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Edge Disjoint Steiner Trees in Graphs without Large Bridges

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Abstract

A set A of vertices of an undirected graph G is called *k-edge-connected in G* if for all pairs of distinct vertices $a, b \in A$ there exist k edge disjoint a, b-paths in G. An *A-tree* is a subtree of G containing A, and an *A-bridge* is a subgraph B of G which is either formed by a single edge with both end vertices in A or formed by the set of edges incident with the vertices of some component of $G - A$.

It is proved that (i) if A is $k \cdot (\ell + 2)$-edge-connected in G and every A-bridge has at most ℓ vertices in $V(G) - A$ or at most $\ell + 2$ vertices in A then there exist k edge disjoint A-trees, and that (ii) if A is k-edge-connected in G and B is an A-bridge such that B is a tree and every vertex in $V(B) - A$ has degree 3 then either A is k-edge-connected in $G - e$ for some $e \in E(B)$ or A is $(k-1)$-edge-connected in $G - E(B)$.

AMS subject classification: 05C70, 05C40.

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1 Introduction

All graphs considered here are supposed to be finite and undirected and may contain loops or multiple edges. For terminology not defined here see [2]. A set A of vertices of a graph G is called *k-edge-connected* in G if for all pairs of distinct vertices $a, b \in A$ there exist k edge disjoint a, b-paths in G. A STEINER *tree with respect to A* or, briefly, an *A-tree* is a subtree of G covering A.

By TUTTE's and NASH-WILLIAMS's base packing theorem for graphs [14, 13] it follows readily that every $2k$-edge-connected graph has a collection of k edge disjoint spanning trees (cf. [2]). It has been conjectured in [6] that there is the following generalization to A-trees (see also [4] and [3]).

Conjecture 1 *[6] Every $2k$-edge-connected set A of vertices in a graph G yields k edge disjoint A-trees.*

Throughout this paper, the empty graph is considered to be a $\emptyset$-tree, and "k edge disjoint A-trees" actually means "a family of k edge disjoint A-trees", so that for $|A| \leq 1$ there exist families of edge disjoint A-trees of *any* required size. It is not difficult to prove that for each ℓ there exists an $f_\ell(k)$ such that every $f_\ell(k)$-edge-connected set A with $|A| \leq \ell$ in some graph G admits a set of k edge disjoint A-trees. The $f_\ell(k)$ derived in [6] is linear in k but exponential in ℓ, whereas from the results in [4] one can obtain a bound which is linear in both ℓ and k, with a good constant. The optimal f_2, which is $f_2(k) = k$, is an immediate consequence of the definitions, whereas determining the optimal f_3, which is $f_3(k) = \lfloor \frac{8k+3}{6} \rfloor$, turned out to be more tedious (see [6] and [4]). In both [6] and [4], conjectures on the optimum value of $f_\ell(k)$ have been made, and from the estimations in [4] it follows that Conjecture 1 is true for $|A| \leq 5$. Similar results hold if $\overline{A} := V(G) - A$ is bounded: Every $(2k + 2\ell)$-edge-connected set A with $|V(G) - A| \leq \ell$ in some graph G admits a set of k edge disjoint A-trees [6].

Recently, Lau proved that every $26k$-edge-connected set A of vertices of some graph G admits k edge disjoint A-trees [7], and a bound of $24k$ is given in his thesis [8, Theorem 3.1.2]. These are the first bounds $f(k)$ which do not involve the size of A. Moreover, Lau's proof yields a polytime approximation algorithm for the Steiner tree packing problem, that is, given G and $A \subseteq V(G)$ with $|A| \geq 2$, find a largest set of edge disjoint A-trees.

If $V(G) - A$ is independent or, equivalently, A is dominating in G, then there is a much better bound, namely $f(k) = 3k$ [3], and if every vertex in $V(G) - A$ has an even degree then $f(k) = 2k$ suffices [6], as conjectured.

First we prove a result which involves more structure of the instance (G, A). An *A-bridge* is a subgraph B of G which is either formed by a single edge with both end vertices in A or formed by the set of edges incident with the vertices of some component of $G - A$. We prove that for all integers $k, \ell \geq 0$ every $k \cdot (\ell + 2)$-edge-connected set A of vertices in a graph G such that every A-bridge has at most ℓ vertices in $V(G) - A$ or at most $\ell + 2$ vertices in A admits a set of k edge disjoint A-trees.

