is incident with a 2-contractible edge. The corresponding statement for 3-connected graphs fails, but it is still true that for an arbitrary vertex x in a 3-connected graph nonisomorphic to K_4 there is a 3-contractible edge at distance 0 or 1 from x (references in [7]).

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No such result holds for 4-connected graphs, as there are 4-connected graphs without 4-contractible edges; these are squares of cycles of length at least 5 and 4-connected line graphs of cubic graphs, and there are no other graphs without 4-contractible edges [3, 9]. As they are all 4-regular, every 4-connected graph G whose average degree $\overline{d}(G)$ is larger than 4 must have at least one 4-contractible edge.

Here we refine these results substantially by showing that the number of 4-contractible edges in a 4-connected graph is at least $|V(G)| \cdot c \cdot (\overline{d}(G) - 4)$ for some constant $c > 0$. We prove that $c \geq \frac{1}{68}$ and construct examples showing $c \leq \frac{1}{10}$.

2 Concepts and preliminary results

For a graph G, let $\kappa(G)$ denote its *(vertex) connectivity*, and let $\mathcal{T}(G) := \{S \subseteq V(G) : G - S$ disconnected and $|S| = \kappa(G)\}$ denote the set of its smallest separating sets. For $T \in \mathcal{T}(G)$, a *T-fragment* is the union of the vertex sets of at least one but not of all components of $G-T$. Note that a given T-fragment F determines T to be $N_G(F)$. If F is a T-fragment then so is $\overline{F} := V(G)-(F \cup T)$. A T-fragment of cardinality 1 is called *trivial*, and $T \in \mathcal{T}(G)$ is *trivial* if there exists a trivial T-fragment, that is, $T = N_G(x)$ for some vertex of degree $\kappa(G)$.

We say that $e \in E(G)$ is *covered* by $T \subseteq V(G)$ if $V(e) \subseteq T$. Note that an edge e of a non-complete graph G of connectivity k is not k-contractible if and only if it is covered by some smallest separating set. We call it *trivially non-k-contractible* if it is covered by some trivial smallest separating set, that is, if the endvertices of e have a common neighbor of degree k.

An $S \in \mathcal{T}$ *crosses* $T \in \mathcal{T}(G)$, if S intersects every T-fragment. It is easy to see that S crosses T if and only if T crosses S, which is in turn equivalent to saying that S intersects at least two components of $G-T$. Furthermore, we call $\mathcal{S} \subseteq \mathcal{T}(G)$ *cross free* if any two members of $\mathcal{S}$ do not cross.

Consider a T-fragment F and an S-fragment A of G. It is well known that if $F \cap A \neq \emptyset$ then

$$|F \cap S| \geq |\overline{A} \cap T|,$$

and if equality holds here then $F \cap A$ is a $T_G(F, A)$-fragment, where

$$T_G(F, A) := (T \cap A) \cup (T \cap S) \cup (F \cap S).$$

For a proof, see [6] or [8]. Applications of these statements to some pair of fragments will be indicated by (*) throughout. In particular, if $F \cap A \neq \emptyset$ and $\overline{F} \cap \overline{A} \neq \emptyset$ then $F \cap A$ is a $T_G(F, A)$-fragment and $\overline{F} \cap \overline{A}$ is a $T_G(\overline{F}, \overline{A})$-fragment.

Let D be a digraph. For $t \in V(D)$, a vertex $s \neq t$ with $ts \in E(D)$ is called an *outneighbor* of t, and we let $N_D^+(t)$ denote the set of all outneighbors of t. Similarly we let $N_D^-(t) := \{s \in V(D) - \{t\} : st \in E(D)\}$.

We call $a \in V(D)$ a *root* of D if for every $t \in V(D)$ there exists a directed a,t-path and D is edge-minimal with respect to this property. If a root exists then it is uniquely determined and we call D a *tree*. Now let D be a tree with root a. It is easy to see that $|N_D^-(a)| = 0$ and $|N_D^-(t)| = 1$ for all $t \in V(D) - \{a\}$. A vertex $s \in V(D)$ is called a *leaf* if $N_G^+(s) = \emptyset$. A vertex $t \in V(D)$ is called a *pseudo-leaf* if it is not a leaf and every $s \in N_G^+(t)$ is a leaf. To *truncate* the pseudo-leaf t means to delete $N_D^+(t)$ from D. A subtree D' of D is called *good* if it can be obtained from D by a sequence of pseudo-leaf truncations. Observe that if $|V(D)| \geq 2$ then D has a pseudo-leaf. Therefore, pseudo-leaf truncation can be used as an inductive device within the set of all good subtrees of D.

The HASSE-*digraph* of a finite partially ordered set $(V, \leq)$ is the digraph on V where there is an edge from s to t if and only if $s < t$ and $s < r < t$ for no $r \in V$. We call $(V, \leq)$ a *tree order* if its HASSE-digraph is a tree. Note that, in this case, the root of the HASSE-digraph is the minimum element of $(V, \leq)$.

Theorem 1 *[6] Let G be a noncomplete graph and $\mathcal{S} \subseteq \mathcal{T}(G)$ such that no two members of $\mathcal{S}$ cross. Among all T-fragments with $T \in \mathcal{S}$, choose an inclusion minimal one, say A.*

Then for each $S \in \mathcal{S}$ there exists a unique component $C(S)$ of $G - S$ with $A \subseteq V(C(S))$, and the partial order on $\mathcal{S}$ defined by

$$S \leq T :\longleftrightarrow V(C(S)) \subseteq V(C(T))$$

is a tree order with minimum element $N_G(A)$.

Let us summarize some properties of the objects in Theorem 1.

Lemma 1 *Let G, $\mathcal{S}$, A, $C(\cdot)$, $\leq$ be as in Theorem 1.*

(i) For $S, T \in \mathcal{S}$, $T \cap \overline{C(S)} \neq \emptyset$ implies $S < T$.

(ii) If $S, T \in \mathcal{S}$ are not comparable with respect to $\leq$ then $\overline{C(S)} \cap \overline{C(T)} = \emptyset$.

(iii) For $S \in \mathcal{S}$, $(\bigcup_{R \leq S} R) \cap (\bigcup_{T \geq S} T) \subseteq S$.

Proof. To prove (i), consider $S, T \in \mathcal{S}$ with $T \cap \overline{C(S)} \neq \emptyset$. Then T is not equal to S, and T cannot intersect $C(S)$. For every $z \in C(S)$, there is a z, A-path P in $C(S)$, and P does not intersect T, hence $z \in C(T)$. It follows $C(S) \subset C(T)$, which proves (i).

To prove (ii), consider $S, T \in \mathcal{S}$ and suppose that $Y := \overline{C(S)} \cap \overline{C(T)}$ is not empty. Then Y is an R-fragment (*), where $R = T_G(\overline{C(S)}, \overline{C(T)}) = (S \cap \overline{C(T)}) \cup (S \cap T) \cup (\overline{C(S)} \cap T)$. If $S = T$ then S, T are trivially comparable, otherwise $S \cap \overline{C(T)} \neq \emptyset$ or $T \cap \overline{C(S)} \neq \emptyset$, implying $T < S$ or $S < T$ by (i). This proves (ii).

To prove (iii), consider $R, S, T \in \mathcal{S}$ such that $R \leq S \leq T$. Then $R \cap \overline{C(S)} = \emptyset$ by (i), and $T \cap C(S) = \emptyset$ since $C(S) \subseteq C(T)$. Consequently, $R \cap T \subseteq S$, and so (iii) follows by the distributive law. □

Our second ingredient is tailored to 4-connected graphs. The following result has already been mentioned in the introduction.

Theorem 2 *[3] [9] Every* 4*-connected graph* G *without any* 4*-contractible edges is either the square of a cycle of length at least* 5 *or the line graph of a cubic essentially* 4*-edge-connected graph. In particular,* G *is* 4*-regular.*

Let $V_4(G)$ denote the set of vertices of degree 4 in G. The following statement is extracted from Claim 1 in the proof of Lemma 4 in [5].

Lemma 2 *Let* w *be a vertex of a* 4*-connected graph* G *such that every edge incident with* w *is not* 4*-contractible. Let* F *be a* T*-fragment of* G *such that* T *contains* w *and a neighbor of* w*. Then* F *is intersected by some triangle which contains* w *and a neighbor of* w *of degree* 4*.*

From this one deduces the following.

