Round about Theta. Part I Prehistory

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There is a huge amount of work on different kinds of theta functions, the theta correspondence, cohomology classes coming from special Schwartz classes via theta distribution, and much more. The aim of this text is to try to find joint construction principles while often leaving aside relevant but cumbersome details.

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1 Riemann and Jacobi Theta Series

1.1 Weil Representation

1.1.1 Our fundamental object is the symplectic group

$$\hat{G} = \mathrm{Sp}(n, \mathbf{R}) := \{g \in \mathrm{M}(2n, \mathbf{R}); {}^t g J g = J := \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}\},$$

i.e. the group with elements

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, A, B, C, D \in \mathrm{M}(n, \mathbf{R}), {}^tAD - {}^tCB = E_n, {}^tAC = {}^tCA, {}^tBD = {}^tDB,$$

and its projective (Segal-Shale-)Weil or oscillator representation ω given as its Schrödinger model on the space $\mathcal{H} = L^2(\mathbf{R}^n)$ by the prescription

$$\begin{aligned} \omega(d(A))f(x) &= |\det A|^{1/2} f({}^tAx) \quad \text{for all } d(A) := \begin{pmatrix} A & \\ & {}^tA^{-1} \end{pmatrix}, A \in \mathrm{GL}(n, \mathbf{R}), \\ \omega(n(B))f(x) &= e^{\pi i {}^txBx} f(x) \quad \text{for all } n(B) := \begin{pmatrix} 1 & B \\ & 1 \end{pmatrix}, B \in \mathrm{Sym}(n, \mathbf{R}), \\ \omega(J)f(x) &= \gamma \hat{f}(x), \hat{f}(x) := \textstyle\int_{\mathbf{R}^n} f(y)\, e^{2\pi i {}^tyx} dy, \end{aligned}$$

where γ will be specified later. This projective representation corresponds to a representation $\tilde{\omega}$ of the twofold cover of $\hat{G}$, the metaplectic group $\tilde{\hat{G}} = \mathrm{Mp}(n, \mathbf{R})$ with elements (g, t), $g \in \hat{G}, t^2 = s(g)^{-1}$, where $s(g)$ as specified in [LV] p.70 will not be needed at the moment.

1.1.2 The Lie algebra of the symplectic group is

$$\hat{\mathfrak{g}} = \mathfrak{sp}(n, \mathbf{R}) := \{X = \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix}; A, B, C \in \mathrm{M}(n, \mathbf{R}), B = {}^tB, C = {}^tC\}.$$

$\mathfrak{sp}$ has dimension $2n^2 + n$ and Cartan decomposition

$$\mathfrak{sp} = \mathfrak{k} + \mathfrak{p}$$

with

$$\mathfrak{k} := \{\begin{pmatrix} A & B \\ -B & A \end{pmatrix}; {}^tB = B, {}^tA = -A\}, \ \mathfrak{p} := \{\begin{pmatrix} A & B \\ B & -A \end{pmatrix}; {}^tB = B, {}^tA = A\}.$$

The complexification $\mathfrak{g}_c$ of $\mathfrak{g} = \mathfrak{sp}$ has the $\mathrm{Ad}J-$eigenspace decomposition

$$\mathfrak{g}_c = \mathfrak{sp}^{(1,1)} + \mathfrak{sp}^{(2,0)} + \mathfrak{sp}^{(0,2)}$$

with

$$\begin{aligned} \mathfrak{sp}^{(1,1)} &:= \{\begin{pmatrix} A & B \\ -B & A \end{pmatrix}; A, B \in \mathrm{M}(n, \mathbf{C}), {}^tB = B, {}^tA = -A\}, \\ \mathfrak{sp}^{(2,0)} &:= \{\begin{pmatrix} A & iA \\ iA & -A \end{pmatrix}; A \in \mathrm{M}(n, \mathbf{C}), {}^tA = A\}, \\ \mathfrak{sp}^{(0,2)} &:= \{\begin{pmatrix} A & -iA \\ -iA & -A \end{pmatrix}; A \in \mathrm{M}(n, \mathbf{C}), {}^tA = A\}. \end{aligned}$$

We take over the notation from Adams ([Ad] p.466)

$$A_{ij} := \begin{pmatrix} E_{ij} & \\ & -E_{ji} \end{pmatrix}, \ U_{ij}^+ := \begin{pmatrix} 0 & B_{ij} \\ 0 & 0 \end{pmatrix}, \ U_{ij}^- := \begin{pmatrix} 0 & 0 \\ B_{ij} & 0 \end{pmatrix}$$

where for $1 \leq i, j \leq n$ E_{ij} is the elementary matrix with zero entries except there is 1 in the ith row and jth column and $B_{ij} := E_{ij} + E_{ji}$ for $i \neq j$, $B_{ii} := E_{ii}$.

1.1.3 For $X \in \mathfrak{g}_c$ we denote by $\hat{X}$ its operator in the derived representation of the Weil representation ω, i.e. we put

$$\hat{X}f(x) := \frac{d}{dt}|_{t=0}(\omega(\exp tX)f)(x) \quad \text{for all } f \in \mathcal{S}(\mathbf{R}^n).$$

One easily comes to

$$\hat{A}_{jk} = x_j\partial_k + \delta_{jk}/2,\ \hat{U}^+_{jk} = 2\pi i x_j x_k, j \neq k,\ \hat{U}^+_{jj} = \pi i x_j^2,$$

and, using appropriate commutation formulae like $[U^+_{jj}, U^-_{jj}] = A_{jj}$,

$$\hat{U}^-_{jj} = -1/(4\pi i)\partial_j^2,\ \hat{U}^-_{jk} = -1/(2\pi i)\partial_j\partial_k \quad \text{for all } j \neq k.$$

Hence, for the complex algebras $\mathfrak{sp}^{(1,1)} = \mathfrak{k}_c$, $\mathfrak{sp}^{(2,0)} =: \mathfrak{p}^+$, $\mathfrak{sp}^{(0,2)} =: \mathfrak{p}^-$, respectively generated by

$$A_{jk} - A_{kj},\ U^+_{jk} - U^-_{jk}$$

and

$$(1/2)(\mp i(A_{jk} + A_{kj})/(1 + \delta_{jk}) + U^+_{jk} + U^-_{jk}) =: \check{U}^{\pm}_{jk},$$

one has

$$\begin{aligned} \hat{\check{U}}^{\pm}_{jj} &= (1/2)(\mp i(x_j\partial_j + (1/2)) + \pi i x_j^2 - (1/4\pi i)\partial_j^2), \\ \hat{\check{U}}^{\pm}_{jk} &= (1/2)(\mp i(x_j\partial_k + x_k\partial_j) + 2\pi i x_j x_k - (1/2\pi i)\partial_j\partial_k). \end{aligned}$$

In $\mathfrak{p}$ one has the Cartan algebra $\mathfrak{h} :=< A_{jj} >_{j=1,\ldots,n}$ and (among others) the relations

$$[A_{jj}, U^{\pm}_{jj}] = \pm 2U^{\pm}_{jj},\ \ [U^+_{jj}, U^-_{jj}] = A_{jj}.$$

We use the Cayley transformation, i.e. conjugation by

$$c = (1/\sqrt{2})\begin{pmatrix} 1_n & i_n \\ i_n & 1_n \end{pmatrix},$$

to introduce

$$H_j := cA_{jj}c^{-1} = -i(U^+_{jj} - U^-_{jj})$$

which obeys the relation

$$[H_j, \check{U}^{\pm}_{jj}] = \pm 2\check{U}^{\pm}_{jj}.$$

We get

$$\hat{H}_j = \pi x_j^2 - (1/4\pi)\partial_j^2.$$

1.1.4 Now we can see that for the Gaussian

$$\varphi_0(x) := e^{-\pi\Sigma x_j^2}$$

one has

$$\begin{aligned} \hat{H}_j\varphi_0 &= (1/2)\varphi_0, \\ \hat{\check{U}}^+_{jj}\varphi_0 &= (i/2)(4\pi x_j^2 - 1)\varphi_0, \\ \hat{\check{U}}^+_{jk}\varphi_0 &= 4\pi i x_j x_k\varphi_0, \quad \text{for all } j \neq k, \\ \hat{\check{U}}^-_{jk}\varphi_0 &= 0, \quad \text{for all } j \neq k, \\ \hat{\check{U}}^-_{jj}\varphi_0 &= 0. \end{aligned}$$

This shows that φ_0 is annihilated by all elements of $\mathfrak{sp}^{(0,2)}$ and reproduced with eigenvalue 1/2 by all elements of $\mathfrak{sp}^{(1,1)}$, i.e. φ_0 is a *vacuum vector* for ω : a vector of lowest weight 1/2 for the Weil representation ω in its Schrödinger model.

1.2 Riemann Thetas

1.2.1 There is a standard way to construct a modular form which in this case comes out like this: One applies the Weil representation ω to φ_0 and averages over all $\ell \in \mathbf{Z}^n$ to get a function

$$\Phi_\theta(g) := \sum_{\ell \in \mathbf{Z}^n} (\omega(g)\varphi_0)(\ell), \; g \in \hat{G},$$

which can be proven to be invariant under the theta subgroup Γ_θ of $\mathrm{Sp}(n, \mathbf{R})$. And up to an automorphic factor this function can be identified with θ, the (zero value) of the Jacobi theta function: With some more (but not all) details, this means the following.

1.2.2 We have the transitive action of $\mathrm{Sp}(n, \mathbf{R})$ on the Siegel half space

$$\mathfrak{H}_n := \{\hat{\tau} \in \mathrm{M}(n, \mathbf{C}); \; {}^t\hat{\tau} = \hat{\tau}, \quad \mathrm{Im}\; \hat{\tau} > 0\}$$

given by

$$g(\hat{\tau}) := (A\hat{\tau} + B)(C\hat{\tau} + D)^{-1} \quad \text{for all } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We take an element $g = g_{\hat{\tau}} \in \hat{G}$ such that

$$g_{\hat{\tau}}(i_n) = \hat{\tau} =: \hat{u} + i\hat{v},$$

namely, using the notation introduced above

$$g_{\hat{\tau}} = n(\hat{u})d(A), \quad \text{with } A^tA = \hat{v}.$$

Then we get

$$(\omega(g_{\hat{\tau}})\varphi)(x) = |\det \hat{v}\,|^{1/4} e^{\pi i^t x \hat{\tau} x}$$

and hence

$$\Phi_\theta(g_{\hat{\tau}}) := |\det \hat{v}\,|^{1/4} \sum_{\ell \in \mathbf{Z}^n} e^{\pi i^t \ell \hat{\tau} \ell}.$$

Here we find the standard theta series

$$\theta(\hat{\tau}) := \sum_{\ell \in \mathbf{Z}^n} e^{\pi i^t \ell \hat{\tau} \ell}$$

1.2.3 The fact that φ_0 is a lowest weight vector annihilated by $\mathfrak{sp}^{(0,2)}$ tanslates into the fact that θ is a holomorphic function in $\hat{\tau} \in \mathfrak{H}_n$ and the fact (which is not so easy to prove) that Φ_θ as a function of $g \in \hat{G}$ is invariant under the theta group Γ_θ translates into the automorphic functional equation

$$\theta(\gamma\hat{\tau}) = \varepsilon(\gamma) \det(C\hat{\tau} + D)^{1/2} \theta(\hat{\tau})$$

for $\gamma \in \Gamma_\theta$, the group of elements

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma := \mathrm{Sp}(n, \mathbf{Z}),$$

where ${}^tCA$ and ${}^tBD$ both have even diagonal entries. $\varepsilon(\gamma)$ is a character of Γ_θ as defined in [LV] p.166. In particular for the group

$$\Gamma_0^0(2) := \{\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma; \;\; B \equiv C \equiv 0 \mod 2\}$$

one has

$$\epsilon(\gamma)^2 = (\frac{-1}{\det D}).$$

This statement is Theorem 2.2.37 in [LV]. We shall call this θ as a function on the Siegel half space the Riemann theta function, though this name is used by Mumford also for the more general function which we introduce now and then call Jacobi theta function.

1.3 Jacobi Thetas

1.3.1 For $\tau \in \mathfrak{H}_n$ and $z \in \mathbf{C}^n$ we get the Jacobi theta function

$$\theta(\tau, z) := \sum_{\ell \in \mathbf{Z}^n} e^{\pi i ({}^t\ell \hat{\tau} \ell + 2\,{}^t z \ell)}.$$

1.3.2 Here we have to extend the symplectic group $\hat{G}$ to its semidirect product with an appropriate Heisenberg group $\mathrm{Heis}(\mathbf{R}^n)$ to come to the Jacobi group $\hat{G}^J$. As a set one has $\mathrm{Heis}(\mathbf{R}^n) = \mathbf{R}^{2n+1}$ and all multiplication laws are fixed by the embedding into the symplectic group $\mathrm{Sp}(n+1, \mathbf{R})$ given by

$$\mathrm{Heis}(\mathbf{R}^n) \ni (\lambda, \mu, \kappa) \longmapsto \begin{pmatrix} 1_n & & & \mu \\ {}^t\lambda & 1 & {}^t\mu & \kappa \\ & & 1_n & -\lambda \\ & & & 1 \end{pmatrix},$$

$$\mathrm{Sp}(n, \mathbf{R}) \ni M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto \begin{pmatrix} A & & B & \\ & 1 & & \\ C & & D & \\ & & & 1 \end{pmatrix}.$$

We write

$$g = (p, q, \kappa)M \ \text{ or } \ g = M(\lambda, \mu, \kappa) \ \in G^J(\mathbf{R}^n).$$

$\mathrm{Heis}(\mathbf{R}^n)$ acts on $\mathbf{R}^{2n}$ via $(x, y) \longmapsto (x + \lambda, y + \mu)$ and G^J acts on $\mathfrak{H}_n \times \mathbf{C}^n$ via

$$(\tau, z) \longmapsto g(\tau, z) := (M(\tau), (z + \tau\lambda + \mu)(C\tau + D)^{-1})$$

where $g = M(\lambda, \mu, \kappa) \in G^J(\mathbf{R}^n), \tau \in \mathfrak{H}_n, z \in \mathbf{C}^n$. For $g = (p, q, \kappa)M$ one has

$$g(i_n, 0) = (\tau = M(i_n), z = \tau p + q).$$

1.3.3 The construction of the Weil representation usually goes via the standard representation of the Heisenberg group which is the Schrödinger representation in the space $\mathcal{H} = L^2(\mathbf{R}^n)$ for real non-zero m and $(\lambda, \mu, \kappa) \in \mathrm{Heis}(\mathbf{R}^n)$ given by

$$(\pi_S^m(\lambda, \mu, \kappa)f)(x) := e^m(\kappa + (2^t x + {}^t\lambda)\mu) f(x + \lambda) \quad \text{ for all } \ f \in \mathcal{H}.$$

Then one has the Schrödinger-Weil representation π_{SW} of G^J given by

$$\pi_{SW}^m((p, q, \kappa)M) := \pi_S^m((p, q, \kappa)\omega(M)$$

1.3.4 It is not difficult to verify that the vacuum vector of the Weil representation $\varphi_0^m(x) = e^{\pi i m {}^t x x}$ is also a vacuum vector of the Schrödinger-Weil representation and one can use it again as done above: For $M_{\hat{\tau}} = n(\hat{u})d(A)$ we get

$$(\pi_{SW}((p, q, \kappa)M_{\hat{\tau}})\varphi_0^m)(x) = |\det \hat{v}\,|^{1/4} e^m(\kappa + {}^t p \hat{\tau} p + {}^t p q) e^{\pi i m ({}^t x \hat{\tau} x + 2({}^t p \hat{\tau} + {}^t q) x)}$$

and

$$\Phi_\theta((p,q,\kappa)M_{\hat\tau}) = \sum_{\ell\in\mathbf{Z}^n} |\det\hat v|^{1/4} e^m(\kappa + {}^tp\hat\tau p + {}^tpq) e^{m\pi i({}^t\ell\hat\tau\ell + 2({}^tp\hat\tau + {}^tq)\ell)}.$$

With $z = p\hat\tau + q$, for $m = 1/2$, up to a factor we find the Jacobi theta function

$$\theta(\tau, z) := \sum_{\ell\in\mathbf{Z}^n} e^{\pi i({}^t\ell\hat\tau\ell + 2{}^tz\ell)}.$$

The properties and the functional equation of this function and its generalizations are discussed with the appropriate details in the books by Igusa ([Ig] p.48f) and by Mumford ([MuIII] p.142). Here we only will record the following observation.

