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A Weaker Version of Lovász' Removable Paths Conjecture

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Abstract

We prove there exists a function $f(k)$ such that for every $f(k)$-connected graph G and for every edge $e \in E(G)$, there exists an induced cycle C containing e such that $G - E(C)$ is k-connected. This proves a weakening of a conjecture of Lovász due to Kriesell.

Key Words : graph connectivity, removable paths, non-separating cycles

1 Introduction

The following conjecture is due to Lovász (see [14]):

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Conjecture 1.1 *There exists a function $f = f(k)$ such that the following holds. For every $f(k)$-connected graph G and two vertices s and t in G, there exists a path P with endpoints s and t such that $G - V(P)$ is k-connected.*

Conjecture 1.1 can alternately be phrased as following: there exists a function $f(k)$ such that for every $f(k)$-connected graph G and every edge e of G, there exists a cycle C containing e such that $G - V(C)$ is k-connected. Lovász also conjectured [9] that every $(k+3)$-connected graph contains a cycle C such that $G - V(C)$ is k-connected. This was proven by Thomassen [13].

Conjecture 1.1 is known to be true in several small cases. In the case $k = 1$, a path P connecting two vertices s and t such that $G - V(P)$ is connected is called a *non-separating path*. It follows from a theorem of Tutte that any 3-connected graph contains a non-separating path connecting any two vertices, and consequently, $f(1) = 3$. When $k = 2$, it was independently shown by Chen, Gould, and Yu [1] and Kriesell [6] that $f(2) = 5$. In [1], the authors also show that in a $(22k + 2)$-connected graph, there exist k internally disjoint non-separating paths connecting any pair of vertices. In [5], Kawarabayashi, Lee, and Yu obtain a complete structural characterization of which 4-connected graphs do not have a path linking two given vertices whose deletion leaves the graph 2-connected.

In a variant of the problem, one can attempt to delete the edges of the path instead of deleting all the vertices. Mader proved [11] that every k-connected graph with minimum degree $k + 2$ contains a cycle C such that deleting the edges of C leaves the graph k-connected. Jackson independently proved the same result when $k = 2$ in [4]. As a corollary to a stronger result, Lemos and Oxley have shown [8] that in a 4-connected graph G, for any edge e there exists a cycle C containing e such that $G - E(C)$ is 2-connected.

Kriesell has postulated the following natural weakening of Conjecture 1.1

Conjecture 1.2 (Kriesell, [7]) *There exists a function $f(k)$ such that for every $f(k)$-connected graph G and any two vertices s and t of G, there exists an induced path P with ends s and t such that $G - E(P)$ is k-connected.*

We answer this question in the affirmative with the following theorem.

Theorem 1.3 *There exists a function $f(k) = O(k^4)$ such that the following holds: for any two vertices s and t of an $f(k)$-connected graph G, there exists an induced $s-t$ path P such that $G-E(P)$ is k-connected.*

Corollary 1.4 *For every $(f(k) + 1)$-connected graph G and for every edge e of G, there exists an induced cycle C containing e such that $G - E(C)$ is k-connected.*

In the proof of Theorem 1.3, we will at several points need to force the existence of highly connected subgraphs using the fact that our graph will have large minimum degree. A theorem of Mader implies the following.

Theorem 1.5 (Mader, [10]) *Every graph of minimum degree $4k$ contains a k-connected subgraph.*

In addition to simply requiring a highly connected subgraph, we will require the subgraph have small boundary. The *boundary* of a subgraph H of a graph G, denoted $\partial_G(H)$, is the set of vertices in $V(H)$ that have a neighbor in $V(G) - V(H)$. We use the following related result of Thomassen. By strengthening the minimum degree condition in Theorem 1.5, we can find a highly connected subgraph that further has a small boundary.

Theorem 1.6 (Thomassen, [15]) *Let k be any natural number, and let G be any graph of minimum degree $> 4k^2$. Then G contains a k-connected subgraph with more than $4k^2$ vertices whose boundary has at most $2k^2$ vertices.*

Given a path P in a graph, and two vertices x and y on P, we denote by xPy the subpath of P starting at vertex x and ending at y. A *separation* of a graph G is a pair (A, B) of subsets of vertices of G such that $A \cup B$ is equal to $V(G)$, and for every edge $e = uv$ of G, either both u and v are contained in A or both are contained in B. The *order* of a separation (A, B) is $|A \cap B|$. Where not otherwise stated, we follow the notation of [2].

