

**HAMBURGER BEITRÄGE  
ZUR MATHEMATIK**

Heft 564

**On the structure of dense graphs with fixed  
clique number**

Heiner Oberkampf, Augsburg

Mathias Schacht, Hamburg

Version September 2015

# ON THE STRUCTURE OF DENSE GRAPHS WITH FIXED CLIQUE NUMBER

HEINER OBERKAMPF AND MATHIAS SCHACHT

ABSTRACT. We study structural properties of graphs with fixed clique number and high minimum degree. In particular, we show that there exists a function  $L = L(r, \varepsilon)$ , such that every  $K_r$ -free graph  $G$  on  $n$  vertices with minimum degree at least  $(\frac{2r-5}{2r-3} + \varepsilon)n$  is homomorphic to a  $K_r$ -free graph on at most  $L$  vertices. It is known that the required minimum degree condition is approximately best possible for this result.

For  $r = 3$  this result was obtained by Łuczak [*On the structure of triangle-free graphs of large minimum degree*, *Combinatorica* **26** (2006), no. 4, 489–493] and, more recently, Goddard and Lyle [*Dense graphs with small clique number*, *J. Graph Theory* **66** (2011), no. 4, 319–331] deduced the general case from Łuczak’s result. Łuczak’s proof was based on an application of Szemerédi’s regularity lemma and, as a consequence, it only gave rise to a tower-type bound on  $L(3, \varepsilon)$ . The proof presented here replaces the application of the regularity lemma by a probabilistic argument, which yields a bound for  $L(r, \varepsilon)$  that is doubly exponential in  $\text{poly}(\varepsilon)$ .

## §1. INTRODUCTION

1.1. **Notation.** We follow the notation from [4] and briefly review some of it below. The graphs we consider here are undirected, simple, and have no loops and for a graph  $G = (V, E)$  we denote by  $V = V(G)$  its *vertex set* and by  $E = E(G) \subseteq \binom{V}{2}$  its *edge set*. The number of vertices is finite and often denoted by  $n = |V|$ . By  $C_r$  and  $K_r$  we denote the *cycle* and the *complete graph/clique* on  $\ell$  vertices. For two adjacent vertices  $x, y$  we simply denote its edge by  $xy$ . If  $x \in X \subseteq V(G)$  and  $y \in Y \subseteq V(G)$  then  $xy$  is an  $X - Y$ -edge. For disjoint sets  $X$  and  $Y$  the set of all  $X - Y$ -edges is a subset of  $E$  and is denoted by  $E_G(X, Y)$ . Moreover, the number of  $X - Y$  edges in  $G$  is denoted by  $e_G(X, Y) = |E_G(X, Y)|$ . and for nonempty  $X$  and  $Y$  the *density* is defined by  $d_G(X, Y) = \frac{e_G(X, Y)}{|X||Y|}$ . The set of *neighbours* of a vertex  $v$  is denoted by  $N_G(v)$  and its size  $d_G(v) = |N_G(v)|$  is the *degree* of a vertex  $v$ , where we sometimes

---

2010 *Mathematics Subject Classification.* 05C35 (primary), 005C75 (secondary).

*Key words and phrases.* extremal graph theory, chromatic thresholds.

The second author was supported through the Heisenberg-Programme of the DFG.

suppress the subscript  $G$  if there is no danger of confusion. We denote the *minimum degree* of  $G$  by  $\delta(G)$ . For a subset  $U \subseteq V$  we define the *common (or joint) neighbourhood* of  $U$  as

$$N(U) = \bigcap_{u \in U} N(u)$$

and we emphasise that this differs from the notation in [4]. For later reference we note that the size of  $N(U)$  can be easily bounded from below in terms of the minimum degree of  $G = (V, E)$  by

$$|N(U)| \geq |U| \cdot \delta(G) - (|U| - 1) \cdot |V|. \quad (1)$$

Moreover, for a subset  $U \subseteq V$  we denote by  $G[U]$  the *induced subgraph on  $U$*  and we write  $G - U$  for  $G[V \setminus U]$ .