1.3.5 Remark: We introduced the Jacobi group $G^J(\mathbf{R}^n)$ as a subgroup of the symplectic group $G^* := \mathrm{Sp}(n+1;\mathbf{R})$. We have

$$g^*(i_{n+1}) = (A^* i_{n+1} + B^*)(C^* i_{n+1} + D^*)^{-1} \text{ for } g^* = \begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix}.$$

If we specialize this for $g^* = (p,q,\kappa)M_{\hat\tau}$, we get

$$((p,q,\kappa)M_{\hat\tau})(i_{n+1}) = \begin{pmatrix} \hat\tau & p\hat\tau + q \\ {}^tp\hat\tau + {}^tq & {}^tp\hat\tau p + {}^tqp + \kappa + i \end{pmatrix}.$$

And if we specialize the standard theta series for G^*

$$\theta_{n+1}(\tau^*) = \sum_{\ell^*\in\mathbf{Z}^{n+1}} e^{\pi i {}^t\ell^*\tau^*\ell^*}$$

for

$$\tau_0^* = \begin{pmatrix} \hat\tau & z \\ {}^tz & a \end{pmatrix},\ a = {}^tp\hat\tau p + {}^tqp + \kappa + i$$

with ${}^t\ell^* = ({}^t\ell, l)$, $\ell\in\mathbf{Z}^n$, $l\in\mathbf{Z}$, we get

$$\theta_{n+1}(\tau_0^*) = \sum_{l\in\mathbf{Z}} e^{\pi i a l^2} \sum_{\ell\in\mathbf{Z}^n} e^{\pi i({}^t\ell\hat\tau\ell + 2l{}^tz\ell)}.$$

Up to the factor $l\in\mathbf{Z}$ in the exponent, we find again the Jacobi theta series. This fits into the framework of the Fourier-Jacobi expansion of a Siegel modular form which is introduced (for the lowest dimensional case) in [EZ] p.72f. From here we can easily take over that each coefficient

$$\phi_{l^2}(\tau, z) := \sum_{\ell\in\mathbf{Z}^n} e^{\pi i({}^t\ell\hat\tau\ell + 2l{}^tz\ell)}$$

has the transformation property of a Jacobi form and using the operator U_l defined in [EZ] p.41 (multiplication of the $z-$variable by l) we can even write

$$\phi_{l^2}(\tau, z) = U_l\,\theta(\tau, z).$$

1.3.6 There are many ways to introduce more general functions of this type. As one can well imagine, all this generalizes rather easily if one takes a rational symmetric

$h \times h-$matrix S belonging to a positive definite quadratic form. We follow [MuIII] p.96f:

1.3.7 Definition: Let $S \in \mathrm{Sym}_h(\mathbf{Q})$ be positive definite, $T \in \mathfrak{H}_n$ and $Z \in M_{n,h}(\mathbf{C})$. Then we put

$$\theta^S(T,Z) := \sum_{N \in M_{n,h}(\mathbf{Z})} e^{\pi i \mathrm{Tr}({}^tNTNS + 2{}^tNZ)}.$$

As it is rather easy to see that for $M, N \in \mathrm{M}_{g,h}(\mathbf{Z})$ one has

$$\theta^S(T, Z + TMS + N)e^{\pi i \mathrm{Tr}({}^tMTMS + 2{}^tMZ)} = \theta^S(T,Z)$$

one is lead to suggest that θ^S is just a Jacobi theta series for a more general situation, namely for the complex torus

$$M_{n,h}(\mathbf{C})/(T\mathrm{M}_{n,h}(\mathbf{Z})S + \mathrm{M}_{n,h}(\mathbf{Z})).$$

To see this we use the identifications given by

$$\mathrm{M}_{n,h}(\mathbf{C}) \longrightarrow \mathbf{C}^{nh},\ \ Z = (Z_1, \dots, Z_h) \longmapsto z = \begin{pmatrix} Z_1 \\ \cdot \\ \cdot \\ Z_h \end{pmatrix},$$

and

$$\tau := T \otimes S = \begin{pmatrix} TS_{11} & \cdot & \cdot & TS_{1h} \\ \cdot & & & \cdot \\ & & & \\ \cdot & & & \cdot \\ TS_{h1} & \cdot & \cdot & TS_{hh} \end{pmatrix}.\ \in \mathrm{M}_{nh,nh}(\mathbf{C}).$$

One has to check that $W = TZS$ translates into $w = \tau z$ and that one has $\mathrm{Tr}{}^tWZ = {}^twz$. Then we get (Lemma 6.2 in [MuIII])

$$M_{n,h}(\mathbf{C})/(T\mathrm{M}_{n,h}(\mathbf{Z})S + \mathrm{M}_{n,h}(\mathbf{Z})) \cong \mathbf{C}^{nh}/(\tau\mathbf{Z}^{nh} + \mathbf{Z}^{nh}).$$

Hence we can see that we have

$$\theta^S(T,Z) = \sum_{n \in \mathbf{Z}^{nh}} e^{\pi i({}^tn\tau n + 2{}^tnz)} = \theta_{nh}(\tau, z).$$

Moreover one can see without too much trouble ([MuIII] Corollary 6.6):

If $S = \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ & & \cdot \\ 0 & & d_h \end{pmatrix}$ and $Z = (z_1, \dots, z_h)$, then we have

$$\theta^S(T,Z) = \Pi_{i=1}^h \theta_n(d_iT, z_i).$$

1.3.8 Thetas belonging to (positive definite) quadratic forms are still more widely generalized by considering spherical harmonic polynomials as coefficients of the exponentials in the series. This is treated for instance in [MuIII] p.145ff. Here we shall come to this later.

2 Hecke and Siegel Theta Series

2.1 Hecke Thetas

2.1.1 It is immediate that one has a convergence problem if one tries to consider thetas for quadratic forms which are not positive definite. For instance, for $\tau \in \mathfrak{H} := \mathfrak{H}_1$ the sum

$$\Sigma_{x_1,x_2\in\mathbf{Z}} e^{2\pi i\tau(x_1^2-12x_2^2)}$$

has no sense. It was Hecke on his way to associate modular forms to real quadratic fields (and a bit later Schoeneberg) who achieved substantial progress in this topic:
Let $K = \mathbf{Q}(\sqrt{D})$ be a real quadratic number field with diiscriminant D and $\mathfrak{o} := \mathfrak{o}_K$ as its ring of integers. For $Q \in \mathbf{N}$ and $\alpha \in \mathfrak{o}$ Hecke defines in [H1] and [H2] the functions of $\tau \in \mathfrak{H}$

$$\vartheta(\tau;\alpha,Q\sqrt{D}) := \sum_{(\mu)} \mathrm{sgn}\mu \; e^{2\pi i\tau\frac{|\mu\mu'|}{QD}}$$

and

$$\vartheta_+(\tau;\alpha,Q\sqrt{D}) := \sum_{(\mu),\mu\mu'>0} \mathrm{sgn}\mu \; e^{2\pi i\tau\frac{|\mu\mu'|}{QD}}$$

Here the prime ' indicates the conjugate in K and the summation $\sum_{(\mu)}$ is meant over a family of elements $\mu \in \mathfrak{o}$, which are congruent $\mod Q\sqrt{D}$ to α and not associated, i.e. do not differ by a unit $\mod Q\sqrt{D}$ as a factor. Hecke's main theorem in this context is a transformation formula $\tau \longmapsto -1/\tau$ for ϑ_+, which is the essential to show that ϑ_+ is a modular form. Without going into further details we state that the idea for his proof is to use the already known transformation property of a standard theta function in two variables.

2.1.2 Hecke discusses the example $\vartheta_+(\tau;1,\sqrt{12})$. He shows that one has

$$\begin{aligned}\vartheta_+(\tau+1;1,\sqrt{12}) &= e^{2\pi i/12}\vartheta_+(\tau;1,\sqrt{12}),\\ \vartheta_+(-1/\tau;1,\sqrt{12}) &= -i\tau\vartheta_+(\tau;1,\sqrt{12}),\end{aligned}$$

and one has the nice relation to the Delta function

$$\vartheta_+(\tau;1,\sqrt{12}) = (\Delta(\tau))^{1/12} = e^{2\pi i\tau/12}\Pi_{i=1}^{\infty}(1-q^n)^2, \; q := e^{2\pi i\tau}.$$

2.2 Siegel Thetas

2.2.1 As we saw, Hecke solved the convergence problem for the theta series for indefinite quadratic forms by summing only over those elements such that the form has positive values. Siegel had the idea to use the *majorant* of a quadratic form to associate to the form a convergent series for which (in [S1] and [S2]) he also could prove a modular property.

2.2.2 One starts with the quadratic form belonging to a non-degenerate symmetric matrix $S \in \mathrm{M}_n(\mathbf{R})$ with signature $\mathrm{sig}\; S = (p,q)$

$$S[x] := {}^t xSx,$$

where as above x is a column. We know that one can find a matrix $C \in \mathrm{GL}(n, \mathbf{R})$ such that

$$S[C] = {}^tCSC = \begin{pmatrix} 1_p & \\ & -1_q \end{pmatrix} =: S_0,$$

i.e. with $x = Cy$ one has

$$S[Cy] = S_0[y] = y_1^2 + \cdots + y_p^2 - (y_{p+1}^2 + \cdots + y_{p+q}^2) =: {y'}^2 - {y''}^2.$$

Siegel now uses the notion of the majorant of $S[x]$ which goes back to Hermite and is a positive definite quadratic form, say $P[x]$, such that $P[x] \geq S[x]$ for all $x \in \mathbf{R}^n$. With C as above, we take $P := (C{}^tC)^{-1}$ and get

$$P[x] = {}^tx(C{}^tC)^{-1}x = {}^tyy.$$

In [S2] 1. Siegel shows that P belongs to a majorant of $S[x]$ if and only if P fulfills the two conditions

$$PS^{-1}P = S, \ {}^tP = P > 0.$$

Moreover, Siegel parametrizes the set $\mathcal{P} := \mathcal{P}(S)$ of these matrices P and shows that the orthogonal group

$$\mathrm{O} := \mathrm{O}(S) = \{A \in \mathrm{M}_n(\mathbf{R});\ {}^tASA = S\}$$

via $(A, P) \longmapsto P[A]$ acts transitively on $\mathcal{P}$. We will come back to this later but now can give Siegel's definition of his Theta function:

2.2.3 Definition: One takes $\tau = u + iv \in \mathfrak{H}$, $R := uS + ivP$ and puts

$$\theta(\tau) := \theta(\tau, P) := \sum_{x \in \mathbf{Z}^n} e^{2\pi i R[x]}.$$

This definition makes sense because Im $R = vP$ is positive definite. For $a \in \mathbf{Q}; as \in \mathbf{Z}$ where $s := \det S$ Siegel also looks at the variant

$$\theta_a(\tau) := \theta_a(\tau, P) := \sum_{x \in \mathbf{Z}^n} e^{2\pi i R[x+a]}.$$

2.2.4 This function θ_a is not a holomorphic function in τ but has a modular behaviour with respect to certain modular substitutions $\tau \longmapsto \hat{\tau} = (a\tau + b)(c\tau + d)^{-1}$ which is given in Hilfssatz 1 in [S2]. The proof again uses the Poisson summation formula. We will not repeat this here but just indicate that an automorphic factor of type

$$(c\tau + d)^{-p/2}(c\bar{\tau} + d)^{-q/2}$$

comes in. The dependence on P resp. on appropriate parameters for P will be discussed later.

2.2.5 There are several generalizations of Siegel's definition, see, for instance, Vignéras [Vi] and in particular Borcherds [Bo]. Moreover, there is an extension in the direction of Jacobi thetas by O. Richter [Ri]:

2.2.6 Definition: Let $S \in \mathrm{Sym}_m(\mathbf{Z})$ be an invertible matrix with even diagonal entries with sig $S = (p, q)$ and such that qS^{-1} for $q \in \mathbf{N}$ is integral and even. Let P be a majorant of $S, \tau = u + iv \in \mathfrak{H}_n, \zeta \in \mathrm{M}_{m,j}(\mathbf{Z})$, and $Z \in \mathrm{M}_{j,n}(\mathbf{C})$. Then one puts

$$\theta_{S,P,\zeta}(\tau, Z) := \sum_{N \in \mathrm{M}_{m,n}} e^{\pi i \mathrm{Tr}(S[N]u + iP[N]v + 2{}^tNS\zeta Z)}.$$

For ζ such that $S\zeta = P\zeta$, Richter proves a transformation formula concerning

$$\Gamma_0^{(n)} := \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n, \mathbf{Z});\ C \equiv 0 \mod q\}.$$

The definition of these theta series is quite natural in the context the autors had. But they also have a representation theoretic background as we will try to elucidate in the sequel.