Lemma 3 *Suppose that* uab *is a triangle in a* 4*-connected graph* G *such that* $u \in V(G) - V_4(G)$ *and* $a, b \in V_4(G)$*. Then one of* a, b *is incident with a contractible edge.*

Proof. Suppose, to the contrary, that all edges incident with a or b are not contractible. Let $T \in \mathcal{T}(G)$ cover ab such that the set $S(T)$ of edges incident with a or b covered by T is as large as possible. Let F be a T-fragment not containing u.

If $u \in T$ then each of a and b has at least one neigbor in each of $F, \overline{F}$. Hence a has a unique neighbor $x \in F$, b has a unique neighbor $y \in F$, and a has a unique neighbor $z \in \overline{F}$. By assumption, az is covered by some $T' \in \mathcal{T}$. T' separates $N_G(a) - \{z\} = \{x, u, b\}$. It follows that $x \neq y$ (for otherwise, $F = \{x\}$ because $N_G(F - \{x\}) \subseteq (T - \{a, b\}) \cup \{x\}$ cannot separate G, and so uby was a triangle). By Lemma 2, applied to $w = a$, axu must be a triangle, so xub is a path, implying that T' contains u and separates x from b, which implies that there is a $t \in T' \cap F$. Now $T' = \{z, u, a, t\}$, and, for any T'-fragment F', if $F' \cap \overline{F}$ was not empty then it was a $\{u, a, z, s\}$-fragment for either $s = b$ or s being the element in $T - \{u, a, b\}$; but a had no neighbor in $F' \cap \overline{F}$, which is impossible. Hence $\overline{F} = \{z\}$ — but then ax is contractible because $N_G(a) - \{x\} = \{u, b, z\}$ is a triangle.

Hence $u \in \overline{F}$. Then $|\overline{F}| > 1$, since u has degree exceeding 4, and so $N_G(\{a, b\}) \cap \overline{F}$ cannot consist of u only (for otherwise $(T - \{a, b\}) \cup \{u\}$ would separate $\overline{F} - \{u\}$ from $F \cup \{a, b\}$, which is absurd). So one of a, b, say, a, has a neighbor $z \in \overline{F} - \{u\}$. Then a has a unique neighbor x in F, and, by Lemma 2 applied to $w = a$, F is intersected by some triangle containing w, which must be abx. Let y be the neighbor of b distinct from a, x, u and note that $S(T) \subseteq \{ab, by\}$. Consider a smallest separating set T' covering az. Since T' must separate $N_G(a) - \{z\}$, which induces a path ubx, $b \in T'$ follows. Hence $\{ab, az\} \subseteq S(T')$. By choice of T, $S(T) = \{ab, by\}$ and $S(T') = \{ab, az\}$. In particular, $y \in T - T'$ and $N_G(\{a, b\}) \cap F = \{x\}$, which implies $F = \{x\}$. Since $ax, by \notin S(T')$ and $xy \in E(G)$, there exists a T'-fragment F' containing x, y. But then $N_G(a) \cap \overline{F'} = N_G(b) \cap \overline{F'} = \{u\}$, which implies that $(T' - \{a, b\}) \cup \{u\}$ separates $\overline{F'} - \{u\} \neq \emptyset$ from $F' \cup \{a, b\}$ — a contradiction. □

Lemma 4 *Suppose that uab is a triangle in a 4-connected graph G such that $b \in V_4(G)$ and $u, a \in V(G) - V_4(G)$. Suppose that A is an S-fragment such that $a \in A$ and $u, b \in S$, and $|\overline{A}| \geq 2$. Then b is incident with a contractible edge.*

Proof. Suppose, to the contrary, that b is not incident with a contractible edge. By Lemma 2, there exists a triangle Δ intersecting A and containing b and a neighbor c of b of degree 4. Since $c \neq b$, b has exactly one neighbor $x \in \overline{A}$. By assumption, bx is covered by some $T \in \mathcal{T}$. T separates $N_G(b) - T$.

Case 1. $\Delta = ubc$

Then $c \in A$, and T spearates a from c. Hence there exists a $t \in A \cap T$, so $T = \{t, u, b, x\}$. Since $\overline{A} \neq \{x\}$, there exists a T-fragment F intersecting $\overline{A}$. By (*), $|F \cap S| = |\overline{F} \cap S| = 1$, and $F \cap \overline{A}$ is an $R := T_G(F, \overline{A})$-fragment, where $b \in R$. But b has no neighbor in $F \cap \overline{A}$.

Case 2. $\Delta = abc$ and $c \in A$.

Then T separates c from u, so $a \in T$. Let F be a T-fragment such that $c \in F$ and $u \in \overline{F}$. It follows that $\overline{A} \cap \overline{F} = \emptyset$ (for otherwise the latter set would be an $R := T_G(\overline{A}, \overline{F})$-fragment, which would not contain a neighbor of $b \in R$). Furthermore, $A \cap \overline{F} = \emptyset$ (for otherwise, $|R := T_G(A, \overline{F})| > 4$ holds, since b has no neighbor in $A \cap \overline{F}$; but then $|T_G(\overline{A}, F)| < 4$, implying that $\overline{A} \subseteq T$. But then $|F \cap S|, |\overline{F} \cap S| \geq 2$, contradicting the fact that $b \in T \cap S$). Hence $\overline{F} \subseteq S$. Since u has degree exceeding 4, $|\overline{F}| \geq 2$. Furthermore, $|T \cap \overline{A}| \geq 2$ (if $|T \cap \overline{A}| \leq 1$, it follows from (*) that $|\overline{A}| = |T \cap \overline{A}| = 1$, which contradicts the assumption that $|\overline{A}| \geq 2$). But then $|F \cap S| \geq |\overline{A} \cap T| \geq 2$, too, which contradicts $b \in T \cap S$.

Case 3. $\Delta = abc$ and $c \in S$.

Then T separates c from u, so $a \in T$. Let A' be one of $A, \overline{A}$, so $|A'| \geq 2$, and let F be a T-fragment. Assume for a while that $A' \cap F \neq \emptyset$. Then the latter set cannot be a $T_G(A', F)$-fragment because it does not contain a neighbor of b. Hence $|F \cap S| > |\overline{A'} \cap T| \geq 1$, and $|A' \cap T| > |\overline{F} \cap S| \geq 1$. Now $\overline{A'} \cap \overline{F} = \emptyset$ by (*), and $\overline{A'} \cap F = \emptyset$ (for otherwise $|\overline{A'} \cap T| > 1$, too, implying $|T| = |A' \cap T| + |S \cap T| + |\overline{A'} \cap T| \geq 2+2+1$, which is impossible). Hence $\overline{A'} \subseteq S$, and $|\overline{A'}| \leq |T| - |T \cap S| - |T \cap A'| \leq 1$, which is absurd. Hence $A' \cap F = \emptyset$, which implies $V(G) \subseteq S \cup T$ as A', F have been choosen arbitrarily; but then $|V(G)| \leq 8 - |S \cap T| \leq 7$, which contradicts $|V(G)| = |A| + |S| + |\overline{A}| \geq 8$. □

3 The main result

For an edge e in a graph G of connectivity k we write $e \to z$ if z has degree k and $N_G(z)$ is the unique member of $\mathcal{T}(G)$ which covers e.

Theorem 3 *Every 4-connected graph G has at least $\frac{1}{34} \cdot (|E(G)| - 2|V(G)|)$ many 4-contractible edges.*

Proof. Let $a(G)$ denote the number of contractible edges of G and let $b(G) := |E(G)| - 2|V(G)|$. For simplicity, we call the 4-contractible edges of G *contractible*, and the others *noncontractible*.

We have to prove that $a(G) \geq \frac{1}{34} b(G)$. Suppose this is not true and take a minimum counterexample G. Then $b(G) > 0$, so G is not 4-regular. Hence $a(G) > 0$ by Theorem 2, thus $b(G) > 34$. In particular, $|V(G)| > 8$, as $b(G) \leq |E(G)| \leq 28$ for $|V(G)| \leq 8$.

Let N be the set of all edges which can be covered by some member of $\mathcal{T}(G)$, let $M \subseteq N$ be the set of all edges which can be covered by some trivial member of $\mathcal{T}(G)$, and let L be the set of edges e with $V(e) \subseteq V_4(G)$.

Choose a sequence $A_1, \ldots, A_k$ of fragments such that every edge in $N - M - L$ is covered by some $N_G(A_i)$ ($i \in \{1, \ldots, k\}$) and such that $(k, |A_1|, \ldots, |A_k|)$ is lexicographically minimal among all these choices. In particular, $2 \leq |A_i| \leq |\overline{A_i}|$, and, as $|V(G)| > 8$, $|\overline{A_i}| > 2$.

