Unifying duality theorems for width parameters in graphs and matroids
I. Weak and strong duality

Reinhard Diestel

Mathematisches Seminar, Universität Hamburg

Sang-il Oum*

KAIST, Daejeon, 305-701 South Korea

June 15, 2014

Abstract

We prove a general duality theorem for width parameters in combinatorial structures such as graphs and matroids. It implies the classical such theorems for path-width, tree-width, branch-width and rank-width, and gives rise to new width parameters with associated duality theorems. The dense substructures witnessing large width are presented in a unified way akin to tangles, as orientations of separation systems satisfying certain consistency axioms.

1 Introduction

There are a number of theorems in the structure theory of sparse graphs that assert a duality between certain 'dense objects' and an overall tree structure. For example, a graph has small tree-width if and only if it contains no large-order bramble. The aim of this paper is to prove one such theorem in a general setting, a theorem that will imply all the classical duality theorems as special cases, but with a unified and simpler proof. Our theory will give rise to new width parameters as well, with dual 'dense objects', and conversely provide dual tree-like structures for notions of dense objects that have been considered before but for which no duality theorems were known.

Amini, Mazoit, Nisse, and Thomassé [1] have also established a theory of dualities of width parameters, which pursues (and achieves) a similar aim. Our theory differs from theirs in two respects: we allow more general separations of a given ground set than just partitions, including ordinary separations of graphs; and our 'dense objects' are modelled after tangles, while theirs are modelled on brambles. Hence while our main results can both be used to deduce those classical duality theorems for width parameters, they differ in substance. And so do their corollaries for the various width parameters, even if they imply the

same classical results. Moreover, while the main results of [1] can easily be deduced from ours (see Section 5.7), the converse seems less clear. And finally, our theory gives rise to duality theorems for new width parameters that can only be expressed in our setup.

All we need in our set-up is that we have a notion of 'separation' for the combinatorial structure to be considered, by which we mean an ordered pair (A, B) of subsets of some ground set V such that $A \cup B = V$.[1] For example, V might be the vertex set of a graph or the ground set of a matroid, and 'separations' would be defined as is usual for graphs and matroids. In order to apply our theorem we may need in addition that there is a submodular function defined on these separations, such as their order, but our main result can be stated without such an assumption.

Our unified treatment of 'dense objects' is gleaned from the notion of tangles in graph minor theory [18], or of ultrafilters in set theory. The idea is as follows. Consider any set S of separations of a given graph or matroid. In order to deserve its name with respect to S, we expect of a 'dense object' that for every separation in S it lies on one side but not the other. For example, if S is the set of all separations (A, B) of a graph G such that $|A \cap B| < k$, then every K_n minor of G with $n \geq k$ will have a branch set in $A \smallsetminus B$ or in $B \smallsetminus A$, but not both. Our dense object $\mathcal{D}$ therefore *orients* every separation in S by choosing exactly one of the two ordered pairs $(A, B), (B, A)$ in such cases,[2] and our paradigm is that this orientation of S is the only information about $\mathcal{D}$ that we ever use. We formalize this by *defining* 'dense objects' as certain orientations of S.

To deserve their name, 'dense objects' cannot be arbitrary orientations of S but have to satisfy some consistency rules. For example, if in a graph G we have two separations $(A, B), (C, D)$ and their inverses in S, and $A \subseteq C$ and $B \supseteq D$, then $\mathcal{D}$ should not orient $\{A, B\}$ towards A by selecting (B, A) and $\{C, D\}$ towards D by selecting (C, D). While this rule will be common to all the 'dense objects' we shall consider, there may be further rules depending on the type of object, so that we can tell them apart. These additional rules will stipulate that the orientation of S given by a dense object $\mathcal{D}$ must not contain certain subsets of S, such as the set $\{(B, A), (C, D)\}$ in the above example. Thus, each type of dense object will be specified by a collection $\mathcal{F}$ of 'forbidden' subsets of S.

The tree-like structure that is dual to a dense object $\mathcal{D}$, i.e., which will exist in a graph or matroid if and only if it contains no instance of $\mathcal{D}$, will be defined by this same collection $\mathcal{F}$ of separation sets forbidden in $\mathcal{D}$. It will typically come as a subset of S that is nested, and which thus cuts up the underlying set in a tree-like way, and the 'stars of separations' by which this tree branches will be *required* to lie in $\mathcal{F}$. Tangles, for example, are defined in this way: with $\mathcal{F}$ the set of all triples $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ of separations whose 'small' sides A_1, A_2, A_3 cover the entire graph or matroid, and branch decompositions, their dual objects, as nested sets of separations branching at precisely such triples.

The following familiar dualities between dense objects and tree structures

---

[1]In fact, we need even less. It would be enough to consider instead of 'separations' any poset with an involution that commutes with its ordering, just as the ordering of separations introduced below satisfies $(A, B) \leq (C, D) \Leftrightarrow (B, A) \geq (D, C)$. It is only for the sake of readability that we are writing this paper in terms of separations, as readers are likely to have graphs or matroids in mind.

[2]Our notational convention will be that we think of (A, B) as pointing towards B.

can be captured in this way, and their duality theorems will follow from our theorem. For graphs, we can capture path-decompositions and blockages [2], tree-decompositions and brambles [19], branch-decompositions and tangles of graphs [18]. For matroids, our framework captures branch-decompositions and tangles [18, 10], as well as matroid tree-decompositions [11] and their dual objects proposed by Amini, Mazoit, Nisse, and Thomassé [1]. Our framework also captures branch-decompositions and tangles of symmetric submodular functions [18, 10], which includes branch-width of graphs and matroids, carving-width of graphs [20], and rank-width of graphs [16].

Since blockages and brambles are not defined in terms of orientations of sets of separations, the duality theorems we obtain when we specify S and $\mathcal{F}$ to capture path- or tree-width (of graphs or matroids) will differ from their known duality theorems. But they will be easily interderivable with these. Since S and $\mathcal{F}$ can be chosen in many other ways too, our results also imply dualities for new width parameters.

Our unifying duality theorem will come in three flavours: as *weak*, *strong*, and *general* duality. Our *weak duality theorem*, presented in Section 3, will be easy to prove but have no direct applications. It will be used as a stepping stone for both the *strong duality theorem* and the *general duality theorem*, our two main results. The strong duality theorem, presented in Section 4, will imply all the classical results mentioned earlier. The general duality theorem, presented in Part II of this paper [9], will be applicable to a much wider set of notions of dense objects, including profiles [13, 4, 5] and k-blocks [7, 6]. But the tree-like structure witnessing their exclusion will be less specific than in the strong theorem, so these two theorems are independent.

2 Terminology and basic facts

A *separation* of a set V is a pair (A, B) of subsets such that $A \cup B = V$. Its *inverse* is the separation (B, A). A set S of separations is *symmetric* if $(B, A) \in S$ whenever $(A, B) \in S$, and *antisymmetric* if $(B, A) \notin S$ whenever $(A, B) \in S$. A symmetric set of separations of a set V is a *separation system* on V.

The separation (A, B) is *proper* if $A, B \neq V$, and *improper* otherwise. The separations of V are partially ordered by

$$(A, B) \leq (C, D) :\Leftrightarrow A \subseteq C \text{ and } B \supseteq D.$$

Note that this is equivalent to $(D, C) \leq (B, A)$, and that (A, B) is proper if and only if (A, B) and (B, A) are incomparable with respect to $\leq$.

Informally, we think of (A, B) as *pointing towards B* and *away from A*. Similarly, if $(A, B) \leq (C, D)$, then (A, B) *points towards* (C, D) and (D, C), while (C, D) *points away from* (A, B) and (B, A).

A set S of separations of V is *nested* if each of them is comparable with every other or its inverse. Thus, two nested separations are either comparable, or point towards each other, or point away from each other. Two separations that are not nested are said to *cross*.

A set of separations is a *star* if they point towards each other (Fig. 1). Thus, S is a *star* if $(A, B) \leq (D, C)$ for distinct $(A, B), (C, D) \in S$. In particular, stars are nested. They need not be antisymmetric, but if not they contain an inverse pair $(A, B), (B, A)$, then any other separation they contain must be improper.
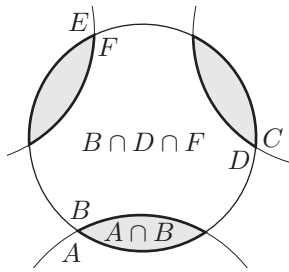
Figure 1: The separations $(A, B), (C, D), (E, F)$ form a 3-star

Let $\mathcal{F} \subseteq 2^S$ be a collection of sets of separations in S, and $S^- \subseteq S$. An *S-tree over* $\mathcal{F}$ and *rooted in* S^- is a pair (T, α) of a tree T with at least one edge and a function $\alpha \colon \vec{E}(T) \to S$ from the set

$$\vec{E}(T) := \{(s, t) : \{s, t\} \in E(T)\}$$

of all *orientations* of edges of T satisfying the following:

(i) For each edge xy of T, if $\alpha(x, y) = (A, B)$ then $\alpha(y, x) = (B, A)$.

(ii) For each internal node t of T, the set $\{\alpha(s, t) : st \in E(T)\}$ is in $\mathcal{F}$.

(iii) For each leaf s of T with neighbour t, say, $\alpha(s, t) \in S^-$.

We say that the separation $\alpha(s, t)$ in (iii) is *associated with*, or simply *at*, the leaf s. The separations at leaves are the *leaf separations* of (T, α).
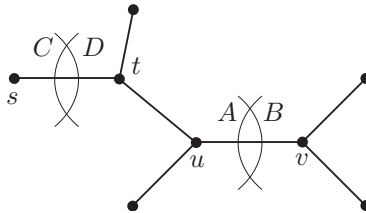

Figure 2: An S-tree with $(C, D) = \alpha(s, t) \leq \alpha(u, v) = (A, B)$

An important example are the S-trees *over stars*: the S-trees over some $\mathcal{F}$ all whose elements are stars of separations. In such an S-tree (T, α) the map α preserves the *natural partial ordering* on $\vec{E}(T)$ defined by letting $(s, t) \leq (u, v)$ if the unique $\{s, t\}$–$\{u, v\}$ path in T starts at t and ends at u. Indeed, the images under α of the oriented stars

$$S_t = \{(s, t) : t \text{ an internal node of } T\},$$

which by (ii) are sets in $\mathcal{F}$, are then stars of separations. This means precisely that α preserves the partial ordering on the sets S_t as induced by $\vec{E}(T)$, which in turn easily implies that α preserves the ordering on all of $\vec{E}(T)$.

Proposition 2.1. *If (T, α) is an S-tree over stars and $\alpha(s, t) = (A, B) = \alpha(s', t')$ for distinct leaves $s, s' \in T$, then (A, B) is improper with $B = V$.*

4

Proof. As s and s' are leaves, we have $(s,t) \leq (t',s')$ in the partial ordering of $\vec{E}(T)$. Hence $(A,B) = \alpha(s,t) \leq \alpha(t',s') = (B,A)$, and thus $A \subseteq B$. As $A \cup B = V$ by definition of separation, this implies $B = V$. □

Recall that stars need not, by definition, be antisymmetric: a star may contain both (A,B) and its inverse (B,A). While it is important for our proofs to allow this, we can always contract an S-tree (T, α) over a collection $\mathcal{F}$ of stars to an S-tree (T', α') over the subcollection $\mathcal{F}' \subseteq \mathcal{F}$ of all antisymmetric stars in $\mathcal{F}$ whose leaf separations are among those for (T, α). Indeed, if T has a node t such that $\alpha(S_t)$ is not antisymmetric, then t has neighbours s_1, s_2 such that $\alpha(s_1, t) = (A,B) = \alpha(t, s_2)$ for some $(A,B) \in S$. Let T' be obtained from T by contracting both the edge ts_2 and the component of $T - s_1t - ts_2$ containing t, and let $\alpha' := \alpha \restriction \vec{E}(T')$. Then (T', α') is again an S-tree over $\mathcal{F}$. Since we can do this whenever some S_t maps to a star of separations that is not antisymmetric, but only finitely often, we must arrive at an S-tree over $\mathcal{F}'$.