A (vertex) *colouring* of a graph  $G = (V, E)$  is a map  $c: V \rightarrow \mathcal{C}$  such that  $c(v) \neq c(w)$ , whenever  $v$  and  $w$  are adjacent. The elements of the set  $\mathcal{C}$  are called the available *colours*. A graph  $G = (V, E)$  is  *$k$ -colourable* if there exists colouring  $c: V \rightarrow [k] = \{1, \dots, k\}$ . The *chromatic number*  $\chi(G)$  is the smallest integer  $k$  such that  $G$  has is  $k$ -colourable.

A *homomorphism* from a graph  $G$  into a graph  $H$  is a mapping  $\varphi: V(G) \rightarrow V(H)$  which preserves adjacencies, i.e. for all pairs  $uv \in E(G)$  we have  $\varphi(u)\varphi(v) \in E(H)$ . We write  $G \xrightarrow{\text{hom}} H$  to indicate that some homomorphism  $\varphi$  exists. We say  $G$  is a *blow-up* of  $H$ , if  $G$  is obtained from  $H$  by replacing every vertex  $x$  of  $H$  by an independent sets  $I_x$  and edges of  $H$  correspond to complete bipartite graphs, i.e.,  $I_x, I_y$  spans a complete bipartite graph in  $G$  if  $xy \in E(H)$  and otherwise  $e_G(I_x, I_y) = 0$ . Clearly, if  $G$  is a blow-up of  $H$  then  $G \xrightarrow{\text{hom}} H$  and  $K_r \subseteq G$  if and only if  $K_r \subseteq H$ .

By  $F \subseteq G$  we mean that  $G$  contains a copy of  $F$ , that is there exists an injective homomorphism from  $F$  into  $G$ . If  $G$  contains no such copy of  $F$  (i.e.,  $F \not\subseteq G$ ), then we say  $G$  is  $F$ -free. For a graph  $F$

$$\text{Forb}(F) = \{G: F \not\subseteq G\}$$

denotes the class of  $F$ -free graphs, i.e. the collection of those graphs  $G$  which do not contain a copy of  $F$ . Moreover, we set

$$\text{Forb}_n(F) = \{G: F \not\subseteq G \text{ and } |V(G)| = n\}.$$

An  $F$ -free graph  $G = (V, E)$  is *maximal  $F$ -free* if the addition of any edge to  $G$  leads to a copy of  $F$  in  $G$ , i.e., for every  $xy \in \binom{V}{2} \setminus E$  we have  $F \subseteq (V, E \cup \{xy\})$ . Finally, we define the join  $G \vee H$  of two graphs  $G$  and  $H$  as the graph with vertex set  $V(G \vee H) = V(G) \cup V(H)$

and edge set

$$E(G \vee F) = E(G) \cup E(F) \cup \{vw : v \in V(G) \text{ and } w \in V(F)\}.$$

**1.2. Chromatic thresholds of graphs.** We are interested in structural properties of graphs  $G \in \text{Forb}(F)$ , which are forced by an additional minimum degree assumption on  $G$ . For example, if  $F$  is a clique and  $\delta(G)$  is sufficiently high, then Turán's theorem [16] assert that  $G$  is  $(r-1)$ -partite and, in particular, the chromatic number of  $G$  is bounded by a constant independent of  $|V(G)|$ . More generally, Andrásfai [2] raised the following question: *For a given graph  $F$  and an integer  $k$ , what is smallest condition imposed on the minimum degree  $\delta(G)$  such any graph  $G \in \text{Forb}(F)$  satisfying this minimum degree condition has chromatic number at most  $k$ ?* Here we are interested in the case when the minimum degree condition yields an upper bound on  $\chi(G)$  independent from the graph  $G$  itself. This leads to the so called *chromatic threshold* for a given graph  $F$

$$\delta_\chi(F) = \inf \{ \alpha \in [0, 1] : \exists k \in \mathbb{N} \text{ s.t. } \chi(G) \leq k \quad \forall G \in \text{Forb}(F) \text{ with } \delta(G) \geq \alpha |V(G)| \}.$$

If  $F' \subseteq F$ , then  $\text{Forb}(F') \subseteq \text{Forb}(F)$ , so obviously  $\delta_\chi(F') \leq \delta_\chi(F)$ . Moreover, it follows from the Erdős–Stone theorem [6] that  $\delta_\chi(F) \leq \frac{\chi(F)-2}{\chi(F)-1}$  for every graph  $F$  with at least one edge.