For all $i \in \{1, \ldots, k\}$, $S_i := N_G(A_i)$ must cover at least one edge from $N - M - L$, and A_i can't occur twice in the sequence — otherwise, we could remove it from the sequence, which decreases k and violates the minimality constraint.

Let $\mathcal{S} := \{S_1, \ldots, S_k\}$.

Claim 1. $\mathcal{S}$ is cross free.

Suppose (reductio ad absurdum) that S_i, S_j do cross for distinct i, j.

First assume that $i < j$, so $|A_i| \leq |A_j|$. If, for $F \in \{A_j, \overline{A_j}\}$, $X := A_i \cap F \neq \emptyset$ and $Y := \overline{A_i} \cap \overline{F} \neq \emptyset$ then X, Y are fragments and every edge covered by S_i or S_j is covered by $N_G(X) = T_G(A_i, F)$ or $N_G(Y) = T_G(\overline{A_i}, \overline{F})$. As $|X| < |A_i|$, replacing A_i, A_j with X, Y at their respective positions in the sequence will violate the minimality constraint. Hence one of $A_i, \overline{A_i}$ is contained in S_j or one of $A_j, \overline{A_j}$ is contained in S_i. If $j < i$ then the latter statement follows symmetrically.

Suppose that $F \in \{A_i, \overline{A_i}\}$ is contained in S_j and consider $F' \in \{A_j, \overline{A_j}\}$. If $F' \cap \overline{F} \neq \emptyset$ then $|S_i \cap F'| \geq |F \cap S_j| = |F| \geq 2$, and if, otherwise, $F' \subseteq S_i$ then $|S_i \cap F'| \geq 2$ holds trivially. Hence $|S_i \cap F'| = |S_i \cap \overline{F'}| = 2$; if $F' \cap \overline{F} \neq \emptyset$ or $\overline{F'} \cap \overline{F} \neq \emptyset$ then $|F| = 2$, and, otherwise, $|F| = 2$ trivially. It follows $F = A_i$.

The argument of the preceeding paragraph works with swapped i, j, too. We may assume without loss of generality that $A_i = \{x, y\} \subseteq S_j$. If $A_j = \{x', y'\} \subseteq S_i$, too, then we may assume, without loss of generality, that $d_G(x) + d_G(y) \geq d_G(x') + d_G(y')$. This choice is designed to simplify some later case analysis.

$A_j \cap S_i = \{a, u\}$, and $\overline{A_j} \cap S_i = \{b, v\}$. Note that there is no edge connecting one of a, u to one of b, v. For simplicity, set $A := A_i = \{x, y\}$ and $S := S_i = \{a, u, b, v\}$.

Subclaim 1.1. There is no $z \in \overline{A}$ such that $\{x, y, a, u, z\}$ or $\{x, y, b, v, z\}$ separates G.

Let $T := \{x, y, a, u, z\}$. Since G is 4-connected, every component of $G - T$ contains a neighbor of $\{x, y\} \subseteq T$, which is either b or v. So $G - T$ has exactly two components. Let $C, \overline{C}$ denote their vertex sets, where $b \in C$ and $v \in \overline{C}$.

Since b, v are not adjacent and S covers a member of $N - M - L$, $au \in N - M - L$ follows. Since b is not adjacent to a or u, $C \neq \{b\}$ follows, so $X := C \cap \overline{A}$ is not empty. As $N_G(X) \subseteq \{b, a, u, z\}$, X is a $\{b, a, u, z\}$-fragment, and as $au \notin M$, $|X| \geq 2$ follows. There exists a b, a-path in $X \cup \{b, a\}$ intersecting X, so X intersects S_j. Analogously, $Y := \overline{C} \cap \overline{A}$ is a $\{v, a, u, z\}$-fragment intersecting S_j, so $|X \cap S_j| = |Y \cap S_j| = 1$.

From $\overline{A_j} \cap X \neq \emptyset$ we deduce $1 = |X \cap S_j| \geq |A_j \cap \{b, a, u, z\}| \geq 2$, which is absurd. So $A_j \cap X \neq \emptyset$, which implies $1 = |X \cap S_j| \geq |\overline{A_j} \cap \{b, a, u, z\}|$, and so b is the unique vertex in $\overline{A_j} \cap (X \cup \{b, a, u, z\})$. Analogously, v is the unique vertex in $\overline{A_j} \cap (Y \cup \{v, a, u, z\})$, and hence $\overline{A_j} = \{b, v\}$ follows. Consequently,

b, v are independent vertices of degree 4, so $N_G(b) = N_G(v) = S_j$ is a trivial member of $\mathcal{T}(G)$, a contradiction.

The same argument works if we swap the roles of A_j and $\overline{A_j}$; hence Subclaim 1.1. follows.

Since S covers a member of $e \in N - M - L$ and since the following arguments will not rely on the fact that $|A_j| \leq |\overline{A_j}|$, we may assume without loss of generality that $au \in N - M - L$ and $a \notin V_4(G)$ from now on.

Subclaim 1.2. The edges xy, bx, by, vx, vy are present in G, the graph $G' := (G - \{x, y\}) + \{ab, av, ub, uv\}$ is 4-connected, and if $\{ux, uy\} \subseteq E(G)$ or $d_G(u) > 4$ then every edge from $E(G') - E(G'(S))$ that is 4-contractible in G' is a 4-contractible edge in G, too.

(Note that if $ax, uy \in E(G)$ then $G' = G/ax/uy$, whereas otherwise, $ay, ux \in E(G)$ and $G' = G/ay/ux$.)

If x has degree 5 then xy, bx, vx in $E(G)$ follows trivially, if x has degree 4 then it can't be adjacent to both a and u, as $au \in N - M - L$, hence $xy, bx, vx \in E(G)$ in either case. Symmetrically, $by, vy \in E(G)$, which proves the first statement of Subclaim 1.2.

Consider a smallest separator T of G'. If some component of $G - T$ does not intersect S then T separates G, too, and $|T| \geq 4$ follows. Otherwise, b, v are in distinct components of $G' - T$, so that $a, u \in T$; hence $T \cup \{x, y\}$ separates G, and $|T| \geq 4$ follows from Subclaim 1.1. Hence G' is 4-connected.

Finally, let $e \in E(G') - E(G'(S))$ and suppose that e is 4-contractible in G'. If it was not 4-contractible in G then there would be a $T \in \mathcal{T}(G)$ with $V(e) \subseteq T$. Observe that T intersects A, for otherwise it would separate G', violating the fact that e is 4-contractible in G'.

If there is some T-fragment F containing y then $\overline{F} \cap S$ is one of $\{a\}, \{u\}$. Now if $\overline{F} \cap \overline{A} \neq \emptyset$ then the latter set is a fragment whose neighborhood covers e (*) and which separates G', too, contradicting the fact that e is 4-contractible in G'. So $\overline{F}$ equals one of $\{a\}, \{u\}$. Since $d_G(a) > 4$, $\overline{F} = \{u\}$. So $d_G(u) \not> 4$ and $\{ux, uy\} \not\subseteq E(G)$, a contradiction.

Hence $y \in T$ and, symmetrically, $x \in T$. Suppose that $|T \cap S| = 1$. Since $|V(G)| > 8$, there exists a T-fragment F such that $F \cap \overline{A} \neq \emptyset$. Then $|F \cap S| \geq |T \cap A| = 2$. Since $|S - T| = 3$, this forces $|F \cap S| = 2$. But then $T_G(F, \overline{A})$ is a member of $\mathcal{T}(G)$ such that $V(e) \subseteq T_G(F, \overline{A})$, and $T_G(F, \overline{A}) \cap A = \emptyset$, a contradiction. Thus $T \cap S = \emptyset$. Therefore, $T \cap \overline{A} = V(e)$. If $T = N_G(s)$ for some $s \in S$ then $s \in \{b, v\}$; as $d_G(s) = d_{G'}(s)$, this contradicts our assumption that e is 4-contractible in G'. Hence $|F \cap S| \geq 2$ and, therefore $|F \cap S| = 2$ for every T-fragment F. Since $|V(G)| > 8$, $X := F \cap \overline{A} \neq \emptyset$ for some T-fragment F, hence X is a $T_G(F, \overline{A})$-fragment of G and of G' covering e, a contradiction.

This proves Subclaim 1.2.

Subclaim 1.3. If sz is not 4-contractible for some $s \in S$ and $z \in \{x, y\}$ such that each vertex in $\{a, u\} - \{s\}$ is adjacent to the vertex in $\{x, y\} - \{z\}$ then $sz \to t$, where t is the unique vertex such that $\{s, t\} \in \{\{a, u\}, \{b, v\}\}$.