We will need the following results on systems of disjoint paths with pre-specified endpoints.

Definition A *linkage* is a graph where every connected component is a path.

A *linkage problem* in a graph G is a set of pairs of vertices in G. We will typically write the linkage problem $\mathcal{L}$ as follows:

$$\mathcal{L} = \{\{s_1, t_1\}, \dots, \{s_k, t_k\}\}.$$

A *solution* to the linkage problem $\mathcal{L} = \{\{s_1, t_1\}, \dots, \{s_k, t_k\}\}$ is a set of pair-wise internally disjoint paths $P_1, \dots, P_k$ such that the ends of P_i are s_i and t_i, and furthermore, if $x \in V(P_i) \cap V(P_j)$ for some distinct indices i and j, then $x = s_i$ or $x = t_i$. A graph G is *strongly k-linked* if every linkage problem $\mathcal{L} = \{\{s_1, t_1\}, \dots, \{s_k, t_k\}\}$ consisting of k pairs in G has a solution. The graph G is *k-linked* if every linkage problem with k pair-wise disjoint pairs of vertices has a solution. We utilize the following theorem:

Theorem 1.7 ([12]) *Every $10k$-connected graph is k-linked.*

Any k-linked graph on at least $2k$ vertices is strongly k-linked. Thus the following statement follows trivially from Theorem 1.7.

Corollary 1.8 *Every $10k$-connected graph is strongly k-linked.*

2 Proof of Theorem 1.3

We prove the theorem with the function $f(k) = 1600k^4 + k + 2$. Let $\mathcal{S}$ be a $2k$-connected subgraph of G such that $G - E(\mathcal{S})$ contains an induced s-t path. To see that such a subgraph $\mathcal{S}$ exists, consider an s-t path P_0 of minimum length. We note that P_0 is an induced path, and, further, that $G - E(P_0)$ has minimum degree $f(k) - 3 > 8k$. By Theorem 1.5, $G - E(P_0)$ contains the desired $2k$-connected subgraph $\mathcal{S}$.

Our goal in the proof of Theorem 1.3 will be to pick an s-t path P which uses no edges of $\mathcal{S}$ and has the following property. For every vertex x of G, in the graph $G - E(P)$ the vertex x has k internally disjoint paths to distinct vertices in $\mathcal{S}$. This will suffice to show that $G - E(P)$ is k-connected. To find such a path, we pick P to maximize the number of vertices with k paths to $\mathcal{S}$, and subject to that, to maximize the number of vertices with $k-1$ paths to $\mathcal{S}$, and so on. This leads to the following definition. For any induced $s - t$ path P such that $E(P)$ is disjoint from $E(\mathcal{S})$, we define the set:

$$S_k = S_k(P) = \{v | \exists k \text{ internally disjoint paths in } G - E(P) \text{ from } v \text{ to } V(\mathcal{S}) \text{ with distinct ends in } V(\mathcal{S})\}.$$

For i between 0 and $k-1$ we define sets S_i where a vertex v is in S_i if v is joined to $V(\mathcal{S})$ by i paths in $G - E(P)$ disjoint except at v and not $i+1$ such paths.

We choose an induced $s-t$ path P disjoint from $E(\mathcal{S})$ so as to lexicographically maximize

$$(S_k, S_{k-1}, ..., S_0).$$

It now suffices to show that for this P, $|S_k| = |V(G)|$. We let $min = min\{i | S_i \neq \emptyset\}$. We will show that if $min < k$, there exists an induced path P^* which avoids $E(\mathcal{S})$ and satisfies the following properties:

(a) for all v in $S_j(P)$, $j > min$, $v \in S_{j^*}(P^*)$ for some $j^* \geq j$,

(b) there exists a v in S_{min} which is in $S_{j^*}(P^*)$ for some $j^* > min$.

This contradicts our choice of P.

To find P^*, observe that there exists a separation (A, B) of $G - E(P)$ of order min with $V(\mathcal{S}) \subseteq A$ and $v \in B - A$. Assume we have chosen such a separation to minimize $|A|$. Let X denote the set $A \cap B$. It follows from our choice of min that every vertex of $B - A$ is contained in S_{min}.

Consider the subgraph of G induced by $B - A$. We note that $G[B - A]$ has minimum degree at least $f(k) - k - 2 = 1600k^4$. By Theorem 1.6, there exists a $20k^2$-connected subgraph F in $G[B - A]$ of size at least $1600k^4$ which has a boundary of size at most $800k^4$.