Let S be a separation system. An *orientation* of S is a subset $O \subseteq S$ that contains, for every $(A,B) \in S$, exactly one of (A,B) and (B,A). A *partial orientation* of S is an orientation of some symmetric subset of S.

A set P of separations is *consistent* if it contains no two separations pointing away from each other: if $(C,D) \leq (A,B) \in P$ implies $(D,C) \notin P$.[3] Note that this does not imply $(C,D) \in P$: it may also happen that P contains neither (C,D) nor (D,C). Note that consistent sets of separations are antisymmetric.

If $P \subseteq S$ is consistent, it is clearly a partial orientation of S. Conversely, if P is an orientation of all of S, it is consistent if and only if it is *closed down* in the partial ordering of S, i.e., if and only if $(C,D) \in P$ whenever $(C,D) \leq (A,B) \in P$ and $(C,D) \in S$.

Whenever $P \subseteq O \subseteq S$ we say that P *extends to* O, and O *extends* P.

Proposition 2.2. *Every consistent partial orientation of a separation system S extends to a consistent orientation of S.*

Proof. Let P be a consistent orientation of a symmetric subset S' of S. We apply induction on $|S \setminus S'|$. For the induction step pick a separation $(A,B) \in S \setminus S'$. Suppose that neither $P \cup \{(A,B)\}$ nor $P \cup \{(B,A)\}$ is consistent. Then there exist $(C_1, D_1), (C_2, D_2) \in P$ such that $(B,A) \leq (C_1, D_1)$ and $(A,B) \leq (C_2, D_2)$. But then $(D_1, C_1) \leq (A,B) \leq (C_2, D_2)$, contradicting our assumption that P is consistent. □

3 Weak duality

Our paradigm in this paper is to capture the notion of a 'dense object' $\mathcal{D}$ in a structure on a set V by orientations of suitable separation systems S on V. Here, 'suitable' means that for every separation in S the object $\mathcal{D}$ should 'lie on' one of its sides but not the other, and S should ideally contain all separations of V for which this is the case.

---

[3]It is a good idea to work with this formal definition of consistency, since the more intuitive notion of 'pointing away from each other' can be counterintuitive. For example, we shall need that no consistent set of separations of V contains a separation of the form (V, A); this follows readily from the formal definition, as $(A,V) \leq (V,A)$, but is less obvious from the informal.

If $\mathcal{D}$ was a concrete subset X of V, for example, such as a set spanning a large complete subgraph in a graph, there would then be a unique orientation O of S that describes $\mathcal{D}$: the set $\{(A, B) \in S : X \subseteq B\}$. What makes the orientations paradigm so attractive, however, is that it is more general than this. For example, a large grid H in a graph G defines a high-order tangle $\mathcal{T}$ – for every small-order separation of G, most of H will lie on one side but not the other – yet the intersection of the 'large sides' B of all the oriented separations $(A, B) \in \mathcal{T}$ will be empty. What the existence of a large grid H in G does imply, however, is that G has no three low-order separations (A_i, B_i) $(i = 1, 2, 3)$ such that $H \subseteq G[A_1] \cup G[A_2] \cup G[A_3]$. So Robertson and Seymour [18] chose this latter property as the defining axiom for a *tangle*.

In this spirit, we seek to define our 'dense objects' as orientations of separation systems S that do not contain certain subsets of S. We say that a partial orientation P of a separation system S *avoids* $\mathcal{F} \subseteq 2^S$ if P has no subset in $\mathcal{F}$, i.e., if $2^P \cap \mathcal{F} = \emptyset$.

Theorem 3.1 (Weak Duality Theorem). *Let S be a separation system of a set V, and let $S^- \subseteq S$ contain every separation of the form $(A, V) \in S$. Let $\mathcal{F}$ be a set of stars in S. Then exactly one of the following holds:*

(i) *There exists an S-tree over $\mathcal{F}$ rooted in S^-.*

(ii) *There exists an $\mathcal{F}$-avoiding orientation of S extending S^-.*

For the easy direction that (i) and (ii) cannot both hold, we do not need the assumption that $\mathcal{F}$ consists of stars, nor that S^- contains all the separations (A, V):

Lemma 3.2. *Let S be a separation system, $S^- \subseteq S$ a partial orientation of S, and $\mathcal{F} \subseteq 2^S$. If there exists an S-tree over $\mathcal{F}$ rooted in S^-, then no orientation of S extending S^- avoids $\mathcal{F}$.*

Proof. Let (T, α) be an S-tree over $\mathcal{F}$ rooted in S^-, and let O be an orientation of S that extends S^-. Then α maps for every edge of T exactly one of its orientations to a separation in O. Let t be a sink in the resulting orientation of T; then $S_t = \{\alpha(s, t) : st \in E(T)\} \subseteq O$. Since leaf separations lie in $S^- \subseteq O$, their inverses do not lie in O, so t is not a leaf. Hence $S_t \in \mathcal{F}$, so O does not avoid $\mathcal{F}$. $\qquad\square$

Proof of Theorem 3.1. By Lemma 3.2, at most one of (i) and (ii) holds. We now show that at least one of them holds. If S^- contains a separation (X, Y) together with its inverse (Y, X), then (T, α) with $T = K_2$ and $\operatorname{im} \alpha = \{(X, Y), (Y, X)\}$ is an S-tree satisfying (i). So we may assume that S^- is a partial orientation of S. We apply induction on $|S| - 2 |S^-|$ to show that (i) or (ii) holds.

If $|S| = 2 |S^-|$, then S^- itself is an orientation of S extending S^-. If (ii) fails, S^- has a subset $\{(A_1, B_1), \ldots, (A_n, B_n)\} \in \mathcal{F}$. Let T be an n-star with centre t and leaves $s_1, \ldots, s_n$. Let $\alpha(s_i, t) = (A_i, B_i)$ and $\alpha(t, s_i) = (B_i, A_i)$, for $i = 1, \ldots, n$. Then (T, α) satisfies (i).

Thus we may assume that S has a separation (X, Y) such that neither (X, Y) nor (Y, X) is in S^-. Our assumption about S^- implies that (X, Y) is a proper separation. Let $S_X^- = S^- \cup \{(Y, X)\}$ and $S_Y^- = S^- \cup \{(X, Y)\}$. Since any orientation of S extending S_X^- or S_Y^- also extends S^-, we may assume that no such orientation avoids $\mathcal{F}$. By the induction hypothesis, there are S-trees

(T_X, ξ) and (T_Y, υ) over $\mathcal{F}$, rooted in S_X^- and S_Y^-, respectively. Unless one of these is in fact rooted in S^-, implying (i), T_X has a leaf x, with neighbour x' say, such that $\xi(x, x') = (Y, X)$, and T_Y has a leaf y, with neighbour y' say, such that $\upsilon(y, y') = (X, Y)$. By Proposition 2.1, x and y are the only such leaves in their trees.

Let T be the tree obtained from the disjoint union of $T_X - x$ and $T_Y - y$ by joining x' to y'. Let $\alpha \colon \vec{E}(T) \to S$ map (x', y') to (X, Y) and (y', x') to (Y, X) and otherwise extend ξ and υ. Then $\xi(x', x) = \alpha(x', y') = \upsilon(y, y')$, so α maps the oriented stars at x' and y' to the same stars of separations in S as ξ and υ did. These lie in $\mathcal{F}$, so (T, α) is an S-tree over $\mathcal{F}$ rooted in S^-. $\qquad \square$

Theorem 3.1, alas, has a serious fault: there are few, if any, sets S and $\mathcal{F} \subseteq 2^S$ such that $\mathcal{F}$ consists of stars in S and the $\mathcal{F}$-avoiding orientations of S capture an interesting notion of 'dense objects'. The reason for this is that we are not requiring these orientations O to be consistent: we allow that O contains separations (D, C) and (A, B) when $(C, D) \leq (A, B)$, which will not usually be the case when O is induced by a meaningful dense object in the way discussed earlier.

So what happens if we strengthen (ii) so as to ask for a consistent orientation of S? The first time the proof breaks down is at the induction start: we now also have to ask that S^- be consistent. As such, this is no big restriction: the separations in S^- are meant to be small, so it is natural to require even that S^- be closed down in $(S, \leq)$ (which is stronger than consistency).

But now we have a problem at the induction step: we have to add not only (X, Y) or (Y, X) to S^-, but their entire *down-closures*

$$\lceil (X, Y) \rceil_S := \{ (U, W) \in S : (U, W) \leq (X, Y) \}$$

and $\lceil (Y, X) \rceil_S$ in S. This, then, spawns more problems: now T_X and T_Y can have many leaf separations not in S^-, not just (Y, X) and (X, Y). Even if each of these occurs only once (which can fail only if they are improper), there is then no obvious way to merge T_X and T_Y into a single S-tree over S^-.

We shall deal with this problem in the next section, in a fairly radical way. We shall aim to 'shift' the separations in the image of ξ to X, and the separations in the image of υ to Y, to turn (T, ξ) into an S-tree essentially on X and (T, υ) into an S-tree essentially on Y. The only leaf separation of the new S-tree for X that is not in S^- will be (Y, X), and the only leaf separation of the new S-tree for Y that is not in S^- will be (X, Y). We can then merge these two S-trees into an S-tree rooted in S^-.

In order for this approach to work, we shall have to impose some conditions on S and $\mathcal{F}$. These conditions are quite stringent, but they are so natural that all choices of S and $\mathcal{F}$ needed to capture the classical notions of 'dense objects' such as tangles and brambles will satisfy them.

4   Strong duality

Let $(X, Y) \leq (U, W)$ be elements of a set S of separations of a set V. Assume that $U \neq V$, and that (W, U) is associated with a leaf w of an S-tree (T, α) over

some set $\mathcal{F} \subseteq 2^S$ of stars.[4] Our aim is to 'shift' (T, α) to a new S-tree (T, α') based on the same tree T, by shifting the separations in the image of α over to X.

Given a separation $(A, B) \leq (U, W)$, let us define (Fig. 3, left)

$$f\!\downarrow_{(X,Y)}^{(U,W)} (A, B) := (A \cap X, B \cup Y) \quad \text{and} \quad f\!\downarrow_{(X,Y)}^{(U,W)} (B, A) := (B \cup Y, A \cap X).$$

This defines a *shifting map* $f\!\downarrow_{(X,Y)}^{(U,W)}$ on the set $S_{(U,W)}$ of separations $(A, B) \leq (U, W)$ and their inverses. Since (W, U) is a leaf separation of (T, α) and $\mathcal{F}$ consists of stars, the image of α lies in $S_{(U,W)}$ (Fig. 3, right). Hence the concatenation

$$\alpha' := f\!\downarrow_{(X,Y)}^{(U,W)} \circ \, \alpha$$

is well defined. However it is not clear for now whether α' takes all its images in S.
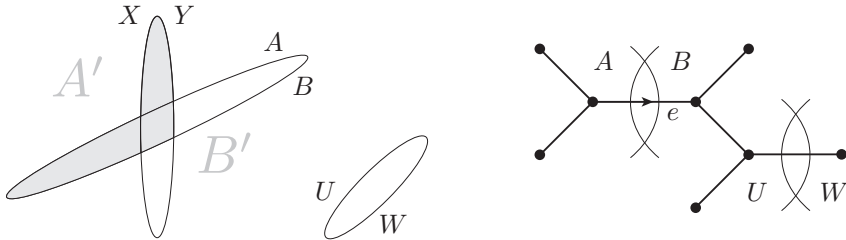

Figure 3: Shifting $\alpha(\vec{e}) = (A, B)$ to $\alpha'(\vec{e}) = (A', B')$

What is immediate, however, is that $f\!\downarrow_{(X,Y)}^{(U,W)}$ maps stars to stars:

Lemma 4.1. *The map $f\!\downarrow_{(X,Y)}^{(U,W)}$ preserves the ordering $\leq$ of separations.*

Proof. Consider two separations $(A, B), (C, D) \leq (U, W)$. If $(A, B) \leq (C, D)$ then $A \subseteq C$ and $B \supseteq D$, and hence also $A \cap X \subseteq C \cap X$ and $B \cup Y \supseteq D \cup Y$. Thus, $f\!\downarrow_{(X,Y)}^{(U,W)} (A, B) \leq f\!\downarrow_{(X,Y)}^{(U,W)} (C, D)$.

