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Duality of ends

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Abstract

We investigate the end spaces of infinite dual graphs. We show that there exists a natural homeomorphism between the end spaces of a graph and its dual, and that this homeomorphism maps thick ends to thick ends. Along the way, we prove that Tutte-connectivity is invariant under taking (infinite) duals.

1 Introduction

In 1932 Whitney [18] introduced the concept of dual graphs: a (multi-)graph G^* is a dual of a finite (multi-)graph G if there exists a bijection $* : E(G) \to E(G^*)$ so that a set $F \subseteq E(G)$ is a circuit of G precisely when F^* is a bond in G^*. Building on work by Thomassen [15, 16], Bruhn and Diestel [1] extended duality to (a superclass of) locally finite graphs. Many properties of dual graphs are retained in infinite graphs, such as Whitney's planarity criterion, the symmetry of the duality relation (i.e. that G is a dual of G^*), and the uniqueness of the dual of 3-connected planar graphs; see [1].

The main aim of this work is the study of a yet unexplored aspect of duality in infinite graphs, namely of the relation between the end space of a graph and the end space of its dual. Our first result states that there exists a homeomorphism between these two spaces that arises in a natural way from the bijection $*$ on the edges.

More precisely, we will demonstrate that, given a pair G, G^* of (infinite) duals, the endvertices of a set $F \subseteq E(G)$ converge towards an end ω of G if and only if the endvertices of F^* converge towards the dual end ω^*. This is the content of Theorem 6, which together with a discussion of the topology involved can be found in Section 3.

Thick ends, those that contain an infinite set of disjoint rays, play an important role in the study of the automorphism group of a graph, see for instance Halin [9]. As our second result, we will prove that thickness is preserved in the dual end:

Theorem 1. *Let G, G^* be a pair of dual graphs, and let ω be an end of G. Then ω is thick if and only if ω^* is thick.*

In order to prove Theorem 1, we make use of a notion of connectivity, introduced by Tutte [17], that coincides with the matroid connectivity of the cycle-matroid of the graph. As a by-product we obtain a generalisation to infinite graphs of the following classical result:

Theorem 2 (Tutte [17])**.** *Let G and G^* be a pair of finite dual graphs, and let $k \geq 2$. Then G is k-Tutte-connected if and only if G^* is k-Tutte-connected.*

Theorem 3. *Let G and G^* be a pair of dual graphs, and let $k \geq 2$. Then G is k-Tutte-connected if and only if G^* is k-Tutte-connected.*

We will define Tutte-connectivity in Section 5 (all other definitions can be found in the next section), but let us remark here that a graph is 3-Tutte-connected if and only if it is 3-connected. Therefore, Theorem 3 has the following consequence:

Corollary 4 (Thomassen [16])**.** *Let G and G^* be a pair of dual graphs. Then G is 3-connected if and only if G^* is 3-connected.*

Duality for infinite graphs has first been explored by Thomassen. Faced with the incongruity that an infinite graph may have infinite cuts as well as finite ones but (in the traditional definition) only finite circuits he chose to ignore infinite cuts. Consequently, G^* is a dual of G, in the sense of Thomassen, if for all *finite* sets $F \subseteq E(G)$, F is a circuit precisely when F^* is a bond. This concept allowed him to prove an infinite version of Whitney's planarity criterion: a 2-connected graph G has a (Thomassen-)dual if and only if it is planar and satisfies

every two vertices of G can be separated by finitely many edges. (†)

However, Thomassen's definition is not completely satisfactory, as the symmetry in taking duals is lost, as well as the uniqueness of the duals of 3-connected graphs. These deficits are ultimately due to the disregard of infinite cuts.

Infinite circuits, which have been proposed by Diestel and Kühn [5, 6, 7], promise a way out of this dilemma. Taking infinite circuits into account led to the more restrictive definition of duals in [1]: there, a set $F \subseteq E(G)$, finite or infinite, is a circuit if and only if F^* is a bond. These duals overcome the drawbacks of Thomassen's definition, i.e. they retain the basic properties of finite duals. We will define and very briefly discuss infinite circuits in the next section.

2 Definitions and preliminaries

All our graphs are allowed to have loops and parallel edges, with the exception of 2-connected graphs, which we require to be loopless, and of 3-connected graphs, which, in addition, cannot have parallel egdes. Otherwise our notation follows Diestel [4].

Let G be a fixed graph. A 1-way infinite path is called a *ray*, a 2-way infinite path is a *double ray*. Two rays are called *equivalent* if there are infinitely many disjoint paths between them. The equivalence classes of rays are the *ends of* G, we denote the set of these by $\Omega(G)$.

The proof of the following lemma, which is not hard, can be found in [4, Lemma 8.2.2].

Lemma 1. *Let G be a connected graph, and let U be an infinite subset of $V(G)$. Then G contains a ray R with infinitely many disjoint R–U paths or a subdivided star with infinitely many leaves in U.*

Diestel and Kühn [5, 6, 7] employed a topological approach to define (finite and infinite) circuits in infinite graphs. For them, a circuit of G is the edge set of a homeomorphic image of the unit circle in a topological space based on G.