3 A Dual Pair and Siegel Thetas

3.1 Dual Pairs

It was Howe who (in [Ho1]) coined the following notion and later contributed essential parts of its discussion.

3.1.1 Definition: A dual reductive pair is a pair of subgroups (G, G') in a symplectic group $\hat{G} = \mathrm{Sp}(n, \mathbf{R})$ such that
i) G is the centralizer of G' in $\hat{G}$ and G' is the centralizer of G in $\hat{G}$.
ii) The actions of G and G' on $\hat{V} := \mathbf{R}^{2n}$ are completely reducible (i.e. every invariant subspace has an invariant complement).

This is only a special case: here one can also replace $\mathbf{R}$ by more general fields. One defines irreducible pairs as those where one can not decompose $\hat{V}$ as the direct sum of two symplectic subspaces each of which is invariant under both G and G'. There is the classification of irreducible pairs done in [MVW]. We won't go into this but just point out that these pairs provide the background for a lot of important relations between different kinds of automorphic forms. Roughly, this goes like this: The Weil representation ω of $\hat{G}$ restricts to representations of the subgroups G and G' and to $G \times G'$. If one has a decomposition of ω where to an irreducible representation of G corresponds exactly one irreducible representation of G', one can hope for a correspondence between automorphic forms belonging to these representations . With more details one has the Howe conjecture making precise statements in this direction. For the moment we will use a small part of the picture to make reappear the Siegel thetas.

3.1.2 We take the orthogonal group $G = \mathrm{O}(p, q) \cong \mathrm{O}(S)$ belonging to the non-degenerate symmetric matrix $S \in \mathrm{M}_n(\mathbf{R})$ with signature sig $S = (p, q)$. Then one can verify easily that G together with $G' = \mathrm{SL}(2, \mathbf{R})$ is a dual pair in $\hat{G} = \mathrm{Sp}(n, \mathbf{R})$.

3.1.3 Remark: We use the embeddings

$$G = \mathrm{O}(p, q) \ni A \longmapsto \hat{A} := \begin{pmatrix} {}^tA^{-1} & \\ & A \end{pmatrix} \in \hat{G} = \mathrm{Sp}(n, \mathbf{R})$$

and with $S_0 = \begin{pmatrix} 1_p & \\ & -1_q \end{pmatrix}$

$$G' = \mathrm{SL}(2, \mathbf{R}) \ni M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \hat{M} := \begin{pmatrix} a1_n & bS_0 \\ cS_0^{-1} & d1_n \end{pmatrix} \in \hat{G} = \mathrm{Sp}(n, \mathbf{R}).$$

These embeddings come as special cases from the following more general consideration:

3.1.4 For the symplectic space $V' \simeq \mathbf{R}^{2m}$ with the action of $G' = \mathrm{Sp}(m, \mathbf{R})$ we take as basis $e_1, \dots, e_m, e'_1, \dots, e'_m$ such that for all $j = 1, \dots, m$ and $J_m = \begin{pmatrix} & 1_m \\ -1_m & \end{pmatrix}$ one has $J_m e_j = e'_j$ and $J_m e'_j = -e_j$. For the orthogonal space $V \simeq \mathbf{R}^n$ with the action of $G = \mathrm{O}(p, q), p + q = n$, we take as basis $v_1, \dots, v_p, v_{p+1}, \dots, v_{p+q}$ such that for all $\alpha = 1, \dots, p$ and $\nu = p + 1, \dots, p + q$ and $S_0 = \begin{pmatrix} 1_p & \\ & -1_q \end{pmatrix}$ one has $S_0 v_\alpha = v_\alpha$ and $S_0 v_\nu = -v_\nu$. Hence $\hat{V} := V \otimes V' \simeq \mathbf{R}^{2mn}$ is a symplectic space with basis $e_j \otimes v_\alpha, e_j \otimes v_\nu$ and $e'_j \otimes v_\alpha, -e'_j \otimes v_\nu$, i.e. $\hat{e}_1, \dots, \hat{e}_{mn}, \hat{e}'_1, \dots, \hat{e}'_{mn}$ where

$$\hat{e}_1 := e_1 \otimes v_1, \dots, \hat{e}_m := e_m \otimes v_1, \dots, \dots, \hat{e}_{mn} := e_m \otimes v_{p+q}$$

and

$$\hat{e}'_1 := e'_1 \otimes v_1, \dots, \hat{e}'_{mp} := e'_m \otimes v_q, \hat{e}'_{mp+1} := -e'_1 \otimes v_{p+1}, \dots, \hat{e}'_{mn} := -e'_m \otimes v_{p+q}.$$

In particular for $m = 1$ we have $\hat{V}$ with basis

$$\hat{e}_j := e_1 \otimes v_j, \ \hat{e}'_j := e'_1 \otimes S_0 v_j, \quad \text{for all } j = 1, \dots, n.$$

The action of G on V and of G' on V' induce naturally actions on $\hat{V}$, in particular those described in the remark above.

3.2 Siegel Thetas as Special Values of Riemann Thetas

3.2.1 We have the standard compact subgroups

$$K := \mathrm{O}(p) \times \mathrm{O}(q), \ K' := \mathrm{SO}(2),$$

and

$$\hat{K} := \{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix}; \ {}^tAA + {}^tBB = 1_n, \ {}^tAB = {}^tBA \} \simeq \mathrm{U}(n).$$

There are the standard maps to the associated homogeneous spaces

$$\hat{G} \longrightarrow \hat{G}/\hat{K} = \mathfrak{H}_n; \ \hat{g} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto \hat{g}(i_n) =: \hat{\tau} = \hat{u} + i\hat{v},$$

$$G' \longrightarrow G'/K' = \mathfrak{H}; \ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto g(i) = \tau = u + iv,$$

and (without a big loss of generality) restricting to $S = S_0$

$$G \longrightarrow G/K =: \mathfrak{D}; \ A \longmapsto (A^tA)^{-1} =: P.$$

This homogeneous space has different realizations which we will discuss later. Here we refer to our remarks in 2.2.2 where, following Siegel, we introduced $\mathfrak{D} = \mathcal{P}$ as the set of majorants of $S = S_0$. The embedding $G \times G' \longrightarrow \hat{G}$ induces a map

$$\mathfrak{D} \times \mathfrak{H} \longrightarrow \mathfrak{H}_n; \ (P, \tau) \longmapsto uS_0 + ivP =: \hat{\tau}_{P,\tau} =: \hat{\tau}_0,$$

which is a consequence of

$$\hat{g}_\tau(i_n) = uS_0 + vi_n \text{ and } \hat{A}(\hat{\tau}) = {}^tA^{-1}\hat{\tau}A^{-1}.$$

3.2.2 Remark: If we specialize the variable $\hat{\tau}$ in the standard Riemann theta series $\theta(\hat{\tau})$ for $\mathrm{Sp}(n, \mathbf{R})$ to $\hat{\tau} = \hat{\tau}_0$, we recover the Siegel theta series

$$\vartheta(\tau, P) = \sum_{\ell \in \mathbf{Z}^n} e^{\pi i {}^t\ell(uS_0 + ivP)\ell}.$$

In a parallel way, one can take the vacuum vector $\varphi_0(x) = e^{\pi \Sigma x_j^2}$ for the Weil representation ω in the Schrödinger model and apply the restriction of ω to $G \times G'$ to construct a function on $G \times G'$ with certain invariance properties. This way we come to

$$\omega(\hat{A} \cdot \hat{g_\tau})\varphi_0(x) = v^{n/4} e^{\pi i {}^tx(uS_0 + iv{}^tA^{-1}A^{-1})x}.$$

3.3 Siegel Theta and its Representation

In [S2] Siegel uses these theta series to study the diophantine problem of integral solutions $x \in \mathbf{Z}^n$ of the quadratic equation

$$S[x + a] = t.$$

Here we won't go into this interesting topic but analyze a bit the relation of the Siegel theta series to the representation theory of the two groups G and G' going into our construction.

3.3.1 In 1.1.2 we discussed the Lie algebra $\hat{\mathfrak{g}}$ of the symplectic group $\hat{G} = \mathrm{Sp}(n, \mathbf{R})$. As a special case we have $\mathfrak{g}' = \mathrm{Lie}\, G', G' = \mathrm{SL}(2, \mathbf{R})$ with

$$\mathfrak{g}' =< F := \begin{pmatrix} & 1 \\ & \end{pmatrix},\ G := \begin{pmatrix} & \\ 1 & \end{pmatrix},\ H := \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} >$$

and the relations

$$[H, F] = 2F,\ [H, G] = -2G,\ [F, G] = H.$$

The complexification is given by

$$\mathfrak{g}'_c =< Z := -i(F - G) = \begin{pmatrix} & -i \\ i & \end{pmatrix}, X_\pm := (1/2)(H \pm i(F + G)) = 1/2 \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix} >$$

and the relations

$$[Z, X_\pm] = \pm 2X_\pm,\ [X_+, X_-] = Z.$$

3.3.2 If we use the embedding of G' into $\hat{G}$ from 3.1.3 and the notation for $\hat{\mathfrak{g}}$ from 1.1.2, we can identify $\mathfrak{g}'$ as a subalgebra of $\hat{\mathfrak{g}}$ as follows

$$Z = -i(F - G) = -i(\sum_{j=1}^{p}(U_{jj}^+ - U_{jj}^-) - \sum_{j=p+1}^{p+q}(U_{jj}^+ - U_{jj}^-))$$

and its realization by the infinitesimal Weil representation $d\omega$

$$\hat{Z} = \pi(x, x) - (1/4\pi)\Delta,$$

where we use the notation indicating the quadratic form given by S_0 and its Laplacian

$$(x,x) := (x,x)_{S_0} := \sum_{j=1}^{p} x_j^2 - \sum_{j=p+1}^{p+q} x_j^2,\ \Delta := \Delta_{S_0} := \sum_{j=1}^{p} \partial_j^2 - \sum_{j=p+1}^{p+q} \partial_j^2.$$

The same way, we have

$$X_\pm := (1/2)(H \pm i(F+G)) = (1/2)(\sum_{j=1}^{n} A_{jj} \pm i(\sum_{j=1}^{p}(U_{jj}^+ + U_{jj}^-) - \sum_{j=p+1}^{p+q}(U_{jj}^+ + U_{jj}^-)))$$

and its realization as a differential operator acting on the Schwartz space $\mathcal{S}(\mathbf{R}^n)$

$$\hat{X}_\pm = (1/2)(E + n/2 \mp (\pi(x,x) + (1/4\pi)\Delta)),$$

where we use the Euler operator

$$E := \sum_{j=1}^{n} x_j \partial_j.$$

3.3.3 The orthogonal group $G = \mathrm{O}(S_0) = \mathrm{O}(p,q)$ has as its Lie algebra

$$\begin{aligned} \mathfrak{o}(p,q) = \{Y = \begin{pmatrix} Y^{11} & Y^{12} \\ Y^{21} & Y^{22} \end{pmatrix};\ Y^{11} &= -{}^tY^{11} \in \mathrm{M}_p(\mathbf{R}), \\ Y^{22} &= -{}^tY^{22} \in \mathrm{M}_q(\mathbf{R}),\ Y^{12} = {}^tY^{21} \in \mathrm{M}_{pq}(\mathbf{R})\}. \end{aligned}$$

One has $\dim \mathfrak{o}(p,q) = n(n-1)/2$. We write

$$\mathfrak{o}(p,q) = \mathfrak{k} + \mathfrak{p};\ \mathfrak{k} = \{\begin{pmatrix} Y^{11} & \\ & Y^{22} \end{pmatrix}\} \simeq \mathfrak{o}(p) \times \mathfrak{o}(q),\ \mathfrak{p} = \{\begin{pmatrix} & Y^{12} \\ {}^tY^{12} & \end{pmatrix}\}.$$

As in 3.1.4 α, β denote indices $1, \dots, p$ and μ, ν indices between $p+1$ and $p+q$. Then $\mathfrak{k}$ is spanned by $n \times n-$matrices of the types

$$E_{\alpha\beta} - E_{\beta\alpha},\ E_{\mu\nu} - E_{\nu\mu}$$

and $\mathfrak{p}$ by those of the type

$$E_{\alpha\mu} + E_{\mu\alpha}.$$

The embedding of G into $\hat{G}$ from 3.1.3 induces an embedding of $\mathfrak{g}$ into $\hat{\mathfrak{g}}$ given by

$$\mathfrak{g} \ni Y \longmapsto \begin{pmatrix} -{}^tY & \\ & Y \end{pmatrix} \in \hat{\mathfrak{g}}.$$

We use this for an identification and hence with the notation from 1.1.2 can realize the elements of $\mathfrak{k}$ in the Weil representation as operators acting on $\mathcal{S}(\mathbf{R}^n)$ by

$$\hat{A}_{\alpha\beta} - \hat{A}_{\beta\alpha} = x_\alpha \partial_\beta - x_\beta \partial_\alpha,\ \hat{A}_{\mu\nu} - \hat{A}_{\nu\mu} = x_\mu \partial_\nu - x_\nu \partial_\mu$$

and the elements of $\mathfrak{p}$ by

$$\hat{A}_{\alpha\mu} + \hat{A}_{\mu\alpha} = x_\alpha \partial_\mu + x_\mu \partial_\alpha.$$

3.3.4 Example $G = \mathrm{O}(2,1)$: To simplify things, we look at the example $p = 2, q = 1$, i.e. $\mathfrak{g} = \mathfrak{o}(2,1) \simeq \mathfrak{sl}(2,\mathbf{R})$. Here we use the notation

$$H := \begin{pmatrix} & 1 & \\ -1 & & \\ & & 0 \end{pmatrix}, \ Y_1 := \begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix}, \ Y_2 = \begin{pmatrix} & & \\ & & 1 \\ & 1 & \end{pmatrix}$$

and have

$$[H, Y_1] = -Y_2,\ [H, Y_2] = Y_1,\ [Y_1, Y_2] = H.$$

With the identification given by the embedding from 3.1.2 above one has

$$H = A_{12} - A_{21},\ Y_1 = -(A_{13} + A_{31}),\ Y_2 = -(A_{23} + A_{32})$$

and the realization as operators for the Weil representation

$$\hat{H} = x_1\partial_2 - x_2\partial_1,\ \hat{Y}_1 = -(x_1\partial_3 + x_3\partial_1),\ \hat{Y}_2 = -(x_2\partial_3 + x_3\partial_2).$$

As usual, we complexify

$$\mathfrak{g}_c =< H_0 := -2iH,\ Y_\pm := Y_1 \pm iY_2 >$$

with

$$[H_0, Y_\pm] = \pm 2Y_\pm,\ [Y_+, Y_-] = H_0$$

and get the corresponding operators

$$\hat{H}_0 = -2i(x_1\partial_2 - x_2\partial_1),\ \hat{Y}_\pm = -(x_1 \pm ix_2)\partial_3 - x_3(\partial_1 \pm i\partial_2).$$

3.3.5 Remark: If we apply these operators to the vacuum vector of the Weil representation ω, the Gaussian

$$\varphi_0(x) = e^{-\pi(x_1^2+x_2^2+x_3^2)},$$

we get

$$\hat{H}_0\varphi_0 = 0,\ \hat{Y}_\pm\varphi_0 = 4\pi x_3(x_1 \pm ix_2)\varphi_0$$

and using the operators obtained in 3.3.2 specialized to $S_0 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$

$$\hat{Z}\varphi_0 = (1/2)\varphi_0,\ \hat{X}_+\varphi_0 = (1 - 2\pi(x_1^2 + x_2^2))\varphi_0,\ \hat{X}_-\varphi_0 = ((1/2) - 2\pi x_3^2)\varphi_0.$$

Hence, for the restricted representation $\omega|_{G\times G'}$, the Schwartz function φ_0 generating the Siegel theta is a vector of weight $(0, 1/2)$ but not a lowest weight vector as it is for the Weil representation of the ambient group $\hat{G}$. It is a natural task to search for a Schwartz function which is a vector of dominant weight for irreducible representations contained as subrepresentations in $\omega|_{G\times G'}$. Before we go into this, we just state the following observation as a byproduct of the small calculations leading to the Remark above.