Suppose $T \in \mathcal{T}(G)$ covers sz. Since b, v are adjacent to x and to y by Subclaim 1.2, it follows by the condition to s, z that $N_G(z) - \{s\}$ has a spanning star centered at the vertex w in $\{x, y\} - \{z\}$. As T separates $N_G(z) - T$, $w \in T$ follows, so $A \subseteq T$. There exists a T-fragment F such that $F \cap S = \{t\}$ for some $t \in S - \{s\}$, so $F \cap \overline{A} = \emptyset$ (as otherwise $|F \cap S| \geq^{(*)} |A \cap T| = 2$), and, consequently, $F = \{t\}$. This proves Subclaim 1.3.

We distinguish three cases, according to the possible degrees of x, y.

Case 1.1. $d_G(x) = d_G(y) = 5$.

Take G' as in Subclaim 1.2. Then, for every $s \in S$, $d_G(s) = d_{G'}(s)$, and sx is 4-contractible if and only if sy is 4-contractible by Subclaim 1.3. Furthermore, ux, uy are 4-contractible by Subclaim 1.3 as $ux \not\to a$.

Hence $a(G) \geq a(G') - |E(G'(S))| + |\{ux, uy\}| \geq a(G') - 6 + 2$. We sharpen this to $a(G) > a(G')$, which will cause a contradiction.

Recall that for each $s \in S$, sx is 4-contractible if and only if sy is 4-contractible (by Subclaim 1.3). Hence, if sx is 4-contractible for all $s \in S$ then $a(G) \geq a(G') - |E(G'(S))| + 8 > a(G')$ follows.

If sx is not 4-contractible in G for some $s \in S$ then $sx \to t$ for some unique $t \in S$ by Subclaim 1.3; as t has degree 4 in G', too, all edges in $E(G'(S))$ nonincident with t are not 4-contractible in G' (so all but at most 3). Hence, if s is the unique $s \in S$ such that sx is not 4-contractible in G then $a(G) \geq a(G') - 3 + 6 \geq a(G')$, and, otherwise, if there exists an $s' \in S - \{s\}$ such that $s'x$ is not 4-contractible in G then $s'x \to t' \not{t}$ and every edge in $E(G'(S))$ not connecting t, t' is not 4-contractible in G', so $a(G) \geq a(G') - 1 + 2 > a(G')$.

Now $b(G) = b(G') + 5 - 2 \cdot 2 = b(G') + 1$. By choice of G, $a(G) \geq a(G') + 1 \geq \frac{1}{34} b(G') + 1 = \frac{1}{34} b(G) - c + 1 > a(G) - \frac{1}{34} + 1$, a contradiction.

Case 1.2. Either $d_G(x) = 5$, $d_G(y) = 4$, or $d_G(x) = 4$, $d_G(y) = 5$

By symmetry of x, y it suffices to analyze the subcase that $d_G(x) = 5$, $d_G(y) = 4$. Note that if $bv \in E(G)$, then $bv \in M$ because it is covered by $N_G(y)$. Thus au

is the unique edge from $N - M - L$ covered by S.

We first consider the case that y is not adjacent to a. The edges sx with $s \neq a$ are not 4-contractible as they are covered by S_j and $N_G(y)$, and uy is 4-contractible by Subclaim 1.3, as $uy \not\to a$.

Take G' as in Subclaim 1.2. Then $d_{G'}(s) = d_G(s)$ for $s \in \{u, b, v\}$, and $b(G) = b(G') + 4 - 2 \cdot 2 = b(G')$.

Now $\{a, u\} \neq A_j$, since $d_G(a) > 4$ but $ay \notin E(G)$, so $X := A_j \cap \overline{A}$ is nonempty and a $T_G(A_j, \overline{A})$-fragment of G whose neighborhood contains a, u and does not intersect A. Hence X is a fragment of G', too, so au is not 4-contractible in G'. Also if $xy \notin N - M - L$, then $N_G(X)$ covers all edges from $N - M - L$ covered by S or S_j, which contradicts the minimality of k. Thus $xy \in N - M - L$, and hence $d_G(b), d_G(v) \geq 5$. Since $d_G(x) = 5$ and $d_G(y) = 4$, it follows from the choice of x and y that $\overline{A_j} \cap \overline{A} \neq \emptyset$. Hence $\overline{A_j} \cap \overline{A}$ is a $T_G(\overline{A_j}, \overline{A})$-fragment of G and G'. Thus if $bv \in E(G)$, then bv is not 4-contractible in G' as well.

We are aiming to show that $a(G) \geq a(G')$. If all three edges ax, by, vy are 4-contractible in G then $a(G) \geq a(G') - |\{ab, av, ub, uv\}| + |\{uy, ax, by, vy\}|$, so the statement follows. If ax is not 4-contractible in G then $ax \to u$ by Subclaim 1.3, so ab, av are not 4-contractible in G', if by is not 4-contractible in G then $by \to v$ by Subclaim 1.3, so ab, ub are not 4-contractible in G', if vy is not 4-contractible in G then $vy \to b$ by Subclaim 1.3, so av, uv are not 4-contractible in G'. Hence, if at most two of ax, by, vy are not 4-contractible in G then at most two of ab, av, ub, uv are 4-contractible in G' and $a(G) \geq a(G') - 2 + 1 + |\{uy\}| \geq a(G')$, and if all of ax, by, vy are not 4-contractible in G then no edge of ab, av, ub, uv is 4-contractible in G' and $a(G) \geq a(G') + |\{uy\}| \geq a(G')$. Hence, in either case $a(G) \geq a(G')$, and, by choice of G, $a(G) \geq a(G') \geq \frac{1}{34} b(G') = \frac{1}{34} b(G) > a(G)$, which is absurd.

Hence it remains to consider the case that y is adjacent to a and, therefore, nonadjacent to u. We may assume that u has degree 4, for otherwise we could swap the roles of a, u. Furthermore, ux, ay are 4-contractible in G by Subclaim 1.3, as neither $ux \to a$ nor $ay \to u$ holds. Note that Claim 2 is not applicable here. In order to proceed similarly as above, we reduce G in a different way.

Subclaim 1.4. We have $xy \in N - M - L$ (so $d_G(b), d_G(v) \geq 5$, and $\overline{A_j} \cap \overline{A} \neq \emptyset$).

For otherwise, the two vertices in $S_j \cap \overline{A}$ form the unique edge e in $N - M - L$ covered by S_j. If $Z := \overline{A} \cap A_j \neq \emptyset$ then Z would be a fragment whose neighborhood covers all the edges from $N - M - L$ covered by S or by S_j, and hence we can replace A_j, A_i by Z in our sequence to obtain a shorter one with the desired properties, contradicting the choice. So $A_j = \{a, u\}$ and u is adjacent to both endvertices of e. Since $d_G(u) = 4$, this contradicts $e \in N - M - L$. Thus $xy \in N - M - L$. Hence $d_G(b), d_G(v) \geq 5$, and it follows from the choice of x and y that $\overline{A_j} \cap \overline{A} \neq \emptyset$, which proves Subclaim 1.4.

Let $G' := G/vx/by$. Then $d_{G'}(u) = d_G(u) = 4$, $d_{G'}(a) = d_G(a) > 4$, $d_{G'}(b) \leq d_G(b)$, $d_{G'}(v) \geq d_G(v)$.

Consider a smallest separating set T of G'. Suppose, to the contrary, that $|T| \leq 3$. Then T does not separate G, so it separates S and hence $T = \{a, v, z\}$ for some $z \in \overline{A}$. Now $\{a, v, z, x\}$ is a smallest separator of G, and there is an $\{a, v, z, x\}$-fragment C such that $u \in C$ and $b, y \in \overline{C}$. Since u has two neighbors in $\overline{A}$, $X := C \cap \overline{A}$ is not empty and, thus, an $\{a, u, v, z\}$-fragment, and since $au \in N - M - L$, $|X| > 1$ follows.

If $\overline{C} = \{b, y\}$ then b has degree 4, as $ab \notin E(G)$. This contradicts Subclaim 1.4.