This generalizes the initially mentioned statement on spanning trees in $2k$-edge-connected graphs (set $\ell = 0$) as well as the situation that $V(G) - A$ is independent, where we have to take $\ell = 1$ and obtain the same bound $3k$ as in [3]. (It also equips us with an $f_\ell(k)$ as above, which is of the same order as the bound from [4], but with a larger constant.)

In the second part of the paper we prove that if A is a k-edge-connected set of vertices in G and B is an A-bridge such that B is a tree and every vertex in $V(B) - A$ has degree 3 then either A is k-edge-connected in $G - e$ for some $e \in E(B)$ or A is $(k-1)$-edge-connected in $G - E(B)$. This provides a short cut for the determination of $f_3(k)$ as in [6], and we show how one would need to generalize it in order to obtain an alternative proof of the statement of the penultimate paragraph.

2 Essential edges in cubic bridges

Given two distinct vertices a, b of a graph G, let us denote by $\lambda_G(a,b)$ the maximum number of edge disjoint a,b-paths in G. We extend this to a mapping $\lambda_G : V(G) \times V(G) \to \mathbb{N} \cup \{0, \infty\}$ by setting $\lambda_G(a,a) := +\infty$ and extend the natural order on $\mathbb{N} \cup \{0\}$ to $\mathbb{N} \cup \{0, \infty\}$ by defining $a \leq +\infty$ for all $a \in \mathbb{N} \cup \{0, \infty\}$. An a,b-*cut* is a set of edges in G which intersects the edge set of every a,b-path. For $A \subseteq V(G)$, an A-*cut* is an a,b-cut for some vertices $a \neq b$ from A. A variant of MENGER's theorem states that for $a \neq b$ in $V(G)$, $\lambda_G(a,b)$ equals the minimum cardinality of an a,b-cut (cf. [2]), so that $A \subseteq V(G)$ with $|A| \geq 2$ is k-edge-connected if and only if there is no A-cut of cardinality less than k. We call an edge $e = xy$ of G *essential (for A being k-edge-connected in G)*, if A is k-edge-connected in G and A is not k-edge-connected in $G - e$. This is equivalent to the statement that A is k-edge-connected in G and e is contained in some A-cut S of cardinality k; it is easy to see that in this case each component of $G - S$ which contains one of x, y must intersect A. A is *minimally k-edge-connected in G* if every edge is essential for A being k-edge-connected in G.

We start with a useful observation whose ancestors can be found in [9] and [11]. For a graph G and $X, Y \subseteq V(G)$, let $E_G(X,Y)$ denote the set of edges xy with $x \in X$, $y \in Y$. (If an edge e is denoted by a word xy then its endvertices are assumed to be x and y.)

Lemma 1 *Let A be a k-edge-connected set of vertices in a graph G. Let $y \in V(G) - A$ be a vertex of degree 3, let xy, yz, wy be the three edges incident with y, let S, T be A-cuts of cardinality k containing xy, yz, respectively, let C be the component of $G - S$ containing x, and let D be the component of $G - T$ containing y.*

Then $C \cap A \subset D \cap A$ or A is k-edge-connected in $G - wy$.

Proof. Let $B_1 := C \cap D$, $B_2 := C \cap \overline{D}$, $B_3 := \overline{C} \cap D$, $B_4 := \overline{C} \cap \overline{D}$, $A_i := B_i \cap A$ for $i \in \{1,2,3,4\}$, and $R_{ij} := E_G(B_i, B_j)$ for $i \neq j$ in $\{1,2,3,4\}$. Then $S = R_{13} \cup R_{14} \cup R_{23} \cup R_{24}$ and $T = R_{12} \cup R_{14} \cup R_{32} \cup R_{34}$. Set $Q_i = E_G(B_i, \overline{B_i}) = \bigcup_{j \in \{1,2,3,4\} - \{i\}} R_{ij}$ for $i \in \{1,2,3,4\}$. So $xy, yz \in Q_3$. Observe that $xy \notin T$ and $wy \notin T$, for otherwise $(T - \{yz, xy\}) \cup \{wy\}$ or $(T - \{yz, wy\}) \cup \{xy\}$ would be an A-cut, which contradicts the connectivity condition to A; in particular, $w, x \in D$, so $xy \in Q_1$. Symmetrically, $w, z \in \overline{C}$ (and $yz \in Q_4$), so $w \in B_3$, and the objects are located as depicted in Figure 1.