We shall introduce the topology in two steps. First we define a topological space $|G|$, whose points are the vertices and ends of G, as well as the interior points on edges of G. In the second step we shall identify some of the points of $|G|$.

So, in order to define $|G|$, see G as endowed with the topology of a 1-complex, so every edge is homeomorphic to the unit interval and a basic open neighbourhood of a vertex consists of the union of half-open edges, one for each incident edge. In order to describe the neighbourhoods of an end ω, pick a finite vertex set S, and denote the component of $G - S$ that contains a ray of ω (and thus a subray for every ray in ω) by $C(S, \omega)$. We say that ω *belongs to* $C(S, \omega)$. A basic open neighbourhood of ω now consists of $C(S, \omega)$, all ends that have a ray in $C(S, \omega)$ and the union of all interior points of edges between S and $C(S, \omega)$. The resulting space $|G|$ is Hausdorff, and in the case of the a locally finite graph called the *Freudenthal compactification* of G.

In non-locally finite graphs, we say that a vertex v *dominates* an end ω, if there are infinitely many paths between v and a ray in ω that pairwise only meet in v. We define an equivalence relation $\sim$ on $|G|$ as follows. For two ends ω and ω', let $\omega \sim \omega'$ if both ω and ω' are dominated by the same vertex. For a vertex v and an end ω, let $v \sim \omega$ if v dominates ω. We denote by $\tilde{G}$ the quotient space of $|G|$ under the equivalence relation $\sim$. In particular, if G is locally finite, then $\tilde{G} = |G|$. Observe furthermore that, if G satisfies (†), then no two vertices of G are identified in $\tilde{G}$.

We shall need to work within both spaces $|G|$ and $\tilde{G}$. In order to distinguish between closures of sets $X \subseteq V(G) \cup E(G)$ in the two spaces, we write $\overline{X}$ for the closure of X in $|G|$, and $\widetilde{X}$ for the closure of X in $\tilde{G}$.

The following basic lemma relates topological connectivity to finite cuts. Its proof is similar to that of Lemma 8.5.5 in [4]. As a convenience we will, for a set F of edges, write $V[F]$ to denote the set of endvertices of the edges in F.

Lemma 2. *Let G be a graph satisfying (†), and let $X \subseteq E(G)$. Then $\widetilde{X}$ is path-connected (as a subspace of $\tilde{G}$) if and only if every finite cut that separates two vertices of $V[X]$ meets X.*

Next, we define *circles* in $\tilde{G}$ as the homeomorphic images of the unit circle. If a circle contains an interior point of an edge then it contains the whole edge. Thus it makes sense to speak of the edge set of a circle, which is called a *circuit*. The homeomorphic image of the unit interval $[0, 1]$ in $\tilde{G}$ is an *arc*. Observe that circuits as well as arcs must contain edges.

For the merits of infinite circuits and the topological cycle space, which is based on this definition, see the overview article by Diestel [3]. Let us remark that a more general approach to cycle spaces has been pursued by Richter and Vella [13], who define (infinite) circuits for a wider range of topological spaces.

Now, we can finally define duals. For this, assume G to satisfy (†)—as Thomassen [16] observed this is a necessary condition for a graph to have a dual (in Thomassen's and thus in our sense as well). We call a graph G^* a *dual* of G if there is a bijection $* : E(G) \to E(G^*)$ so that a (finite or infinite) set $F \subseteq E(G)$ is a circuit of G precisely when F^* is a bond in G^*. (A *bond* is a minimal non-empty cut.)

The dual G^* then can be seen to satisfy (†) as well. So, the class of graphs with (†) is closed under taking duals, unlike the class of locally finite graphs.

Whenever we speak of duals we will therefore tacitly assume that the original graph (and then automatically the dual too) satisfies (†). We refer to [1] for more details.

We list two properties of duals, that we shall need throughout the paper.

Lemma 3. *Let G and G^* be a pair of dual graphs. Then G is 2-connected if and only if G^* is 2-connected.*

The lemma follows easily from the fact that a every two edges lie in a common circuit if and only if the graph is 2-connected, which is the case precisely when every two edges lie in a common bond. Variants of this lemma can be found in Thomassen [15] as well as in [1].

Theorem 5. [1] *Let G^* be a dual graph of a graph G. Then G is also a dual of G^*.*

3 Discussion

Our main aim in this paper is twofold: given two dual graphs G and G^*, we firstly shall demonstrate that there is a bijection between the ends of G and the ends of G^*, which arises in a natural way; secondly, we shall prove that if an end ω of G is thick, i.e. if it contains infinitely many disjoint rays, then the dual end ω^* of G^* is thick, too. We shall make our aims more precise in what follows.

Let us start with the bijection we wish to define between the end spaces of dual graphs. Our mapping will be an extension of the bijection $* : E(G) \to E(G^*)$ on the edges (and we will therefore, slightly abusing notation, denote it with $*$ as well). More precisely, we aim at a bijection $*$ between $\Omega(G)$ and $\Omega(G^*)$, so that for all $F \subseteq E(G)$, the endvertices of F converge against an end ω of G if and only if the endvertices of F^* converge against ω^*.