Hence $|\overline{C}| > 2$, so $Y := \overline{C} \cap \overline{A}$ is not empty and, thus, a $\{a, b, v, z\}$-fragment. As both $N_G(X), N_G(Y)$ contain $a \in A_j$ and $v \in \overline{A_j}$, S_j must intersect X, Y. Hence $|X \cap S_j| = |Y \cap S_j| = 1$. Since $\overline{X} \cap S_j \supseteq (Y \cap S_j) \cup \{x, y\}$, this implies $|\overline{X} \cap S_j| = 3$. Similarly, $|\overline{Y} \cap S_j| = 3$. From $|X| > 1$ we now deduce that either $A_j \cap X \neq \emptyset$, which implied $|A_j \cap N_G(X)| \geq^{(*)} 3$, or that $\overline{A_j} \cap X \neq \emptyset$, which implied $|\overline{A_j} \cap N_G(X)| \geq^{(*)} 3$. As the latter is not true, we deduce $|A_j \cap N_G(X)| \geq 3$ and $\overline{A_j} \cap X = \emptyset$, so $z \in A_j$. Now $|N_G(Y) \cap A_j| = |N_G(Y) \cap \overline{A_j}| = 2$, implying that $Y \cap A_j = Y \cap \overline{A_j} = \emptyset$ (*). Since $z \in A_j$, we now obtain $\overline{A_j} \cap \overline{A} = (\overline{A_j} \cap X) \cup (\overline{A_j} \cap Y) = \emptyset$, which contradicts Subclaim 1.4.

Hence we proved that G' is 4-connected. Now consider an edge $e \in E(G') - E(G'(S))$ and suppose that it is 4-contractible in G' but not in G. Then $V(e)$ is contained in some $T \in \mathcal{T}(G)$ of cardinality 4, which does not separate G' and, therefore separates S. So $x \in T$.

If $T = N_G(s)$ for some $s \in S$ then $d_G(s) = 4$ and hence $s = u$ by Subclaim 1.4. But then since $d_{G'}(u) = d_G(u) = 4$, e covered by $N_{G'}(u)$ would not be 4-contractible in G'.

Hence $T \neq N_G(s)$ for all $s \in S$. If $y \in T$ then $|F \cap S| = 2$ for every T-fragment F, and hence $T \cap \overline{A} = V(e)$. As $|V(G)| > 8$, there exists a T-fragment F such that $F \cap \overline{F}$ is not empty and, therefore, a fragment whose neighborhood contains $V(e)$ and does not intersect A, contradicting the fact that e is 4-contractible in G'.

Hence $y \in F$ for some T-fragment F and, therefore, $\overline{F} \cap S = \{u\}$. As $T \neq N_G(u)$, $\overline{F} \cap \overline{A}$ is not empty and, therefore, a fragment whose neighborhood contains $V(e)$ and does not intersect A, again a contradiction.

Hence we proved that every edge in $E(G') - E(G'(S))$ which is 4-contractible in G' is 4-contractible in G, too.

We claim that $a(G) > a(G')$.

As $\overline{A_j} \cap \overline{A}$ is not empty by Subclaim 1.4, and, therefore, a fragment whose neighborhood does not intersect A and contains b, v, the edge bv (if it exists) is not 4-contractible in G'. As av is covered by $N_{G'}(u)$, it is not 4-contractible in G' either, so $E(G'(S))$ has at most three 4-contractible edges. Since both by, vy are 4-contractible in G by Subclaims 1.3 and 1.4, $a(G) \geq a(G') - 3 + 4 > a(G')$ follows.

As $b(G) = b(G') + 4 - 2 \cdot 2$ if $bv \notin E(G)$ and $b(G) = b(G') + 5 - 2 \cdot 2$ if $bv \in E(G)$ we deduce $b(G') \geq b(G) - 1$, and $a(G) \geq a(G') + 1 \geq \frac{1}{34} b(G') + 1 \geq \frac{1}{34} b(G) - \frac{1}{34} + 1 > a(G) - \frac{1}{34} + 1$, a contradiction.

Case 1.3. $d_G(x) = d_G(y) = 4$.

We are coming back to S_j here. S_j must cover an edge $e \in N - M - L$. As $xy \notin N - M - L$, $S_j \cap \overline{A_j} = V(e)$ and e is the unique edge in $N - M - L$ covered by S_j. If bv was an edge then it would be in M, so au is the unique edge in $N - M - L$ covered by S. Furthermore, $X := \overline{A} \cap A_j$ is not empty, as $d_G(a) > 4$ and a is not adjacent to both x and y. As $|\overline{A} \cap S_j| = |A_j \cap S|$, X is a fragment whose neighborhood $V(e) \cup \{a, u\}$ covers all edges from $N - M - L$ that are covered by S, S_j. Hence we may replace $A_i = A, A_j$ in our sequence with X to obtain a shorter one with the desired properties — which contradicts our choice.

This proves Claim 1.

Let $X := \bigcup_{i=1}^{k} E(G(S_i))$ be the set of edges covered by one of $S_1, \ldots, S_k$. Let $P := \{(u, a) : ua \in E(G) - X, u \in V(G) - V_4(G)\}$ and let $Q := \{(x, y) : xy \in E(G) \text{ is 4-contractible}\}$. We establish a map $\varphi : P \to Q$ according to the following rules. The stages of the choice process are labelled for later reference.

Consider (u, a) in P.

1st choice. If ua is contractible then set $\varphi(u, a) := (u, a)$.

Otherwise, ua is trivially noncontractible because ua is not covered by some S_i; hence u, a have a common neighbor b of degree 4.

2nd choice. If a has degree 4 then, by Lemma 3, we may choose a contractible edge xy with $x \in \{a, b\}$ such that $|\{b\} - \{x\}| \cdot d_G(y)$ is as large as possible, and set $\varphi(u, a) := (x, y)$. That is, we take $x = a$ if possible, and in this case we take y of largest possible degree.

Otherwise, a has degree exceeding 4, and we look at the edge ub instead of ua.

3rd choice. If ub is contractible then set $\varphi(u, a) := (u, b)$.

So we may assume that ub is noncontractible; in contrast to ua, ub could well be covered by some S_i.

4th choice. If ub is covered by some S_i then b is incident with some contractible edge bz, $z \neq u$. This follows directly from Lemma 4, applied to S_i for S. We choose z in such a way that $d_G(z)$ is minimal and set $\varphi(u,a) := (b,z)$.

Final choice. Hence we may assume that ub is trivially noncontractible, implying that u, b have a common neighbor c of degree 4. Clearly, $c \neq a$, as a has degree exceeding 4. It follows from Lemma 3 again that there exists a contractible edge xy with $x \in \{b, c\}$, where $y \neq u$. We choose it in such a way that $(|\{b\} - \{x\}|, d_G(y))$ is lexicographically minimal, and set $\varphi(u,a) := (x,y)$.

We say that (x,y) is ith choice for (u,a) if it has been chosen in the ith part of the rule.

Claim 2. $|\varphi^{-1}(x,y)| \leq 4$ for each $(x,y) \in Q$. In particular, $|P| \leq 4|Q|$.

If x has degree exceeding 4 then $|\varphi^{-1}(x,y)| \leq 4$, for if $\varphi(u,a) = (x,y)$ then either first choice applied to $(u,a) = (x,y)$, or the third choice applied to (u,a) where $u = x$ and a is one of at most 3 common neighbors of u and y.

So we may assume that x has degree 4. If $\varphi(u,a) = (x,y)$ then the second, the fourth, or the final choice applied to (u,a), where u is a neighbor of x of degree exceeding 4 distinct from y such that ux is noncontractible.

Let $U := N_G(x) - V_4(G) - \{y\}$.

Subclaim 2.1. If $|U| = 3$ then $|\varphi^{-1}(x,y)| \leq 4$

Let $u \in U$. If (x,y) is second choice for some (u,a) then $(x,y) = (a,b)$ and $y \in V_4(G)$ follows (b as in the choice rule), since from the fact that y is the only neighbor of x with degree 4, it follows that $\{a,b\} = \{x,y\}$, and hence the rule in the 2nd choice implies $a = x$. Similarly, if (x,y) is final choice for (u,a) then $(x,y) = (b,c)$ and $y \in V_4(G)$ follows (b, c as in the choice rule). Hence either $a = x$ (2nd choice), or a has degree exceeding 4 and is one of the three neighbors of x distinct from u (4th or final choice).

Let $U = \{u_1, u_2, u_3\}$. Suppose that $u_1u_2 \in E(G) - X$ and, for each $i \in \{1,2\}$, (x,y) is the fourth choice for some (u_i, a_i) with $a_i \in (U - \{u_i\}) \cup \{y\}$. We prove that Subclaim 2.1. holds in this situation and the symmetric ones, which we will therefore call *nice*.