If $(A, B) \leq (D, C)$ the assertion is trivial, since $f\!\downarrow_{(X,Y)}^{(U,W)}$ decreases A and C while increasing B and D. Up to re-naming, this covers all cases to consider. $\qquad\square$

As remarked before, Lemma 4.1 implies that $f\!\downarrow_{(X,Y)}^{(U,W)}$ maps stars to stars.[5] It also implies that all leaf separations of (T, α), other than (W, U), get smaller in the transition to α'.[6] Indeed, if $\alpha(s, t) = (A, B)$ with $s \neq w$ a leaf of T, then $(A, B) \leq (U, W)$ and hence

$$f\!\downarrow_{(X,Y)}^{(U,W)} (A, B) = (A \cap X, B \cup Y) \leq (A, B).$$

It remains to ensure that α' takes its image in S if α does. The following condition on S will ensure the existence of a separation (X, Y) for which this is the case. Let us say that $(X, Y) \in S$ is *linked* to $(U, W) \in S$ if $(X, Y) \leq (U, W)$ and

$$(A \cap X, B \cup Y) \in S$$

---

[4]Thus, $(W, U) \in \lceil(Y, X)\rceil$, and in the context of the last section T would be T_X.

[5]This will help us show that (T, α') is over $\mathcal{F}$ if (T, α) is.

[6]This will help us show that (T, α') is rooted in S^- if (T, α) is.

for all $(A, B) \in S$ with $(A, B) \leq (U, W)$. Let us call S *separable* if for every pair $(W', U') \leq (U, W)$ of separations in S there exists $(X, Y) \in S$ such that (X, Y) is linked to (U, W) and (Y, X) is linked to (U', W').

Finally, we need a condition on $\mathcal{F}$ to ensure that the shifts of stars that occur as images under α of oriented stars at nodes of T are not only again stars but are also again in $\mathcal{F}$.[5] Let us say that a separation $(X, Y) \in S$ is $\mathcal{F}$-*linked* to $(U, W) \in S$ with $U \neq V$ if (X, Y) is linked to (U, W) and the image under $f\!\downarrow_{(X,Y)}^{(U,W)}$ of any star $S' \subseteq S_{(U,W)}$ in $\mathcal{F}$ that contains a separation (A, B) with $(B, A) \leq (U, W)$ is again in $\mathcal{F}$. We say that S is $\mathcal{F}$-*separable* if for every pair $(W', U') \leq (U, W)$ of separations in S, with $U, U' \neq V$, there exists $(X, Y) \in S$ such that (X, Y) is $\mathcal{F}$-linked to (U, W) and (Y, X) is $\mathcal{F}$-linked to (U', W'). And a set $\mathcal{F}$ of stars in S is *closed under shifting* if whenever $(X, Y) \in S$ is linked to $(U, W) \in S$ with $U \neq V$ it is even $\mathcal{F}$-linked to (U, W).

The following observation is immediate from the definitions:

Lemma 4.2. *If S is separable and $\mathcal{F}$ is closed under shifting, then S is $\mathcal{F}$-separable.* $\qquad\square$

In Section 5 we shall see that for all sets $\mathcal{F}$ describing classical 'dense objects', such as tangles and brambles (as well as many others), the usual separation systems S are $\mathcal{F}$-separable. In many cases, $\mathcal{F}$ will even be closed under shifting, in which case we will simply prove this stronger property.

We now have all the ingredients needed to shift an S-tree:

Lemma 4.3. *Let $\mathcal{F} \subseteq 2^S$ be a set of stars, let $S^- \subseteq S$ be closed down in S, and let (T, α) be an S-tree over $\mathcal{F}$ rooted in S^-. Let (W, U) be associated with a leaf w of T, where $U \neq V$, let (X, Y) be a proper separation in S that is $\mathcal{F}$-linked to (U, W), and let $\alpha' := f\!\downarrow_{(X,Y)}^{(U,W)} \circ \alpha$. Then (T, α') is an S-tree over $\mathcal{F}$ rooted in $S^- \cup \{(Y, X)\}$, in which (Y, X) is associated with w but with no other leaf of T, and (W, U) is not a leaf separation unless $(W, U) = (Y, X)$.*

Proof. Since $\mathcal{F}$ consists of stars, the map α preserves the natural ordering on $\vec{E}(T)$. In this ordering, every edge tt' of T has an orientation $(t, t') \leq (u, w)$, where u is the unique neighbour of the leaf w. Since $\alpha(u, w) = (U, W)$, this means that α maps $\vec{E}(T)$ to $S_{(U,W)}$. As $f\!\downarrow_{(X,Y)}^{(U,W)}$ is defined on this set, the map α' is well defined. And its image lies in S, because (X, Y) is linked to (U, W).

From Lemma 4.1 we know that $f\!\downarrow_{(X,Y)}^{(U,W)}$ maps stars to stars. Since (X, Y) is $\mathcal{F}$-linked to (U, W), it maps stars from $\mathcal{F}$ that are images under α of oriented stars at nodes of T to $\mathcal{F}$. Hence (T, α'), like (T, α), is an S-tree over $\mathcal{F}$.

In (T, α'), the separation at the leaf w is $\alpha'(w, u) = f\!\downarrow_{(X,Y)}^{(U,W)}(W, U) = (Y, X)$. Now consider a leaf $s \neq w$ of T, adjacent to t say, with $\alpha(s, t) = (A, B) \in S^-$. As $(s, t) \leq (u, w)$ in the natural ordering on $\vec{E}(T)$ we have $(A, B) \leq (U, W)$, so

$$\alpha'(s, t) = f\!\downarrow_{(X,Y)}^{(U,W)}(A, B) = (A \cap X, B \cup Y) \leq (A, B).$$

As S^- is closed down in S, this means that $\alpha'(s, t) \in S^-$. But it also means that $\alpha'(s, t) \neq (Y, X)$, since $(A \cap X, B \cup Y) = (Y, X)$ would imply $Y \subseteq X$, contradicting our assumption that (X, Y) is a proper separation. Thus, w is the unique leaf of T associated with (Y, X) under α', and (T, α') is rooted in $S^- \cup \{(Y, X)\}$.

Finally, if $\alpha'(s,t) = (W,U)$ then $U = B \cup Y \supseteq Y$ while $U \supseteq X$ by assumption, giving $U = V$ contrary to our assumption. $\qquad\square$

We can now strengthen our weak duality theorem so as to give consistent orientations, provided that S is $\mathcal{F}$-separable and S^- is closed down in S.

Theorem 4.4 (Strong Duality Theorem). *Let S be a separation system of a set V, and let $\mathcal{F} \subseteq 2^S$ be a set of stars. Let S^- be a down-closed subset of S containing all its separations of the form (A,V). If S is $\mathcal{F}$-separable, then exactly one of the following holds:*

(i) *There exists an S-tree over $\mathcal{F}$ rooted in S^-.*

(ii) *There exists a consistent $\mathcal{F}$-avoiding orientation of S extending S^-.*

Proof. By Lemma 3.2 it suffices to prove (i) assuming that (ii) fails. As in the proof of Theorem 3.1 we may assume that S^- is a partial orientation of S, and apply induction on $|S| - 2\,|S^-|$. At the induction start, S^- orients all of S. Since S^- is a down-closed partial orientation, it is consistent, so our assumption that (ii) fails means that S^- has a subset in $\mathcal{F}$. This defines an S-tree (T, α) with T a star as in the proof of Theorem 3.1, proving (i).

For the induction step, we now assume that S^- does not orient all of S. So there exist separations $(W_1, U_1) \le (U_2, W_2)$ in $S \smallsetminus S^-$, possibly equal, whose inverses are also not in S^-. Let us choose these with (W_1, U_1) minimal and (U_2, W_2) maximal. Note that $U_1, U_2 \ne V$, by our assumption about S^-.

Let $S_1^- = S^- \cup \{(W_1, U_1)\}$ and $S_2^- = S^- \cup \{(W_2, U_2)\}$. To show that S_1^- is closed down in S, consider a separation $(A,B) < (W_1, U_1)$ in S. By the minimality of (W_1, U_1), either (A,B) or (B,A) is in S^-. It cannot be (B,A), because then also $(U_1, W_1) < (B,A)$ would be in S^- (which is closed down in S by assumption), contrary to our choice of (W_1, U_1). Similarly, the maximal choice of (U_2, W_2) implies that S_2^- is closed down in S.

Since any orientation extending S_i^- also extends S^-, our assumption that (ii) fails implies for $i = 1, 2$ that no consistent $\mathcal{F}$-avoiding orientation of S extends S_i^-. By the induction hypothesis, there exists an S-tree (T_i, α_i) over $\mathcal{F}$ rooted in S_i^-. We may assume that (T_i, α_i) is not rooted in S^-, so T_i has a leaf w_i such that $\alpha_i(w_i, u_i) = (W_i, U_i)$, where u_i is the unique neighbour of w_i in T_i.

Since S is $\mathcal{F}$-separable and $(W_1, U_1) \le (U_2, W_2)$, there exists $(X_1, X_2) \in S$ such that (X_1, X_2) is $\mathcal{F}$-linked to (U_2, W_2) and (X_2, X_1) is $\mathcal{F}$-linked to (U_1, W_1). Then $(W_1, U_1) \le (X_1, X_2) \le (U_2, W_2)$. Our assumption that (W_1, U_1), $(W_2, U_2) \notin S^-$ but S^- is closed down in S thus implies that neither (X_1, X_2) nor (X_2, X_1) lies in S^-. Since S^- contains every improper separation or its inverse, this means that (X_1, X_2) is a proper separation. By Lemma 4.3, there are S-trees (T_i, α_i') over $\mathcal{F}$ rooted in $S_i^- \cup \{(X_i, X_{3-i})\}$, in which (X_i, X_{3-i}) is associated with w_i but with no other leaf of T_i, and all other leaf separations lie in S^-.

Let T be the tree obtained from the disjoint union of $T_1 - w_1$ and $T_2 - w_2$ by joining u_1 to u_2. Let $\alpha \colon \vec{E}(T) \to S$ map (u_2, u_1) to (X_1, X_2) and (u_1, u_2) to (X_2, X_1), and otherwise extend the α_i'. Then (T, α) satisfies (i), as in the proof of Theorem 3.1. $\qquad\square$

5 Applications of strong duality

In this section we show that the separation systems usually considered for graphs and matroids are all separable, and that the collections $\mathcal{F}$ needed to capture 'dense objects' such as tangles, brambles and blockages are closed under shifting. This will make our strong duality theorem imply the classical duality theorems for graphs and matroids. We also obtain some interesting new such theorems.

Let us call a separation system a *universe* if for any two of its separations (A, B) and (C, D) it also contains $(A \cap C, B \cup D)$. For instance, the set of all partitions of the ground set of a matroid is a universe, and so is the set of all vertex separations of a graph (which does not normally include all its vertex partitions).

We shall call a real function $(A, B) \mapsto |A, B|$ on a universe $\mathcal{U}$ an *order function* if it is symmetric and submodular, that is, if $|A, B| = |B, A|$ and

$$|A \cap C, B \cup D| + |A \cup C, B \cap D| \leq |A, B| + |C, D|$$

for all $(A, B), (C, D) \in \mathcal{U}$. We then call $|A, B|$ the *order* of the separation (A, B). Given a universe $\mathcal{U}$ with an order function, our focus will often be on the subsystem

$$S_k = \{(A, B) \in \mathcal{U} : |A, B| < k\}$$

for some positive integer k.

Lemma 5.1. *Every such S_k is separable.*

Proof. Given two separations $(W', U') \leq (U, W)$ in S_k, choose $(X, Y) \in S_k$ of minimum order with $(W', U') \leq (X, Y) \leq (U, W)$. We show that (X, Y) is linked to (U, W); the proof that (Y, X) is linked to (U', W') is analogous.