By our choice of min, there exist $|X|$ disjoint paths from X to F in the graph $G - E(P)$ restricted to the set B. We choose $|X|$ such paths internally disjoint from F. Let X' be the endpoints of the paths in F. Let $\mathcal{L}_1$ be the linkage problem $\{\{x, y\} | x, y \in X', x \neq y\}$ consisting of every pair of vertices of X'.

For every vertex $x \in X$, $x \in S_t$ for some value of $t = t(x)$. There exist paths $Q_1^x, \dots, Q_{t(x)}^x$ in $G - E(P)$ disjoint except for the vertex x each having one endpoint in $\mathcal{S}$ and the other endpoint equal to x. Let Q be a path in G with endpoints u and v. A vertex $x \in V(F) \cap V(Q)$ is Q-*extremal* if either uQx or xQv contains no vertex of $V(F)$ other than the vertex x. We let $\mathcal{Q}$ be the set of paths $\{Q_i^x | x \in X, 1 \leq i \leq t(x)\}$. Note, two distinct $Q_1, Q_2 \in \mathcal{Q}$ are not necessarily disjoint. A vertex $x \in V(F)$ is $\mathcal{Q}$-*extremal* if there exists a path $Q \in \mathcal{Q}$ such that x is Q-extremal. Let Y' be the set of $\mathcal{Q}$-extremal vertices in $V(F)$), and let $\mathcal{L}_2$ be the natural linkage problem induced by $\mathcal{Q}$:

$$\mathcal{L}_2 = \{\{x, y\} | x, y \in Y' \text{ and } \exists Q \in \mathcal{Q} \text{ such that } x \text{ and } y \text{ are } Q\text{-extremal}\}$$

Observe that while a vertex in X may have many neighbors in $V(F) - \partial_{G[B-A]}(F)$, the only edges of G with one end in $A - B$ and the other end in $V(F) - \partial_{G[B-A]}(F)$ are contained in P. It follows that either X' or Y' may contain vertices of $V(F) - \partial_{G[B-A]}(F)$. See Figure 1.

Recall that the size of the boundary of F is at most $800k^4$ in $G[B - A]$. It follows from the connectivity of G that there exists a matching of size three from $V(F) - X' - Y' - \partial_{G[B-A]}(F)$ to $A - X$ using only edges of P. Let aa', bb' and cc' be three edges forming such a matching where the vertices a, b, and c lay in $V(F) - X' - Y' - \partial_{G[B-A]}(F)$. By our choice of (A, B) to minimize $|A|$, there exist $|X| + 1$ disjoint paths from $X \cup \{a'\}$ to $V(\mathcal{S})$ in $G - E(P)$ (and similarly for $X \cup \{b'\}$ and $X \cup \{c'\}$).

By Theorem 1.7, the graph F is strongly $2k^2$-linked. Fix vertices s^* and s' as follows. Let s^* be a vertex in $V(F) - X' - Y'$ such that s^* has a neighbor s' on P in G and furthermore, assume that s^* and s' are chosen so that s' is as close to s on P as possible. Similarly, we define t^* and t' such that t^* is a vertex of $V(F) - X' - Y'$ with a neighbor t' as close to t as possible. The vertices s^*

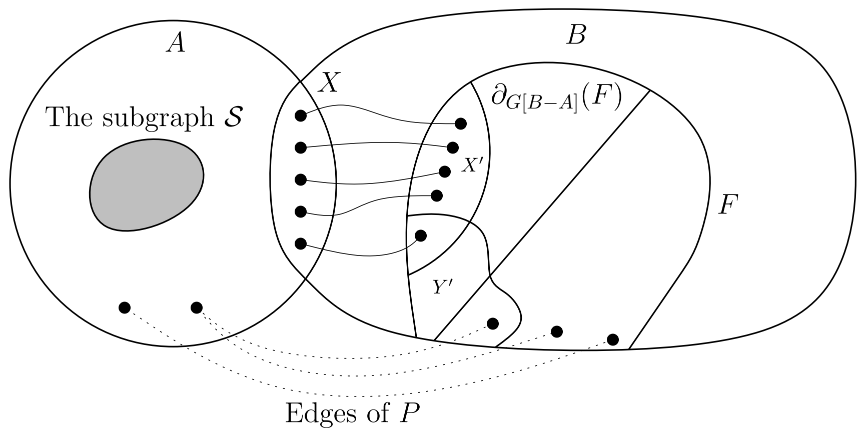

Figure 1: An example of the separation (A,B) with the subgraphs $\mathcal{S}$ and F and possible sets X' and Y'.