For  $F = K_3$  it was shown in [5] that  $\delta_\chi(K_3) \geq 1/3$ . In the other direction, Thomassen [14] obtained a matching upper bound and, therefore,  $\delta_\chi(K_3) = 1/3$ . In fact, Erdős and Simonovits [5] asked whether all triangle-free graphs  $G$  with  $\delta(G) \geq (1/3 + o(1))|V(G)|$  are 3-colorable. This was answered negatively by Häggkvist [8], but recently Brandt and Thomassé [3] showed that the chromatic number of such graphs is bounded by 4.

Nikiforov [12] and Goddard and Lyle [7] extended those results from triangles to  $r$ -cliques and showed for every  $r \geq 3$  that

$$\delta_\chi(K_r) = \frac{2r-5}{2r-3} \tag{2}$$

and, in fact,  $\chi(G) \leq r+1$  for every  $K_r$ -free graph  $G$  with  $\delta(G) > \frac{2r-5}{2r-3}|V(G)|$ .

In the case when  $F$  is an odd cycle of length at least five it was shown by Thomassen [15] that the chromatic threshold is zero and Łuczak and Thomassé [11] proved that  $\delta_\chi(F) \notin (0, 1/3)$  for all graphs  $F$  and that  $\delta_\chi(F) = 0$  if  $F$  is nearly bipartite (a graph is nearly bipartite if it is triangle-free and it admits a vertex partition into two parts such that one part is independent and the other part induces a graph with maximum degree one). Recently, Allen et al. [1] extended of the work of Łuczak and Thomassé and determined the chromatic threshold for every graph  $F$ .

**1.3. Homomorphism thresholds of graphs.** Viewing  $\chi(G) \leq k$  as the property that  $G \xrightarrow{\text{hom}} K_k$ , one may ask for a graph  $G \in \text{Forb}(F)$ , whether  $K_k$  can be replaced by a graph  $H$  of bounded size (independent of  $G$ ), that is  $F$ -free itself. More precisely, in [14] Thomassen posed the following question: *Given a fixed constant  $c$ , does there exist a finite family of triangle-free graphs such that every triangle-free graph on  $n$  vertices and minimum degree greater than  $cn$  is homomorphic to some graph of this family?* To formalise this question we define the *homomorphism threshold* of a graph  $F$

$$\delta_{\text{hom}}(F) = \inf \left\{ \alpha \in [0, 1] : \exists k \in \mathbb{N} \text{ s.t. } \forall G \in \text{Forb}(F) \text{ with } \delta(G) \geq \alpha|V(G)| \right. \\ \left. \exists H \in \text{Forb}_k(F) \text{ with } G \xrightarrow{\text{hom}} H \right\}.$$

Thomassen then asked to determine  $\delta_{\text{hom}}(K_3)$ . Since  $G \xrightarrow{\text{hom}} H$ , implies  $\chi(G) \leq |V(H)|$ , we clearly have

$$\delta_{\text{hom}}(F) \geq \delta_{\chi}(F).$$

In [10] Łuczak proved  $\delta_{\text{hom}}(K_3) = 1/3$  and, hence, for the triangle  $K_3$  the homomorphic and the chromatic threshold are equal. Recently, Goddard and Lyle [7] extended Łuczak's result showing, that  $K_r$ -free graphs with minimum degree bigger than  $\frac{2r-5}{2r-3}$  are homomorphic to the join  $K_{r-3} \vee H$ , where  $H$  is a triangle-free graph with  $\delta(H) > |V(H)|/3$ . Consequently, we have for every  $r \geq 3$

$$\delta_{\text{hom}}(K_r) = \delta_{\chi}(K_r) = \frac{2r-5}{2r-3}. \quad (3)$$

Łuczak's proof in [10] was based on Szemerédi's regularity lemma [13]. We give a different proof of (3), which avoids the regularity lemma and uses a simple probabilistic argument.

**Theorem 1.1.** *For every integer  $r \geq 3$  we have*

$$\delta_{\text{hom}}(K_r) = \frac{2r-5}{2r-3}.$$

It seems an interesting open question to determine the homomorphism threshold for other graphs than cliques. In particular, the case of odd cycles of length at least five seems to be a first interesting open case and we put forward the following question.