3.4 Intermezzo: The Gaussian and $U(\mathfrak{g}_c)-$Modules

We stay with the example $G = \mathrm{O}(2,1)$ though a generalization should be easy.

3.4.1 Remark: If we apply the operators for the derived representation of the restriction $\omega|_{G\times G'}$ to the vector φ_1 with

$$\varphi_1(x) := e^{-\pi(x_1^2+x_2^2-x_3^2)} = e^{-\pi(x,x)}$$

we get

$$\hat{H}_0\varphi_1 = 0,\ \hat{Y}_\pm\varphi_1 = 0$$

and

$$\hat{Z}\varphi_1 = (3/2)\varphi_1,\ \hat{X}_+\varphi_1 = (1/2)(3-4\pi(x,x))\varphi_1,\ \hat{X}_-\varphi_1 = 0.$$

Hence, φ_1, which obviously is not a Schwartz function, has the properties of a lowest weight vector of weight 0 for $\omega|_G$ and weight 3/2 for $\omega|_{G'}$.
One has

$$\varphi_1(x) = e^{2\pi x_3^2}\varphi_0(x)$$

and (from 1.1.3) $\hat{U}_{33}^+ = \pi i x_j^2$. Then the Remark above also reflects in the formal calculation where we identify the elements of $\mathfrak{g}_c'$ with their images given by the embedding into $\hat{\mathfrak{g}}$:

3.4.2 Proposition: Let v_0 be an element of an $U(\hat{\mathfrak{g}}_c)-$module such that

$$H_j v_0 = (1/2)v_0 \quad \text{for all } j=1,\dots,n \text{ and } \check{U}^- v_0 = 0 \quad \text{for all } \check{U}^- \in \mathfrak{sp}^{(0,2)},$$

and

$$S := \sum_{l=1}^{\infty}(-2iU_{33}^+)^l/l!.$$

Then one has

$$ZSv_0 = (3/2)Sv_0,\ X_-Sv_0 = 0.$$

Proof: We recall

$$Z = -i(\sum_j \varepsilon_j(U_{jj}^+ - U_{jj}^-)),\ \varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = 1,$$

$$X_\pm = -(1/2)(\sum_j A_{jj} \pm i\sum_j \varepsilon_j(U_{jj}^+ + U_{jj}^-))$$

and abbreviate

$$U_{33}^+ =: V,\ U_{33}^- =: U,\ A_{33} =: A.$$

V commutes with all U_{jj}^+ and with U_{jj}^- for $j = 1,2$ and one has $[V,U] = A$. Moreover, V commutes with $A_{jj}, j = 1,2$ and one has $[V,A] = -2V$. By induction, one easily verifies for $l \in \mathbf{N}$ as relations in $U(\mathfrak{g}_c)$

$$AV^l = V^lA + 2lV^l \text{ and } UV^l = V^lU - lV^{l-1}A - l(l-1)V^{l-1}.$$

Hence we get

$$AS = SA + \sum 2l(-2i)^lV^l/l! = SA - 4iSV \text{ and } US = SU + 2iSA + 4SV.$$

and

$$\begin{aligned} X_-Sv_0 &= (1/2)(S(\sum A_{jj} - 4iSV - i(\sum \varepsilon_j(U^+_{jj} + U^-_{jj}) - 2iSA - 4SV)v_0 \\ &= (1/2)(S(\sum \varepsilon_j(A_{jj} - i(U^+_{jj} + U^-_{jj})v_0 = 0, \end{aligned}$$

as v_0 has the property $\check{U}^-jjv_0 = 0$ for $j = 1, 2, 3$. Similarly, one has

$$ZS = -i(S(\sum \varepsilon_j(U^+_{jj} - U^-_{jj})) + 2iSA + 4SV).$$

Here we use that we have the relations $-i(U^+_{jj} - U^-_{jj})v_0 = 1/2$ for all j and $\check{U}^-_{jj}v_0 = (iA_{jj} + U^+_{jj} + U^-_{jj})v_0 = 0$ leading to $(iA_{jj} + 2U^+_{jj})v_0 = (i/2)v_0$ and get

$$ZSv_0 = (3/2)v_0.$$

3.4.3 From the Remark 3.4.1 above one would expect to have also $H_0Sv_0 = Y_\pm v_0 = 0$. Here again we identify $\mathfrak{g} = \mathfrak{o}(2, 1)$ with its image in $\hat{\mathfrak{g}}_c$. One has

$$H_0 = -2iH,\ Y_\pm = Y_1 \pm iY_2$$

with

$$H = A_{12} - A_{21},\ Y_1 = -(A_{13} + A_{31},\ Y_2 = -(A_{23} + A_{32}).$$

H commutes with $V = U^+_{33}$. One has

$$[Y_1, U^+_{33}] = -U^+_{13},\ [Y_2, U^+_{33}] = -U^+_{23},$$

and hence

$$Y_1V^l = V^lY_1 - lv^{l-1}U^+_{13},\ Y_2V^l = V^lY_2 - lv^{l-1}U^+_{23}.$$

We get

$$H_0S = SH_0,\ \text{and}\ Y_\pm S = S(Y_\pm + 2i(U^+_{13} \pm iU^+_{23}))$$

and see that for the relations $H_0Sv_0 = 0$, $Y_\pm Sv_0 = 0$ one needs the conditions

$$A_{12}v_0 = A_{21}v_0,\ ,\ \text{and}\ (A_{jk} + A_{kj})v_0 = 2iU^+_{jk}v_0.$$

These conditions are fulfilled if we take $v_0 = \varphi_0$ and the realization by the Weil representation but certainly there is more background to this simple discussion.

3.4.4 It should be interesting to see an explicit decomposition of the $U(\hat{\mathfrak{g}}_c)$ module belonging to the Weil representation into its irreducible $(U(\mathfrak{g}_c) \times U(\mathfrak{g}'_c))-$modules. I don't know whether this is done someplace. But a somewhat equivalent task is easily accessible and we will describe this now.

4 Decomposition of the Weil Representation and Rallis-Schiffmann Thetas

4.1 Weil Representation $\tilde{\omega}_S$

We present explicit material (partially going back to Rallis and Schiffmann [RS1-3]) collected by M. Vergne in [LV] concerning the decomposition of the Weil representation ω as a representation of the dual pair $G = \mathrm{O}(p, q), G' = \mathrm{Sp}(m, \mathbf{R})$, in particular for $m = 1$. In this section we are interested in the discrete spectrum of ω as representation of $G \cdot G'$. It is a special case of a more general conjecture by Howe that the restriction of ω induces a one-one correspondence between the irreducible unitary (discrete series) representations of G and G'. This has been proved by work of Howe, Rallis, Schiffmann, and Strichartz ([Ho3], [RS1], [Str]). We shall follow essentially the presentation given in [LV] p.205ff.

4.1.1 We restrict our treatment to the case $G' = \mathrm{SL}(2, \mathbf{R}), G = \mathrm{O}(S) = \mathrm{O}(p, q)$ and denote by $\tilde{\omega}_S$ the restriction to $G \cdot \tilde{G}'$ of the Weil representation $\tilde{\omega}$ of the metaplectic cover $\tilde{\hat{G}}, \hat{G} = \mathrm{Sp}(n, \mathbf{R}), n = p + q$. In order to describe the decomposition of $\tilde{\omega}$ we have to fix a lot of notation and reproduce some elements of the representation theory of G' and G though the reader will probably know most of this.

4.2 Discrete Series of $G' = \mathrm{SL}(2, \mathbf{R})$

This group has three types of unitary irreducible representations , the discrete, the principal, and the complementary series representations .

4.2.1 Here we only report on the discrete series π_k^+ and π_k^- for $k \in \mathbf{N}, k \geq 2$. π_k^+ is a lowest weight representation with a lowest weight vector v which, using our notation from 3.6, is characterized by

$$d\pi_k^+(Z)v = kv,\ d\pi_k^+(X_-)v = 0.$$

Similarly π_k^- has a vector v_- of highest weight $-k$ characterized by

$$d\pi_k^-(Z)v_- = kv_-, d\pi_k^-(X_+)v_- = 0.$$

Standard models for these representations come from the space $\mathcal{O}(\mathfrak{H})$ of holomorphic functions on the upper half plane, resp. the space $\bar{\mathcal{O}}(\mathfrak{H})$ of antiholomorphic functions with prescriptions

$$(\pi_k(g^{-1})f) = (c\tau + d)^{-k} f(g(\tau)); \quad \text{for all } f \in \mathcal{O}(\mathfrak{H}),\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G',$$

resp.

$$(\bar{\pi}_k(g^{-1})f) = (c\bar{\tau} + d)^{-k} f(g(\tau)); \quad \text{for all } f \in \bar{\mathcal{O}}(\mathfrak{H}),\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G'.$$

If one restricts to the spaces $L^2_{hol}(\mathfrak{H}, d\mu_k)$ resp. $L^2_{antihol}(\mathfrak{H}, d\mu_k)$ of holomorphic resp. antiholomorphic funtions on $\mathfrak{H}$ with finite norm for the measure $d\mu_k(\tau) = v^{k-2}dudv$, one has unitary representations .

4.2.2 Remark: It is an easy but interesting exercise to verify that

$$\psi_w(\tau) := (\bar{\tau} - w)^{-k},\ w \in \mathfrak{H},$$

is a lowest weight vector for $\bar{\pi}_k$ which fulfills the *fundamental relation*

$$\bar{\pi}_k(g)\psi_w = (cw + d)^{-k}\psi_{g(w)}.$$

We use the coordinization

$$G' = \mathrm{SL}(2,\mathbf{R}) \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = n(u)t(v)r(\vartheta),$$

with

$$n(u) := \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix},\ t(v) := \begin{pmatrix} v^{1/2} & \\ & v^{-1/2} \end{pmatrix},\ r(\vartheta) := \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$$

and the standard notation $j(g,\tau) = (c\tau + d)$, $j_k(g,\tau) = (c\tau + d)^{-k}$. Hence, we have $g(i) = u + iv = \tau, j(g,i) = (ci + d) = v^{-1/2}e^{-i\vartheta}$ and we put

$$g_\tau := n(u)t(v).$$

(This coordinization goes back to Lang's book $\mathrm{SL}_2(\mathbf{R})$. Vergne in [LV] and some other authors use $u(\theta) := r(-\theta)$ for the parametrization of SO(2))

For $U \in \mathfrak{g}'$ we denote by $\mathcal{R}_U$ the right invariant operator on $\mathcal{C}^\infty(G')$ given by

$$\mathcal{R}_U\Phi(g) = \frac{d}{dt}\Phi(exp(-tU)g)|_{t=0}$$

and by $\mathcal{L}_U$ the left invariant operator given by

$$\mathcal{L}_U\Phi(g) = \frac{d}{dt}\Phi(g(exp(tU)))|_{t=0}.$$

4.2.3 For a moment we replace u, v by x, y. A standard calculation (see for instance Lang [La] p.113) leads to

$$\begin{aligned} \mathcal{R}_Z &= i((1 + x^2 - y^2)\partial_x + 2xy\partial_y + y\partial_\vartheta) \\ \mathcal{R}_{X_\pm} &= \pm(i/2)(((x \pm i)^2 - y^2)\partial_x + 2(x \pm i)y\partial_y + y\partial_\vartheta). \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_Z &= -i\partial_\vartheta \\ \mathcal{L}_{X_\pm} &= \pm(i/2)e^{\pm 2i\vartheta}(2y(\partial_x \mp i\partial_y) - \partial_\vartheta). \end{aligned}$$

If $v_0 \in V$ is a lowest weight vector of weight k for the representation (π, V), one has $\pi(r(\vartheta))v_0 = e^{ik\vartheta}v_0$ and one can see that

$$(ci + d)^k\pi(g)v_0 = v^{-k/2}\pi(g_\tau)v_0$$

depends only on τ, hence we abbreviate it by v_τ.

4.2.4 Remark: The lowest weight representation (π, V) is isomorphic to a subrepresentation of $(\bar{\pi}_k, \bar{\mathcal{O}}(\mathfrak{H}))$, the isomorphism being obtained by sending v_w to ψ_w.