In the space $\tilde{G}$, which is instrumental in the definition of duality, the accumulation points of vertex sets are the identification classes of ends. Recall that any two ends that cannot be separated by finitely many edges, are identified, giving rise to larger equivalence classes of rays called *edge-ends* by some authors (e.g. Hahn, Laviolette and Širáň [8]). So, should we not search for a bijection of the edge-ends rather than of the ends?

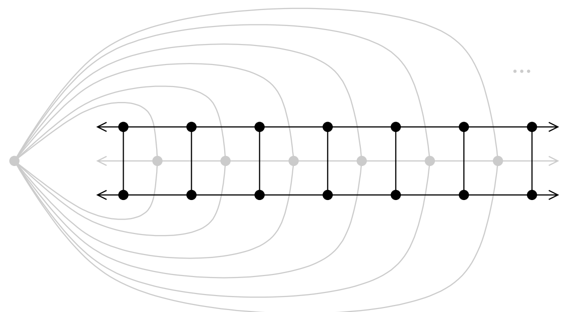

Figure 1: No correspondence between edge-ends of duals

Figure 1 demonstrates that there is no hope for a bijection between edge-ends (even without any structural requirements). The double ladder has two edge-ends, while its dual graph has only one edge-end.

The reason that this attempt fails lies in the nature of duals. The existence of finite edge-cuts between (edge-)ends will not be preserved in the dual. In fact, such a (minimal) cut corresponds to a circuit in the dual, which need not separate anything. By contrast, a vertex-separation whose deletion results in two sufficiently large sides does, in some sense, carry over to the dual graph; this is the essence of Theorem 3 and will be more explored in Section 5.

Our bijection will thus be between the ends of G and G^*. This means that we will work in $|G|$, since any two identified ends cannot be distinguished topologically in $\tilde{G}$. Endowing $\Omega(G)$ resp. $\Omega(G^*)$ with the subspace topology of $|G|$ resp. $|G^*|$, we will show the existence of a bijection $\Omega(G) \to \Omega(G^*)$, which is structure-preserving in the sense above. Moreover, we will see that $*$ is a homoeomorphism:

Theorem 6. *Let G and G^* be 2-connected dual graphs. Then there is a homeomorphism $* : \Omega(G) \to \Omega(G^*)$, where the two spaces are endowed with the subspace topology of $|G|$ resp. $|G^*|$, so that*

$$\text{for all } F \subseteq E(G) \text{ and ends } \omega \text{ it holds that } \omega \in \overline{F} \text{ if and only if } \omega^* \in \overline{F^*}. \quad (1)$$

We remark that the requirement that G and G^* are 2-connected cannot be dropped. This is illustrated by the example of the double ray. Every dual of the double ray is a graph whose edge set is the union of countably many loops, and thus contains no end at all.

We shall prove Theorem 6 in the next section.

Let us now turn to our second objective: showing that our bijection $*$ preserves thickness. This will be achieved in Theorem 1. Again, we are confronted with the question why focus on preserving (vertex-)thickness instead of "edge-thickness", i.e. the existence of infinitely many edge-disjoint rays in an end.

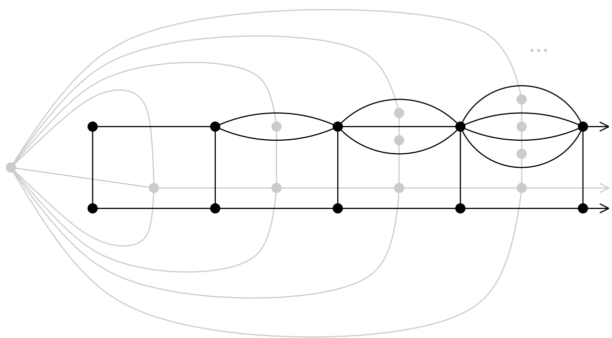

Figure 2: Edge-thick end with edge-thin dual end

This is answered by Figure 2, which shows a graph that has a single edge-thick end while the unique end of its dual graph does not even possess two edge-disjoint rays. The reason is the same as above: although (or because) the notion of duals is based on edges and operations with edges, the existence of (small) edge-separators is not preserved in the dual.

Since not all vertex-separators are preserved in the dual, connectivity is not an invariant of (finite or infinite) duals, as we have already remarked in the introduction. But, the related notion of *Tutte-connectivity* is. We defer to Section 5 for the definition; suffice it to say here that there are two reasons why a graph may have low Tutte-connectivity: Either it has a small vertex-separator or it contains a small circuit. In Section 5, we prove that Tutte-connectivity is an invariant of infinite duals, too (Theorem 3).

Theorem 3 is an important step on our way to proving Theorem 1. Our proof of Theorem 3 differs from the usual proof of its finite version, Theorem 2, which is done in two steps. First, one shows that Tutte-connectivity coincides with the connectivity of the cycle-matroid of the graph. Then one observes that matroid connectivity is invariant under duality.