By definition, there exist $S_i \in \mathcal{S}$ covering u_ix for $i \in \{1,2\}$. Since u_1u_2 not contained in X, there exist S_i-fragments F_i for $i \in \{1,2\}$ such that $u_1 \in S_1 \cap F_2$ and $u_2 \in S_2 \cap F_1$. Since S_1, S_2 do not cross, we conclude that $\overline{F_1} \subseteq F_2$ and

$\overline{F_2} \subseteq F_1$. Since x must have neighbors in each of $\overline{F_1}, \overline{F_2}$, $u_3 \in \overline{F_i}$ and $y \in \overline{F_{3-i}}$ for some $i \in \{1, 2\}$.

If (x, y) was a choice for some (u_3, a) then it is fourth choice as u_3, y are not adjacent, so there exists an $S_3 \in \mathcal{S}$ covering u_3x and separating $N_G(x) - \{u_3\} = \{u_1, u_2, y\}$, thus separating y from u_1 and u_2; but this is impossible since S_3 does not intersect F_i, as S_3, S_i do not cross.

If (x, y) was a choice for some (u_i, a) then it is fourth choice and $a \in \{u_3, u_{3-i}\}$, since u_i, y are not adjacent.

If (x, y) was second choice for some (u_{3-i}, a) then $a = x$, if it was final choice for some (u_{3-i}, a) then $a = u_i$, and if it was fourth choice for some (u_{3-i}, a) then $a = u_i$ or $a = y$. Observe that the latter case implies that $y \in V(G) - V_4(G)$, so that (x, y) can not be second choice (for (u_{3-i}, a) at the same time. Hence $\varphi^{-1}(x, y) \subset \{(u_i, u_3), (u_i, u_{3-i}), (u_{3-i}, x), (u_{3-i}, u_i), (u_{3-i}, y)\}$, which accomplishes the discussion of the nice situation.

Now if y has degree 5 then it can only be fourth choice, and it follows straightforward that if $|\varphi^{-1}(x, y)| \geq 5$ then there is a good situation. Hence we may assume that y has degree 4, implying that (x, y) is not a choice for any (u_i, y).

Without loss of generality, there exists an $\ell \in \{0, 1, 2, 3\}$ such that, for $i \in \{1, 2, 3\}$, (x, y) is choice for some (u_i, a) if and only if $i \leq \ell$. If $\ell \leq 1$ then $|\varphi^{-1}(x, y)| \leq 4$ follows from the initial paragraph of the proof of the actual subclaim. If $\ell = 3$ then y is not adjacent to all of u_1, u_2, u_3, since otherwise $N_G(\{x, y\}) = \{u_1, u_2, u_3\}$, violating 4-connectivity. Say, y is not adjacent to u_1. Then (x, y) is fourth choice for some (u_1, a), where $a \in \{u_2, u_3\}$, so $a = u_2$ without loss of generality. There exists an $S_1 \in \mathcal{S}$ covering u_1x. Now we may assume that (x, y) is not fourth choice for some (u_2, a), for otherwise we had a good situation. So $u_2y \in E(G)$, but then $u_3y \notin E(G)$ (for otherwise $y \in S_1$ because S_1 separates $N_G(x) - S_1$; so S_1 covers xy — but xy is contractible). So (x, y) is fourth choice for (u_3, a), where $a \in \{u_1, u_2\}$. Now $a \neq u_2$ (for otherwise $u_2 \in S_1$ because S_1 separates $N_G(x) - S_1$, so S_1 covers u_1u_2 — but $u_1u_2 \notin X$). Hence $a = u_1$. But then, again, we have a good situation.

It remains to consider the case $\ell = 2$. Suppose that $|\varphi^{-1}(x, y)| \geq 5$. Then $u_1u_2 \in E(G) - X$. If (x, y) is not fourth choice then both u_1, u_2 are adjacent to y; so u_3 is not adjacent to y (for otherwise, $N_G(\{x, y\}) = \{u_1, u_2, u_3\}$, contradicting 4-connectedness). Thus $N_G(x) - \{u_3\} = \{u_1, u_2, y\}$ induces a complete graph, and hence xu_3 is contractible. Since $d_G(u_3) > d_G(y)$, this implies that the second choice for (u_i, x) must be (x, u_3) for $i \in \{1, 2\}$. Hence $\varphi^{-1}(x, y) \subseteq \{(u_1, u_2), (u_1, u_3), (u_2, u_1), (u_2, u_3)\}$, and we are done.

Hence (x, y) is fourth choice for, say, (u_1, a), and we may assume that it is not fourth choice for any (u_2, a'), for otherwise we had a nice situation. Hence u_1x is covered by some $S_1 \in \mathcal{S}$, and $u_2y \in E(G)$. But then $u_3y \notin E(G)$

(for otherwise $y \in S_1$ because S_1 separates $N_G(x) - S_1$, but xy can not be covered by xy since xy is contractible). Now if (x, y) is choice for some (u_1, a) then $a \in \{u_2, u_3\}$, and if it is choice for some (u_2, a') then $a' \in \{x, u_1, u_3\}$. Hence $\varphi^{-1}(x, y) \subseteq \{(u_1, u_2), (u_1, u_3), (u_2, x), (u_2, u_1), (u_2, u_3)\}$. Assume, to the contrary, that equality holds here. Then $u_2u_3 \in E(G)$, which forces $u_2 \in S_1$. Therefore $u_2x \in X$, which implies $(u_2, x) \notin \varphi^{-1}(x, y)$, a contradiction.

This proves Subclaim 2.1.

The next subclaim deals rules out a special situation in the final choice.

Subclaim 2.2. If (x, y) is final choice for some (u, a) where $|\{b\} - \{x\}| > 0$ (b as in the final choice rule) then $|\varphi^{-1}(x, y)| \leq 4$.

Let b, c be as in the final-choice-rule and let d denote the neighbor of b distinct from u, a, c. The minimality constraint there implies that every edge incident with b is noncontractible. Let T be a smallest separating set covering bd. Then $u \in T$ as T separates the path auc formed by $N_G(b) - \{d\}$. There is a T-fragment F such that a is the unique neighbor of b in F and c is the unique neighbor of b in $\overline{F}$. By Lemma 2, applied to $w = b$, a is adjacent to d and d has degree 4. In view of Lemma 4, we have $|\overline{F}| = 1$. Thus $\overline{F} = \{c\}$. Since xy is contractible but $cd = xd$ is not, we have $d \neq y$, so $N_G(c = x) = \{y, u, b, d\}$. If $ud \in E(G)$, then $N_G(\{b, d\}) = \{a, u, c\}$, a contradiction. Thus $ud \notin E(G)$.

Now it is easy to conclude that $\varphi^{-1}(x, y) \subseteq \{(u, a), (u, b), (u, c), (u, y)\}$: Consider $(u', a') \in \varphi^{-1}(x, y)$; then $u' \in N_G(x) - V_4(G)$ where $u'x$ is noncontractible, which implies $u' = u$; if $a' \notin \{b, c, y\}$ then a' is a neighbor of u in F, so (x, y) must be final choice for (u, a') as $x = c$ is not adjacent to a'. Let b', c' denote the respective vertices b, c as in the final-choice-rule; consequently, $c' = c$, b' is a common neighbor of u, a', c, hence $b' \in \{y, b\}$. If $b' = y$ then we would have chosen (y, x) rather than (x, y) when chosing $\varphi(u, a')$, so $b' = b$. As a is the unique neighbor of $b \in F$, $a' = a$ follows.

This proves Subclaim 2.2.

By Subclaim 2.2, we may assume that if (x, y) has been chosen for (u, a) then either $x = a$ or a is a common neighbor of u and x. Hence, if $|U| \leq 1$, then $|\varphi^{-1}(x, y)| \leq 4$ holds, and it suffices to consider the case that $|U| = 2$.

Let $U = \{u_1, u_2\}$ and let z denote the neighbor of x distinct from u_1, u_2, y. By the preceeding paragraph, $\varphi^{-1}(x, y) \subseteq \{(u_1, x), (u_1, y), (u_1, z), (u_1, u_2), (u_2, x), (u_2, y), (u_2, z), (u_2, u_1)\}$

If (x, y) is choice for some (u_i, y) then it can't be 2nd choice because of the maximality constraint in the 2nd-choice-rule; therefore, y has degree exceeding 4.

Case 2.1. z is adjacent to both u_1, u_2.

Let d denote the neighbor of z distinct from u_1, u_2, x. Then zd is contractible (for if, otherwise, zd was covered by some smallest separating set T then $x \in T$ follows; for some T-fragment F, $\{x, z\}$ had only one neighbor u in F, which is among u_1, u_2; as F is not trivial, $(T - \{x, y\}) \cup \{u\}$ separates $F - \{u\}$ from $\overline{F} \cup \{x, y\}$, which is impossible).