and t^* are well defined since a, b, and c all have a neighbor on P in G. Without loss of generality, we may assume that $b \neq s^*, t^*$. Let v be a vertex of $V(F) - X' - Y' - \{s^*, t^*\}$. Now consider the linkage problem

$$\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \{\{v, x\} | x \in X'\} \cup \{\{v, b\}, \{s^*, t^*\}\} .$$

The linkage problem $\mathcal{L}$ has at most $\binom{k}{2} + k(k-1) + k + 2 \leq 2k^2$ pairs, and so there exists a solution $\mathcal{R}$ in F. Let $R \in \mathcal{R}$ be the path with ends s^* and t^*. We now define P^* to be the shortest induced subpath of $sPs's^*Rt^*t'Pt$. We claim that P^* is the desired path violating our choice of P. Let $S_i^* = S_i(P^*)$ for $i = 0, \dots, k$.

To complete the proof, it now suffices to verify the following claim.

Claim 1 $(S_k^*, \dots, S_0^*)$ *is lexicographically greater than* $(S_k, \dots, S_0)$

Proof. We begin with the observation that by construction and the choice of s^* and t^*, there exists a subpath $\overline{R}$ of R with ends $\overline{s}$ and $\overline{t}$ such that $P^* = sPs'\overline{s}\overline{R}\overline{t}t'Pt$. Furthermore, it follows that $E(P[A]) \supseteq E(P^*[A])$ and $E(P^*) - E(P) \subseteq E(F) \cup \{s'\overline{s}, t'\overline{t}\}$. It follows that $E(P^*) \cap E(\mathcal{S}) = \emptyset$ since the edges $s'\overline{s}$ and $t'\overline{t}$ each have at least one endpoint in F and F and $\mathcal{S}$ are disjoint. .

For any vertex $u \in V(G)$ such that $u \in S_i$ for some $i > min$, it suffices now to show that u has i internally disjoint paths from u to distinct vertices in $\mathcal{S}$ to imply that $u \in S_j^*$ for some $j \geq i$. To see this, first observe that the vertex u must be contained in A. Assume as a case that $u \in A - X$. In the graph $G - E(P)$, there exist i internally disjoint paths $N_1, \dots, N_i$ each with a distinct end in $\mathcal{S}$ and the other endpoint equal to u. Then any path N_l with at most one vertex in X does not contain any edge of $(G - E(P))[B]$ and consequently does not use any edges of P^*. Any path N_l that does use at least two vertices of X has a first and last vertex in X. There exists a linkage from X to X' avoiding the edges of P^*, and consequently a path in $\mathcal{R}$ connecting the ends in X' avoiding edges of P^*. It follows that $u \in S_j^*$ for some $j \geq i$.

We now assume $u \in X$. One path from u to $\mathcal{S}$ can be found as above by following the linkage from X to X' and using a path in the solution to the linkage problem $\mathcal{L}_1$. However, as many as i of the paths ensuring that $u \in S_i$ may have used edges contained in $B - A$. Thus the solution to the linkage problem $\mathcal{L}_2$ will ensure that u has i internally disjoint paths to distinct vertices in

$\mathcal{S}$ in $G - E(P^*)$. Let $Q_1^u, \ldots, Q_i^u$ be the internally disjoint paths linking u to distinct vertices of $\mathcal{S}$ contained in $\mathcal{Q}$. As in the previous paragraph, any path that uses at most one vertex of $V(F)$ will still exist in $G - E(P^*)$. If Q_l^u uses at least two vertices of $V(F)$, then by the fact that $\mathcal{R}$ contains a solution to the linkage problem $\mathcal{L}_2$, there exists a path of $\mathcal{R}$ rerouting Q_l^u to avoid any edge of P^*.

We now will see that the vertex $v \in V(F)$ lies in S_j^* for some $j > min$. The vertex v has $|X|$ internally disjoint paths in F to X' that avoid $E(P^*)$ and an additional path to the vertex b. Then X' is linked to X avoiding $E(P)$, and as a consequence, avoiding $E(P^*)$. Furthermore, by construction, the edge bb' is not contained in $E(P^*)$. Finally, our choice of separation (A, B) ensures that $X \cup \{b'\}$ sends $|X| + 1$ disjoint paths to $V(\mathcal{S})$ avoiding edges of P^* to prove that $v \in S_j^*$ for some $j > min$. This completes the proof of the claim. □

This completes the proof of Theorem 1.3.