If we want to use this approach for Theorem 3 as well, we first have to answer two questions. Which notion of infinite matroids should we use? And how do we define higher connectivity in a matroid?

The first question is easy to answer. Although it is sometimes claimed that there is no proper concept of an infinite matroid that provides duality and the existence of bases at the same time, B-matroids, as defined by Higgs [10], accomplish that (see also Oxley [11]). Moreover, one can prove that duality in B-matroids is compatible with taking dual graphs. While the second problem, the definition of higher connectivity, can also be overcome in a satisfactory way, its solution together with the introduction of B-matroids would take quite a bit of time and effort. Therefore, we will, in Section 5, present a matroid-free proof of Theorem 1.

4 $*$ induces a homeomorphism on the ends

Before we are able to prove Theorem 6, we need two lemmas.

Lemma 4. *Let G be a 2-connected graph satisfying (†). If U is an infinite set of vertices then $\overline{U}$ contains an end of G.*

Proof. Suppose otherwise. Then there is no ray R in G with infinitely many disjoint R–U paths. So, an application of Lemma 1 yields a subdivided star S that contains an infinite subset U' of U. We delete the centre of S and apply Lemma 1 again, this time to U', which yields another subdivided star S' with infinitely many leaves in U'. But then, the centre of S and the centre of S' are infinitely connected, contradicting (†). □

Lemma 5. *Let G be a 2-connected graph, and let X and Y be two sets of edges such that $\overline{X} \cap \overline{Y} \cap \Omega(G) \neq \emptyset$. Then there are infinitely many (edge-)disjoint finite circuits each of which meets both X and Y.*

Proof. Let $\mathcal{Z}$ be an $\subseteq$-maximal set of finite disjoint circuits so that each $C \in \mathcal{Z}$ meets both X and Y, and suppose that $|\mathcal{Z}|$ is finite. Putting $Z := \bigcup \mathcal{Z}$, we pick for every two $x, y \in V[Z]$ for which it is possible an x–y path $P_{x,y}$ that is edge-disjoint from Z. Denote by Z' the union of Z with the edge sets of all these paths, and observe that still $|Z'| < \infty$.

We claim that for every component K of $G - V[Z']$ it holds that

$$\textit{for every } v, w \in N(K) \textit{ there is a } v\text{–}w \textit{ path in } G[Z'] - Z. \tag{2}$$

Indeed, by construction, there are $x, y \in V[Z]$ and (possibly trivial) v–x resp. w–y paths Q_v resp. Q_w in $G[Z'] - Z$. Then x and y are connected through $K \cup Q_v \cup Q_w \subseteq G - Z$. Hence in $P_{x,y} \cup Q_v \cup Q_w \subseteq G[Z'] - Z$ we find a v–w path. This proves (2).

Now, because $\overline{X} \cap \overline{Y}$ contains an end, there exists a component K of $G - V[Z']$ which contains infinitely many vertices of both $V[X]$ and $V[Y]$. Choose edges $e_X, e_Y \in E(K) \cup E(K, G-K)$ so that $e_X \in X$, and $e_Y \in Y$. Since G is 2-connected, there is a finite circuit C which contains both e_X and e_Y. The maximality of $\mathcal{Z}$ implies that C meets Z in at least one edge. In particular, C contains the edge sets of (possibly identical) $N(K)$-paths P_X and P_Y so that $e_X \in E(P_X)$, and $e_Y \in E(P_Y)$.

Being connected, K contains a $V(P_X)$–$V(P_Y)$ path P. Thus, we find in $P \cup P_X \cup P_Y$ an $N(K)$-path P' with $e_X, e_Y \in E(P')$. By (2), there exists a path R in $G[Z'] - Z$ between the endvertices of P'. Now, $E(P') \cup E(R)$ is a circuit that meets both X and Y but is edge-disjoint from Z, a contradiction to the maximality of $\mathcal{Z}$. $\square$

Proof of Theorem 6. We start by claiming that for each $F \subseteq E(G)$ and each end ω of G the following is true:

$$\textit{if } \overline{F} \cap \Omega(G) = \{\omega\} \textit{ then } \overline{F^*} \textit{ contains exactly one end.} \tag{3}$$

Suppose the claim is not true. By Lemma 4, this cannot be because $\overline{F^*}$ fails to contain an end; rather there must be (at least) two ends, α_1 and α_2, in $\overline{F^*}$. Take a finite connected subgraph T of G^* so that $V(T)$ separates α_1 and α_2 in G^*. For $i = 1, 2$, denote by K_i the component of $G^* - T$ to which α_i belongs, and set $X_i^* := (E(K_i) \cup E(K_i, T)) \cap F^*$. Since each of the X_i^* is infinite, it follows from Lemma 4 that $\overline{X_i}$ contains an end. As $\overline{X_i} \subseteq \overline{F}$, this end must be ω. Hence, Lemma 5 yields disjoint finite circuits $C_1, C_2, \ldots$ in G each of which meets X_1 as well as X_2.