Observe that $\varphi^{-1}(x, y) \subseteq \{(u_1, x), (u_1, y), (u_1, u_2), (u_2, x), (u_2, y), (u_2, u_1)\}$, since, by the maximality constraint in the 2nd-choice-rule, we choose (z, d) for (u_i, z) rather than (x, y). We thus may assume $u_1u_2 \in E(G)$ (for otherwise $\varphi^{-1}(x, y) \subseteq \{(u_1, x), (u_1, y), (u_2, x), (u_2, y)\}$), We may assume that for some $i \in \{1, 2\}$, (x, y) is a choice for both (u_i, y) and (u_i, u_{3-i}) (for otherwise $|\varphi^{-1}(x, y)| \leq 4$, too); but this yields a contradiction: Without loss of generality, $i = 1$; it follows that y has degree exceeding 4. Then xz is not contractible, for otherwise, according to the minimality constraints in the 4th- and final-choice-rule, respectively, we would have choosen (x, z) rather than (x, y) for (u_1, u_2). So let T be a separator covering xz. As T separates $N_G(x) - \{z\}$, it must contain u_1, and there is a T-fragment F such that $u_2 \in F$ and $y \in \overline{F}$. Then d is the unique neighbor of z in $\overline{F}$, and u_2 is the unique neighbor of x and of z in F. Consequently, $(T - \{x, z\}) \cup \{u_2\}$ separates $F - \{u_2\}$ from the other vertices, contradicting the 4-connectedness of G.

So $|\varphi^{-1}(x, y)| \leq 4$ in Case 2.1.

Case 2.2. z is adjacent to none of u_1, u_2.

We may assume that (x, y) is choice for at least one of $(u_1, y), (u_2, y)$, for otherwise $\varphi^{-1}(x, y) \subseteq \{(u_1, x), (u_1, u_2), (u_2, x), (u_2, u_1)\}$. Hence y has degree exceeding 4. But then, for each $i \in \{1, 2\}$, u_i, x have no common neighbor of degree 4, hence (x, y) is not a choice for (u_i, x), implying that $\varphi^{-1}(x, y) \subseteq \{(u_1, y), (u_1, u_2), (u_2, y), (u_2, u_1)\}$.

Case 2.3. z is adjacent to exactly one of u_1, u_2.

Say, $u_2z \in E(G)$.

(*) If $u_1y, u_1u_2 \in E(G)$ then (x, y) is not a choice for (u_2, z).

Suppose, to the contrary, that (x, y) is a choice for (u_2, z). Then it is a 2nd choice, and, by the maximality constraint in the 2nd-choice-rule, all edges incident with z are noncontractible. Let T be a smallest separating set covering zx. Since T separates $N_G(x) - \{z\}$, $u_1 \in T$ follows. There exists a T-fragment F such that y is the unique neighbor of x in F and u_2 is the unique neighbor of x in $\overline{F}$. As $\overline{F}$ is not trivial, u_2 can't be the unique neighbor of z in $\overline{F}$ (for otherwise $(T - \{x, z\}) \cup \{u_2\}$ would separate G), hence z has only one neighbor in F, say, d, and only one neighbor in T, which is x. By Lemma 2, applied to $w = z$, it follows that x, z and $d = y$ form a triangle. But then xy is not contractible, as it is covered by $N_G(z)$. This proves (*).

Suppose that $\{(u_1, x), (u_2, z)\} \subseteq \varphi^{-1}(x, y)$. Then u_1x is noncontractible and u_1, x must have a common neighbor of degree 4, which must be y. Hence (x, y) can't be choice for $(u_1, y), (u_2, y)$. We thus may assume that $u_1u_2 \in E(G)$, for otherwise $\varphi^{-1}(x, y) \subseteq \{(u_1, x), (u_2, x), (u_2, z)\}$. Now (*) applies, yielding a contradiction.

Hence it follows that at most one of $(u_1, x), (u_2, z)$ is in $\varphi^{-1}(x, y)$. We thus may assume that $u_1u_2 \in E(G)$ and that at least one of $(u_1, y), (u_2, y)$ is in $\varphi^{-1}(x, y)$ (otherwise, $|\varphi^{-1}(x, y)| \leq 4$). In particular, y has degree exceeding 4. Now if $(u_1, x) \in \varphi^{-1}(x, y)$ then u_1x is not contractible and u_1, x have a common neighbor of degree 4, which is impossible.

Hence $(u_1, x) \notin \varphi^{-1}(x, y)$.

We may assume that $(u_2, y) \in \varphi^{-1}(x, y)$ (for otherwise, $u_1y \in E(G)$, and, by (*), $\varphi^{-1}(x, y) \subseteq \{(u_1, y), (u_1, u_2), (u_2, x), (u_2, u_1)\}$).

In particular, $u_2y \in E(G)$. If (x, y) was a choice for (u_2, z) then, as in the proof of (*), all edges incident with z are noncontractible. Let again T be a smallest separating set covering zx. Since T separates $N_G(x) - \{z\}$, $u_2 \in T$, and there exists a T-fragment F such that y is the unique neighbor of x in F and u_1 is the unique neighbor of x in $\overline{F}$. Let p be the unique neighbor of z in F and let q be the unique neighbor of z in $\overline{F}$. Note that $p \neq y$, as xy is contractible and, thus, not covered by $N_G(z)$, and that $q \neq u_1$ as $u_1z \notin E(G)$. By Lemma 2, applied to $w = z$, we deduce that z, p, u_2 form a triangle where p has degree 4 and that z, q, u_2 form a triangle where q has degree 4. Let T' be a smallest separating set covering zp. As $N_G(z) - \{p\}$ induces a path qu_2x, $u_2 \in T'$ follows. Let F' be a T'-fragment such that x is the unique neighbor of z in F' and q is the unique neighbor of z in $\overline{F'}$. As there exists an x, q-path whose inner vertices are in $\overline{F}$, T' intersects $\overline{F}$. Hence T, T' cross and $T' = \{u_2, z, p\} \cup (T' \cap F')$. Therefore, $y \notin T'$, which implies $y \in F \cap F'$, and $T_G(F, F') = \{u_2, z, x, p\}$. However, z has no neighbor in $F \cap F'$, so $\{u_2, x, p\}$ separates G, a contradiction.

Hence $(u_2, z) \notin \varphi^{-1}(x, y)$. Now assume, to the contrary, that $\varphi^{-1}(x, y) = \{(u_1, y), (u_1, u_2), (u_2, x), (u_2, y), (u_2, u_1)\}$. Observe that (x, y) is a 4th or a final choice for (u_1, y). From the minimality constraints in the corresponding rules we deduce that xz is noncontractible, for otherwise we would have chosen (x, z) rather than (x, y).

But xz is contractible, because $N_G(x) - \{z\}$ is a triangle u_1u_2y and cannot be separated by any set covering xz.

This proves Claim 2.

Let $Q_4 := Q \cap \{(x, y) : x \in V_4(G)\}$ and let $K := \{(x, y) : xy \in X,\ x \in V(G) - V_4(G)\}$.

Claim 3. $6(|P| + |Q_4|) \geq |K|$.

Recall that, by Claim 1, $\mathcal{S}$ is cross free. Observe that A_1 is inclusion minimal among all T-fragments with $T \in \mathcal{S}$. Hence we may apply Theorem 1 (with A_1 for A), and obtain $C(\cdot)$ and a tree order $(\mathcal{S}, \leq)$ as there. Let D be the Hasse digraph of $(\mathcal{S}, \leq)$.

For a good subtree D' of D and $u \in V(G)$, let $\mathcal{S}(D', u) := \{S \in V(D') : u \in S\}$, and let $\mathcal{S}^*(D', u)$ denote the maximal elements of $\mathcal{S}(D', u)$ with respect to $\leq$. Furthermore, let the subgraph $G_{D'}$ of G defined by $V(G_{D'}) := \bigcup_{S \in V(D')} S$ and $E(G_{D'}) := \bigcup_{S \in V(D')} E(G(S)) \cap (X - L)$. If $u \in V(G) - V_4(G)$ then let $\psi(D', u) := |\mathcal{S}^*(D', u)|$, if $u \in V_4(G) \cap V(G_{D'})$ and u has at least one neighbor in $G_{D'}$ then let $\psi(D', u) := 1$. In all other cases, set $\psi(D', u) := 0$.

We first look at some properties of these sets when $D' = D$. Let $R(u) := \{(u, x) : ux \in E(G) - X\}$, and let $Q(u) := \{(u, x) : ux \in E(G) - N\}$.

Subclaim 3.1. $|R(u)| \geq |\mathcal{S}^*(D, u)|$ for each $u \in V(G)$.

