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**Quickly proving the
ANDRÁSFAI–ERDŐS–SÓS–Theorem.**

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QUICKLY PROVING THE ANDRÁSFAI–ERDŐS–SÓS–THEOREM.

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ABSTRACT. Given an integer $r \geq 2$, an important theorem first proved by B. ANDRÁSFAI, P. ERDŐS, and V. T. SÓS states that any K_{r+1} -free graph on n vertices whose minimum degree is greater than $(3r - 4)n/(3r - 1)$ is r -colourable, and determines the graphs that are extremal in this context. The purpose of this note is to give an alternative proof of this result using a different idea.

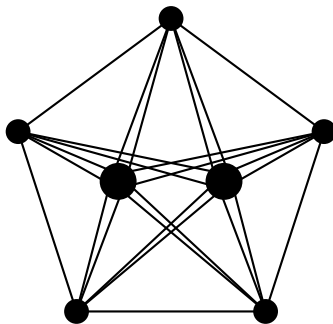
The origin of extremal graph theory, i.e. TURÁN'S Theorem (see [2]), gives us an exact answer to the question how many edges are required for a graph on n vertices so that it necessarily contains a K_{r+1} , a clique on $r + 1$ vertices. As it turns out, that number is approximately $\frac{r-1}{2r} \cdot n^2$ with an error of $\pm O_r(1)$ and one way to show TURÁN'S Theorem proceeds by repeatedly removing vertices of lowest degree. Taking that route one usually realizes that every graph on n vertices the minimum degree of which is greater than $(r - 1)n/r$ contains a K_{r+1} . This prompts the further question as to whether every graph destitute of $(r + 1)$ -cliques and whose minimum degree is just a bit smaller than that needs to be structured in a way that makes the absence of $(r + 1)$ -cliques “easy to see”. In this vein, B. ANDRÁSFAI, P. ERDŐS, and V. T. SÓS proved the following result in 1974:

Theorem 1. (see [1]) *Given an integer $r \geq 2$, each graph on n vertices whose minimum degree is larger than $(3r - 4)n/(3r - 1)$ is either r -colourable or contains a K_{r+1} .*

They also described the corresponding extremal graphs. To introduce those, take any integers $r \geq 2$ and $k \geq 1$, and define a graph $\mathcal{H}_{r,k}$ as follows: its vertex set is the disjoint union of five sets P_1, \dots, P_5 of size k whose indices are considered modulo 5 as well as of $r - 2$ further sets Q_1, \dots, Q_{r-2} of size $3k$. Each of those $r + 3$ sets is required to be independent and each vertex from some Q_i is adjacent to all other vertices except for those coming from the same Q_i . Moreover each vertex from some A_i is adjacent to all vertices from $A_{i\pm 1}$ but to none of the vertices from $A_{i\pm 2}$.

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The stronger version of Theorem 1 mentioned above reads

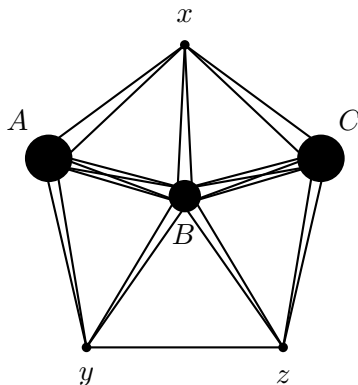
Theorem 2. (also from [1]) *Let $r \geq 2$ be an integer and let \mathcal{G} be a graph on n vertices whose minimum degree is at least $(3r - 4)n/(3r - 1)$. Then either \mathcal{G} is r -colourable, or it contains a K_{r+1} , or n is divisible by $3r - 1$ and \mathcal{G} is isomorphic to $\mathcal{H}_{r,n/(3r-1)}$.*

As far as we know the only published proof of those results appeared in [1] and the purpose of this note is just to give another such proof. For the sake of exposition, we will first phrase the argument so as to yield Theorem 1 only and then we shall briefly elaborate on a few additional steps that may be used to obtain Theorem 2 as well.

Proof of Theorem 1. A.) We argue indirectly and fix for a given $r \geq 2$ a counterexample $\mathcal{G} = (V, E)$ on n vertices that has subject to these conditions the maximal number of edges. Evidently the addition of any further edge to \mathcal{G} would result in the presence of a K_{r+1} . It is plain that \mathcal{G} cannot be a complete multipartite graph and thus it contains three distinct vertices x, y , and z such that $xy, xz \notin E$ but $yz \in E$.

B.) Next we explain what we mean by a *configuration of intensity k* in \mathcal{G} , where $k \in \{0, 1, \dots, r - 1\}$, and why such a thing has to exist: such a configuration is a triple (A, B, C) of disjoint sets of vertices from $\mathcal{G} - \{x, y, z\}$ satisfying the following three properties:

- (K1) A, B , and C are cliques in \mathcal{G} of sizes $r - 1 - k, k$, and $r - 1 - k$, respectively.
- (K2) Each vertex from B is adjacent to every other vertex from $A \cup B \cup C \cup \{x, y, z\}$.
- (K3) Every vertex from A is adjacent to x and y , and every vertex from C has x and z among its neighbours.



To find such a configuration in our graph \mathcal{G} we exploit that its edge maximality entails in view of $xy \notin E$ the existence of a $(r - 1)$ -clique M consisting of common neighbours of x and y . Similarly there is a $(r - 1)$ -clique N whose members are joined to both x and z . Setting now $A = M - N$, $B = M \cap N$, and $C = N - M$ we have found a configuration in \mathcal{G} .