We claim that each of the bonds C_i^* contains an edge of T. Indeed, let M_1 and M_2 be the two components of $G^* - C_i^*$. Since C_i^* meets both X_1^* and X_2^*, each M_j contains a vertex in $K_1 \cup T$ and a vertex in $K_2 \cup T$. As, for $j = 1, 2$, M_j is connected it follows that $V(M_j) \cap V(T) \neq \emptyset$. So, since T is connected, there is an M_1–M_2 edge in $E(T)$, i.e. $C_i^* \cap E(T) \neq \emptyset$, for each $i \in \mathbb{N}$. This yields a contradiction since the C_i^* are disjoint but T is finite. Therefore, Claim (3) is established.

Now, we define $* : \Omega(G) \to \Omega(G^*)$. Given an end $\omega \in \Omega(G)$, pick any set $F \subseteq E(G)$ with $\overline{F} \cap \Omega(G) = \{\omega\}$ (choose, for instance, the edge set of a ray in ω). Define $\omega^* = \omega^*(F)$ to be the, by (3), unique end in $\overline{F^*}$. To see that this mapping is well-defined, i.e. that it does not depend on the choice of F, consider a second set $D \subseteq E(G)$ as above, and observe that $\omega^*(D) = \omega^*(D \cup F) = \omega^*(F)$. Since G is a dual of G^* (Theorem 5), we may apply (3) to G^* and see that $*$ is a bijection and satisfies (1).

Next, we prove that $* : \Omega(G) \to \Omega(G^*)$ is continuous. For this, let an end $\omega^* \in \Omega(G^*)$ and an open neighbourhood $U^* \subseteq \Omega(G^*)$ of ω^* be given. Then there exists a finite vertex set $S \subseteq V(G^*)$, and a component K of $G^* - S$ so that $W^* := K \cap \Omega(G^*) \subseteq U^*$.

Setting $F^* := E(G^*) \setminus (E(K) \cup E(S, K))$, we observe that $W^* = \Omega(G^*) \setminus \overline{F^*}$. Hence, by (1), $W = \Omega(G) \setminus \overline{F}$. So, W is an open neighbourhood of ω whose image is contained in U^*. Finally, by interchanging the roles of G and G^* we see that the inverse of $*$ is continuous as well. $\square$

5 Tutte-connectivity

In this and in the next section, we are concerned with how (Tutte-)connectivity is preserved in the dual. The main idea underlying our proofs is the duality of spanning trees: given a pair of finite connected dual graphs G and G^*, a set D is the edge set of a spanning tree of G, if and only if $E(G^*) \setminus D^*$ is the edge set of a spanning tree in G^*.

For a pair of infinite graphs, the situation is slightly more complicated. In fact, if $E(G^*) \setminus D^*$ is the edge set of a spanning tree, then $(V(G), D)$ might very well be disconnected—topologically, however, $\widetilde{D}$ (the closure of D in $\tilde{G}$) is always connected.

Moreover, $\widetilde{D}$ forms a *topological spanning tree* (TST for short) of $\tilde{G}$: a path-connected circuit-free subspace of $\tilde{G}$ that contains all vertices of G, and every edge of which it contains an interior point. For more on the relation between spanning trees in G and G^* see [1]. TSTs were first introduced by Diestel and Kühn in [7], where it is proved that $\tilde{G}$ always has a TST provided G is connected.

We will use the tree duality implicitly in the key lemma, Lemma 8, below. The next two lemmas help to relate the tree duality to vertex separations.

Lemma 6. *Let G be a graph satisfying (†), let T be a subgraph that does not contain any circuits, and let $U \subseteq V(T)$ such that $0 < |U| < \infty$. Then there exists a set $F \subseteq E(T)$ of size at most $|U| - 1$ so that every arc in $\widetilde{T}$ between two vertices in U meets F.*

Proof. We use induction on $|U|$. The assertion is trivial for $|U| = 1$, so for the induction step asume that $|U| > 1$. Choose $v \in U$, then by the induction assumption there is a set $D \subseteq E(T)$ such that each vertex w of $U \setminus \{v\}$ lies in a different path-component K_w of $\widetilde{T} - D$. If there is no vertex $w \in U \setminus \{v\}$ such that $v \in K_w$, we are done, so assume there is such a w.

Observe that there exists exactly one v–w arc A in $\widetilde{T} - D$. Indeed, if there were two, then it is easy to see that the edge set of their union would contain a circuit. Now, choose any edge e on A, and set $F := D \cup \{e\}$. Clearly, F is as desired, which completes the proof. $\square$

Lemma 7. *Let H be a connected graph, let $F \subseteq E(H)$, and let $W \subseteq V(H)$. If every W-path in H meets F then $|F| \geq |W| - 1$.*

Proof. Since no two vertices of W can lie in the same component of $H - F$, we deduce that $H - F$ has at least $|W|$ components. As each deletion of a single edge increases the number of components by at most one, $H - F$ can have at most $|F| + 1$ components. $\square$

Let us now introduce the notion of Tutte-connectivity, see Tutte [17]. For finite graphs, the Tutte-connectivity coincides with the connectivity of the cycle-matroid of the graph. We remark that for $k \in \{2, 3\}$, a graph is k-Tutte-connected if and only if it is k-connected. For greater k the two notions of connectivity are not equivalent.