If $A_2 \neq \emptyset \neq A_3$ then Q_2, Q_3 are A-cuts, so $|Q_2|, |Q_3| \geq k$. From $|Q_2| + |Q_3| \leq$

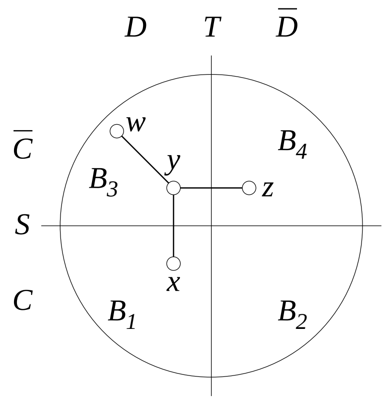

Figure 1: Location of the objects in Lemma 1.

$|R_{21}|+|R_{23}|+|R_{24}| + |R_{31}|+|R_{32}|+|R_{34}| = |S|+|T|-2|R_{14}| \leq 2k$ we deduce $|Q_2| = |Q_3| = k$, implying that $(Q_3 - \{xy, yz\}) \cup \{wy\}$ is an A-cut, again a contradiction.

It follows that one of A_2, A_3 is empty. As S, T are A-cuts, each of $C, \overline{C}, D, \overline{D}$ intersects A, implying $A_1 \neq \emptyset \neq A_4$. Now $|Q_1|, |Q_4| \geq k$, and from $|Q_1|+|Q_4| \leq |R_{12}|+|R_{13}|+|R_{14}| + |R_{41}|+|R_{42}|+|R_{43}| = |S|+|T|-2|R_{23}| \leq 2k$ we deduce $|Q_1| = |Q_4| = k$. Take $a \in A_1$ and $b \in \overline{D} \cap A$. There exists a set $\mathcal{P}$ of k edge disjoint a,b-paths. Since Q_1 is an a,b-cut, $xy \in E(\mathcal{P}) := \bigcup_{P \in \mathcal{P}} E(P)$, and since T is an a,b-cut, $yz \in E(\mathcal{P})$. As y has degree 3, y is contained in precisely one path $P \in \mathcal{P}$, and $xy, yz \in E(P)$, so $wy \notin E(\mathcal{P})$. Taking $a, a' \in A_1, b, b' \in \overline{D} \cap A$ we thus proved that there exist k edge disjoint a,b-paths and k edge disjoint a,b'-paths and k edge disjoint a',b-paths in $G-wy$. Given $R \subseteq E(G-wy)$ with $|R| < k$, at least one of the paths of each system survives in $G-wy-R$, so that R is neither an a,a'-cut nor a b,b'-cut nor an a,b-cut in $G-wy$. It follows that $A - A_3$ is k-edge-connected in $G-wy$.

If $A_3 = \emptyset$ then A is k-edge-connected in $G-wy$, and if $A_3 \neq \emptyset$ then $A_2 = \emptyset$ and $C \cap A = A_1 \subset A_1 \cup A_3 = D \cap A$. $\square$

Lemma 1 inductively extends to paths as follows.

Lemma 2 *Let A be a k-edge-connected set of vertices in a graph G.*

Let $P = v_0, v_1, \ldots, v_\ell$ be a path of length $\ell \geq 2$ such that for $i \in \{1, \ldots, \ell-1\}$, v_i is a vertex in $V(G) - A$ of degree 3 and the three edges incident with v_i are essential for A being k-edge-connected. Let S, T be A-cuts of cardinality k containing $v_0v_1, v_{\ell-1}v_\ell$ respectively, let C be the component of $G-S$ containing v_0 and D be the component of $G-T$ containing $v_{\ell-1}$.

Then $C \cap A \subset D \cap A$.

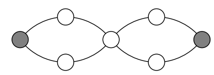

Figure 2: Necessity of the degree condition in Theorem 1.

Proof. For $i \in \{2, \dots, \ell - 1\}$, let T_i be a cut of cardinality k containing $v_{i-1}v_i$. Set $T_1 := S$ and $T_\ell := T$. Let C_i be the component of $G - T_i$ containing v_{i-1}, so $C_1 = C$ and $C_\ell = D$. By Lemma 1, applied to v_{i-1}, v_i, v_{i+1}, T_i, T_{i+1} for x, y, z, S, T we deduce $C_i \cap A \subset C_{i+1} \cap A$. □

Lemma 2 enables us to reduce cycles in "internally 3-regular" A-bridges, yielding the main result of this section.