If one has a group acting transitively on a space and an automorphic factor j for this action, there is a standard procedure to define an appropriate lift of functions on the space to functions living on the group. In our case, we look at the action of G' on $\mathfrak{H}$ and the automorphic factor $j_k(g,\tau) = (c\tau + d)^{-k}$ and for a function $f : \mathfrak{H} \longrightarrow \mathbf{C}$ define the lift $\varphi_k : f \longmapsto \Phi_f$ by $\Phi_f(g) := j_k(g,i)f(g(i))$. One has the fundamental fact (see for instance [La] IX §5):

4.2.5 Proposition: The function f is holomorphic if and only if

$$\mathcal{L}_{X_-}\Phi_f = 0.$$

We denote

$$M(G', k) := \{\Phi \in \mathcal{C}(G'); \Phi(gr(\vartheta)) = e^{ik\vartheta}\Phi(g) \quad \text{for all } \ g \in G',\, r(\vartheta) \in K'\}$$

and by λ_k the representation of G' given on $M(G', k)$ by (inverse) left translation $\lambda_k(g_0)\Phi(g) = \Phi(g_0^{-1}g)$. Then it is not difficult to prove

4.2.6 Proposition: The lifting map φ_k intertwines the representations π_k and λ_k.

There is the inverse map I_k to φ_k given by associating to $\Phi \in M(G', k)$ the function $f(\tau) = v^{-k/2}\Phi(g_\tau)$.

4.2.7 Up to now the index k was an integer and our group $G' = \mathrm{SL}(2, \mathbf{R})$. We also will have to work with the metaplectic cover $\tilde{G}' = \mathrm{Mp}(2, \mathbf{R})$ and halfinteger k. As in this report we follow Vergne in [LV] p.184, we look at the universal cover $\mathcal{G}$ of G' in the form

$$\mathcal{G} := \{(g, \varphi_g); g \in G', \varphi \in \mathcal{O}(\mathfrak{H}) \text{ such that } e^{\varphi_g(\tau)} = j(g, \tau)\}.$$

We denote by *pr* the projection $\mathcal{G} \longrightarrow G'$. and, for $\alpha \in \mathbf{R}$, by $\pi_{\alpha,0}$ the representation of $\mathcal{G}$ given on $\mathcal{O}(\mathfrak{H})$ by

$$\pi_{\alpha,0}((g, \varphi)^{-1}f(\tau) = e^{-\alpha\varphi(\tau)}f(g(\tau)).$$

For $\alpha = k \in \mathbf{Z}$ the projection *pr* intertwines $\pi_{k,0}$ with the representation π_k of G' given above.

For $J := \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ one has the one parameter subgroup

$$K' = \mathrm{SO}(2) = \{r(\vartheta) = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}\}$$

of G' and, above in $\mathcal{G}$, the group

$$\mathcal{K} = \{\delta(\vartheta) := (r(\vartheta), \varphi_\vartheta); \varphi_\vartheta(i) = -i\vartheta\}.$$

For $\alpha \in \mathbf{R}$ we define

$$M(\mathcal{G}, \alpha) := \{\Phi \text{ analytic on } \mathcal{G};\ \Phi(\tilde{g}\delta(\vartheta)) = e^{i\alpha\vartheta}\Phi(\tilde{g}) \quad \text{for all } \ \tilde{g} \in \mathcal{G}\}.$$

and denote by λ_α the representation of $\mathcal{G}$ given on $M(\mathcal{G}, \alpha)$ by left inverse multiplication $\lambda_\alpha(\tilde{g}_0)\Phi = \Phi(\tilde{g}_0^{-1}\tilde{g})$. We put $a_\alpha((g, \varphi) := e^{\alpha\varphi(i)}$. One can verify easily that for $\Phi \in M(\mathcal{G}, \alpha)$ the function $I_\alpha\Phi$ given by

$$I_\alpha(\tilde{g}) := a_\alpha(\tilde{g})\Phi(\tilde{g})$$

is invariant by right translation by $\delta(\vartheta)$, Hence one can find a function on $\mathfrak{H}$ still denoted by $I_\alpha\Phi$ such that $I_\alpha\Phi(\tilde{g}) = I_\alpha\Phi(pr(\tilde{g}))(i))$. From Lemma 2.3.13 in [LV] we take over the following fact.

4.2.8 Remark: I_α intertwines the representations λ_α and $\pi_{\alpha,0}$

4.3 Elementary Thetas and their Construction

Though it is somewhat redundant in view of what we already did in 1.2.2, we want to illustrate the notions we just introduced and we describe again a bit more precisely a construction of the simplest theta functions.

4.3.1 The Weil representation of G' on $\mathcal{H} = L^2(\mathbf{R})$ as a special case given by the formulae in 1.1.1 decomposes into two irreducible subrepresentations $\omega_{even} =: \pi_{W,+}, \omega_{odd} =: \pi_{W,-}$ namely those given by the even and the odd functions. As already shown in 1.1.4, the even function $\varphi_0 = e^{-\pi x^2}$ is a lowest weight vector of weight 1/2 for ω_{even} and a small calculation shows that

$$\varphi_0'(x) := xe^{-\pi x^2}$$

is a lowest weight vector of weight 3/2 for ω_{odd}.

4.3.2 Vergne's Program: As Vergne states in [LV] p.179, one can construct a modular form of weight k and character χ for a discrete subgroup Γ of G' (or $\mathcal{G}$) by achieving two steps.

1.) Find lowest a lowest weight vector $v \neq 0$ of weight k in the space $\mathcal{H}$ of a representation π.

2.) Find a functional $\theta \in \mathcal{H}'$ such that

$$\pi(\gamma)\theta = (\chi(\gamma))^{-1}\theta \quad \text{for all } \gamma \in \Gamma.$$

Then, using the map I_k resp I_α introduced above, one can transfer $\theta(\pi(g)v)$ to a holomorphic function f living on $\mathfrak{H}$ with the properties wanted.

In our case we have the lowest weight vectors $v = \varphi_0$ and $v = \varphi_0'$ of weights $k = 1/2$ and $k = 3/2$ for the subrepresentations of the Weil representation . For the determination of the semi invariant functional fitting to certain subgroups Vergne goes a long way back to the Schrödinger and lattice representation of the Heisenberg group. To abbreviate things, here we simply take the theta distribution $\theta = \sum_{n\in\mathbf{Z}} \delta_n$ and get

$$\theta(\omega(g_\tau)\varphi_0) = v^{1/4} \sum_{n\in\mathbf{Z}} e^{\pi i\tau n^2}, \; I_{1/2}\theta(\omega(g_\tau)\varphi_0) = \sum_{n\in\mathbf{Z}} e^{\pi i\tau n^2} = \theta(\tau)$$

and

$$\theta(\omega(g_\tau)\varphi_0') = v^{3/4} \sum_{n\in\mathbf{Z}} ne^{\pi i\tau n^2}, \; I_{3/2}\theta(\omega(g_\tau)\varphi_0') = \sum_{n\in\mathbf{Z}} e^{\pi i\tau n^2}.$$

The first function comes out as our fundamental theta function with the precise transformation property given as follows (see for instance [LV] p.204)

4.3.3 Theorem: For every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbf{Z}$, $ad - bc = 1$, $ac \equiv 0 \mod 2$, $bd \equiv 0 \mod 2$ one has

$$\theta(\gamma(\tau)) = \lambda(\gamma)(c\tau + d)^{1/2}\theta(\tau)$$

where $(c\tau+d)^{1/2}$ is the principal determination of $(c\tau+d)^{1/2}$ and λ is a rather complicated character given in [LV] p.201. In particular, if c is even, $\lambda(\gamma) = \varepsilon_d^{-1}(\frac{2c}{d})$. For d an odd integer ε_d is defined to be

$$\begin{aligned} \varepsilon_d &= 1 \text{ if } d \equiv 1 \mod 4 \\ &= i \text{ if } d \equiv -1 \mod 4. \end{aligned}$$

The second function apparently is identically zero. But one still gets reasonable functions by taking a character $\psi \mod N$ and $t \in \mathbf{Z}$ defining

$$\theta^+_{\psi,t}(\tau) := \sum_{n\in\mathbf{Z}} \psi(n) e^{2\pi i\tau t n^2} \text{ for } \psi \text{ even}$$

and

$$\theta^-_{\psi}(\tau, t) := \sum_{n\in\mathbf{Z}} \psi(n) n e^{2\pi i\tau t n^2} \text{ for } \psi \text{ odd}$$

with

$$\theta^+_{\psi,t}(\gamma(\tau)) = \psi(d)(\frac{t}{d})\varepsilon_d^{-1}(c\tau+d)^{1/2}\theta^+_{\psi,t}(\tau) \quad \text{for all } \gamma \in \Gamma_0(4N^2t),$$

and

$$\theta^-_{\psi,t}(\gamma(\tau)) = \psi(d)(\frac{t}{d})\varepsilon_d^{-1}(c\tau+d)^{3/2}\theta^-_{\psi,t}(\tau) \quad \text{for all } \gamma \in \Gamma_0(4N^2t).$$

4.4 The Theta Miracle

We add an observation which for me plays the role of a *Theta Miracle*. In 1.3.2 and 1.3.3 we extended the Weil representation of G' resp. $\mathcal{G}$ with the Schrödinger representation of the Heisenberg group to the representation π_{SW} of the Jacobi group G^J. If we take our vacuum vector of lowest weight 1/2, namely $\varphi_0(x) = e^{-\pi x^2}$, we get

$$\varphi_{g^J}(x) := \pi_{SW}((p,q,k)g_\tau r(\vartheta))\varphi_0(x) = v^{1/4}e^{\pi i\vartheta/2}e^{\pi i(\kappa+p^2\tau+pq+x^2\tau+2(p\tau+q)x)}.$$

We will see that $\Phi_0(g^J) := \varphi_{g^J}(0)$ directly can be interpreted as a lowest weight vector.

4.4.1 To explain this and prepare some more material useful later, we treat the Lie algebra of the Jacobi group. As in [BeS] or [Ya], we describe the Lie algebra $\mathfrak{g}^J$ as a subalgebra of $\mathfrak{sp}(2,\mathbf{R})$ by

$$G(x,y,z,p,q,r) = \begin{pmatrix} x & 0 & y & q \\ p & 0 & q & r \\ z & 0 & -x & -p \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and denote

$$X = G(1,0,\dots,0),\dots,R = G(0,\dots,0,1).$$

We get the commutators

$$\begin{array}{lll} [X,Y] = 2Y, & [X,Z] = -2Z, & [Y,Z] = X, \\ [X,P] = -P, & [X,Q] = Q, & [P,Q] = 2R, \\ [Y,P] = -Q, & [Z,Q] = -P, & \end{array}$$

all others are zero. Hence, we have the complexified Lie algebra given by

$$\mathfrak{g}^J_c = < Z_1, X_\pm, Y_\pm, Z_0 >$$

where as in [BeS]

$$Z_1 = -i(Y-Z),\ Z_0 = -iR,$$

$$X_\pm = (1/2)(X \pm i(Y+Z)),\ Y_\pm = (1/2)(P \pm iQ)$$

with the commutation relations

$$[Z_1, X_\pm] = \pm 2X_\pm,\ [Z_0, Y_\pm] = \pm Y_\pm, \text{ etc.}$$

4.4.2 The complexified Lie algebra $\mathfrak{g}_c^J$ of the Jacobi group is realized (see [BeS] p.12) by the left invariant differential operators

$$\begin{aligned}
&\mathcal{L}_{Z_0} = i\partial_\kappa\\
&\mathcal{L}_{Y_\pm} = (1/2)y^{-1/2}e^{\pm i\vartheta}(\partial_p - (x \pm iy)\partial_q - (p(x+iy)+q)\partial_\kappa)\\
&\mathcal{L}_{X_\pm} = \pm(i/2)e^{\pm 2i\vartheta}(2y(\partial_x \mp i\partial_y) - \partial_\vartheta)\\
&\mathcal{L}_Z = -i\partial_\vartheta
\end{aligned}$$

acting on differentiable functions $\phi = \phi(g)$ with the coordinates coming from

$$g = (p, q, \kappa)n(x)t(y)r(\vartheta).$$

where

$$n(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, t(a) = \begin{pmatrix} a^{1/2} & \\ & a^{-1/2} \end{pmatrix},\ r(\vartheta) = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

with

$$\alpha = \cos\vartheta,\ \beta = \sin\vartheta,$$

and

$$g(i, 0) = (\tau, z) = (x + iy, p\tau + q).$$

As usual, we put

$$\begin{aligned}
N' &= \{n(x), x \in \mathbf{R}\}, A' = \{t(a), a \in \mathbf{R}_{>0}\},\\
K' &= \mathrm{SO}(2) = \{r(\vartheta), \vartheta \in \mathbf{R}\}, M = \{\pm E\}.
\end{aligned}$$

4.4.3 We introduce the standard automorphic factor for $g^J = n(u)t(v)r(\vartheta)(\lambda, \mu, \kappa) \in G^J$

$$j_{m,k}(g^J, (\tau, z)) := (c\tau + d)^{-k}e^m(\kappa - \frac{c(z + \lambda\tau + \mu)^2}{c\tau + d} + \lambda^2\tau + 2\lambda z + \lambda\mu)$$

4.4.4 Proposition: The function

$$\Phi_0(g^J) = v^{1/4}e^{i\vartheta/2}e^{\pi i(\kappa + pz)},\ g^J = (p, q, \kappa)n(u)t(v)r(\vartheta)$$

fulfills the relations

$$\mathcal{L}_{Z_0}\Phi_0 = (1/2)\Phi_0,\ \mathcal{L}_Z\Phi_0 = (1/2)\Phi_0,\ \mathcal{L}_{Y_-}\Phi_0 = 0,\ \mathcal{L}_{X_-}\Phi_0 = 0,$$

i.e. Φ_0 spans in the restriction of the right regular representation of G^J a representation of lowest weight $(1/2, 1/2)$ for the subgroup $K^J := \{(0, 0, \kappa)r(\vartheta); \kappa, \vartheta \in \mathbf{R}\}$ of G^J.

4.4.5 It is easy to see that one has $\Phi_0(g^J) = j_{1/2,1/2}(g^J, (i, 0))$ and that Φ_0 is invariant under left multiplication by elements of the group $N^J := \{(0, q, 0)n(u); q, u \in \mathbf{R}\}$. Now, if we want invariance under the group $\Gamma^J := \mathrm{SL}(2, \mathbf{Z}) \ltimes Z^3$ or an appropriate subgroup, we try the averaging

$$\Theta(g^J) := \sum_{\ell \in \mathbf{Z}} \Phi((\ell, 0, 0)g^J),$$

which has the same outcome as the application of the theta distribution from 4.3.2 to $\varphi_{g^J}(x)$: We get

$$\begin{aligned}\Theta(g^J) &= \sum_{\ell\in\mathbf{Z}} \pi_{SW}((p,q,k)g_\tau r(\vartheta))\varphi_0(\ell)\\ &= v^{1/4}e^{\pi i\vartheta/2}e^{\pi i(\kappa+p^2\tau+pq)}\sum_{\ell\in\mathbf{Z}} e^{\pi i(\tau\ell^2+2(p\tau+q)\ell)}\\ &= j_{1/2,1/2}(g^J,i)\theta(\tau,z).\end{aligned}$$

As usual, using the Poisson summation formula, one can see ([Be1]) that (by the *theta miracle*) our construction has a built in invariance property with respect to the element $(0,0,0)J\in\Gamma^J$ and hence for the whole theta group. We cite from [MuI] p.32.