Definition 7. *A k-*Tutte-separation *of a graph G is a partition (X, Y) of $E(G)$ so that $|X|, |Y| \geq k$ and so that at most k vertices of G are incident with edges in both of X and Y.*
*We say that a graph G is k-*Tutte-connected *if G has no ℓ-Tutte-separation for any $\ell < k$.*

Consider a k-Tutte-separation (X, Y) in a (2-connected) graph G with a dual G^*. To prove that Tutte-connectivity is invariant under taking duals, we would ideally like to see that (X^*, Y^*) is a k-Tutte-separation in G^*. This, however, is not always true—if the two sides of the separation do not induce connected subgraphs of G^*, then the number of vertices in $V[X^*] \cap V[Y^*]$ can be much higher than k. Thus we will strengthen the requirements and lessen our expectations. By demanding $G[Y] - V[X]$ to be connected, we shall be able to guarantee that at least $G^*[Y^*]$ is connected. Moreover, we will be content with finding an ℓ-Tutte-separation of G^* for some $\ell \leq k$ that is derived from (X^*, Y^*).

The statement of the next lemma, which accomplishes just that, is a bit more general than we need for Theorem 3, as we shall reuse it for Theorem 1.

Lemma 8. *Let G and G^* be a pair of 2-connected dual graphs, and let (X, Y) be a k-Tutte-separation such that $C_Y := G[Y] - V[X]$ is non-empty and connected, and such that $Y = E(C_Y) \cup E(C_Y, V[X])$. Then*

(i) *there exists a component L of $G^*[X^*]$ so that $(E(L), E(G^*) \setminus E(L))$ is an ℓ-Tutte-separation for some $\ell \leq k$; and*

(ii) *for each component K of $G^*[X^*]$ with $|E(K)| \geq k$ it holds that $(E(K), E(G^*) \setminus E(K))$ is a k-Tutte-separation.*

Proof. First, we prove that

$$\widetilde{G^*[Y^*]} \text{ is path-connected in } \tilde{G}^*. \tag{4}$$

Suppose that this is not the case. Then we can write Y as the disjoint union of two sets Y_1 and Y_2 so that there is no Y_1^*–Y_2^* arc in $\tilde{G}^*$ that only uses edges from Y^*.

In particular, there is no circle in $\tilde{G}^*$ that only uses edges from Y^* and meets both Y_1^* and Y_2^*. Equivalently, there is no bond in G that only uses edges from Y, and meets both Y_1 and Y_2.

However, since C_Y is connected and since every edge in Y is incident with a vertex in C_Y, there is a vertex $x \in V(C_Y)$ which is incident with both Y_1 and Y_2. Observe that the cut B_x of G, which consists of all edges incident with x, is a subset of Y. As G is 2-connected, B_x is a bond, which yields the desired contradiction and thus proves (4).

Now, set $U := V[X] \cap V[Y]$ and $W := V[X^*] \cap V[Y^*]$. Observe that each vertex in W is incident with both X^* and Y^*. So, if $|W|$ is infinite, then Lemma 4 implies that $\overline{X^*} \cap \overline{Y^*}$ contains an end, while $\overline{X} \cap \overline{Y}$ does not (as X and Y are finitely separated by U). This contradicts Theorem 6. We have thus shown that

$$|W| \text{ is finite.} \tag{5}$$

Let T_X be the edge set of a maximal topological spanning forest of $\widetilde{G}[X]$, i.e. the union of TSTs of the spaces $\tilde{C}$ corresponding to the components C of $G[X]$.

We point out that every circuit of G that lies entirely in X is a circuit of $G[X]$. It follows that T_X does not contain any circuits of G.

Next, we prove that

$$\textit{every } W\textit{-path in } G^*[X^*] \textit{ meets } T_X^*. \tag{6}$$

Suppose there is a W-path whose edge set D^* lies in $X^* \setminus T_X^*$. By (4), there is a circuit C^* of G^* with $C^* \cap X^* = D^*$. Thus, C is a bond in G, and hence D is a finite cut of $G[X]$. Consequently, D contains a bond B of $G[X]$, which then is completely contained in one component K_B of $G[X]$. As $B \subseteq D \subseteq X \setminus T_X$, the intersection of B with T_X is empty, but $\widetilde{T_X}$, restricted to $\widetilde{K_B}$, is path-connected, a contradiction to Lemma 2. This proves (6).