Consider $S \in \mathcal{S}^*(D, u)$. Then $u \in S$ must have a neighbor $x_S \in \overline{C(S)}$; ux_S is not covered by some $T \in \mathcal{S}$, for otherwise $S < T$ by (i) of Lemma 1, contradicting the maximality of S. Hence $(u, x_S) \in R(u)$. By (ii) of Lemma 1, the sets $\overline{C(S)}$, $S \in \mathcal{S}^*(D, u)$ are pairwise disjoint, and hence the (u, x_S), $S \in \mathcal{S}^*(D, u)$, are pairwise distinct. This proves Subclaim 3.1.

Subclaim 3.2. $Q(u) \neq \emptyset$ for each $u \in V_4(G)$ with at least one neighbor in G_D.

Let x be a neighbor of u in G_D. Assume, to the contrary, that $Q(u) = \emptyset$. Since $ux \in X$, there exists a member S_0 of $\mathcal{S}$ which covers ux. Choose a nontrivial smallest separating set S and an S-fragment F with $u \in S$ and $F \subseteq \overline{C(S_0)}$ so that F is inclusion minimal. Let a be a neighbor of u in F. If $ua \in N - M$ and if we let T be a nontrivial smallest separating set covering ua, then since $S \cap T \neq \emptyset$, S and T do not cross (see the first three paragraphs of the proof of Claim 1), and hence we see that there exists a T-fragment F' such that $F' \subseteq F$ by arguing as in the proof of (i) of Lemma 1, a contradiction. Thus $ua \in M$. Since $x \in S_0 \subseteq S \cup \overline{F}$, $x \neq a$. Since $ua \in M$, it follows that u, a have a common neighbor c of degree 4. Since $uc \in L$, $uc \notin E(G_D)$, so $x \notin \{a, c\}$.

Now choose a nontrivial smallest separating set R and an R-fragment B with $u \in R$ and $B \subseteq C(S)$ such that B is inclusion minimal. Recall that $|B| \geq 2$.

Let b be a neighbor of $u \in B$. Arguing as in the preceding paragraph, we see that $ub \in M$ and $x \neq b$.

It follows that u, b have a common neighbor d of degree 4, and, again, $x \notin \{b, d\}$. Since a, c, x are distinct, b, d, x are distinct, and $a \neq b$, we deduce that $c = d$. But then either $(S - \{u, c\}) \cup \{a\}$ separates $F - \{a\}$ from all other vertices, or $(T - \{u, d\}) \cup \{b\}$ separates $B - \{b\}$ from all other vertices, a contradiction.

This proves Subclaim 3.2.

Subclaim 3.3. $\sum_{u \in V(G_{D'})} \psi(D', u) \geq |E(G_{D'})|/3$ for all good subtrees of D.

We prove this by induction on $|D'|$. For $V(G_{D'}) = \{S\}$ we observe $d_{G_{D'}}(u) \leq 3$ for every $u \in S$, and hence $\sum_{u \in V(G_{D'})} \psi(D', u) \geq |\{u \in V(G_{D'}) : d_{G_{D'}}(u) \geq 1\}|$ $\geq \sum_{u \in V(G_{D'})} d_{G_{D'}}(u)/3 \geq |E(G_{D'})|/3$.

For $|V(G_{D'})| \geq 2$, take any pseudo-leaf T of $G_{D'}$ and let D'' be obtained from D' by truncating T. By (iii) of Lemma 1, $\bigcup N^+_{D'}(T) \cap V(G_{D''}) \subseteq T$, and hence $|E(G_{D'})| - |E(G_{D''})| \leq \sum_{u \in V(G_{D'}) - V(G_{D''})} d_{G_{D'}}(u)$. The right hand side is bounded from above by $\sum_{R \in N^+_{D'}(T)} \sum_{u \in R - V(G_{D''}) - V_4(G)} d_{G_D(R)}(u)$ + $\sum_{u \in V_4(G) \cap (V(G_{D'}) - V(G_{D''}))} d_{G_{D'}}(u)$. Obviously, $d_{G_D(R)}(u) \leq 3$ for all $R \in \mathcal{S}$; since every vertex $u \in R \in N^+_{D'}(T)$ has a neighbor in $\overline{C(R)}$, which is not in $V(G_{D'})$, $d_{G_{D'}}(u) \leq d_G(u) - 1$ holds. Hence we may estimate each term of the sums by 3, thus obtaining $|E(G_{D'})|/3 - |E(G_{D''})|/3 \leq \sum_{R \in N^+_{G_{D'}}(T)} |R - V(G_{D''}) - V_4(G)| + \sum_{u \in V_4(G) \cap (V(G_{D'}) - V(G_{D''}))} \psi(D', u)$.

For each $u \in V(G_{D'}) - V(G_{D''})$ it follows $\{R \in N^+_{G'_D}(R) : u \in R\} \subseteq \mathcal{S}^*(D', u)$; so $\sum_{R \in N^+_{G_{D'}}(T)} |R - V(G_{D''}) - V_4(G)| = \sum_{u \in V(G_{D'}) - V(G_{D''}) - V_4(G)} |\{R \in N^+_{G'_D}(R) : u \in R\}| \leq \sum_{u \in V(G_{D'}) - V(G_{D''}) - V_4(G)} \psi(D, u)$. Therefore, $|E(G_{D'})|/3 - |E(G_{D''})|/3$ $\leq \sum_{V(G_{D'}) - V(G_{D''})} \psi(D', u)$.

Since $\psi(D'', u) \leq \psi(D', u)$ for every $u \in V(G_{D''})$, we obtain by the induction hypothesis $|E(G_{D'})|/3 \leq \sum_{u \in V(G_{D''})} \psi(D'', u) + \sum_{u \in V(G_{D'}) - V(G_{D''})} \psi(D', u) \leq$ $\sum_{u \in V(G_{D'})} \psi(D', u)$.

This proves Subclaim 3.3.

Now, for $Q_4(u) := \{(x, y) \in Q_4 : x = u\}$, $|P| + |Q_4| = \sum_{u \in V(G) - V_4(G)} |R(u)| +$ $\sum_{u \in V_4(G)} Q_4(u) \geq \sum_{u \in V(G_D)} \psi(D, u) \geq |E(G_D)|/3 \geq \sum_{u \in V(G_D) - V_4(G)} d_{G_D}(u)/6$ $= |K|/6$. This proves Claim 3.

Let's put the inequalities of Claim 2 and Claim 3 together. On the one hand, $|K| + |P| = |\{(x, y) : xy \in E(G), x \in V(G) - V_4(G)\}| = \sum_{x \in V(G) - V_4(G)} d_G(x)$

$= 2|E(G)| - 4|V_4(G)| = \geq 2|E(G)| - 4|V(G)| = 2b(G)$. On the other hand, $|K| + |P| \leq 6|P| + 6|Q_4| + |P| \leq 7|P| + 6|Q| \leq 34|Q| = 34 \cdot 2a(G)$. Hence $a(G) \geq \frac{1}{34}b(G)$, contradicting our assumption that G is a counterexample to the statement. □

4 A lower bound for the optimal constant

We now construct graphs showing that we can't expect a constant better that $\frac{1}{5}$ in Theorem 3. Let $\ell > 4$ be an integer such that $\ell - 1$ is divisible by 3 and $\ell \cdot (\ell - 1)$ is divisible by 12. Set $m := \binom{\ell}{2}$. Then, by the results in [4], we can partition K_ℓ into $m/6$ many copies of K_4. For $i \in \{1, \ldots, m/6\}$, let $\{a_i, b_i, c_i, d_i\}$ denote the vertex sets of either copy. Let G_ℓ be obtained from K_ℓ by adding $m/6$ many disjoint new 4-cycles $C_i := p_i q_i r_i s_i p_i$, $i \in \{1, \ldots, m_6\}$, and connecting each C_i to K_ℓ by adding the edges $a_i p_i, a_i q_i, b_i p_i, b_i q_i$ and $c_i r_i, c_i s_i, d_i r_i, d_i q_i$. Then G_ℓ is 4-connected, has $\ell + 4 \cdot m/6$ vertices, has $m + 12 \cdot m/6$ edges, but has only $2m/6$ many contractible edges, namely the edges $q_i r_i$ and $s_i p_i$ for each $i \in \{1, \ldots, m/6\}$. Hence the ratio of $|E(G_\ell)| - 2|V(G_\ell)|$ and the number of contractible edges of G_ℓ tends to $\frac{1}{5}$ as ℓ tends to infinity, proving that we can't expect a constant larger than $\frac{1}{5}$ in Theorem 3.

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