3 An Approach to Conjecture 1.1

We make the following conjecture:

Conjecture 3.1 *There exists a function $f = f(k)$ such that the following holds. Let G be an $f(k)$-connected graph and let s, t and v be three distinct vertices of G. Then G contains an $s - t$ path P and a k-connected subgraph H such that $v \in V(H)$ and furthermore, H and P are disjoint.*

We will see that Lovász' conjecture in fact follows from Conjecture 3.1

Theorem 3.2 *If Conjecture 3.1 is true, then Conjecture 1.1 is true.*

Proof. Let $f(k)$ be a function satisfying Conjecture 3.1. We show the existence of a function $g(k)$ satisfying Conjecture 1.1, where $g(k)$ will be any function sufficiently large to make the necessary inequalities of the proof true.

Let s and t be two fixed vertices of a $g(k)$-connected graph G, and let F be a maximal k-connected subgraph that does not separate s and t. To see that such a subgraph F must exist, consider a shortest path P from s to t. Every vertex not contained in P can have at most three neighbors on P, and so the minimum degree of $G - V(P)$ must be strictly greater than $4k$. Theorem 1.5 implies that there exists a k-connected subgraph that does not separate s and t.

A *block* is a maximal 2-connected subgraph. Every connected graph G has a *block decomposition* $(T, \mathcal{B})$ where T is a tree and $\mathcal{B} = \{B_v | v \in V(T)\}$ is a collection of subsets of vertices of G indexed by the vertices of T such that the following hold:

i. for every $v \in V(T)$, $G[B_v]$ is either an edge or a block of G,

ii. for every edge uv of T, $|B_v \cap B_u| = 1$, and

iii. every edge of G is contained in B_v for some $v \in V(T)$.

Observe that for any edge $uv \in E(T)$, the vertex in $B_u \cap B_v$ is a cut vertex of the graph. See [2] for more details.

Consider a block decomposition $(T, \mathcal{B})$ of the component of $G - F$ containing s and t. Assume there exists a leaf v of T such that such that $B_v - u$ does not contain either s or t (where the vertex u separates $B_v - \{u\}$ from the rest of $G - F$). Then deleting any vertex of $B_v - \{u\}$ does not separate

s and t. If any such vertex x in $B_v - \{u\}$ had k neighbors in F, then $F \cup x$ would be a k-connected graph that does not separate s and t, contradicting our choice of F. It follows that $G[B_v - \{u\}]$ has minimum degree at least $g(k) - k$. We assume $g(k)$ satisfies the following inequality:

$$g(k) - k \geq 4k^2.$$

By Theorem 1.6, we conclude $G[B_v - u]$ has a k-connected subgraph H whose boundary has at most $2k^2$ vertices. It follows that there exists a matching of size at least k from $V(H) - \partial_{G[B_v]}(H)$ to $V(F)$ in G. This is a contradiction, since then $H \cup F$ is a larger k-connected subgraph that does not separate s from t.

By the same argument as above, $G - F$ has exactly one component. It follows that the block decomposition $(T, \mathcal{B})$ of $G - F$ has T equal to a path. Let the blocks of the decomposition be $B_0, \ldots, B_l$ with $B_i \cap B_{i+1} = v_i$. Then we may assume that $s \in B_0$ and $t \in B_l$. Moreover, for all $i = 0, \ldots, l-1$, it follows that $v_i \neq v_{i+1}$, and $s \neq v_0$ and $t \neq v_{l-1}$.

Now assume there exists a block B_i which is non-trivial, i.e. not a single edge. Let $s' = s$ if $i = 0$, and $s' = v_{i-1}$ otherwise. Similarly, let $t' = t$ if $i = l$ and $t' = v_i$ otherwise. Observe that any vertex v of $B_i - \{s', t'\}$ does not separate s' from t', and so, as above, v cannot have more than k neighbors in F, lest we contradict our choice of F. It follows that $G[B_i - \{s', t'\}]$ has minimum degree at least $g(k) - k - 1$. We assume that

$$g(k) - k - 1 > 4f(k+1)^2.$$

Then $G[B_i] - \{s', t'\}$ contains an $f(k+1)$-connected subgraph F' with boundary at most $2f(k+1)^2$. Moreover, by the connectivity of G, there exist $f(k+1)$ vertices $u_1, \ldots, u_{f(k+1)} \in V(F') - \partial_{G[B_i - \{s', t'\}]}(F')$ such that each has a distinct neighbor in F (in the graph G).