Consider any $(A, B) \in S_k$ with $(A, B) \leq (U, W)$. By submodularity,

$$|A \cap X, B \cup Y| + |A \cup X, B \cap Y| \leq |A, B| + |X, Y|.$$

Notice that

$$(W', U') \leq (A \cup X, B \cap Y) \leq (U, W).$$

By the choice of (X, Y) this implies $|A \cup X, B \cap Y| \geq |X, Y|$. Hence

$$|A \cap X, B \cup Y| \leq |A, B| < k,$$

and thus $(A \cap X, B \cup Y) \in S_k$ as desired. $\qquad\square$

For the remainder of this section, whenever we consider a graph $G = (V, E)$ we let $\mathcal{U}$ be its universe of *vertex separations*, the set of pairs (A, B) of vertex sets A, B such that $A \cup B = V$ and G has no edge between $A \setminus B$ and $B \setminus A$. We then take $|A, B| := |A \cap B|$ as our order function for $\mathcal{U}$, and put

$$S_k^- := \{(A, B) \in S_k : |A| < k\}.$$

This is obviously closed down in S_k, and S_k is separable by Lemma 5.1.

We remark that any consistent orientation O of S_k must extend the subset of S_k^- consisting of its separations of the form (A, V). This is because otherwise O would contain (V, A), with $(A, V) \leq (V, A) \in O$ violating consistency.

5.1 Branch-width and tangles of graphs

Robertson and Seymour [18] introduced branch-width and tangles for graphs, and more generally hypergraphs. For the sake of readability we only treat simple graphs here (no parallel edges or loops), but our results extend readily to multigraphs and hypergraphs with the obvious adaptations.

Let $G = (V, E)$ be a finite graph. A *tangle of order k* in G is (easily seen to be equivalent to) an $\mathcal{F}$-avoiding orientation of S_k extending S_k^- for

$$\mathcal{F} := \big\{\{(A_1, B_1), (A_2, B_2), (A_3, B_3)\} \subseteq S_k : G[A_1] \cup G[A_2] \cup G[A_3] = G\big\}.$$

(The three separations $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ need not be distinct.) Notice that any orientation of S_k that avoids this $\mathcal{F}$ is consistent, since for any pair of separations $(C, D) \leq (A, B)$ we have $G[D] \cup G[A] \supseteq G[B] \cup G[A] = G$ and hence $\{(D, C), (A, B)\} \in \mathcal{F}$.

Since our duality theorems, so far, only work with $\mathcal{F}$ consisting of stars of separations, let us consider the set $\mathcal{F}^*$ of those sets in $\mathcal{F}$ that are stars. Using submodularity we can easily show that an $\mathcal{F}^*$-avoiding orientation of S_k in fact avoids all of $\mathcal{F}$ – but only if we assume consistency:

Lemma 5.2. *Every consistent $\mathcal{F}^*$-avoiding orientation O of S_k avoids $\mathcal{F}$.*

Proof. Suppose O has a subset $S = \{(A_1, B_1), (A_2, B_2), (A_3, B_3)\} \in \mathcal{F}$. We show that we can replace one of its elements (A_i, B_i) with a smaller separation (A_i', B_i') in O so that the resulting set $S' \subseteq O$ is again in $\mathcal{F}$. This will contradict the finiteness of G.

As O avoids $\mathcal{F}^*$, we know that S is not a star, so $(A_1, B_1) \not\leq (B_2, A_2)$, say. By submodularity, either $(A_1 \cap B_2, B_1 \cup A_2)$ or $(A_2 \cap B_1, B_2 \cup A_1)$ is in S_k. We assume that $(A_1', B_1') := (A_1 \cap B_2, B_1 \cup A_2) \in S_k$; the other case is analogous. Clearly, $(A_1', B_1') \leq (A_1, B_1)$ as well as $(A_1', B_1') \leq (B_2, A_2)$. By our assumption of $(A_1, B_1) \not\leq (B_2, A_2)$ we thus have $(A_1', B_2') < (A_1, B_1)$.

Since O is an orientation of $S_k \ni (A_1', B_1')$, its consistency and the fact that $(A_1', B_1') \leq (A_1, B_1) \in O$ imply $(A_1', B_1') \in O$. To complete the proof it remains to show that $S' = \{(A_1', B_1'), (A_2, B_2), (A_3, B_3)\} \in \mathcal{F}$. But any vertex or edge of $G[A_1]$ that is not in $G[A_1']$ lies in $G[A_2]$, so this follows from the fact that $S \in \mathcal{F}$. □

Lemma 5.3. *$\mathcal{F}^*$ is closed under shifting.*

Proof. This is easy from the definitions. Informally, when we shift a star

$$\{(A_1, B_1), (A_2, B_2), (A_3, B_3)\} \subseteq S_{(U,W)}$$

with $(B_1, A_1) \leq (U, W)$, we replace A_i with $A_i \cap X$ for $i \geq 2$ but A_1 with $A_1 \cup Y$. As any vertex or edge that is not in $G[X]$ lies in $G[Y]$, this means that $\bigcup_i G[A_i] = G$ remains unchanged. □

The following technical but easy lemma provides the link between our S-trees and branch-decompositions as defined by Robertson and Seymour [18]:

Lemma 5.4. *For every integer $k \geq 3$,[7] a graph G has branch-width less than k if and only if G has an S_k-tree over $\mathcal{F}^*$ rooted in S_k^-.*

Proof. Let us prove the forward implication first. We may assume that G has no isolated vertices, because we can easily add a leaf in an S_k-tree corresponding to an isolated vertex.

Suppose (T, L) is a branch-decomposition of width $< k$. For each edge $e = st$ of T, let T_s and T_t be the components of $T - e$ containing s and t, respectively. Let $A_{s,t}$ and $B_{s,t}$ be the sets of vertices incident with an edge in $L^{-1}(V(T_s))$ and $L^{-1}(V(T_t))$, respectively. Thus, $B_{s,t} = A_{t,s}$.

For all adjacent nodes $st \in T$ let $\alpha(s,t) := (A_{s,t}, B_{s,t})$. Since G has no isolated vertices these $\alpha(s,t)$ are separations, and since (T, L) has width $< k$ they lie in S_k. For each internal node t of T and its three neighbours s_1, s_2, s_3, every edge of G has both ends in one of the $G[A_{s_i,t}]$, so

$$S_t := \{\alpha(s_1, t), \alpha(s_2, t), \alpha(s_3, t)\} \in \mathcal{F}.$$

As $A_{s_i,t} \subseteq A_{t,s_j} = B_{s_j,t}$ for all $i \neq j$, the set S_t is a star. This proves that (T, α) is an S_k-tree over $\mathcal{F}^*$. As $k \geq 3$, it is rooted in S^-.

Now let us prove the converse. We may again assume that G has no isolated vertex. Let (T, α) be an S_k-tree over $\mathcal{F}^*$ rooted in S_k^-. For each edge e of G, let us orient the edges st of T towards t whenever $\alpha(s, t) = (A, B)$ is such that B contains both ends of e. If e has its ends in $A \cap B$, then we choose arbitrary orientation of st. As T has fewer edges then nodes, there exists a node $t =: L(e)$ such that every edge at t is oriented towards t.

Let us choose an S_k-tree (T, α) and $L \colon E(G) \to V(T)$ so that the number of leaves in $L(E(G))$ is maximized, and subject to this with $|V(T)|$ minimum. We claim that, for every edge e of G, the node $t = L(e)$ is a leaf of T. Indeed, if not, let us extend T to make $L(e)$ a leaf. If t has degree 2, we attach a new leaf t' to t and put $\alpha(t', t) = (V(e), V(G))$ and $L(e) = t'$, where $V(e)$ denotes the set of ends of e. If t has degree 3 then, by definition of $\mathcal{F}$, there is a neighbour t' of t such that $e \in G[A]$ for $(A, B) = \alpha(t', t)$. As $t = L(e)$, this means that e has both ends in $A \cap B$. Subdivide the edge tt', attach a leaf t^* to the subdividing vertex t'', put $\alpha(t', t'') = \alpha(t'', t) = (A, B)$ and $\alpha(t^*, t'') = (V(e), V(G))$, and let $L(e) = t^*$. In both cases, (T, α) is still an S_k-tree over $\mathcal{F}^*$, rooted in S_k^-.

By the minimality of $|V(T)|$, every leaf of T is in $L(E(G))$, since we could otherwise delete it. Moreover, no node of t of T has degree 2, since contracting an edge at t while keeping α unchanged on the remaining edges would leave an S_k-tree over $\mathcal{F}^*$. (Here we use that G has no isolated vertices, and that $L(e) \neq t$ for every edge e of G.) Hence L is a bijection from $E(G)$ to the set of leaves of T, and T is a ternary tree. Thus, (T, L) is a branch-decomposition of G, clearly of width less than k. $\qquad\square$

We can now derive, and extend, the Robertson-Seymour [18] duality theorem for tangles and branch-width:

---

[7]This restriction is made necessary by a quirk in the notion of branch-width. This is 2 for all non-trivial trees other than stars, but 1 for stars $K_{1,n}$. For a clean duality theorem, however, it should be 2 also for stars: every graph with at least one edge has a tangle of order 2, because we can orient all separations in S_2 towards a fixed edge. Similarly, the branch-width of a disjoint union of edges is 0, but its tangle number is 2.

Theorem 5.5. *The following are equivalent for finite graphs $G \neq \emptyset$ and $k > 0$:*

(i) *G has a tangle of order k.*

(ii) *S_k has an $\mathcal{F}$-avoiding orientation extending S_k^-.*

(iii) *S_k has a consistent $\mathcal{F}^*$-avoiding orientation extending S_k^-.*

(iv) *G has no S_k-tree over $\mathcal{F}^*$ rooted in S_k^-.*

(v) *G has branch-width at least k, or $k \leq 2$ and G is a disjoint union of stars and isolated vertices and has at least one edge.*

Proof. If $k = 1$, then all statements are true for all $G \neq \emptyset$. If $k = 2$, they are all true if G has an edge, and all false if not. Assume now that $k \geq 3$.

(i)↔(ii) is immediate from the definition of a tangle (as earlier).

(ii)→(iii) is immediate from the definition of $\mathcal{F}$; the converse is Lemma 5.2.

(iii)↔(iv) is Theorem 4.4.

(iv)↔(v) is Lemma 5.4. □

5.2 Tree-width and brambles of graphs

We now apply our strong duality theorem to yield a duality theorem for tree-width in graphs. Its dual 'dense objects' will be orientations of S_k, like tangles, and thus different from brambles (or 'screens'), the dual objects in the classical tree-width duality theorem of Seymour and Thomas [19].

This latter theorem, which ours easily implies, says that a finite graph either has a tree-decomposition of width less than $k-1$ or a bramble of order at least k, but not both. The original proof of this theorem is as mysterious as the result is beautiful. The shortest known proof is given in [8] (where we refer the reader also for definitions), but it is hardly less mysterious. A more natural, if slightly longer, proof was given recently by Mazoit [15]. The proof by our strong duality theorem, as outlined below, is perhaps the most basic proof one can have.[8]

Consider a finite graph $G = (V, E)$, with sets of vertex separations $S_k^- \subseteq S_k$ for some integer $k > 0$ as defined at the start of Section 5. Let

$$\mathcal{F}_k := \left\{ S \subseteq S_k \mid S = \{(A_i, B_i) : i = 0, \ldots, n\} \text{ is a star with } \left| \bigcap_{i=0}^n B_i \right| < k \right\}.$$

We have seen that S_k is separable (Lemma 5.1). To apply Theorem 4.4 we thus only need the following lemma – whose proof contains the only bit of magic now left in the tree-width duality theorem:

Lemma 5.6. *$\mathcal{F}_k$ is closed under shifting.*

Proof. Let $(X, Y) \leq (U, W)$ be separations in S_k such that (X, Y) is linked to (U, W). Let

$$S = \{(A_i, B_i) : i = 0, \ldots, n\}$$

be a star in $\mathcal{F}_k \cap S_{(U,W)}$, with $(B_0, A_0) \leq (U, W)$. Then

$$(A_i, B_i) \leq (B_0, A_0) \leq (U, W) \text{ for all } i \geq 1. \tag{1}$$

We have to show that

$$S' = \{(A_i', B_i') : i = 0, \ldots, n\} \in \mathcal{F}_k$$

for $(A_i', B_i') := f\!\downarrow_{(X,Y)}^{(U,W)}(A_i, B_i)$.

---

[8]For example, we do not need Menger's theorem, as all the other proofs do.

From Lemma 4.1 we know that S' is a star. Since (X, Y) is linked to (U, W), we have $S' \subseteq S_k$ by (1). It remains to show that $\left| \bigcap_{i=0}^n B_i' \right| < k$. The trick will be to rewrite this intersection as the intersection of the two sides of a suitable separation that we know to be in S_k.