Theorem 1 *Let A be a k-edge-connected set of vertices in a graph G and let B be an A-bridge such that every vertex $x \in V(B) - A$ has degree 3 in G and the three edges incident with x are essential for A being k-edge-connected.*

Then every A-cut of cardinality k contains at most one edge of B. In particular, B is a tree whose end vertices are the vertices of $V(B) \cap A$.

Proof. Suppose, to the contrary, that S is an A-cut of cardinality k containing $e \neq f$ in $E(B)$. Then we may choose a path $P = v_0, v_1, \dots, v_\ell$ of length $\ell \geq 2$ with $v_i \in V(B) - A$ for all $i \in \{1, \dots, \ell - 1\}$ such that $E(P) \cap S = \{v_0v_1, v_{\ell-1}v_\ell\}$. Let C be the component of $G - S$ containing v_0 and let D be the component of $G - S$ containing $v_{\ell-1}$. By Lemma 2, applied to $T = S$, $C \cap A \subset D \cap A$. However, $D = \overline{C}$ by choice of P, a contradiction.

As every cycle intersects every cut in an even number of edges and every edge of B is contained in some A-cut of cardinality k, B is a tree, and, as no vertex of $V(B) \cap A$ separates the bridge B, the second part of the statement follows. □

The degree condition to $V(B) - A$ in Theorem 1 is necessary, as the graph G in Figure 2 shows. The vertices in A are colored black, and A is minimally 2-edge-connected in G. G itself is the unique A-bridge, and every A-cut must intersect it twice.

3 Graphs without large binary bridges

We proceed with a consequence of the following Theorem from [3]. Basic hypergraph terminology can be found in [1]. A hypergraph is called *k-partition-connected* for some integer $k \geq 0$ if

$$e_G(\mathcal{P}) \quad \geq \quad k \cdot (|\mathcal{P}| - 1) \tag{1}$$

holds for every partition $\mathcal{P}$ of $V(G)$, where $e_G(\mathcal{P})$ denotes the number of edges of G which intersect at least two distinct members of $\mathcal{P}$. Observe that every 1-partition-connected hypergraph is connected.

Theorem 2 *[3] A hypergraph is k-partition-connected if and only if it has k edge disjoint 1-partition-connected spanning subhypergraphs.*

From Theorem 2 we deduce the following.

Theorem 3 *Let $r \geq 2$ and A be an rk-edge-connected set of vertices in some graph G such that $X := V(G) - A$ is independent in G and $d_G(x) \leq r$ for every $x \in X$.*

Then there exist k edge disjoint A-trees in G which are pairwise disjoint on X.

Proof. Without loss of generality we may assume that A is independent in G, for subdividing every edge in $E(G(A))$ once and adding the subdivision vertices to X keeps the conditions to the new instances $G', X', A' = A$ alive, and if G' admits a set of k edge disjoint A-trees pairwise disjoint on X' then we may construct easily a system of k edge disjoint A-trees of G disjoint on X.

The set family $(e_x := N_G(x))_{x \in X}$ constitutes a hypergraph H on A. Let $\mathcal{P}$ be a partition of $V(H) = A$ into at least two classes. Let Y denote the set of vertices in X which have neighbors in at least two members of $\mathcal{P}$, and for $P \in \mathcal{P}$, let $a(P)$ denote the number of edges in G which connect a vertex in P to some vertex in Y. Since A is rk-edge-connected in G, $a(P) \geq rk$ for all $P \in \mathcal{P}$, and since $d_G(x) \leq r$ for all $x \in X$ we deduce $r \cdot e_H(\mathcal{P}) \geq \sum_{x \in Y} d_G(x) = \sum_{P \in \mathcal{P}} a(P)$ $\geq rk|\mathcal{P}|$, so (1) holds.

By Theorem 2, H admits k edge disjoint (1-partition-) connected spanning subhypergraphs $H_1, \dots, H_k$. Let $X_i := \{x \in X : e_x \in E(H_i)\}$. Then the graphs $G(X_i \cup A)$, $i \in \{1, \dots k\}$ are connected subgraphs and pairwise disjoint on X. Choose a spanning tree of each $G(X_i \cup A)$. This produces k edge disjoint A-trees in G disjoint on X. □

The basic reduction technique to prove the following result is to *split* pairs of edges at some vertex in a graph G. A *splitting at x* is a pair $p = (wx, xy)$ of distinct edges. The graph $G(wx, xy) = G(p)$ obtained from $G - \{wx, xy\}$ by adding a single new *bypass edge* from w to y is also said to be obtained from G by *performing* p. A splitting p at x is *admissible* if $\lambda_{G(p)}(a, b) = \lambda_G(a, b)$ for all $a, b \in V(G) - \{x\}$. Mader's Splitting Lemma [10, 12] can be stated as follows.