Attempt to find a path from s' to t' in $G[B_i - V(F')]$. If such a path exists, then F' does not separate s' from t' in $G[B_i]$, and the subgraph induced by $V(F \cup F')$ contradicts our choice of F to be as large as possible. It follows that F' does separate s from t in $G - F$. Let $\overline{P}$ be a path in $G[B_i]$ with ends s' and t'. Let $\overline{s}$ be the vertex of $V(\overline{P}) \cap V(F')$ closest to s' on $\overline{P}$. Similarly, let $\overline{t}$ be the vertex of $V(\overline{P}) \cap V(F')$ closest to t' on $\overline{P}$. We define a new graph $\overline{F}$ with vertex set $V(\overline{F})$ equal to $V(F') \cup \overline{v}$ where $\overline{v}$ is a new vertex representing the subgraph F. The edge set of $\overline{F}$ is given by $E(\overline{F}) = E(F') \cup \{\overline{v}u_i | i = 1, \ldots, f(k+1)\}$. Then $\overline{F}$ is an $f(k+1)$-connected graph, so by our assumption that f is a function satisfying Conjecture 3.1, there exists a $(k+1)$-connected subgraph H of $\overline{F}$ containing the vertex $\overline{v}$, and moreover, $F' - H$ contains a path from $\overline{s}$ to $\overline{t}$. By construction, $H - \overline{v}$ is a k-connected subgraph of $G[B_i]$ that does not separate s from t, and moreover, there exists a matching of size k from $H - \overline{v}$ into the vertices of F. It follows that $G[V(F) \cup V(H) - \{\overline{v}\}]$ is a subgraph violating our choice of F to be a maximum k-connected subgraph not separating s from t. This contradicts our assumption that the block decomposition of $G - F$ contained a non-trivial block. It follows that $G - F$ is an induced $s - t$ path, completing the proof. □

Conjecture 3.1 is closely related to the following strengthening of Conjecture 1.1 due to Thomassen.

Conjecture 3.3 (Thomassen, [15]) *For every $l, t \in \mathbb{N}$ there exists $k = k(l, t) \in \mathbb{N}$ such that for all k-connected graphs G and $X \subseteq V(G)$ with $|X| \leq t$, the vertex set of G can be partitioned into non-empty sets S and T such that $X \subseteq S$, each vertex in S has at least l neighbors in T and both $G[S]$ and $G[T]$ are l-connected subgraphs.*

As the conjecture originally appeared, t was assumed to be equal to l. We have introduced the additional parameter to discuss partial progress on the conjecture.

Observation 3.4 *If $\forall l \geq 0, 0 \leq t \leq 2$ there exists a positive integer $k = k(l,t)$ satisfying Conjecture 3.3, then Conjecture 1.1 is true.*

Proof. Let l be any positive integer, $k = k(l,2)$ be as in Conjecture 3.3, and let G be a k-connected graph. Then there exists a partition (A,B) of the vertices of G such that $s,t \in A$, $G[A]$ and $G[B]$ are l-connected graphs, and, furthermore, every vertex of A has at least l neighbors in B. Then if P is a path in $G[A]$ connecting s and t, $G - V(P)$ is an l-connected graph. Thus $f(l) = k(l,2)$ is a function satisfying Conjecture 1.1. □

Kühn and Osthus [3] have proven Conjecture 3.3 is true when the integer t is restricted to 0. A consequence of Theorem 3.2 is the following corollary.

Corollary 3.5 *If $\forall l \geq 0, 0 \leq t \leq 1$ there exists a positive integer $k = k(l,t)$ satisfying Conjecture 3.3, then Conjecture 1.1 is true.*

Proof. Let l be a positive integer and let $k = k(l+2,1)$ be the value given by Conjecture 3.3. Then let G be a k-connected graph, and let v, s, and t be given as in Conjecture 3.1. Let (A,B) be a partition of $V(G)$ such that $G[A]$ and $G[B]$ are $(l+2)$-connected, and furthermore, that $v \in A$. Then $G[A - \{s,t\}]$ is an l-connected subgraph containing v that does not separate s and t, as desired. □

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