By (1) we have $(A_0', B_0') = (A_0 \cup Y, B_0 \cap X)$, while $(A_i', B_i') = (A_i \cap X, B_i \cup Y)$ for $i \geq 1$. Since the (A_i, B_i) are separations, i.e. in $\mathcal{U}$, so is $\left(\bigcup_{i=1}^n A_i, \bigcap_{i=1}^n B_i \right)$. As trivially $(B_0, V) \in \mathcal{U}$, this implies that, for $B^* := \bigcap_{i=1}^n B_i$, also

$$\left(\bigcup_{i=1}^n A_i \cup B_0, \ \bigcap_{i=1}^n B_i \cap V \right) = (B_0, B^*) \in \mathcal{U}.$$

Since $S \in \mathcal{F}_k$ we have $|B_0 \cap B^*| = \left| \bigcap_{i=0}^n B_i \right| < k$, so $(B_0, B^*) \in S_k$ (Fig. 4). As
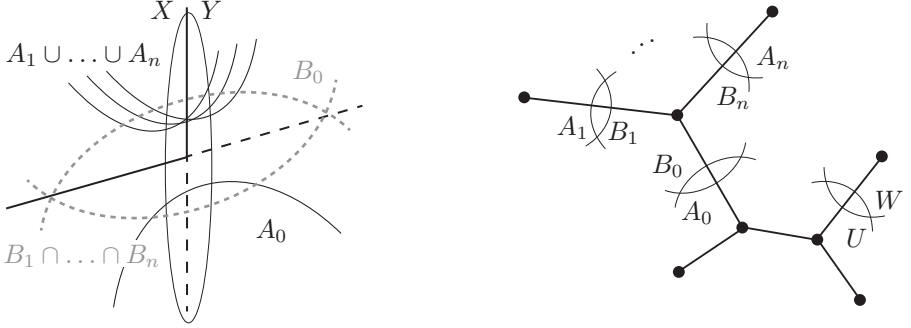

Figure 4: Shifting the separation (B_0, B^*)

also $(B_0, B^*) \leq (U, W)$ by (1), the fact that (X, Y) is linked to (U, W) therefore implies that $(B_0 \cap X, B^* \cup Y) \in S_k$. But then

$$\left| \bigcap_{i=0}^n B_i' \right| = \left| (B_0 \cap X) \cap \bigcap_{i=1}^n (B_i \cup Y) \right| = |B_0 \cap X, B^* \cup Y| < k,$$

which means that $S' \in \mathcal{F}_k$. $\qquad \square$

What remains now is just easy translation work between our terminology and the terms in which tree-width duality is traditionally cast.

Lemma 5.7. G has an S_k-tree (T, α) over $\mathcal{F}_k$ rooted in S_k^- if and only if it has a tree-decomposition $(T, \mathcal{V})$ of width less than $k - 1$.

Proof. Given any S-tree (T, α) of G over a set $\mathcal{F}$ of stars , let $\mathcal{V} = (V_t)_{t \in T}$ be defined by letting

$$V_t := \bigcap \{ B : (A, B) = \alpha(s, t), \ st \in E(T) \} ; \qquad (2)$$

it is easy to check [7] that $(T, \mathcal{V})$ is a tree-decomposition of G with adhesion sets $V_t \cap V_{t'} = A \cap B$ whenever $(A, B) = \alpha(t, t')$. If $\mathcal{F} = \mathcal{F}_k$ as earlier, we have $|V_t| < k$ at all interior nodes $t \in T$. And if (T, α) is rooted in S_k^-, its leaf separations $(A, B) = \alpha(s, t)$ satisfy $|V_s| = |A| < k$. Hence, in our case, $(T, \mathcal{V})$ has width less than $k - 1$.

Conversely, given a tree-decomposition $(T, \mathcal{V})$ with $\mathcal{V} = (V_t)_{t \in T}$, say, define $\alpha \colon \vec{E}(T) \to S_k$ as follows. We may assume that T has at least two nodes: if it has only one, t say, we add another, t', with $V_{t'} = V_t = V$. Given $t_1 t_2 \in E(T)$, let T_i be the component of $T - t_1 t_2$ containing t_i, and put $U_i := \bigcup_{t \in V(T_i)} V_t$ $(i = 1, 2)$. Then let $\alpha(t_1, t_2) := (U_1, U_2)$. One easily checks [8] that $U_1 \cap U_2 = V_{t_1} \cap V_{t_2}$, so α takes its values in S_k if $(T, \mathcal{V})$ has width $< k - 1$. Moreover, every part V_t satisfies (2), so if $(T, \mathcal{V})$ has width $< k - 1$ then (T, α) is over $\mathcal{F}_k$ and rooted in S_k^-. $\qquad\square$

The translation between orientations of S_k and brambles in a graph G is more interesting. Before the notion of a bramble was introduced in [19] (under the name of 'screen'), Robertson and Seymour had looked for an object dual to small tree-width that was more akin to our orientations of S_k: maps β assigning to every set X of fewer than k vertices one component of $G - X$. The question was how to make these choices consistent, so that they would define the desired 'dense object' dual to small tree-width. The obvious consistency requirement, that $\beta(Y) \subseteq \beta(X)$ whenever $X \subseteq Y$, is easily seen to be too weak, while asking that $\beta(X) \cap \beta(Y) \neq \emptyset$ for all X, Y turned out to be too strong. In [19], Seymour and Thomas then found a requirement that worked: that any two such sets, $\beta(X)$ and $\beta(Y)$, should *touch*: that either they share a vertex or G has an edge between them. Such maps β are now called *havens*, and it is easy to show that G admits a haven of *order* k (one defined on all sets X of less than k vertices) if and only if G has a bramble of order at least k.

The notion of 'touching' was perhaps elusive because it appeals directly to the structure of G, its edges: it is not be phrased purely in terms of set containment. It turns out, however, that it can be phrased in such terms after all, as the consistency of orientations of S_k:

Lemma 5.8. *G has a bramble of order at least k if and only if S_k has a consistent $\mathcal{F}_k$-avoiding orientation extending S_k^-.*

Proof. Let $\mathcal{B}$ be a bramble of order at least k. For every $(A, B) \in S_k$, since $A \cap B$ is too small to cover $\mathcal{B}$ but every two sets in $\mathcal{B}$ touch and are connected, exactly one of the sets $A \smallsetminus B$ and $B \smallsetminus A$ contains an element of $\mathcal{B}$. Thus,

$$O = \{ (A, B) \in S_k : B \smallsetminus A \text{ contains an element of } \mathcal{B} \}$$

is an orientation of S_k; it is clearly consistent and extends S_k^-.

To show that O avoids $\mathcal{F}_k$, let $S = \{(A_1, B_1), \ldots, (A_n, B_n)\} \in \mathcal{F}_k$ be given. Then $\left| \bigcap_{i=1}^n B_i \right| < k$, so some $C \in \mathcal{B}$ avoids this set and hence lies in the union of the sets $A_i \smallsetminus B_i$. But these sets are disjoint, since S is a star. Hence C lies in one of them, $A_1 \smallsetminus B_1$ say, putting (B_1, A_1) in O. But then $(A_1, B_1) \notin O$, so $S \not\subseteq O$ as claimed.

Conversely, let O be a consistent $\mathcal{F}_k$-avoiding orientation of S_k extending S_k^-. We shall define a bramble $\mathcal{B}$ containing for every set X of fewer than k vertices exactly one component of $G - X$, and no other sets. Such a bramble will have order at least k, since no such set X covers it.

Given X, let $C_1, \ldots, C_n$ be the vertex sets of the components of $G - X$. Consider the separations (A_i, B_i) with $A_i = C_i \cup N(C_i)$ and $B_i = V \smallsetminus C_i$. Since

$$S_X := \{ (A_i, B_i) : i = 1, \ldots, n \}$$

is a star in $\mathcal{F}_k$, not all the (A_i, B_i) lie in O. So $(B_i, A_i) \in O$ for some i, and since O is consistent this i is unique. Let us make C_i an element of $\mathcal{B}$.

It remains to show that every two sets in $\mathcal{B}$ touch. Given $C, C' \in \mathcal{B}$, there are sets X and X' such that S_X contains a separation (A, B) with $A = C \cup N(C)$ and $(B, A) \in O$, and likewise for C'. If C and C' do not touch, then $C' \subseteq B \smallsetminus A$ and hence $A' \subseteq B$ (Fig. 5), and similarly $A \subseteq B'$. Hence $(A, B) \le (B', A') \in O$ but also $(B, A) \in O$, contradicting the consistency of O. □

$A \smallsetminus B = C$ $C' = A' \smallsetminus B'$
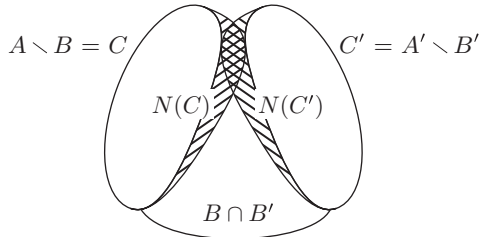
$N(C)$ $N(C')$

$B \cap B'$

Figure 5: If C, C' do not touch, then (B, A) and (B', A') are inconsistent.

We can now prove, and extend, the tree-width duality theorem of Seymour and Thomas [19]:

Theorem 5.9. *The following are equivalent for all finite graphs G and $k > 0$:*

(i) *G has a bramble of order at least k.*

(ii) *S_k has a consistent $\mathcal{F}_k$-avoiding orientation extending S_k^-.*

(iii) *G has no S_k-tree over $\mathcal{F}_k$ rooted in S_k^-.*

(iv) *G has tree-width at least $k - 1$.*

Proof. (i)↔(ii) is Lemma 5.8.
    (ii)↔(iii) holds by Lemma 5.6 and Theorem 4.4.
    (iii)↔(iv) is Lemma 5.7. □

5.3   Path-width and blockages of graphs

A *path-decomposition* of a graph G is a tree-decomposition of G whose decomposition tree is a path. The *path-width* of G is the least width of such a tree-decomposition. Since the S_k-trees in Theorem 5.9 are just another description of tree-decompositions of width $< k - 1$, its equivalence of (i), (ii) and (iii) immediately yields a duality theorem for path-width if we replace $\mathcal{F}_k$ with

$$\mathcal{F}_k^{(2)} := \{ S \subseteq S_k \mid S = \{(A_1, B_1), (A_2, B_2)\} \text{ is a star with } |B_1 \cap B_2| < k \},$$

its subset of stars of order 2.

Instead of the brambles in Theorem 5.9 (i), Bienstock, Robertson, Seymour and Thomas [2] propose a more tangle-like kind of dense object dual to path-width which they call 'blockages'. They show that G has path-width at least $k - 1$ if and only it contains a blockage of order $k - 1$ (see below for definitions).[9] In this section we deduce their result from our strong duality theorem.

---

[9]They go on to show that any graph with a blockage of order $k - 1$, and hence every graph of path-width at least $k - 1$, contains every forest of order k as a minor – a corollary perhaps better known than their path-width duality theorem as such.

Given a set X of vertices in $G = (V, E)$, let us write $\partial(X)$ for the set of vertices in X that have a neighbour outside X. A *blockage of order* $k - 1$, according to [2], is a collection $\mathcal{B}$ of sets $X \subseteq V$ such that

(B1) $|\partial(X)| < k$ for all $X \in \mathcal{B}$;

(B2) $X' \in \mathcal{B}$ whenever $X' \subseteq X \in \mathcal{B}$ and $|\partial(X')| < k$;

(B3) if (X_1, X_2) is a separation of G and $|X_1 \cap X_2| < k$, then $\mathcal{B}$ contains exactly one of X_1, X_2.