Lemma 3 *[10, 12] If x is a nonseparating vertex of the graph G of degree distinct from $0, 1, 3$ then there exists an admissible splitting at x.*

Now we are prepared to prove the main result of this section.

Theorem 4 *Let $\ell \geq 0$ and A be a $k \cdot (\ell + 2)$-edge-connected set of vertices in some graph G such that every A-bridge has at most ℓ vertices in $V(G) - A$ or at most $\ell + 2$ vertices in A.*

Then there exist k edge disjoint A-trees.

Proof. We prove this by induction on $|E(G)| + |V(G)|$. If there is an admissible splitting p at some vertex $x \in V(G) - A$ then A is $k \cdot (\ell + 2)$-edge-connected in $G(p)$, and the vertex set of every A-bridge in $G(p)$ is contained in the vertex set of some A-bridge of G; by induction, $G(p)$ has k edge disjoint A-trees, and from these one easily obtains k edge disjoint A-trees in G. Hence there is no such splitting and, by Lemma 3 we may assume that every vertex in $V(G) - A$ either separates G or has degree $0, 1$ or 3. If $x \in V(G) - A$ has degree 0 or 1 then we apply induction to $G - x$ straightforwardly.

Now suppose that $x \in V(G) - A$ separates G and let $\mathcal{C}$ be the set of components of $G - x$. If there is a $C \in \mathcal{C}$ not containing vertices from A then we apply induction to $G - C$ straightforwardly. Otherwise, we take any $C \in \mathcal{C}$ and observe that for any $a \in A \cap C \neq \emptyset$ and any $b \in A - C \neq \emptyset$ there exist $k \cdot (\ell + 2)$ edge disjoint a, b-paths; since each of them contains x, $A' := (A \cap C) \cup \{x\}$ is $k \cdot (\ell + 2)$-edge-connected in $G' := G(C \cup \{x\})$. Since every A'-bridge B' of G' is a subgraph of some A-bridge of G which, moreover, contains at least one vertex from $A - A'$ if B' contains x, we may apply induction to obtain k edge disjoint A'-trees $T_{C,1}, \ldots, T_{C,k}$ in G' — and so $(\bigcup_{C \in \mathcal{C}} T_{C,i})_{i \in \{1,\ldots,k\}}$ is the desired family of edge disjoint A-trees.

Hence every vertex in $V(G) - A$ has degree 3. Furthermore, we may assume that every edge e is essential for A being $k \cdot (\ell + 2)$-edge-connected in G, as otherwise A is $k \cdot (\ell + 2)$-edge-connected in $G - e$ and the statement follows inductively. By Theorem 1, every A-bridge is a tree such that its vertices from A have degree 1 and its vertices from $V(G) - A$ have degree 3. As the number of end vertices of such a tree equals 2 plus the number of its non-end-vertices, the conditions to the A-bridges imply that it has at most $\ell + 2$ end vertices.

Let G' be obtained from G by contracting each component of $G - A$ to a single vertex. A remains $k \cdot (\ell + 2)$-connected in G', $X := V(G') - A$ is independent, and $d_{G'}(x) \leq \ell + 2$ for every $x \in X$. By Theorem 3, there exist k edge disjoint A-trees in G' which are disjoint on X, and from these one easily obtains k edge disjoint A-trees in G. □

For $\ell = 0$, Theorem 4 states that every $2k$-edge-connected graph admits k edge disjoint spanning trees. For $\ell = 1$ we deduce the existence of k edge disjoint A-trees if A is $3k$-edge-connected and $V(G) - A$ is independent, which was a Corollary in [3]. Equivalently, one could say that every $3k$-edge-connected dominating set A admits k edge disjoint A-spanning trees.

Furthermore, if $|A|$ is bounded from above by some ℓ and is $k \cdot \ell$-edge-connected then G admits k edge disjoint A-trees, as every A-bridge contains at most ℓ

vertices from A. This bound has the same order of magnitude than the one in [4], but a larger constant.