Next, Lemma 6 yields a set $F \subseteq T_X$ of at most $|U| - 1$ edges so that every U-arc in $\widetilde{T_X} \subseteq \tilde{G}[X]$ meets F. This means that every circuit C of G with $C \cap X \subseteq T_X$ meets F. Thus, every bond B^* of G^* with $B^* \cap X^* \subseteq T_X^*$ meets F^*. Hence, denoting by $\mathcal{K}$ the set of components of $G^*[X^*]$, we obtain that

$$\textit{for every } K \in \mathcal{K}\textit{, the graph } H_K := K - (T_X^* \setminus F^*) \textit{ is connected.} \tag{7}$$

Now, for every $K \in \mathcal{K}$, observe that by (6), every W-path in H_K meets F^*. So, by (7), we may apply Lemma 7 to H_K. Doing so for each $K \in \mathcal{K}$, we obtain that $|F^*| \geq |W| - |\mathcal{K}|$. On the other hand, $|F^*| = |F| \leq |U| - 1$ by the choice of F, implying that

$$|W| \leq |U| + |\mathcal{K}| - 1. \tag{8}$$

Suppose that for every $K \in \mathcal{K}$, it holds that $|V(K) \cap W| > |E(K)|$. Then

$$|W| = \sum_{K \in \mathcal{K}} |V(K) \cap W| \geq \sum_{K \in \mathcal{K}} (|E(K)| + 1) = |X^*| + |\mathcal{K}|.$$

As $|X^*| = |X| \geq |U|$, we obtain that $|W| \geq |U| + |\mathcal{K}|$. This yields a contradiction to (8), since by (5), $|W|$ is finite. Therefore, there exists an $L \in \mathcal{K}$ with

$$\ell := |V(L) \cap W| \leq |E(L)|.$$

Observe that if we can show now that $\ell \leq k$, then it follows that $(E(L), E(G^*) \setminus E(L))$ is an ℓ-Tutte-separation of G^*, as desired for (i). So, in order to prove (i), and (ii), it suffices to prove that for each $K \in \mathcal{K}$ it holds that

$$|V(K) \cap W| \leq |U|.$$

Suppose otherwise. Then there exists an $M \in \mathcal{K}$ such that

$$|W| = \sum_{K \in \mathcal{K}} |V(K) \cap W| \geq (|U| + 1) + \sum_{K \in \mathcal{K},\, K \neq M} |V(K) \cap W|.$$

Because G is 2-connected, so is G^* (Lemma 3). Thus $|V(K) \cap W| \geq 1$ for every $K \in \mathcal{K}$, resulting again in $|W| \geq |U| + |\mathcal{K}|$, a contradiction, as desired. □

Theorem 3. *Let G and G^* be a pair of dual graphs, and let $k \geq 2$. Then G is k-Tutte-connected if and only if G^* is k-Tutte-connected.*

Proof. We show that if G has a k-Tutte-separation (X, Y), then G^* has an ℓ-Tutte-separation for some $\ell \leq k$. By Theorem 5, this is enough to prove the theorem.

First, assume that $V[Y] \setminus V[X] \neq \emptyset$. Let K be a component of $G[Y] - V[X]$, and set $Z := E(K) \cup E(K, G - K)$. As $E(K, G - K)$ contains at least one edge for each vertex in $N(K)$, it follows that $|Z| \geq |N(K)|$. Thus, $(Z, E(G) \setminus Z)$ is a k'-Tutte-separation of G for $k' := |N(K)| \leq k$. We can now apply Lemma 8 (i) to obtain the desired ℓ-Tutte-separation of G^*.

So, we may assume that $V[Y] \setminus V[X] = \emptyset$. Then, since $|Y| \geq k$, there is a circuit C in Y, say of length $\ell \leq k$. Hence, C^* is a bond of size ℓ in G^*; let K_1 and K_2 be the components of $G^* - C^*$. Now,

$$|E(K_1 \cup K_2)| = |X^*| + |Y^*| - |C^*| \geq 2k - \ell.$$

Thus, we can partition C^* into C_1^* and C_2^* so that each $Z_i^* := E(K_i) \cup C_i^*$ has cardinality at least ℓ.

In order to show that (Z_1^*, Z_2^*) is an ℓ-Tutte-separation of G^* it remains to check that $U := V[Z_1^*] \cap V[Z_2^*]$ has cardinality at most ℓ. To this end, consider a vertex $v \in U$, and let j be such that $v \in V(K_j)$. Then v is incident with an edge $e_v^* \in C_{3-j}^*$, whose other endvertex lies in K_{3-j}, because C^* is a cut. This defines an injection from $U \to C^*$, which implies $|U| \leq |C^*| \leq \ell$, as desired. $\square$

6 The dual preserves the degrees

In this section we will use Lemma 8 in order to prove a quantitative version of Theorem 1, that relates the 'degree' of an end ω to the degree of its dual end ω^*.

For an end ω, define $m(\omega)$ to be the supremum of the cardinalities of sets of disjoint rays in ω; Halin [9] showed that this supremum is indeed attained. In [2] and in [14] the number of vertex- (or edge-)disjoint rays in an end has been successfully used to serve as the degree of an end in a locally finite graph (whether vertex- or edge-disjoint rays should be considered depends on the application). This motivates the definition of the degree $d(\omega) := m(\omega)$ of an end ω of a locally finite graph.