To deduce the duality theorem of [2] from our strong duality theorem, we just need to translate blockages into orientations of S_k:

Theorem 5.10. *The following are equivalent for all finite graphs G and $k > 0$:*

(i) G *has a blockage of order $k - 1$.*

(ii) S_k *has a consistent $\mathcal{F}_k^{(2)}$-avoiding orientation extending S_k^-.*

(iii) G *has no S_k-tree over $\mathcal{F}_k^{(2)}$ rooted in S_k^-.*

(iv) G *has path-width at least $k - 1$.*

Proof. Since $\mathcal{F}_k^{(2)}$ is just the restriction of the $\mathcal{F}_k$ from Section 5.2 to stars with two separations, it is closed under shifting by Lemma 5.6. By Lemmas 5.1 and 4.2 this implies that S_k is $\mathcal{F}_k^{(2)}$-separable. Theorem 4.4, therefore, yields the equivalence of (ii) and (iii), which is equivalent to (iv) by Lemma 5.7.

(i)→(ii): Suppose that G has a blockage $\mathcal{B}$ of order $k - 1$. By (B2) and (B3),

$$O = \{ (X, Y) \in S_k : X \in \mathcal{B} \}$$

is a consistent orientation of S_k.

For a proof that O extends S_k^- it suffices to show that $\mathcal{B}$ contains every set X of order $< k$. Consider the separation $(X, V) \in S_k$. If $V \in \mathcal{B}$, then also $X \in \mathcal{B}$ by (B2), contradicting (B3). Hence $V \notin \mathcal{B}$, and thus $X \in \mathcal{B}$ by (B3).

It remains to show that O avoids $\mathcal{F}_k^{(2)}$. Given $\{(A_1, B_1), (A_2, B_2)\} \in \mathcal{F}_k^{(2)}$ suppose that $(A_1, B_1) \in O$. As $|B_1 \cap B_2| < k$ by definition of $\mathcal{F}_k^{(2)}$, the separation (B_1, B_2) lies in S_k. We now use (B3) to deduce from our assumption of $A_1 \in \mathcal{B}$ that $B_1 \notin \mathcal{B}$, and hence $B_2 \in \mathcal{B}$, and hence $A_2 \notin \mathcal{B}$. Thus, $(A_2, B_2) \notin O$.

(ii)→(i): Let O be a consistent $\mathcal{F}_k^{(2)}$-avoiding orientation of S_k extending S_k^-. We claim that

$$\mathcal{B} := \{ X : (X, Y) \in O \}$$

is a blockage of order $k - 1$. Clearly, $\mathcal{B}$ satisfies (B1).

Let (X_1, X_2) be as in (B3). Then $(X_1, X_2) \in S_k$, so (X_1, X_2) or its inverse lies in O; say $(X_1, X_2) \in O$. Then $X_1 \in \mathcal{B}$. If also $X_2 \in \mathcal{B}$, there exists Y_2 such that $(X_2, Y_2) \in O$. Then $(X_1 \cap Y_2, X_2)$ is a separation of order $< k$. As $(X_1 \cap Y_2, X_2) \leq (X_1, X_2) \in O$ and O is consistent, we have $(X_1 \cap Y_2, X_2) \in O$. Then $\{(X_1 \cap Y_2, X_2), (X_2, Y_2)\}$ is a star in $\mathcal{F}_k^{(2)}$, contradicting our assumption.

Given $X' \subseteq X \in \mathcal{B}$ as in (B2), with $(X, Y) \in O$ say, let $Y' := \partial(X') \cup (V \smallsetminus X')$ and $Z := \partial(X) \cup (V \smallsetminus X)$. Then $Z \subseteq Y$ and hence $|X \cap Z| \leq |X \cap Y| < k$, so $(X, Z) \in S_k$. By (B3) we have $Z \notin \mathcal{B}$ and hence $(Z, X) \notin O$, so $(X, Z) \in O$. Since O is consistent and $(X', Y') \leq (X, Z)$, we thus obtain $(X', Y') \in O$ and hence $X' \in \mathcal{B}$, as desired. $\square$

5.4 Branch-width and tangles for set separations with arbitrary submodular order functions: carving width, rank width, and matroid tangles

The concepts of branch-width and tangles were introduced by Robertson and Seymour [18] not only for graphs but more generally for hypergraphs. As the order of a separation (A, B) they already considered, instead of $|A \cap B|$, also arbitrary symmetric submodular order functions $|A, B|$ and proved the relevant lemmas more generally for these. Geelen, Gerards, Robertson, and Whittle [10] applied this explicitly to a submodular connectivity function.

Our aim in this section is to derive from Theorem 4.4 a duality theorem for branch-width and tangles in arbitrary separation universes with an order function, as introduced at the start of Section 5. This will imply the above branch-width duality theorems for hypergraphs and matroids, as well as their cousins for carving width [20] and rank-width of graphs [16].

Let $\mathcal{U}$ be any universe of separations of some set E of at least two elements, with an order function $(A, B) \mapsto |A, B|$. Let $k > 0$ be an integer, and consider

$$S_k = \big\{(A, B) \in \mathcal{U} : |A, B| < k\big\} \text{ and } S_k^- = \big\{(A, B) \in S_k : |A| \leq 1\big\}.$$

Let us call an orientation of S_k a *tangle of order k* if it extends S_k^- and avoids

$$\mathcal{F} = \big\{\{(A_1, B_1), (A_2, B_2), (A_3, B_3)\} \subseteq S_k : A_1 \cup A_2 \cup A_3 = E\big\},$$

where $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ need not be distinct; in particular, tangles are consistent. This extends the existing notions of tangles for hypergraphs and matroids, with their edge set or ground set as E, partitions as separations, and the appropriate order functions.

Let $\mathcal{F}^* \subseteq \mathcal{F}$ be the set of stars in $\mathcal{F}$. As in Lemma 5.3, it is easy to prove that $\mathcal{F}^*$ is closed under shifting. We have the following analogue of Lemma 5.2:

Lemma 5.11. *Every consistent $\mathcal{F}^*$-avoiding orientation of S_k avoids $\mathcal{F}$.* $\quad\square$

Let us say that $\mathcal{U}$ has *branch-width $< k$* if there exists an S_k-tree over $\mathcal{F}^*$ that is rooted in S_k^-. As before, this definition agrees with the usual ones when $\mathcal{U}$ is a hypergraph or matroid. By Lemmas 5.1 and 5.11, Theorem 4.4 now specializes as follows:

Theorem 5.12. *Given a separation universe $\mathcal{U}$ with an order function, and $k > 0$, the following assertions are equivalent:*

(i) *$\mathcal{U}$ has a tangle of order k.*

(ii) *S_k has a consistent $\mathcal{F}^*$-avoiding orientation extending S_k^-.*

(iii) *$\mathcal{U}$ does not have branch-width $< k$.* $\quad\square$

5.5 Matroid tree-width

Hliněný and Whittle [11, 12] generalized the notion of tree-width from graphs to matroids.[10] Our aim in this section is to specialize our strong duality theorem to a duality theorem for tree-width in matroids.

---

[10]In our matroid terminology we follow Oxley [17].

Let $M = (E, I)$ be a matroid with rank function r. Its *connectivity function* is defined as
$$\lambda(X) := r(X) + r(E \smallsetminus X) - r(M).$$

We consider the universe $\mathcal{U}$ of all bipartitions (X, Y) of E. Since
$$|X, Y| := \lambda(X) = \lambda(Y)$$

is submodular and symmetric, it is an order function on $\mathcal{U}$.

A *tree-decomposition* of M is a pair (T, τ), where T is a tree and $\tau \colon E \to V(T)$ is any map. Let t be a node of T, and let $T_1, \ldots, T_d$ be the components of $T - t$. Then the *width* of t is the number
$$\sum_{i=1}^{d} r(E \smallsetminus F_i) - (d - 1)\, r(M),$$

where $F_i = f^{-1}(V(T_i))$. (If t is the only node of T, we let its width be $r(M)$.) The *width* of (T, τ) is the maximum width of the nodes of T. The *tree-width* of M is the minimum width over all tree-decompositions of M.

Matroid tree-width generalizes the tree-width of graphs in the expected way:

Theorem 5.13 (Hliněný and Whittle [11, 12])**.** *The tree-width of a finite graph containing at least one edge equals the tree-width of its cycle matroid.*

In order to specialize Theorem 4.4 to a duality theorem for tree-width in matroids, we consider
$$S_k = \{\, (A, B) \in \mathcal{U} : |A, B| < k \,\} \quad \text{and} \quad S_k^- = \{\, (A, B) \in \mathcal{U} : r(A) < k \,\} \subseteq S_k.$$

Since λ is symmetric and submodular, S_k is separable by Lemma 5.1. Let
$$\mathcal{F}_k := \Big\{\, S \subseteq \mathcal{U} \mid S = \{(A_i, B_i) : i = 0, \ldots, n\} \text{ is a star with } n \geq 1$$
$$\text{and } \sum_{i=0}^{n} r(B_i) - n\, r(M) < k \,\Big\}.$$

We remark that requiring $S \subseteq S_k$ in the definition of $\mathcal{F}_k$ would not spare us a proof of the following lemma, which we shall need in the proof of Lemma 5.16.

Lemma 5.14. *Every $S \in \mathcal{F}_k$ is a subset of S_k.*

Proof. We show that every star $S = \{(A_i, B_i) : i = 0, \ldots, n\} \subseteq \mathcal{U}$ satisfies
$$\lambda(A_i) \leq \sum_{i=0}^{n} r(B_i) - n\, r(M)$$

for all $i = 0, \ldots, n$; if $S \in \mathcal{F}_k$, this implies that $|A_i, B_i| < k$ for all i, as desired.

Since S is a star we have $A_i \subseteq B_j$ whenever $i \neq j$, and in particular $A_{i+1} \subseteq B_i^* := B_1 \cap \ldots \cap B_i$ for $i = 1, \ldots, n - 1$. Hence $B_i^* \cup B_{i+1} \supseteq E$. Submodularity of the rank function now gives
$$r(B_i^*) + r(B_{i+1}) \geq r(B_i^* \cap B_{i+1}) + r(B_i^* \cup B_{i+1}) = r(B_{i+1}^*) + r(M)$$

for each $i = 1, \ldots, n-1$. Summing these inequalities over $i = 1, \ldots, n-1$ yields

$$r(B_1) + \ldots + r(B_n) \geq r(B_1 \cap \ldots \cap B_n) + (n-1)\, r(M).$$

Hence for $i = 0$ (and by renumbering also for $i = 1, \ldots, n$), the fact that S is a star, and hence $A_0 \subseteq B_1 \cap \ldots \cap B_n$, implies

$$
\begin{aligned}
\sum_{i=0}^{n} r(B_i) - n\, r(M) &\geq r(B_0) + r(B_1 \cap \ldots \cap B_n) - r(M) \\
&\geq r(B_0) + r(A_0) - r(M) \\
&= \lambda(A_0).
\end{aligned}
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In order to apply Theorem 4.4, we have to prove that S_k is $\mathcal{F}_k$-separable:

Lemma 5.15. S_k is $\mathcal{F}_k$-separable.

Proof. Let $(W', U') \leq (U, W)$ be separations in S_k. Choose $(X, Y) \in S_k$ with $(W', U') \leq (X, Y) \leq (U, W)$ and $|X, Y|$ minimum. We claim that (X, Y) is $\mathcal{F}_k$-linked to (U, W) and (Y, X) is $\mathcal{F}_k$-linked to (U', W'). By symmetry, it is enough to prove that (X, Y) is $\mathcal{F}_k$-linked to (U, W).

The proof of Lemma 5.1 shows that (X, Y) is linked to (U, W).[11] Now let

$$S = \big\{(A_i, B_i) : i = 0, \ldots, n\big\}$$

be a star in $\mathcal{F}_k \cap S_{(U,W)}$, with $(B_0, A_0) \leq (U, W)$. Then

$$(A_i, B_i) \leq (B_0, A_0) \leq (U, W) \quad \text{for all } i \geq 1. \qquad (3)$$

We have to show that

$$S' = \big\{(A_i', B_i') : i = 0, \ldots, n\big\} \in \mathcal{F}_k$$

for $(A_i', B_i') := f\!\downarrow_{(X,Y)}^{(U,W)} (A_i, B_i)$.

From Lemma 4.1 we know that S' is a star. Since (X, Y) is linked to (U, W), we have $S' \subseteq S_k$ by (3). It remains to show that

$$r(X \cap B_0) + \sum_{i=1}^{n} r(Y \cup B_i) - n\, r(M) = \sum_{i=0}^{n} r(B_i') - n\, r(M) < k. \qquad (4)$$

For $i = 1, \ldots, n$ let us abbreviate $A_i^* := A_1 \cup \ldots \cup A_i$ and $B_i^* := B_1 \cap \ldots \cap B_i$.