In [5] it as been shown that if A is minimally k-edge-connected in G and every vertex in $V(G)-A$ has odd degree then $|V(G)| \leq (k+1)|A|-2k$ [5, Theorem 6] Let me briefly sketch an alternative proof, relying on Theorem 1 and the methods of the preceeding proof. We perform induction on $|E(G)|+2\sum_{b\in A} d_G(b)$. If there occur cut vertices anywhere in G then we can eliminate them similarly as in the proof of Theorem 4. By performing admissible splittings at vertices from $V(G)-A$ we can achieve that every vertex in $V(G)-A$ has degree 3, as these splittings keep A *minimally* k-edge-connected in G. Now every bridge is binary by Theorem 1, which implies that for each $b \in A$, there are $d_G(b)$ edge disjoint $b, A-\{b\}$-paths, each in a distinct A-bridge. If $d_G(b) \geq k+2$ we thus may perform an admissible splitting at b keeping A minimally k-edge-connected. If $d_G(b) = k+1$ then we perform an admissible splitting at b, which keeps $A-\{b\}$ minimally k-edge-connected, and consider the bypass edge h of the splitting. Subdividing h by a new vertex y and adding a new edge from y to b produces a new graph in which A is minimally k-edge-connected. Hence we may transform the instance to a new one with possibly more vertices where every vertex in A has degree k and every A-bridge is binary. Now consider the A-bridges in G, say $B_1, \ldots, B_\ell$. Then $\ell \geq k$, and so $|V(G)-A| = \sum_{i=1}^{\ell} |V(B_i)-A| = \sum_{i=1}^{\ell} |V(B_i) \cap A| - 2\ell \leq k \cdot |A| - 2k$, which implies the statement.

4 Removing a single binary bridge

Let G be a graph and $A \subseteq V(G)$. Let us call an A-bridge B *binary* if B is a tree and the vertices in $V(B)-A$ have degree 3 (those of $V(B) \cap A$ must have degree 1 since B is both an A-bridge and a tree). Hence Theorem 1 implies that if every edge of B is essential for A being k-edge-connected and every vertex in $V(B)-A$ has degree 3 then B is binary. In this section we prove that if every edge of a binary bridge is essential for A being k-edge-connected in G then $G-E(B)$ is $(k-1)$-edge-connected. This is far from being true for arbitrary A-bridges: They might disconnect A although each of its edges is essential, as it is shown by replacing every edge xy of a tree on at least 3 vertices whose end vertices constitute A with k distinct edges connecting x, y (Figure 2 displays the case where the tree is a path of length 2 and $k = 2$).

We prefix the following lemma.

Lemma 4 *Let A be a set of vertices in some graph G, B be an A-bridge, and $x, y \in V(G)-(A \cup V(B))$.*

If A is k-edge-connected in $G-E(B)$ and there exist k edge disjoint x, y-paths in G then there exist k edge disjoint x, y-paths in $G-E(B)$.

Proof. For suppose, to the contrary, that there exists an $\{x, y\}$-cut T in $G - E(B)$ with $|T| < k$, and let C be the component of $(G - E(B)) - T$ containing x. Then $y \in \overline{C}$. As A is k-edge-connected in $G - E(B)$, $A \subseteq C$ or $A \subseteq \overline{C}$, and we may assume by symmetry that $A \subseteq \overline{C}$. However, there exist k edge disjoint x, y-paths in G, and since $x \notin A \cup V(B)$, each of them contains an $x, A \cup \{y\}$-path in $G - E(B)$, which must intersect T — a contradiction. □

Theorem 5 *Let A be a k-edge-connected set of vertices of some graph G and suppose that B is a binary A-bridge such that every edge of B is essential for A being k-edge-connected.*

Then A is $(k-1)$-edge-connected in $G - E(B)$.

Proof. For an edge $xy \in E(B)$, let $\mathcal{T}(x, y)$ denote the set of A-cuts of cardinality k which contain xy. Then $\mathcal{T}(x, y) \neq \emptyset$, since xy is essential. For $T \in \mathcal{T}(xy)$, let $C(x, y, T)$ denote the component of $G - T$ containing x, and let $\mathcal{C}(x, y) := \{C(x, y, T) : T \in \mathcal{T}(x, y)\}$. Furthermore, let $A(x, y)$ denote the set of endvertices in the component $B(x, y)$ of $B - xy$ which contains x. As the intersection of any $T \in \mathcal{T}(x, y)$ with $E(B)$ equals $\{xy\}$ by Theorem 1, $C(x, y, T) \cap V(B) \cap A = A(x, y)$. It is well-known and easy to see that $\mathcal{C}(x, y)$ is closed under intersection, hence there is a unique minimal element in $\mathcal{C}(x, y)$ with respect to $\subseteq$, namely $C(x, y) := \bigcap \mathcal{C}(x, y)$. Let $T(x, y) := E_G(C(x, y), \overline{C(x, y)})$ denote the corresponding A-cut from $\mathcal{T}(x, y)$.