4.4.6 Theorem: For $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_\theta$ one has

$$\theta(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d})=\zeta(c\tau+d)^{1/2}e^{\pi icz^2/(c\tau+d)}\theta(\tau,z)$$

with

$$\begin{aligned}\zeta &= i^{(d-1)/2}(\tfrac{c}{|d|}) && c \text{ even, } d \text{ odd}\\ &= e^{\pi ic/4}(\tfrac{d}{c}) && d \text{ even, } c \text{ odd.}\end{aligned}$$

4.4.7 From [MuI] p.227 we take over that one has the same relation for the more general function

$$\theta\left[\begin{array}{c}a\\b\end{array}\right](\tau,z):=\sum_{n\in\mathbf{Z}}e^{\pi i((n+a)^2\tau+2(n+a)(z+b))}.$$

If one takes here the differentiation ∂_z and afterwards puts $z=0$, one gets

$$\frac{\partial\theta\left[\begin{array}{c}a\\b\end{array}\right]}{\partial z}(\gamma(\tau),0)=\zeta(c\tau+d)^{3/2}\frac{\partial\theta\left[\begin{array}{c}a\\b\end{array}\right]}{\partial z}(\tau,0)$$

i.e.

$$\frac{\partial\theta\left[\begin{array}{c}a\\b\end{array}\right]}{\partial z}(\tau,0)=2\pi i\sum_{n\in\mathbf{Z}}(n+a)e^{\pi i[(n+a)^2\tau+2(n+a)b)}$$

is a modular form of weight 3/2. We observe that the function φ_0' reappears by a differentiation of a function stemming from φ_0.

4.5 Representations of the Orthogonal Groups

We start by looking at the compact group $G=\mathrm{O}(p)$ belonging to a positive definite form S on $V=\mathbf{R}^p$. G acts on functions f on V via $\lambda(g)f(x):=f(g^{-1}x)$ and similarly on the space $\mathcal{P}:=\mathbf{C}[x_1,\dots,x_p]$ of polynomials. One identifies V with its dual V^* via S. and this identification extends to an identification of $\mathcal{P}$ the space $\mathcal{D}$ of differential operators with constant coefficients on $\mathbf{R}^p$ via $P[x_1,\dots,x_p]\longmapsto P(\partial_1,\dots,\partial_p)$. Hence $\mathcal{P}$ is provided with an $\mathrm{O}(P)$-invariant hermitian inner product given by

$$<P,Q>:=(P(\partial_x)\bar{Q})(0).$$

4.5.1 Proposition: The Laplace operator $\Delta_S := \sum_{j=1}^{p} \partial_j^2$ commutes with the action of G and the space

$$\mathcal{H}^m = \mathcal{H}^m(\mathbf{R}^p) := \{P \in \mathcal{P}; \Delta_S P = 0, \deg P = m\}$$

of harmonic polynomials of degree m is the space of an irreducible unitary representation δ_m of $\mathrm{O}(p)$.

One has

$$\dim \mathcal{H}^m(\mathbf{R}^p) = \binom{m+p-1}{p-1} - \binom{m+p-3}{p-1}.$$

If $p = 1$, this is zero for $m \geq 2$.

Moreover, it is a standard fact that each polyomial P can be written as a sum $P = \sum S^j P_j$ where the P_j are harmonic and $S = S(x) := S(x, x)$. As the space

$$\{P(x)e^{-\pi S(x)};\ P \in \mathcal{P}\}$$

is dense in $L^2(\mathbf{R}^p)$, one has a decomposition

$$L^2(\mathbf{R}^p) = \oplus_{m \in \mathbf{N}_0} L^2(\delta_m) \cong \oplus_{m \in \mathbf{N}_0} \mathcal{H}^m(\mathbf{R}^p)$$

where $L^2(\delta_m)$ indicates the isotypic component of type δ_m in $L^2(\mathbf{R}^p)$ (and m is only 0 or 1 for $p = 1$).

4.5.2 If S^{p-1} denotes the sphere in $\mathbf{R}^p$, one has as well (see for instance [Kn] p.81)

$$\mathcal{C}^\infty(S^{p-1}) \cong \oplus_{m \in \mathbf{N}_0} \mathcal{H}^m(\mathbf{R}^p).$$

Now, we will take a look at the non-compact case $G = \mathrm{O}(p, q), p \geq q, q \geq 1$.

4.5.3 It is clear that each representation of $G = \mathrm{O}(p, q)$ will decompose under the compact subgroup $K = \mathrm{O}(p) \times \mathrm{O}(q)$ and we shall have to find a way to analyse this decomposition. We take again the action of $g \in G$ given on functions f on $\mathbf{R}^{p+q} = \mathbf{R}^n$ by $\lambda(g)f(w) := f(g^{-1}w)$. We write

$$\mathbf{R}^n = \mathbf{R}^{p+q} \ni w = (x, y),\ x \in \mathbf{R}^p, y \in \mathbf{R}^q;\ r(x) := (x_1^2, \dots, x_p^2)^{1/2},\ r(y) := (y_1^2, \dots, y_q^2)^{1/2}$$

and

$$\begin{aligned} \mathcal{X}^0 &:= \{(x, y) \in \mathbf{R}^{p+q};\ r(x)^2 - r(y)^2 = 0\}, \\ \mathcal{X}^\pm &:= \{(x, y) \in \mathbf{R}^{p+q};\ r(x)^2 - r(y)^2 \gtrless 0\}, \\ \mathcal{X}^t &:= \{(x, y) \in \mathbf{R}^{p+q};\ r(x)^2 - r(y)^2 = t^2\} \text{ for } t > 0. \end{aligned}$$

G acts transitively and one can construct representations on spaces of functions living on each one of these sets. Though it is not the most appropriate one for our later application, we briefly follow the nice and accessible presentation given in [HT], which is based on the cone $\mathcal{X}^0$. There are fundamental papers by Strichartz [Str] and Rallis-Schiffmann [RS1-3], which use $\mathcal{X}^+$ and the hyperboloid $\mathcal{X}^t$. We will come to this later but for now study spaces of homogeneous functions on the ($G-$invariant) light cone $\mathcal{X}^0$: For $a \in \mathbf{C}$ denote by $S^a(\mathcal{X}^0)$ the space

$$S^a(\mathcal{X}^0) := \{f \in \mathbf{C}^\infty(\mathcal{X}^0);\ f(tw) = t^a f(w),\ w \in \mathcal{X}^0,\ t \in \mathbf{R}_{>0}\}.$$

Since $\mathrm{O}(p,q)$ commutes with scalar dilatations, it is clear that $S^a(\mathcal{X}^0)$ will be invariant under the action given by λ. To study the structure of $S^a(\mathcal{X}^0)$ as $\mathrm{O}(p,q)-$module we consider the action of the compact subgroup $K = \mathrm{O}(p)\times\mathrm{O}(q)$. We take the tensor product of the harmonic $m-$forms $\mathcal{H}^m(\mathbf{R}^p)$ and the $n-$forms $\mathcal{H}^n(\mathbf{R}^q)$ and its embedding into $S^a(\mathcal{X}^0)$ given by

$$j_a = j_{a,m,n} : \mathcal{H}^m(\mathbf{R}^p) \otimes \mathcal{H}^n(\mathbf{R}^q) \longrightarrow S^a(\mathcal{X}^0);\ h_1 \otimes h_2 \longmapsto h_1(x)h_2(y)r(x)^{2b}$$

with $m + n + 2b = a$. Then one has (Lemma 2.2 in [HT])

4.5.4 Lemma: As an $\mathrm{O}(p) \times \mathrm{O}(q)$ module, the space $S^a(\mathcal{X}^0)$ decomposes into a direct sum

$$S^a(\mathcal{X}^0) \simeq \sum_{m,n\geq 0} j_a(\mathcal{H}^m(\mathbf{R}^p) \otimes \mathcal{H}^n(\mathbf{R}^q))$$

of mutually inequivalent irreducible representations .

One refers to the spaces $j_a(\mathcal{H}^m(\mathbf{R}^p)\otimes\mathcal{H}^n(\mathbf{R}^q))$ as the $K-$*types* of $S^a(\mathcal{X}^0)$. To understand the representation one has to study how $\mathrm{O}(p,q)$ transforms one $K-$type into another. For this purpose one can use the *ladder operators* given by the generators of the Lie algebra $\mathfrak{p}$ from 3.3.3

$$\hat{A}_{\alpha\mu} + \hat{A}_{\mu\alpha} = x_\alpha\partial_\mu + x_\mu\partial_\alpha.$$

in their action on functions on $\mathbf{R}^{p+q}$. By a routine calculation (carried out in [HT] §6.1) this action can be transfered to maps between the $K-$types and, for $q > 1$, one gets (Lemma 2.3 in [HT])

4.5.5 Lemma: For each pair (m, n), there are maps

$$T^{\pm,\pm}_{m,n} : \mathfrak{p} \otimes (\mathcal{H}^m(\mathbf{R}^p) \otimes \mathcal{H}^n(\mathbf{R}^q)) \longrightarrow (\mathcal{H}^{m\pm 1}(\mathbf{R}^p) \otimes \mathcal{H}^{n\pm 1}(\mathbf{R}^q)),$$

which are independent of a and which are nonzero as long as the target space is nonzero, such that the action of $Y \in \mathfrak{p}$ on the $K-$type $j_a(\mathcal{H}^m(\mathbf{R}^p) \otimes \mathcal{H}^n(\mathbf{R}^q))$ is described by the formula

$$\begin{array}{rcl}\rho(Y)j_a(\phi) & = & (a-m-n)j_a(T^{++}_{m,n}(Y\otimes\phi)) \\ & & +(a-m+n+q-2)j_a(T^{+-}_{m,n}(Y\otimes\phi)) \\ & & +(a+m-n+p-2)j_a(T^{-+}_{m,n}(Y\otimes\phi)) \\ & & +(a+m+n+p+q-4)j_a(T^{--}_{m,n}(Y\otimes\phi),\end{array}$$

where $\phi \in \mathcal{H}^m(\mathbf{R}^p) \otimes \mathcal{H}^n(\mathbf{R}^q)$.

One sees easily that the *transition coefficients* relating the different $K-$types are never zero if a is not an integer and hence one has an irreducible $\mathrm{O}(p,q)$ module. In [HT] one finds a detailed study what happens for integer a and which representations are unitary.

4.5.6 As we are particularly interested in this special case, we finish by reproducing some results concerning the case $q = 1$ omitted above. When $q = 1$, the light cone $\mathcal{X}^0$ is not connected. One denotes

$$\mathcal{X}^{0\pm} := \{(x,y) \in \mathcal{X}^0;\ y = \pm r(x)\}.$$

Each subspace $\mathcal{X}^{0\pm}$ is stabilized by a subgroup $\mathrm{O}^+(p,1)$ of index 2 in $\mathrm{O}(p,1)$. And analyzing $S^a(\mathcal{X}^0)$ as an $\mathrm{O}(p,1)$ module is the same as analyzing the space of even functions

$S^{a+}(\mathcal{X}^0) \simeq S^a(\mathcal{X}^{0+})$ as an $\mathrm{O}^+(p,1)$ module. For $a \in \mathbf{C}$ we can define embeddings

$$j_a = j_{a,m} : \mathcal{H}^m(\mathbf{R}^p) \longrightarrow S^a(\mathcal{X}^{0+});\ h \longmapsto h(x,y)y^{a-m}.$$

One has

$$S^a(\mathcal{X}^{0+}) \simeq \sum_{m \geq 0} j_a(\mathcal{H}^m(\mathbf{R}^p))$$

and parallel to Lemma 4.5.5 the action of $\mathfrak{p}$ in this case leads to

$$(x_j \partial_y + y \partial_{x_j})(hy^{a-m}) = (a-m)T_j^+(h)y^{a-m-1} + (a+m+p-2)T_j^-(h)y^{a-m+1}.$$

One concludes that $S^a(\mathcal{X}^{0+})$ is always irreducible except when a is an integer, with either $a \geq 0$ or $a \leq -p+1$, in which case $S^a(\mathcal{X}^{0+})$ has two constituents, one of which is finite dimensional. In [HT] there is a discussion of the unitarity of the representations . In this degenerate case we have unitarity for $a \in \mathbf{C}$ with Re $a = -(p-1)/2, a \in (-(p-1), 0)$, or $a \in \mathbf{N}$.

4.6 Decomposition of the Weil Representation Associated to a Positive Definite Quadratic Form

We use the notation (V, S) to denote an orthogonal space and (V', B) resp. $(\hat{V} := V \otimes V', \hat{B} := S \otimes B)$ to denote symplectic spaces with standard bases as in 3.1.4, $\dim V = n, \dim V' = 2m$. ω is the Weil representation of $\hat{G} = \mathrm{Sp}(mn, \mathbf{R}) \simeq \mathrm{Sp}(S \otimes B)$. (resp. $\tilde{\omega}$ for the metaplectic cover $\tilde{G}'$) as in 1.1 and ω_S its restriction to $\mathrm{O}(S) \times \mathrm{Sp}(B)$ resp. $\tilde{\omega}_S$ for the metaplectic cover. For positive definite S the decomposition of $\tilde{\omega}_S$ is described in Kashiwara-Vergne [KV]. Here we restrict to $n = p$ and $m = 1$ and follow [LV] p.209f.