**Question 1.** What is  $\delta_{\text{hom}}(C_{2\ell+1})$  for  $\ell \geq 2$ ?

A somewhat related question concerns the homomorphism threshold for forbidden families of graphs. Note that the definitions of  $\text{Forb}(F)$  and  $\delta_{\text{hom}}(F)$  straight forwardly extend from one forbidden graph  $F$  to forbidden families  $\mathcal{F}$  of graphs. In view of Question 1 it is natural

to consider the family  $\mathcal{C}_{2\ell+1} = \{C_3, \dots, C_{2\ell+1}\}$  of odd cycles of length at most  $2\ell + 1$  and we close this introduction with the following open question.

**Question 2.** What is  $\delta_{\text{hom}}(\mathcal{C}_{2\ell+1})$  for  $\ell \geq 2$ ?

## §2. SIMPLE OBSERVATIONS

For an integer  $r \geq 3$  and  $\varepsilon > 0$  the following subclass of  $\text{Forb}(K_r)$  will play a prominent rôle

$$\mathcal{F}(r, \varepsilon) = \{G = (V, E) \in \text{Forb}(K_r) : \delta(G) \geq \left(\frac{2r-5}{2r-3} + \varepsilon\right)|V|\},$$

since Theorem 1.1 asserts that there exists some function  $L = L(r, \varepsilon)$  and  $H \in \text{Forb}(L, K_r)$  such that for every  $G \in \mathcal{F}(r, \varepsilon)$  we have  $G \xrightarrow{\text{hom}} H$ . Note that  $\mathcal{F}(r, \varepsilon)$  contains only graphs on at least  $r - 2$  vertices. We begin with a few observations concerning common neighbourhoods in maximal  $K_r$ -free graphs of given minimum degree  $\delta(G)$ .

**Proposition 2.1.** *For  $r \geq 3$  let  $G = (V, E)$  be a maximal  $K_r$ -free graph. If two distinct vertices  $u, v \in V$  are non-adjacent, then  $|N(u) \cap N(v)| \geq r\delta(G) - (r - 2)|V|$ .*

*Proof.* Since  $G = (V, E)$  is maximal  $K_r$ -free and  $uv \notin E$ , the joint neighbourhood  $N(u) \cap N(v)$  induces a  $K_{r-2}$ . Applying (1) to the  $r - 2$  vertices  $w_1, \dots, w_{r-2}$  that span  $K_{r-2}$  in the joint neighbourhood of  $u$  and  $v$  yields  $N(\{w_1, \dots, w_{r-2}\}) \geq (r - 2)\delta(G) - (r - 3)|V|$ . Moreover, since  $N(\{w_1, \dots, w_{r-2}\})$  must be disjoint from  $N(u) \cup N(v)$ , we obtain

$$\begin{aligned} |V| &\geq (r - 2)\delta(G) - (r - 3)|V| + |N(u) \cup N(v)| \\ &= (r - 2)\delta(G) - (r - 3)|V| + |N(u)| + |N(v)| - |N(u) \cap N(v)| \end{aligned}$$

and the proposition follows. □

In the proof of the last proposition we used the observation, that the neighbourhood of any two non-adjacent vertices in a maximal  $K_r$ -free graph induces a  $K_{r-2}$ . Next we note that for maximal  $K_r$ -free graphs in  $\mathcal{F}(r, \varepsilon)$ , we can strengthen this observation and ensure that the clique  $K_{r-2}$  is disjoint from an arbitrary given small set of vertices.

**Proposition 2.2.** *For  $r \geq 3$  and  $\varepsilon > 0$ , let  $G = (V, E)$  be a maximal  $K_r$ -free graph from  $\mathcal{F}(r, \varepsilon)$ . If two distinct vertices  $u, v \in V$  are non-adjacent in  $G$  and  $U \subseteq V$  satisfies  $|U| < \varepsilon|V|$ , then  $K_{r-2} \subseteq G[(N(u) \cap N(v)) \setminus U]$ .*