Let $\ell := \lceil (k-1)/2 \rceil$.

Claim 1. For each $xy \in E(B)$, $A(x, y)$ is ℓ-edge-connected in $C(x, y) - E(B)$.

We perform induction on $|A(x, y)|$. The statement is trivially true if $|A(x, y)| = 1$. If $|A(x, y)| > 1$ then $x \in V(B) - A$ and there exist distinct x_1, x_2 in $C(x, y) \cap N_G(x)$. $A(x, y)$ is the disjoint union of the two nonempty sets $A_1 := A(x_1, x)$ and $A_2 := A(x_2, x)$. Since $A_i \subseteq D_i := C(x, y) \cap C(x_i, x)$ and $A(y, x) \subseteq \overline{C(x, y)} \cap \overline{C(x_i, x)}$ we deduce that $E_G(D_i, \overline{D_i})$ is an A-cut of cardinality k (similar to the argument in the proof of Lemma 1). Since $x_i \in D_i$ and $x \in \overline{D_i}$, $D_i \in \mathcal{C}(x_i, x)$ follows — so $D_i = C(x_i, x)$ by minimality of $C(x_i, y)$. Therefore, $C(x_i, x) \cap \overline{C(x, y)} = D_i \cap \overline{C(x, y)} = \emptyset$ for $i \in \{1, 2\}$. By induction, each of A_1, A_2 is ℓ-edge-connected in $C(x, y) - E(B) \supseteq (C(x_1, x) - E(B)) \cup (C(x_2, x) - E(B))$, and it suffices to prove that there exist ℓ edge disjoint A_1, A_2-paths in $C(x, y) - E(B)$.

In G, there exist k edge disjoint A_1, A_2-paths. Since the collection of their edges must cover $T(x_1, x) \cup T(x_2, x)$, $x_1 x x_2$ is a subpath of one of them, and xy is contained in neither of them. It follows that there exists a set $\mathcal{P}$ of $k-1$ A_1, A_2-paths in $G - (E(B(x, y)) \cup \{xy\})$. Since every path in $\mathcal{P}$ which intersects $\overline{C(x, y)}$ must contain two edges in $T(x, y) - \{xy\}$, there are at most $\lfloor (k-1)/2 \rfloor$ such paths. Consequently, $\mathcal{P}$ contains at least $|\mathcal{P}| - \lfloor (k-1)/2 \rfloor = \ell$ A_1, A_2-paths in $C(x, y)$ not intersecting $E(B)$, which proves Claim 1.

Claim 2. For each $xy \in E(B)$, $A(x, y)$ is $(k-1)$-edge-connected in $G - E(B)$.

Again, the statement is trivially true for $|A(x,y)| = 1$. Again, if $|A(x,y)| > 1$ then $x \in V(B) - A$, there exist distinct x_1, x_2 in $C(x,y) \cap N_G(x)$, and $C(x_i, x)$ and $\overline{C(x,y)}$ are disjoint for $i \in \{1,2\}$; by induction, each of $A_1 := A(x_1, x)$ and $A_2 := A(x_2, x)$ is $(k-1)$-edge-connected in $G - E(B)$, and it suffices to prove that there exist $k-1$ edge disjoint A_1, A_2-paths in $G - E(B)$.

As in the proof of Claim 1, we find a set $\mathcal{P}$ of $k-1$ edge disjoint A_1, A_2-paths in $G - (E(B(x,y)) \cup \{xy\})$. We consider these paths as being oriented from A_1 towards A_2.

For $P \in \mathcal{P}$, let $\mathcal{Q}(P)$ be the set of oriented subpaths of P of length at least two whose endvertices are in $C(x,y)$ and whose internal vertices are in $\overline{C(x,y)}$. Let $q := |\{P \in \mathcal{P} : \mathcal{Q}(P) \neq \emptyset\}|$, let $\mathcal{R}(P)$ denote the set of components of $P - E(\mathcal{Q}(P))$, and let $\mathcal{Q} := \bigcup_{P \in \mathcal{P}} \mathcal{Q}(P)$.