C.) For the remainder of the proof, we fix a configuration $\mathfrak{K} = (A, B, C)$ in \mathcal{G} the intensity k of which is maximum. As $B \cup \{y, z\}$ cannot be a $(r + 1)$ -clique, we must have $k \leq r - 2$ and consequently none of the sets A and C is empty. Let us assign the *weight* 3 to all vertices from B , the *weight* 1 to all vertices from A and C as well as to the vertices x , y , and z , and the *weight* 0 to all other vertices of \mathcal{G} ; finally the *weight* of an edge is defined to be the sum of the weight of its endvertices. The total weight of all vertices amounts to $2r + k + 1$, which is at most $3r - 1$. By our minimum degree condition, the sum of the weights of all edges can thus be estimated to be

$$(1) \quad > (2r + k + 1) \cdot \frac{3r-4}{3r-1} \cdot n \geq (2r + k - 2)n.$$

Therefore there is a vertex $u \in V$ the weights of whose neighbours sum up to at least $2r + k - 1$. To say the same thing more verbosely, this sum is by at most 2 units smaller than its theoretical maximum value, i.e. the sum of the weights assigned to all vertices. In particular, u sends an edge to all members of B . If u were also adjacent to all vertices from A , then neither ux nor uy could be edges of \mathcal{G} for neither $A \cup B \cup \{u, x\}$ nor $A \cup B \cup \{u, y\}$ is an $(r + 1)$ -clique. But this in turn implied that $B \cup C \cup \{u, z\}$ had to be an $(r + 1)$ -clique by our choice of u . This argument reveals that there has to be some $a \in A$ satisfying $ua \notin E$ and for similar reason there exists $c \in C$ with $uc \notin E$. Now by its very choice, the vertex u has to be adjacent to all vertices from $A \cup B \cup C \cup \{x, y, z\}$ except for a and c . Hence

$$(A - \{a\}, B \cup \{u\}, C - \{c\})$$

is a configuration in \mathcal{G} the intensity of which exceeds k . This contradicts our choice of \mathfrak{K} and Theorem 1 is thereby proved. \blacksquare

Proof of Theorem 2. As indicated above, we shall now mention some amendments that can be made to the above proof if one is also interested in demonstrating Theorem 2. So take an integer $r \geq 2$ as well as a graph \mathcal{G} on n vertices that is neither containing a K_{r+1} nor r -colourable and whose minimum degree is at least $(3r - 4)n/(3r - 1)$. If n were not divisible by $3r - 1$, this fraction were not integral and Theorem 1 yielded a contradiction. Thus $\ell = n/(3r - 1)$ is an integer and our aim is to show $\mathcal{G} \approx \mathcal{H}_{r,\ell}$. Since the addition of a further edge edge to $\mathcal{H}_{r,\ell}$ always results in the presence of a K_{r+1} , we may again assume that \mathcal{G} is edge maximal and thus containing three vertices x , y , and z as above. Thus if the concept of a configuration gets introduced as above we may look again at a configuration (A, B, C) of maximum intensity, say $k \in \{0, 1, \dots, r - 2\}$. Set $K = A \cup B \cup C \cup \{x, y, z\}$.

Repeating the above degree estimate (1) we can prepare for the same contradiction as before unless $k = r - 2$, each vertex from K has degree $(3r - 4)\ell$, and each vertex $u \in V$ has the property that the sum of the weights of its neighbours is exactly $2r + k - 2$. So $A = \{p_1\}$ and $C = \{p_3\}$ for some distinct vertices p_1 and p_3 . It is convenient to relabel $x = p_2$, $y = p_5$, and $z = p_4$ and to consider those indices modulo 5. Let $B = \{q_1, \dots, q_{r-2}\}$. The absence of $(r + 1)$ -cliques and the above information regarding the distribution of weights discloses that there are no further edges between the vertices from K except for those postulated by (K1), (K2), and (K3) and that each $u \in V - K$ is adjacent either to all vertices from K except for some q_j or to all vertices from K except for three cyclically not consecutive vertices among p_1, \dots, p_5 . In other words if we denote the neighbourhood of a vertex x by $N(x)$ and set

$$P_i = \{p_i\} \cup \{x \in V - K \mid N(x) \cap K = K - \{p_i, p_{i+2}, p_{i+3}\}\},$$

for $i = 1, \dots, 5$ as well as

$$Q_j = \{q_j\} \cup \{x \in V - K \mid N(x) \cap K = K - \{q_j\}\},$$

for $j = 1, \dots, r - 2$, then $(P_1, \dots, P_5, Q_1, \dots, Q_{r-2})$ forms a partition of V . If $v \in P_i$ for some i , then the adjacencies in $K \cup \{v\} - \{p_i\}$ are analogous to those in K and all arguments we have so far done with K may be carried out for that ‘‘configuration’’ as well. The same remark applies to $K \cup \{w\} - \{q_j\}$ whenever $w \in Q_j$, where $1 \leq j \leq r - 2$. Combining all those arguments we infer that \mathcal{G} is $(3r - 4)\ell$ -regular, that all vertices from Q_j are adjacent to all vertices from $V - Q_j$, and that all vertices from P_i are adjacent to all vertices from $P_{i\pm 1}$. The absence of $(r + 1)$ -cliques reveals that there can be no further edges in \mathcal{G} . Hence we have $|Q_j| = 3\ell$ for all j by regularity and similarly $|P_{i+1}| + |P_{i-1}| = 2\ell$ for all i . It is now plain that the partition $(P_1, \dots, P_5, Q_1, \dots, Q_{r-2})$ witnesses $\mathcal{G} \approx \mathcal{H}_{r,\ell}$. \blacksquare

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