Now, if G and G^* are a dual pair of 2-connected locally finite graphs, then it will turn out that $m(\omega) = m(\omega^*)$ for every end ω of G. In non-locally finite graphs we need to be a bit more careful: Figure 2 indicates that dominating vertices should be taken into account.

For an end $\omega \in \Omega(G)$ and a finite vertex set S, we say that $U \subseteq V(G)$ *separates S from ω* if U meets every ray in ω that starts in S. We define here the *degree* $d(\omega)$ of an end $\omega \in \Omega(G)$ to be the minimal number k such that for each finite set $S \subseteq V(G)$, we can separate S from ω in G by deleting at most k vertices from G. If there is no such k, we set $d(\omega) := \infty$. Lemma 9 will show that this definition is consistent with the one given above for locally finite graphs.

So, denote by $\mathrm{dom}(\omega)$ the number of vertices that dominate an end ω (possibly infinite). Note that the graphs we are interested in, namely those that satisfy (†), are such that $\mathrm{dom}(\omega) \in \{0, 1\}$ for every end ω.

Lemma 9. *Let G be a graph and let $\omega \in \Omega(G)$. Then $d(\omega) = m(\omega) + \mathrm{dom}(\omega)$.*

Proof. It is easy to see that $d(\omega)$ is at least $m(\omega) + \mathrm{dom}(\omega)$. For the other direction, we may assume that $\mathrm{dom}(\omega) < \infty$. Denote by D the set of vertices that dominate ω. As D is a finite set, there is an obvious bijection between the ends of $G - D$ and G, which we will tacitly use.

We observe first that for any finite vertex set T, there exists a finite T–ω separator T' in $G - D$ that is contained in $C_{G-D}(T, \omega)$. Indeed, otherwise, by Menger's theorem[1], $G[T \cup C(T, \omega)] - D$ contains infinitely many paths between T and some ray in ω that are pairwise disjoint except possibly in T. As T is finite, this implies that $T \setminus D$ contains a vertex which dominates ω, contradicting our choice of D.

Now, consider an arbitrary finite set $S \subseteq V(G)$. Starting with $S_0 := S \setminus D$ we can choose inductively finite vertex sets S_i so that $S_i \subseteq V(C_{G-D}(S_{i-1}, \omega))$ is an S_{i-1}–ω separator in $G - D$, and has minimal cardinality with that property. Since $S_i \subseteq V(C_{G-D}(S_{i-1}, \omega))$, all the S_i are pairwise disjoint.

Applying Menger's theorem repeatedly between S_{i-1} and S_i we obtain a set $\mathcal{R}$ of disjoint rays in ω of cardinality at least $|S_1|$. As $S_1 \cup D$ separates S from ω in G, we have shown that S can be separated from ω by at most $|S_1| + |D| \leq m(\omega) + \mathrm{dom}(\omega)$ vertices, thus proving the lemma. □

We remark that Lemma 9 can be obtained easily from results of Polat [12]; we chose to provide the proof nevertheless since the statement of Polat's results together with the necessary adaptions would have taken about as much time and space.

Theorem 8. *Let G and G^* be a pair of 2-connected dual graphs, and let ω be an end of G. Then $d_G(\omega) = d_{G^*}(\omega^*)$.*

Proof. First assume that $d(\omega) \leq k$, where $k \in \mathbb{N}$ is a finite number. We wish to show that ω^* has vertex-degree $\leq k$, too.

So, let a finite vertex set $T \subseteq V(G^*)$ be given. Pick a finite edge set F^* of cardinality at least k so that $T \subseteq V[F^*]$ and so that F^* induces a connected graph. Now, since $d(\omega) \leq k$ there is a set $U \subseteq V(G)$ of cardinality at most k that separates (the finite set) $V[F]$ from ω. If C is the component of $G - U$ to which ω belongs then set $Y := E(C) \cup E(C, U)$ and $X := E(G) \setminus Y$. Because $k \geq |U| = |V[X] \cap V[Y]|$, and because $|Y| = \infty$ and $|X| \geq |F| \geq k$, it follows that (X, Y) is a k-Tutte-separation.

Since $G^*[F^*] \subseteq G^*[X^*]$ is connected, there is a component K of $G^*[X^*]$ that contains all of F^*. As $|F^*| \geq k$, Lemma 8 (ii) implies that $(E(K), E(G^*) \setminus E(K))$ is a k-Tutte-separation. Moreover, as $\omega \notin \overline{X}$, it follows that $\omega^* \notin \overline{K}$. Thus, $N_{G^*}(G^* - K)$ is a vertex set of cardinality $\leq k$ that separates $T \subseteq V[F^*]$ from ω^*, as desired.

In conclusion, since G is also a dual of G^* (Theorem 5), it follows that $d(\omega) = d(\omega^*)$ if either of ω and ω^* has finite degree. In the remaining case, we trivially have $d(\omega) = \infty = d(\omega^*)$. □

The theorem in conjunction with Lemma 9 immediately yields Theorem 1.

[1] We use here, and below, that the cardinality version of Menger's theorem holds in infinite graphs. This can easily be deduced from Menger's theorem for finite graphs, see for instance [4, Section 8.4]

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