4.6.1 From 3.3.2 we repeat the formulae for the derived representation of ω_S on $\mathcal{S}(\mathbf{R}^n)$ restricted to $\mathfrak{g}'_c = (2, \mathbf{C}) =< Z, X_\pm >$

$$\hat{Z} = \pi(x,x) - (1/4\pi)\Delta,\ \hat{X}_\pm = (1/2)(E + n/2 \mp (\pi(x,x) + (1/4\pi)\Delta)),\ E := \sum_{j=1}^n x_j \partial_j,$$

and from 3.3.3 for the restriction to $\mathfrak{g} = \{Y \in \mathrm{M}(p, \mathbf{R}); Y = -{}^tY\}$

$$\hat{Y}_{\alpha\beta} = x_\alpha \partial_\beta + x_\beta \partial_\alpha.$$

As $\tilde{G}$ and G' commute in $\tilde{\hat{G}}$, the subspace $L^2(\delta_m)$ of functions in $L^2(V) \simeq L^2(\mathbf{R}^n)$ of type δ_m with respect to $\mathrm{O}(S)$ is stable by $\tilde{G} \times \mathrm{O}(S)$ and one has the multiplicity free decomposition

$$L^2(V) = \oplus_{m \in \mathbf{N}_0} L^2(\delta_m).$$

From 4.5.1 we take a harmonic polynomial of degree m and put

$$\varphi_P(x) := P(x)\varphi_0(x),\ \varphi_0(x) := e^{-\pi {}^txx} = e^{-\pi S(x)}.$$

After a short calculation we get

4.6.2 Proposition:
$$\hat{Z}\varphi_P = (m + p/2)\varphi_P,\ \hat{X}_-\varphi_P = 0.$$

Hence $\varphi_P \in L^2(\delta_m)$ is a lowest weight vector for the discrete series representation $\pi^+_{m+p/2} =: \alpha$ of $G' = \mathrm{SL}(2, \mathbf{R})$ resp. its covering group $\mathcal{G}$ and we have the decomposition of $\tilde{\omega}_S$ in unitary irreducible representations of $\mathrm{O}(p) \times \tilde{\mathrm{SL}}(2, \mathbf{R})$ as

$$\tilde{\omega}_S = \oplus_m (\bar{\pi}_{(p/2)+m} \otimes \delta_m).$$

Following the same program as we did in 4.3, now we look at

$$(\omega_S(g_\tau)\varphi_P)(x) = v^{\alpha/2} P(x) e^{\pi i \tau S(x)}$$

and
$$\varphi_{P,\tau}(x) := I_\alpha(\omega_S(g_\tau)\varphi_P(x) = P(x)e^{\pi i \tau S(x)}.$$

We have (at least for integral α) the relation

$$\omega_S(g)\varphi_{P,\tau} = (c\tau + d)^{-\alpha}\varphi_{P,g(\tau)};\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and applying the theta distribution we get for the positive quadratic form S the theta function with harmonic coefficient P

$$\theta_P(\tau) := \sum_{x \in \mathbf{Z}^p} P(x) e^{\pi i \tau S(x)}$$

for which one may expect some behavior of a modular form of weight α.

4.6.3 As we did in 4.4, here we also can construct a Jacobi theta function living on the Jacobi group adapted to this case

$$G^J_n := \mathrm{SL}(2, \mathbf{R}) \ltimes \mathrm{Heis}(\mathbf{R}^n).$$

Analogously extending the previous notation, we get

$$\omega_S((p, q, \kappa)g_\tau r(\vartheta))\varphi_P(x) = e^{\pi i(\kappa + 2({}^t x + {}^t p)q + \tau S(x+p))} P(x + p) v^{\alpha/2} e^{i\alpha}$$

and hence

$$\omega_S((p, q, \kappa)g_\tau r(\vartheta))\varphi_P(0) = e^{\pi i(\kappa + 2{}^t pq + \tau S(p))} P(p) v^{\alpha/2} e^{i\alpha} =: \Phi_P(g^J_n)$$

a lowest weight vector living on G^J_n which as in 4.4.5 may be used to introduce

$$\begin{aligned} \Theta_P(g^J_n) &:= \textstyle\sum_{\ell \in \mathbf{Z}^n} \Phi_P((\ell, 0, 0)g^J_n) \\ &= e^{\pi i(\kappa + {}^t p(\tau S p + q))} v^{\alpha/2} e^{i\alpha\vartheta} \textstyle\sum_{\ell \in \mathbf{Z}^p} P(p + \ell) e^{\pi i(\tau S(\ell) + 2{}^t \ell(\tau S p + q)} \end{aligned}$$

For $P = 1$ with $\hat{\tau}_S := \tau S$, $\hat{z}_S := \tau Sp + q$ we see that this Θ_1 is up to an automorphic lifting factor the special value (already coming up in 1.3.7) of the $\dim n-$Jacobi theta function

$$\theta(\hat{\tau}, \hat{z}) = \sum_{\ell \in \mathbf{Z}^n} e^{\pi i({}^t \ell \hat{\tau} \ell + 2{}^t \ell \hat{z})},$$

which is discussed in [MuIII] Section 8. It is interesting to analyze what happens if P is a polynomial of higher degree. For this here we refer to [MuIII] Section 9 treating Jacobi theta functions in spherical harmonics.

4.6.4 We go back to 4.6.2 and for later use we reproduce from [LV] p.222 the introduction of an operator $\bar{\mathcal{F}}_m$ which intertwines the representation $\tilde{\omega}_S$ of $\mathrm{O}(S) \times \mathcal{G}$ on $L^2(\delta_m)$ with the representation $\bar{\pi}_\alpha \otimes \delta_m$ on the space $\bar{\mathcal{O}}(\mathfrak{H}) \otimes \mathcal{H}^m \simeq \bar{\mathcal{O}}(\mathfrak{H}, \mathcal{H}^m)$. This $\bar{\mathcal{F}}_m$ is defined by

$$< \bar{\mathcal{F}}_m \psi(\tau), Q >:= \int_V e^{-\pi i \bar{\tau} S(\xi)} \psi(\xi) Q(\xi) d\xi \quad \text{for all } \psi \in L^2(\delta_m),\ Q \in \mathcal{H}^m.$$

$\bar{\mathcal{F}}_m$ is injective and has the property

$$\bar{\mathcal{F}}_m \varphi_{P,\tau}(w) = c(\bar{w} - \tau)^{-\alpha} \otimes P.$$

4.7 Decomposition of the Weil Representation Associated to an Indefinite Quadratic Form

We take $G = \mathrm{O}(V) \simeq \mathrm{O}(p,q), G' = \mathrm{SL}(2, \mathbf{R})$ and the covering $\mathcal{G}$. Adapting the notation to the one used here, we reproduce from [LV] 2.5.26

4.7.1 Theorem: The discrete spectrum of the representation $\tilde{\omega}_S$ of $G \times \mathcal{G}$ is given as follows

A) For $p > 1, q > 1$

$$(\tilde{\omega}_S)_d = \oplus_{\alpha > 1} (\hat{\delta}_\alpha \otimes \bar{\pi}_\alpha) \oplus \oplus_{\beta > 1} (\hat{\delta}_\beta \otimes \pi_\beta)$$

where $\alpha, \beta \in \mathbf{Z}$ if $(p-q)/2 \in \mathbf{Z}$ and $\alpha, \beta \in (1/2)\mathbf{Z}$ if $(p-q)/2 \in (1/2)\mathbf{Z}$. The representation $\hat{\delta}_\alpha$(resp. $\hat{\delta}_\beta$) is a irreducible representation of $\mathrm{O}(p,q)$. Its restriction to $\mathrm{O}(p) \times \mathrm{O}(q)$ is

$$\hat{\delta}_\alpha = \oplus_{k,m} \delta_k \otimes \delta_m,\ k - m + (p-q)/2 = \alpha + 2j,\ j \geq 0$$

resp.

$$\hat{\delta}_\beta = \oplus_{k,m} \delta_k \otimes \delta_m,\ m - k + (q-p)/2 = \beta + 2j,\ j \geq 0.$$

B) For $p > 1, q = 1$

$$(\tilde{\omega}_S)_d = \oplus_{\alpha > 1} (\hat{\delta}_\alpha \otimes \bar{\pi}_\alpha)$$

with

$$\hat{\delta}_\alpha = \oplus_{k,m=0,1} \delta_k \otimes \delta_m,\ k - m + (p-1)/2 = \alpha + 2j,\ j \geq 0.$$

C) For $p = q = 1$

$$(\tilde{\omega}_S)_d = 0.$$

We will discuss this theorem and show a way to get lowest weight vectors for these representations . The first thing to remark is that there is a kind of precursor to this theorem going back to Gutkin and Repka treating the decomposition of the tensor product of two discrete series representations of $\mathrm{SL}(2, \mathbf{R})$.

4.7.2 Remark: The product $\bar{\pi}_\alpha \otimes \pi_\beta, \alpha \geq \beta$ contains discretely the sum

$$\oplus_{\alpha - \beta - 2j > 1, j \in \mathbf{Z}}\ \bar{\pi}_{\alpha - \beta - 2j}.$$

4.7.3 Next, we write according to the orthogonal decomposition $V = V_1 \oplus V_2$, $V_1 \simeq \mathbf{R}^p, V_2 \simeq \mathbf{R}^q$

$$L^2(V) = L^2(V_1) \otimes L^2(V_2).$$

Then from 4.7.1 we know that $\tilde{\omega}_S$ as a representation of $\tilde{\mathrm{SL}}(2,\mathbf{R}) \times \mathrm{O}(p) \times \mathrm{O}(q)$ is isomorphic to

$$\oplus_{k,m}\ \bar{\pi}_{(p/2)+k} \otimes \pi_{(q/2)+m} \otimes \delta_k \otimes \delta_m.$$

We put

$$d := k + p/2 - (m + q/2)$$

and will see that, when $d > 1$, the representation $\bar{\pi}_{k+p/2} \otimes \pi_{m+q/2}$ contains $\bar{\pi}_d$ by analyzing the shape of a lowest weight vector v_0 of ω_S :

4.7.4 From 3.3.2 we recall the formulae for $\mathfrak{sl}(2,\mathbf{R})_c$

$$\begin{aligned} \hat{Z} &= \pi(x,x) - (1/(4\pi))\Delta \\ \hat{X}_\pm &= (1/2)(E + n/2 \mp (\pi(x,x) + (1/(4\pi))\Delta)) \end{aligned}$$

where

$$(x,x) = S(x) := \sum_{\alpha=1}^{p} x_\alpha^2 - \sum_{\mu=p+1}^{p+q} x_\mu^2,\ \Delta = \sum_{\alpha=1}^{p} \partial_{x_\alpha}^2 - \sum_{\mu=p+1}^{p+q} \partial_{x_\mu}^2,\ E = \sum_{j=1}^{p+q} x_j \partial_{x_j}.$$

and from 3.3.3 for $\mathfrak{o}(p,q)$ with $1 \le \alpha, \beta \le p,\ p+1 \le \mu, \nu \le p+q$

$$\hat{Y}_{\alpha\beta} = x_\alpha \partial_{x_\beta} - x_\beta \partial_{x_\alpha},\ \hat{Y}_{\mu\nu} = x_\mu \partial_{x_\nu} - x_\nu \partial_{x_\mu},\ \hat{Y}_{\alpha\mu} = x_\alpha \partial_{x_\mu} + x_\mu \partial_{x_\alpha}.$$

Our $\tilde{\omega}_S$ on $\mathcal{H} = L^2(\mathbf{R}^n)$ is simultaneously a representation of $G = \mathrm{O}(p,q)$ and $\tilde{G}' = \mathrm{Mp}(2,\mathbf{R})$. Hence it is natural to look at subspaces of functions which are invariant under both groups and we take

$$\mathcal{H}_+ := \{\varphi \in \mathcal{H}; \varphi|_{\mathcal{X}^-} = 0\}$$

We recall the remarks 3.3.5 and 3.4.1 and look at functions

$$\varphi = \psi\varphi_1,\ \varphi_1(x) = e^{-\pi S(x)},\ \psi \in \mathcal{C}^\infty(\mathbf{R}^n)$$

for x with $S(x) \ge 0$ and $\varphi = 0$ for $S(x) < 0$. By a small calculation we get

4.7.5 Remark: One has

$$\hat{Z}\varphi = (-(1/4\pi))\Delta\psi + E\psi + (n/2)\psi)\varphi_1,\ \hat{X}_-\varphi = (1/(8\pi))\Delta\psi\varphi_1.$$

We see that $\varphi = \psi e^{-\pi S(x)}$ is a lowest weight vector of weight λ if one has

$$\Delta\psi = 0,\ E\psi + (n/2)\psi = \lambda\psi,\ \varphi = \psi e^{-\pi S(x)} \in \mathcal{H}.$$

Hence, we have to look for ψ fulfilling these conditions. If ψ is a homogeneous polynomial of degree m which is annihilated by the Laplacian Δ belonging to the indefinite form S, this could lead to the weight $m + n/2$. But that's not the true story. A refined discussion of the solution of these equations and the representations showing up in the decomposition is done by Rallis and Schiffmann in [RS3] with a summary of their results given in [RS1] or [RS2]. As already done above, here we follow the version given by Vergne in [LV] p.225f.

4.7.6 Remark: By some easy calculation one has with Δ, E, and S as in 4.7.5

$$\Delta(S^\alpha f) = S^\alpha \Delta f + 4\alpha S^{\alpha-1}(E + (n/2) + \alpha - 1)f,\ f \in \mathcal{C}^\infty(\mathbf{R}^n)$$

and with

P_1 a harmonic polynomial of degree k in $x_1, \dots, x_p$,
$S_1 = \sum_{\alpha=1}^{p} x_\alpha^2$,
P_2 a harmonic polynomial of degree m in $x_{p+1}, \dots, x_{p+q}$,
$S_2 = \sum_{\mu=1+p}^{p+q} x_\mu^2$,

and

$\psi := P_1 P_2 S_1^\alpha S_2^\beta S^\gamma$

for $S_1(x) \neq 0$ and $S_2(x) \neq 0$ one has

$$\Delta\psi = 0$$

if

1) $(p+q)/2 + \gamma - 1 + 2\alpha + 2\beta + k + m = 0$ and
2) $\alpha(p/2 + \alpha - 1 + k) = 0$ and
3) $\beta(q/2 + \beta - 1 + m) = 0$.

If we put $\beta = 0$ we already get the first statement in the following theorem.

4.7.7 Theorem: For

$$\begin{array}{rcllll} \psi_{P_1,P_2} & := & P_1 P_2 S_1^{k+(p-2)/2} S^{(p-q)/2+k-m-1} & \text{for all } x & \text{with } S(x) > 0 \\ & = & 0 & \text{for all } x & \text{with } S(x) < 0 \end{array}$$

and the *Rallis-Schiffmann function*

$$\varphi_{P_1,P_2} := \psi_{P_1,P_2} e^{-\pi S(x)}$$

one has with $d := k - m + (p-q)/2$

1) $\Delta\psi_{P_1,P_2} = 0,\ (E + n/2)\psi_{P_1,P_2} = d \cdot \psi_{P_1,P_2}$
2) $\varphi_{P_1,P_2} \in L^2(\mathbf{R}^n)$ if $d > 1$,
3) $\varphi_{P_1,P_2} \in L^1(\mathbf{R}^n)$ if $k - m > q$,
4) If $p+q > 2$ and $k-m > q$, then φ_{P_1,P_2} is in $L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ and is continuous.