By submodularity of the rank function, we have

$$
\begin{aligned}
r(X \cap B_0) + r(X \cup B_0) &\leq r(X) + r(B_0) \\
\text{and} \quad r(Y \cup B_i) + r(Y \cap B_i) &\leq r(Y) + r(B_i) \quad \text{for } i = 1, \ldots, n.
\end{aligned}
$$

For our proof of (4) we need that the sum of the first terms in these $n + 1$ inequalities is at most the sum of the last terms, or equivalently that the sum

---

[11]Technically, we do not need this fact at this point and could use Lemma 5.14 to deduce it from the fact that all S' as below lie in $\mathcal{F}_k$. But that seems heavy-handed.

of the second terms is at least the sum of the third terms. We show the latter, that

$$r(X \cup B_0) + \sum_{i=1}^{n} r(Y \cap B_i) \geq r(X) + n\, r(Y)\,. \tag{5}$$

Since S is a star we have $A_i \subseteq B_j$ whenever $i \neq j$. Hence $A_n^* \subseteq B_0$, giving

$$r(X \cup B_0) \geq r(X \cup A_n^*)\,, \tag{6}$$

and $A_{i+1} \subseteq B_i^*$ for $i \geq 1$. Hence $B_i^* \cup B_{i+1} \supseteq E$. By submodularity, this implies

$$r(Y \cap B_i^*) + r(Y \cap B_{i+1}) \geq r(Y \cap B_{i+1}^*) + r(Y)$$

for each $i = 1, \ldots, n-1$, by induction on i. By summing this for $i = 1, \ldots, n-1$, we obtain

$$\sum_{i=1}^{n} r(Y \cap B_i) \geq r(Y \cap B_n^*) + (n-1)\, r(Y)\,. \tag{7}$$

Since (X, Y) and (A_n^*, B_n^*) are bipartitions of E, so is $(X \cup A_n^*, Y \cap B_n^*)$. As $|X \cup A_n^*, Y \cap B_n^*| < |X, Y|$ would contradict our choice of (X, Y),[12] we thus have $|X \cup A_n^*, Y \cap B_n^*| \geq |X, Y|$, and therefore

$$r(X \cup A_n^*) + r(Y \cap B_n^*) \geq r(X) + r(Y)\,. \tag{8}$$

Adding up inequalities (6), (7), (8) we obtain (5), proving (4). $\qquad\square$

Lemma 5.16. M has an S_k-tree over $\mathcal{F}_k$ rooted in S_k^- if and only if it has tree-width less than k.

Proof. For the forward implication, consider any S_k-tree (T, α) of M. Given $e \in E$, orient every edge st of T, with $\alpha(s, t) = (A, B)$ say, towards t if $e \in B$, and let τ map e to the unique sink of T in this orientation. Then (T, τ) is a tree-decomposition of M. If (T, α) is over $\mathcal{F}_k$ and rooted in S_k^-, the decomposition is easily seen to have width less than k.

Conversely, let (T, τ) be a tree-decomposition of M of width less than k. If $r(M) < k$ then $\{(\emptyset, E), (E, \emptyset)\} \subseteq S_k^-$. So there is a 2-node S_k-tree rooted in S_k^-, which is vacuously over $\mathcal{F}_k$. We may thus assume that $r(M) \geq k$, and so T has at least two nodes. For each edge $e = st$ of T, let T_s and T_t be the components of $T - e$ containing s and t, respectively. Let

$$\alpha(s, t) := \left(\tau^{-1}(T_s), \tau^{-1}(T_t) \right) \in \mathcal{U}.$$

Since every node t has width less than k, its star $\{\, \alpha(s, t) : st \in E(T) \,\}$ of separations is in $\mathcal{F}_k$ unless t is a leaf, in which case $\alpha(t, s) \in S_k^-$. By Lemma 5.14 this implies that $\alpha(\vec{E}(T)) \subseteq S_k$, so (T, α) is an S_k-tree over $\mathcal{F}_k$ rooted in S_k^-. $\quad\square$

Theorem 4.4 now yields the following duality theorem for matroid tree-width.

---

[12]Since $(X \cup A_n^*, Y \cap B_n^*) < (X, Y) < k$ in this case, $(X \cup A_n^*, Y \cap B_n^*)$ would be in S_k.

Theorem 5.17. *Let $M = (E, I)$ be a matroid with the rank function r, and let k be an integer. Then the following statements are equivalent:*

(i) *M has tree-width at least k.*

(ii) *M has no S_k-tree over $\mathcal{F}_k$ rooted in S_k^-.*

(iii) *S_k has a consistent $\mathcal{F}_k$-avoiding orientation extending S_k^-.*

(iv) *There exists a collection $\mathcal{T}$ of subsets of E satisfying the following:*

- *$\lambda(A) < k$ for all $A \in \mathcal{T}$;*
- *for all $A \subseteq E$ with $\lambda(A) < k$, either $A \in \mathcal{T}$ or $E \smallsetminus A \in \mathcal{T}$ but not both;*
- *whenever $B \subseteq A \in \mathcal{T}$ with $\lambda(B) < k$, we have $B \in \mathcal{T}$;*
- *whenever $A_1, \ldots, A_n \in \mathcal{T}$ are disjoint,*

$$\sum_{i=1}^{n} r(E \smallsetminus A_i) - (n-1)\, r(M) \geq k.$$

Proof. Item (iv) is merely spells out the meaning of (iii). $\qquad\square$

5.6 Tree-decompositions of small adhesion

In this subsection we illustrate the versatility of Theorem 4.4 to deduce a duality theorem for a new width parameter, one that measures both the width and the adhesion of a tree-decomposition.

Recall that the *adhesion* of a tree-decomposition $(T, \mathcal{V})$ of a graph $G = (V, E)$ is the largest size of an attachment set, the number $\max_{st \in E(T)} |V_s \cap V_t|$. (If T has only one node t, we set the adhesion to $|V_t| = |V|$.) Trivially if a tree-decomposition has width $< k - 1$, then it has an adhesion $< k$.

The idea now is to have a duality theorem for graphs whose tree structures are the tree-decompositions of adhesion $< k$ and width less than $w - 1 \geq k - 1$. For $w = k$ this should default to the usual duality between tree-width and havens or brambles, as discussed in Section 5.2.

Let S_k and S_k^- be as defined at the beginning of Section 5, with $\mathcal{U}$ the universe of all vertex separations of G equipped with the usual order function. Let

$$\mathcal{F}_w = \left\{ S \subseteq S_k \mid S = \{(A_i, B_i) : i = 0, \ldots, n\} \text{ is a star with } \left| \bigcap_{i=0}^{n} B_i \right| < w \right\},$$

as in Section 5.2. To apply Theorem 4.4, we only need the following lemma.

Lemma 5.18. *S_k is $\mathcal{F}_w$-separable.*

Proof. Let $(W', U') \leq (U, W)$ be separations in S_k. Choose $(X, Y) \in S_k$ so that $(W', U') \leq (X, Y) \leq (U, W)$ and $|X, Y| = |X \cap Y|$ is minimum. We claim that (X, Y) is $\mathcal{F}_w$-linked to (U, W), and that (Y, X) is $\mathcal{F}_w$-linked to (U', W'). By symmetry, it is enough to prove that (X, Y) is $\mathcal{F}_w$-linked to (U, W). The proof of Lemma 5.1 shows that (X, Y) is linked to (U, W).

Let

$$S = \{(A_i, B_i) : i = 0, \ldots, n\}$$

be a star in $\mathcal{F}_w \cap S_{(U,W)}$, with $(B_0, A_0) \leq (U, W)$. Then

$$(A_i, B_i) \leq (B_0, A_0) \leq (U, W) \text{ for all } i \geq 1. \tag{9}$$

We have to show that

$$S' = \left\{(A'_i, B'_i) : i = 0, \ldots, n\right\} \in \mathcal{F}_w$$

for $(A'_i, B'_i) := f\!\downarrow_{(X,Y)}^{(U,W)} (A_i, B_i)$.

From Lemma 4.1 we know that S' is a star. Since (X,Y) is linked to (U,W), we have $S' \subseteq S_k$ by (9). It remains to show that $\left|\bigcap_{i=0}^{n} B'_i\right| < w$. As in Lemma 5.6, we shall prove this by rewriting the intersection of all the B'_i as an intersection of the two sides of a suitable separation, and use submodularity and the choice of (X,Y) to show that this separation has size $< w$.

By (9) and the definition of $f\!\downarrow_{(X,Y)}^{(U,W)}$, the separations in S' are as follows:

$$(A'_0, B'_0) = (A_0 \cup Y, B_0 \cap X), \text{ while } (A'_i, B'_i) = (A_i \cap X, B_i \cup Y) \text{ for } i \geq 1. \quad (10)$$

Since the (A_i, B_i) are separations, i.e. in $\mathcal{U}$, so is $(\bigcup_{i=1}^{n} A_i, \bigcap_{i=1}^{n} B_i)$. As trivially $(B_0, V) \in \mathcal{U}$, this implies that, for $B^* := \bigcap_{i=1}^{n} B_i$, also

$$\left(\bigcup_{i=1}^{n} A_i \cup B_0 \, , \, \bigcap_{i=1}^{n} B_i \cap V\right) = (B_0, B^*) \in \mathcal{U}.$$

Note that

$$|B_0, B^*| = |B_0 \cap B^*| = \bigcap_{i=0}^{n} B_i < w \quad (11)$$

since $S \in \mathcal{F}_w$.

Since $(B_0, B^*) \leq (U, W)$ by (9), and also $(X, Y) \leq (U, W)$, we further have

$$(W', U') \leq (X, Y) \leq (X \cup B_0, Y \cap B^*) \leq (U, W).$$

So $(X \cup B_0, Y \cap B^*)$ was a candidate for the choice of (X, Y). As (X, Y) was chosen, we thus have $|X \cup B_0, Y \cap B^*| \geq |X, Y|$. As

$$|X \cup B_0, Y \cap B^*| + |X \cap B_0, Y \cup B^*| \leq |X, Y| + |B_0, B^*|$$

by the submodularity of the order function, we deduce that

$$|B_0 \cap X, B^* \cup Y| \leq |B_0, B^*| < w,$$

by (11). Hence by (10),

$$\left|\bigcap_{i=0}^{n} B'_i\right| = \left|(B_0 \cap X) \cap \bigcap_{i=1}^{n} (B_i \cup Y)\right| = |B_0 \cap X, B^* \cup Y| < w,$$

which means that $S' \in \mathcal{F}_w$. □

The following translation lemma is proved like Lemma 5.7:

Lemma 5.19. *G has an S_k-tree over $\mathcal{F}_w$ rooted in S_k^- if and only if it has a tree-decomposition of width less than $w - 1$ and adhesion less than k.*

Theorem 4.4 and our two lemmas imply the following duality theorem:

Theorem 5.20. *The following are equivalent for all finite graphs G and $k > 0$:*

(i) *S_k has a consistent $\mathcal{F}_w$-avoiding orientation extending S_k^-.*

(ii) *G has no S_k-tree over $\mathcal{F}_w$ rooted in S_k^-.*

(iii) *G has no tree-decomposition of width $< w - 1$ and adhesion $< k$.* □

5.7 Weakly Submodular Partition Functions

Amini, Mazoit, Nisse and Thomassé [1], and Lyaudet, Mazoit and Thomassé [14], proposed a framework to unify duality theorems in graph minor theory which, unlike ours, is based exclusively on partitions. Their work, presented to us by Mazoit in the summer of 2013, inspired us to look for possible simplifications, for generalizations to separations that are not partitions, and for applications to tangle-like dense objects not covered by their framework. Our findings are presented in this paper and its sequel [9]. Although our approach differs from theirs, we remain indebted to Mazoit and his coauthors for this inspiration.

Since the applications of our strong duality theorem include the applications of [1], it may seem unnecessary to ask whether our result also implies theirs directly. However, for completeness we address this question now.