Let H be obtained from $G(\overline{C(x,y)})$ by adding two new vertices a, b, taking the union with the paths from $\mathcal{Q}$ and redirecting their initial edges from a towards the respective old second vertices and their terminal edges from the resepective penultimate vertices towards b. By construction, there exist q edge disjoint a,b-paths in H, and since their initial and terminal edges correspond to pairwise distinct edges in $T(x,y) - \{xy\}$, $q \leq \lfloor (k-1)/2 \rfloor \leq \ell$.

Now $B(y,x)$ is an $A(y,x)$-bridge in H, and $A(y,x)$ is q-edge-connected in $H - E(B)$ by Claim 1 applied to yx for xy (since $\overline{C(x,y)} \supseteq C(y,x)$). Applying Lemma 4 to the appropriate objects we find a set $\mathcal{S}$ of q edge disjoint a,b-paths in $H - B(y,x)$, and the set of edges in G corresponding to the edges of the paths in $\mathcal{S}$ together with the edges of the paths in the sets $\mathcal{R}(P)$ form a subgraph of $G - E(B)$ which contains $k-1$ edge disjoint A_1, A_2-paths.

This proves Claim 2.

Claim 3. $A \cap V(B)$ is $(k-1)$-edge-connected in $G - E(B)$.

Take $u \neq v$ in $A \cap V(B)$. There exists an u,v-path P in B, and if it has length less than 2 then $|E(B)| = 1$ and we find $k-1$ edge disjoint u,v-paths in $G - E(B)$ by trivial reasons. If, otherwise, P contains a vertex x in $V(B) - A$ then x has a neighbor $y \in V(B) - V(P)$, and, consequently, $u, v \in A(x,y)$, and Claim 2 yields $k-1$ edge disjoint u,v-paths, proving Claim 3.

For an arbitrary $u \in A - V(B)$, we consider $v \in A \cap V(B)$. As there exist k edge disjoint u,v-paths in G, there exist k edge disjoint $u, A \cap V(B)$-paths in $G - E(B)$. Together with Claim 3 this proves the statement of the Theorem. $\square$

Theorem 5 provides a two pages short cut in the argument of [6] showing that any $\lfloor \frac{8k+3}{6} \rfloor$-edge-connected set $\{a,b,c\}$ of vertices admits k edge disjoint $\{a,b,c\}$-trees (that is, $f_3(k) \leq \lfloor \frac{8k+3}{6} \rfloor$, cf. introduction): Performing induction on k, we first reduce the problem to the case that every $\{a,b,c\}$-bridge of the given graph G is binary, just as in the proof of Theorem 4; if $V(G) = \{a,b,c\}$ then the statement follows by a result on spanning trees from [6], and otherwise there is a further vertex x with edges xa, xb, xc which constitute a binary $\{a,b,c\}$-bridge B and, at the same time, an $\{a,b,c\}$-tree. By Theorem 5, $G-E(B)$ is $(\lfloor \frac{8k+3}{6} \rfloor - 1)$-edge-connected and hence $\lfloor \frac{8(k-1)+3}{6} \rfloor$-edge-connected; so there are $k-1$ edge disjoint $\{a,b,c\}$-trees in G by induction, and they form, together with B, the desired family of k edge disjoint $\{a,b,c\}$-trees of G.

It seems to be a difficult problem to generalize Theorem 5 to the deletion of more than one A-bridge. It could be possible that under the assumptions that A is minimally k-edge-connected in G and that there are only binary A-bridges, A remains $(k-\ell)$-edge-connected in any graph obtained from G by removing the edges of any ℓ A-bridges. For $k=2$ this is true by Theorem 5, but it is not clear why for some 3-edge-connected set A there cannot exist B, B' such that A is disconnected in $G-(E(B) \cup E(B'))$. The case $\ell = k-1$ is particularly interesting, as it would imply that the edge connectivity of A is inherited to the "bridge hypergraph" if all bridges are binary:

Problem 1 *Is it true that if A is a minimally k-edge-connected set of vertices in some graph G and every A-bridge is a binary tree then the hypergraph H on A whose edges are formed by the family of sets of endvertices of A-bridges in G is k-edge-connected?*

An affirmative answer to Problem 1 would yield an alternative proof for Theorem 4: After reduction to the case that A is minimally $k \cdot (\ell+2)$-edge-connected in G and every A-bridge is binary, as in the present proof of Theorem 4, we would know that H as in Problem 1 is $k \cdot (\ell+2)$-edge-connected and every edge of H had at most $\ell+2$ vertices. By another result of [3], H is k-partition-connected and thus admits k (1-partition-) connected spanning subhypergraphs, which easily yield the desired family of k edge disjoint A-trees in G.

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