As already said, item 1) is clear from our previous remarks. For the other statements we refer to [LV] p.228/9.

4.7.8 Corollary: Hence the Rallis-Schiffmann function is a lowest weight vector of weight d for the representation of $\tilde{G}'$ and of type k and m with respect to the action of $\mathrm{O}(p) \times \mathrm{O}(q)$.

Vergne observes that moreover for the indefinite case one has an analogy to the definite formulae in 4.6.2. Using the operator $\mathcal{F}_m$ introduced in 4.6.4 one sees that the operator $\bar{\mathcal{F}}_k \otimes \mathcal{F}_m$ intertwines the representation $\tilde{\omega}_S$ of $\tilde{G}' \times \mathrm{O}(p) \times \mathrm{O}(q)$ restricted to $L^2(\delta_k \otimes \delta_m)$ with the representation

$$\bar{\pi}_{(p/2)+k} \otimes \pi_{(q/2)+m} \otimes \delta_k \otimes \delta_m.$$

The representation $\bar{\pi}_{(p/2)+k} \otimes \pi_{(q/2)+m}$ acts on the space of functions $F(\bar{\tau}_1, \tau_2)$ antiholomorphic in τ_1 and holomorphic in τ_2 by

$$((\bar{\pi}_{(p/2)+k} \otimes \pi_{(q/2)+m})(g^{-1})F)(\bar{\tau}_1, \tau_2) = (c\bar{\tau}_1 + d)^{-((p/2)+k)} (c\tau + d)^{-((q/2)+m)} F(g(\bar{\tau}_1), g(\tau_2)).$$

A function antiholomorphic in one and holomorphic in the other variable is is entirely determined by its restriction to the diagonal $(\bar{\tau}, \tau)$. Hence it is natural to consider the representation $\bar{\pi}_{\alpha,\beta}$ of $\tilde{\mathrm{SL}}(2, \mathbf{R})$ acting on all functions on $\mathfrak{H}$ by

$$(\bar{\pi}_{\alpha,\beta}(g^{-1})f)(w) := (c\bar{\tau}_1 + d)^{-\alpha)}(c\tau + d)^{-\beta} f(\tau(g)),\ w \in \mathfrak{H}.$$

The operator

$$\varphi \longmapsto ((\bar{\mathcal{F}}_k \otimes \mathcal{F}_m)\varphi)(\bar{\tau}, \tau)$$

intertwines the representation $\tilde{\omega}_S|_{L^2(\delta_k \otimes \delta_m)}$ with $\bar{\pi}_{(p/2)+k,(q/2)+m} \otimes \delta_k \otimes \delta_m$. We still denote this operator as $(\bar{\mathcal{F}}_k \otimes \mathcal{F}_m)$ and introduce another operator by

$$f(w) \longmapsto (Mf)(w) := (\operatorname{Im} w)^{-((q/2)+m)} f(w),$$

which intertwines the representation $\bar{\pi}_{d,0},\ d = ((p/2) + k - ((q/2) + m)$ acting on the functions on $\mathfrak{H}$ with the representation $\bar{\pi}_{(p/2)+k,q/2)+m}$. The representation $\bar{\pi}_d$ is naturally contained in $\bar{\pi}_{d,0}$, thus in $\bar{\pi}_{(p/2)+k,(q/2)+m}$. From 4.2.2 we deduce that the function on $\mathfrak{H}$ given by

$$\psi'_\tau(w) = (\operatorname{Im} w)^{-((q/2)+m)}(\bar{w} - \tau)^{-d}$$

verifies the relation (excuse the double meaning of the letter d)

$$\bar{\pi}_{(p/2)+k,(q/2)+m}(g)\psi'_\tau = (c\tau + d)^{-d}\psi'_{g(\tau)}.$$

As to be seen by some calculation as in [LV] p.234 the variant of the Rallis-Schiffmann function from 4.7.7

$$\varphi_{P_1 P_2,\tau}(x) := P_1 P_2 S_1^{-(k+(p-2)/2)} S^{d-1} e^{\pi i \tau S(x)},\ \tau \in \mathfrak{H}$$

has the following properties.

4.7.9 Theorem: For $d > 1$ and harmonic polynomials P_1, P_2 of degree k resp. m in p resp. q variables one has

$$\varphi_{P_1 P_2,\tau}(x) \in L^2(\delta_k \otimes \delta_m)$$

and

$$((\bar{\mathcal{F}}_k \otimes \mathcal{F}_m)\varphi_{P_1 P_2 \tau}(x) = \psi'_\tau \otimes P_1 \otimes P_2.$$

Hence $\varphi_{P_1 P_2,\tau}(x)$ fulfills the fundamental formula

$$\omega_S(g)\varphi_{P_1 P_2,\tau}(x) = j(g, \tau)^{-d}\varphi_{P_1 P_2 g(\tau)}(x)$$

which makes it a candidate for the production of a modular form of weight d via a theta distribution.

4.7.10 Remark: We remind that there is a direct way from the lowest weight vector $\varphi_{P_1 P_2}$ to $\varphi_{P_1 P_2 \tau}(x)$. From the formulae of the Weil representation with our matrix g_τ transforming i to $\tau \in \mathfrak{H}$ as already used in 4.3.2 we have in this case

$$\omega(g_\tau)\varphi_{P_1 P_2}(x) = v^d \varphi_{P_1 P_2 g(\tau)}(x).$$

4.7.11 Up to now we treated the Rallis-Schiffmann functions only with respect to their behaviour concerning the group $\tilde{\mathrm{SL}}(2, \mathbf{R})$. But one can proceed similarly concerning the orthogonal group. Here we take as an example the case $p = 2, q = 1$. In 3.3.4 we determined operators for the complexified algebra

$$\hat{H}_0 = -2i(x_1\partial_2 - x_2\partial_1),\ \hat{Y}_\pm = -(x_1 \pm ix_2)\partial_3 - x_3(\partial_1 \pm i\partial_2).$$

For

$$\varphi = \psi\varphi_1,\ \varphi_1(x) = e^{-S(x)}$$

one gets

$$\hat{H}_0\varphi = -2i(x_1\psi_{x_2} - x_2\psi_{x_1})\varphi_1,\ \hat{Y}_\pm\varphi = -(x_1\psi_{x_3} + x_3\psi_{x_1} \mp i(x_2\psi_{x_3} + x_3\psi_{x_2})\varphi_1.$$

For the Rallis-Schiffmann function from Theorem 4.7.7 in this case one has only two choices for the polynomial P_2, namely $P_2(x) = 1$ or $= x$. We take $P_2 = 1$ and get

$$\varphi_{P_1,P_2} := \psi_{P_1,P_2}e^{-\pi S(x)}$$

with

$$\begin{array}{rclll} \psi_{P_1,P_2} & := & P_1S_1^kS^{k-1/2} & \text{for all } x & \text{with } S(x) > 0 \\ & = & 0 & \text{for all } x & \text{with } S(x) < 0 \end{array}$$

As a homogeneous harmonic polynomial P_1 in two variables one can take $P_1(x_1, x_2) := (x_1 \pm ix_2)^k$. Then we have

$$\varphi_\pm^k := \psi_\pm^k\varphi_1,\ \psi_\pm^k := (x_1 \mp ix_2)^{-k}S(x)^{k-1/2}$$

and by the formula above get

$$\hat{H}_0\varphi_\pm^k = \pm 2k\varphi_\pm^k;\ \hat{Y}_-\varphi_+^k = 0,\ \hat{Y}_+\varphi_-^k = 0.$$

Hence we get a refinement of corollary 4.7.8.

4.7.12 Proposition: φ_+^k is simultaneously a lowest weight vector of weight $2k$ for a representation $\hat{\delta}_\alpha, \alpha = k + 1/2$ of $G = \mathrm{O}(2, 1)$ and of weight $k + 1/2$ for the representation $\bar{\pi}_\alpha$ of $\tilde{G}' = \mathrm{Mp}(2, \mathbf{R})$.

In this example we finally have the corner stone for the construction of a theta function living simultaneously on the orthogonal and the metaplectic group and, hence, apt to produce a correspondence between automorphic forms belonging to these groups.

4.7.13 We pursue this a bit reproducing the construction of the theta function leading to the Shimura correspondence from [LV] p.268f.

As orthogonal space we consider the vector space

$$E := \mathrm{Sym}_2(\mathbf{R}) = \{y = \begin{pmatrix} y_1 & y_3 \\ y_3 & y_2 \end{pmatrix}; y_1, y_2, y_3 \in \mathbf{R}\}$$

with the quadratic form $S'(y) = -2\det y$ and the associated bilinear form

$$S'(y, y') = 2y_3y_3' - y_1y_2' - y_2y_1'.$$

The transformation

$$y_1 = x_3 + x_1,\ y_2 = x_3 - x_1,\ y_3 = x_2$$

leads to our usual signature $(2,1)$ situation

$$S'(y) = 2(x_1^2 + x_2^2 - x_3^2).$$

The group $\mathrm{SL}(2,\mathbf{R})$ acts on E by $g \cdot y := gy^tg$. This action leaves $S(y)$ stable and - as well known - leads to a surjective map $\phi : \mathrm{SL}(2,\mathbf{R}) \longrightarrow \mathrm{O}(2,1)^0$ with $\ker\phi = \{\pm 1_2\}$. This map is given by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \phi(g) := \begin{pmatrix} a^2 & b^2 & 2ab \\ c^2 & d^2 & 2cd \\ ac & bd & ad+bc \end{pmatrix}.$$

We use ϕ to transfer the restriction $\tilde{\omega}_S$ of the Weil representation from $\mathrm{O}(2,1) \times \tilde{G}'$ to a representation of $\mathrm{SL}(2,\mathbf{R}) \times \tilde{G}'$ where the representation of the first group, abbreviated by $\tilde{G}$, is given by $\varphi(y) \longmapsto \varphi(\phi(g^{-1})y)$. Our Rallis-Schiffmann function written in the $y-$coordinates looks like

$$\varphi_+(y) = ((1/2)(y_1 - y_2) - iy_3)^{-k} S'(y) e^{-\pi S'(y)}.$$

For $z = x + iy$ and $g = g_z = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix}$ one has

$$\phi(g_z^{-1}) = \begin{pmatrix} y^{-1} & x^2y^{-1} & -2xy^{-1} \\ 0 & y & 0 \\ 0 & -x & 1 \end{pmatrix}$$

and $\phi(g_z^{-1})$ transforms

$$((1/2)(y_1 - y_2) - iy_3 \longmapsto (1/(2y))(y_1 + z^2y_2 - 2zy_3) = -S'(y, Q(z));\ Q(z) = \begin{pmatrix} z^2 & z \\ z & 1 \end{pmatrix}.$$

4.7.14 Hence, using 4.7.10, we get by application of the Weil representation to our Rallis-Schiffmann function expressed in the $y-$coordinates

$$(\omega(g_z \cdot g'_\tau)\varphi_{P_1P_2})(y) = (-2y)^k v^{k+(1/2)/2} S'(y, Q(z)) S'(y)^{k-1/2} e^{\pi i \tau S'(y)}.$$

for $S'(y) > 0$ and by zero for $S'(y) \le 0$. We put

$$\varphi^k(z,\tau)(y) := S'(y, Q(z)) S'(y)^{k-1/2} e^{\pi i \tau S'(y)}$$

and have a function with the fundamental relation

$$\tilde{\omega}(g, g')\varphi^k(z,\tau) = j(g,z)^{-2k} j(g',\tau)^{-(k+1/2)} \varphi^k(g(z), g'(\tau)).$$

By a suitable skillful averaging procedure (as in [LV] p.272f) one comes to a theta function in both variables z, τ : We consider the groups

$$\Gamma_0(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix};\ c \equiv 0 \mod N\}.$$

and

$$\Gamma_0(0, 2N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix};\ b \equiv 0 \mod 2N\}.$$

$\tilde{\Gamma}_0(N)$ denotes the inverse image of $\Gamma_0(N)$ in $\tilde{\mathrm{SL}}(2,\mathbf{R})$.
Let ψ be character mod $4N$ and with a slight abuse of notation also the character

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \psi(d)$$

of $\Gamma_0(4N)$. λ denotes the theta character, i.e. for $\tilde{\gamma} \in \tilde{\Gamma}_0(4)$

$$\lambda(\tilde{\gamma}) := \varepsilon_d(\frac{c}{d}),$$

and u a function on $\mathbf{Z}/N\mathbf{Z}$ satisfying $u(aj) = \psi(a)u(j)$. The one has the central statement.

4.7.15 Theorem(Theorem 2.7.17 in [LV]): For $k > 1$ the function Ω_u given by

$$\Omega_u(z,\tau) := \sum_{y \in \mathbf{Z}^3; S'(y) > 0} u(y_1)\varphi^k(z,\tau)(y)$$

is a holomorphic function of (z,τ), which is

- modular in τ with respect to $\tilde{\Gamma}_0(4N)$, with character $\lambda\psi$, of weight $k + (1/2)$

and

- modular in z with respect to $\Gamma_0(0, 2N)$, with character ψ^{-2}, of weight $2k$.

For a function u on $\mathbf{Z}/N\mathbf{Z}$ the Fourier transform $\hat{u}$ is defined by

$$\hat{u}(m) := \sum_{h \in \mathbf{Z}/N\mathbf{Z}} u(h) e^{-2\pi i m h/N}.$$

If one chooses $u = u_0$ with $\hat{u}_0 = \bar{\psi}$ the Petersson inner product of a certain cusp form of weight $k + (1/2)$ with Ω_{u_0} produces an automorphic form of weight $2k$ and thus establishes a version of the *Shimura correspondence.* This is only an example of much more material which has been obtained in a similar fashion. We refer to the other chapters of [LV] and, for instance to [RS2].

4.7.16 There is the challenge to try to extend parts of this to pairs consisting of an orthogonal group and a Jacobi group or even an euclidean group and a Jacobi group. But this is no longer prehistory and has to appear again later.

The following list contains some items (but not all) which don't belong to Part I and will be needed in the other Parts of this text.

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