A *partition* of a finite set E is a set of disjoint subsets of E, possibly empty, whose union is E. We write $\mathcal{P}(E)$ for the set of all partitions of E. In [1], any function $\mathcal{P}(E) \to \mathbb{R} \cup \{\infty\}$ is called a *partition function* of E. We abbreviate $\Psi(\{A_1, \ldots, A_n\})$ to $\Psi(A_1, \ldots, A_n)$, but note that the partition remains unordered. A partition function Ψ is called *weakly submodular* in [1] if, for every pair $(\mathcal{A}, \mathcal{B})$ of partitions of E and every choice of $A_0 \in \mathcal{A}$ and $B_0 \in \mathcal{B}$, one of the following holds with $\mathcal{A} =: \{A_0, \ldots, A_n\}$ and $\mathcal{B} =: \{B_0, \ldots, B_m\}$:

(i) there exists a set F such that $A_0 \subseteq F \subseteq A_0 \cup (E \smallsetminus B_0)$ and $\Psi(A_0, \ldots, A_n) > \Psi(F, A_1 \smallsetminus F, \ldots, A_n \smallsetminus F)$;

(ii) $\Psi(B_0, \ldots, B_m) \geq \Psi(B_0 \cup (E \smallsetminus A_0), B_1 \cap A_0, \ldots, B_m \cap A_0)$.

Let us translate this to our framework. Given $A \subseteq E$, let $\bar{A} := E \smallsetminus A$. Then $\mathcal{U} := \{(A, \bar{A}) : A \subseteq E\}$ is a universe. Given a partition function Ψ of E, let

$$S_k = \{(A, \bar{A}) \in \mathcal{U} : \Psi(A, \bar{A}) < k\} \quad \text{and} \quad S_k^- = \{(A, \bar{A}) \in S_k : |A| \leq 1\}.$$

Every partition $\mathcal{A} = \{A_0, \ldots, A_n\}$ defines a star $\{(A_0, \bar{A}_0), \ldots, (A_n, \bar{A}_n)\} \subseteq \mathcal{U}$, which we denote by $S(\mathcal{A})$. Let

$$\mathcal{F}_k := \{S(\mathcal{A}) : \mathcal{A} \in \mathcal{P}(E) \text{ and } \Psi(\mathcal{A}) < k\}.$$

If all the stars in $\mathcal{F}_k$ are subsets of S_k, we call Ψ *monotone*. All the weakly submodular partition functions used in [1] for applications are monotone, and we do not know whether any exist that are not.

Lemma 5.21. *If Ψ is weakly submodular, then S_k is $\mathcal{F}_k$-separable.*

Proof. Let $(W', U') \leq (U, W)$ be separations in S_k. Choose $(X, Y) \in S_k$ so that $(W', U') \leq (X, Y) \leq (U, W)$, with $\Psi(X, Y)$ minimum. We claim that (X, Y) is $\mathcal{F}_k$-linked to (U, W) and (Y, X) is $\mathcal{F}_k$-linked to (U', W'). By symmetry, it is enough to prove that (X, Y) is $\mathcal{F}_k$-linked to (U, W).

We first prove that (X, Y) is linked to (U, W). Let (A, B) be a separation in S_k such that $(A, B) \leq (U, W)$. Since Ψ is weakly submodular, one of the following holds:

(i) there exists F such that $X \subseteq F \subseteq X \cup A$ and $\Psi(X, Y) > \Psi(F, \bar{F})$;

(ii) $\Psi(B, A) \geq \Psi(B \cup Y, A \cap X)$.

Since $W' \subseteq X \subseteq F \subseteq X \cup A \subseteq U$, we have $(W', U') \leq (F, \bar{F}) \leq (U, W)$. So (i) does not hold, by the choice of (X, Y). So by (ii), $(A \cap X, B \cup Y) \in S_k$. This proves that (X, Y) is linked to (U, W).

Now let us show that stars can be shifted. Let

$$S = \{(A_i, B_i) : i = 0, \ldots, n\}$$

be a star in $\mathcal{F}_k \cap S_{(U,W)}$, with $(B_0, A_0) \leq (U, W)$. Thus, $\Psi(A_0, \ldots, A_n) < k$. We have to show that

$$S' = \left\{ (A'_i, B'_i) : i = 0, \ldots, n \right\} \in \mathcal{F}_k$$

for $(A'_i, B'_i) := f\downarrow^{(U,W)}_{(X,Y)} (A_i, B_i)$. Since $(A_i, B_i) \leq (B_0, A_0) \leq (U, W)$ for $i \geq 1$, we have $(A'_0, B'_0) = (A_0 \cup Y, B_0 \cap X)$, while $(A'_i, B'_i) = (A_i \cap X, B_i \cup Y)$ for $i \geq 1$.

By the minimal choice of (X, Y), there exists no F such that $X \subseteq F \subseteq X \cup B_0$ and $\Psi(X, Y) > \Psi(F, \bar{F})$ (as earlier). Applying the weak submodularity of Ψ with (X, Y) and $(A_0, \ldots, A_n)$, we deduce that

$$\Psi(A'_0, \ldots, A'_n) = \Psi(A_0 \cup Y, \ A_1 \cap X, \ \ldots, A_m \cap X) \leq \Psi(A_0, \ldots, A_n) < k.$$

Thus, $S' \in \mathcal{F}_k$. $\qquad\square$

In [1], a *k-bramble* for a weakly submodular partition function Ψ of E is a non-empty set $\mathcal{B}$ of pairwise intersecting subsets of E that contains an element from every partition $\mathcal{A}$ of E with $\Psi(\mathcal{A}) < k$. It is *non-principal* if it contains no singleton set $\{e\}$. In our terminology, Amini et al. [1] prove that there exists a non-principal k-bramble for Ψ if and only if there is no S_k-tree over $\mathcal{F}_k$ and rooted in S_k^-; they call this a 'partitioning k-search tree'.

Now any k-bramble $\mathcal{B}$ defines an orientation O of S_k: given $(A, B) \in S_k$ exactly one of A, B must lie in $\mathcal{B}$, and if B does we put (A, B) in O. Clearly O is consistent and avoids $\mathcal{F}_k$, and if $\mathcal{B}$ is non-principal it extends S_k^-. Conversely, given an orientation O of S_k, let $\mathcal{B} := \{B : (A, B) \in O\}$. If O is consistent, no two elements of $\mathcal{B}$ are disjoint. If O extends S_k^-, then $\mathcal{B}$ is non-principal. And finally, if Ψ is monotone and O avoids $\mathcal{F}_k$, then $\mathcal{B}$ contains an element from every partition $\mathcal{A} = \{A_1, \ldots, A_n\}$ of E with $\Psi(\mathcal{A}) < k$: since $S(\mathcal{A}) \in \mathcal{F}_k$ there is $(A_i, \bar{A}_i) \in S(\mathcal{A}) \smallsetminus O$, which means that $(\bar{A}_i, A_i) \in O$ and thus $A_i \in \mathcal{B}$.

Lemma 5.21 and Theorem 4.4 thus imply the duality theorem of Amini et al. [1] for monotone weakly submodular partition functions:

Theorem 5.22. *The following are equivalent for all monotone weakly submodular partition functions Ψ of a finite set E and $k > 0$:*

  (i) *There exists a non-principal k-bramble for Ψ.*

 (ii) *S_k has a consistent $\mathcal{F}_k$-avoiding orientation extending S_k^-.*

(iii) *There exists no S_k-tree over $\mathcal{F}_k$ rooted in S_k^-.*

(iv) *There exists no partitioning k-search tree.*

5.8   Further applications

It would be interesting to see whether other natural 'dense objects' than those discussed so far can be described in a tangle-like way, as orientations of a suitable set S of separations of a graph G that has no subset in some set $\mathcal{F} \subseteq 2^S$ of forbidden stars.

Answering a question of the first author, Bowler [3] answered this in the negative for complete minors, a natural candidate. Using the terminology of [8] for minors H of G, let us say that a separation (A, B) of G *points to* an $IH \subseteq G$ if this IH has a branch set in $B \smallsetminus A$ but none in $A \smallsetminus B$. A set of separations *points to* a given IH if each of its elements does. Clearly, for every $IK_k \subseteq G$ exactly one of (A, B) and (B, A) in S_k points to this IK_k.

Theorem 5.23. *For every $k \geq 5$ there exists a graph G such that for no $\mathcal{F} \subseteq 2^{S_k}$ are the consistent $\mathcal{F}$-avoiding orientations of S_k precisely the orientations of S_k that point to some $IK_k \subseteq G$.*

To prove this, Bowler considered as G a TK_k obtained by subdividing every edge of a K_k exactly once. He constructed an orientation O such that every star $S \subseteq O$ points to an IK_k but the entire O does not. This O, then, avoids every $\mathcal{F}$ consisting only of stars not pointing to any IK_k. But any $\mathcal{F} \subseteq 2^{S_k}$ such that the orientations of S_k pointing to an IK_k are precisely the consistent $\mathcal{F}$-avoiding ones must consist of stars not pointing to an IK_k, since any star that does is contained in the unique orientation of S_k pointing to the IK_k to which it points.

However, K_k minors can be captured by $\mathcal{F}$-avoiding orientations of S_k if we do not insist that $\mathcal{F}$ contain only stars but allow it to contain *weak stars*: sets of separations that pairwise either cross or point towards each other (formally: consistent antichains in S_k). In [9] we prove a duality theorem for orientations of separation systems avoiding such collections $\mathcal{F}$ of weak stars.

Acknowledgement

References

[1] O. Amini, F. Mazoit, N. Nisse, and S. Thomassé. Submodular partition functions. *Discrete Appl. Math.*, 309(20):6000–6008, 2009.

[2] D. Bienstock, N. Robertson, P. Seymour, and R. Thomas. Quickly excluding a forest. *J. Combin. Theory Ser. B*, 52(2):274–283, 1991.

[3] N. Bowler. Presentation at *Hamburg workshop on graphs and matroids*, Spiekeroog 2014.

[4] J. Carmesin, R. Diestel, M. Hamann, and F. Hundertmark. Canonical tree-decompositions of finite graphs I. Existence and algorithms. *J. Combin. Theory Ser. B*, to appear.

[5] J. Carmesin, R. Diestel, M. Hamann, and F. Hundertmark. Canonical tree-decompositions of finite graphs II. The parts. Preprint 2013.

[6] J. Carmesin, R. Diestel, M. Hamann, and F. Hundertmark. *k*-Blocks: a connectivity invariant for graphs. Preprint 2013.

[7] J. Carmesin, R. Diestel, F. Hundertmark, and M. Stein. Connectivity and tree structure in finite graphs. *Combinatorica*, 34(1):1–35, 2014.

[8] R. Diestel. *Graph Theory*. Springer, 4th edition, 2010.

[9] R. Diestel and S. Oum. Unifying duality theorems for width parameters, II. General duality. Preprint 2014.

[10] J. Geelen, B. Gerards, N. Robertson, and G. Whittle. Obstructions to branch-decomposition of matroids. *J. Combin. Theory (Series B)*, 96:560–570, 2006.

[11] P. Hliněný and G. Whittle. Matroid tree-width. *European J. Combin.*, 27(7):1117–1128, 2006.

[12] P. Hliněný and G. Whittle. Addendum to matroid tree-width. *European J. Combin.*, 30:1036–1044, 2009.

[13] F. Hundertmark. Profiles. An algebraic approach to combinatorial connectivity. arXiv:1110.6207, 2011.

[14] L. Lyaudet, F. Mazoit, and S. Thomassé. Partitions versus sets: a case of duality. *European J. Combin.*, 31(3):681–687, 2010.

[15] F. Mazoit. A simple proof of the tree-width duality theorem. arXiv:1309.2266, 2013.

[16] S. Oum and P. Seymour. Approximating clique-width and branch-width. *J. Combin. Theory Ser. B*, 96(4):514–528, 2006.

[17] J. Oxley. *Matroid Theory*. Oxford University Press, 1992.

[18] N. Robertson and P. Seymour. Graph minors. X. Obstructions to tree-decomposition. *J. Combin. Theory (Series B)*, 52:153–190, 1991.

[19] P. Seymour and R. Thomas. Graph searching and a min-max theorem for tree-width. *J. Combin. Theory (Series B)*, 58(1):22–33, 1993.

[20] P. Seymour and R. Thomas. Call routing and the ratcatcher. *Combinatorica*, 14(2):217–241, 1994.