*Proof.* Given  $u, v$  and  $U$  as stated, we first consider any set of  $r - 3$  vertices  $w_1, \dots, w_{r-3} \in V$  and owing to (1) we have

$$N(\{w_1, \dots, w_{r-3}\}) \geq (r - 3)\delta(G) - (r - 4)|V|.$$

Moreover, since  $u$  and  $v$  are non-adjacent Proposition 2.1 tells us that

$$|N(u) \cap N(v)| \geq r\delta(G) - (r - 2)|V|.$$

Consequently, the joint neighbourhood of  $u, v$  and  $w_1, \dots, w_{r-3}$  satisfies

$$\begin{aligned} |N(\{u, v, w_1, \dots, w_{r-3}\})| &\geq N(\{w_1, \dots, w_{r-3}\}) - (|V| - |N(u) \cap N(v)|) \\ &\geq (2r - 3)\delta(G) - (2r - 5)|V| \end{aligned}$$

and the minimum degree condition from  $G \in \mathcal{F}(r, \varepsilon)$  implies that

$$|N(\{u, v, w_1, \dots, w_{r-3}\})| \geq (2r - 3)\varepsilon|V| \geq 3\varepsilon|V| > |U|.$$

Summarising, we have shown that any collection of  $r - 3$  vertices together with  $u$  and  $v$  have a joint neighbour outside of  $U$ . Selecting  $w_1$  from  $(N(u) \cap N(v)) \setminus U$  and inductively  $w_{i+1}$  from  $N(\{u, v, w_1, \dots, w_i\}) \setminus U$  for  $i = 1, \dots, r - 3$  yields the desired clique on  $w_1, \dots, w_{r-2}$ .  $\square$

The last observation asserts that any sufficiently large subset of vertices induces a  $K_{r-2}$  in a graph  $G$  from  $\mathcal{F}(r, \varepsilon)$ .

**Proposition 2.3.** *For  $r \geq 3$  and  $\varepsilon > 0$  let  $G = (V, E)$  be a graph from  $\mathcal{F}(r, \varepsilon)$ . If  $Z \subseteq V$  satisfies  $|Z| \geq (\frac{2r-6}{2r-3} + \varepsilon)|V|$ , then  $K_{r-2} \subseteq G[Z]$ .*

*Proof.* Similarly as in the proof of Proposition 2.2 we consider an arbitrary set of  $(r - 3)$  vertices  $w_1, \dots, w_{r-3} \in V$  and from (1) we infer

$$\begin{aligned} |N(\{w_1, \dots, w_{r-3}\}) \cap Z| &\geq (r - 3)\delta(G) - (r - 4)|V| - (|V| - |Z|) \\ &\geq \left( (r - 3)\frac{2r - 5}{2r - 3} + \frac{2r - 6}{2r - 3} - (r - 3) \right) |V| + (r - 2)\varepsilon|V| \\ &= (r - 2)\varepsilon|V| > 0. \end{aligned}$$

Consequently, any set of  $k - 3$  vertices has a joint neighbour in  $Z$ . Hence, selecting  $w_1$  in  $Z$  and inductively  $w_{i+1}$  from  $N(\{w_1, \dots, w_i\}) \cap Z$  for  $i = 1, \dots, r - 3$  yields the desired clique on  $w_1, \dots, w_{r-2}$ .  $\square$

## §3. PROOF OF THE MAIN RESULT

In the proof of Theorem 1.1 we partition the vertex set of a maximal  $K_r$ -free graph  $G \in \mathcal{F}(r, \varepsilon)$  into a bounded number of stable sets and show that any two such independent sets are spanning complete or empty bipartite graphs. Consequently,  $G$  is a blow-up of a  $K_r$ -free graph of bounded size, which is equivalent to the property that  $G$  has a  $K_r$ -free homomorphic image of bounded size.

We obtain the independent sets in two steps: Roughly speaking, in the first step we consider a random subset  $X \subset V(G)$  of bounded size and partition the vertices of  $V(G)$  according to their neighbourhood in  $X$ . However, since  $X$  has only bounded size, a small (but linear sized) set of vertices may have no or only a few neighbours in  $X$  and we deal with those vertices in the second step, by considering the neighbourhood into the independent sets from the first step.

*Proof of Theorem 1.1.* Let  $r \geq 3$ . Owing to (2) we have  $\delta_X(K_r) = \frac{2r-5}{2r-3}$  and since by definition  $\delta_X(K_r) \leq \delta_{\text{hom}}(K_r)$ , we have to prove the matching upper bound on  $\delta_{\text{hom}}(K_r)$ . Let  $\varepsilon > 0$  and set

$$m = \lceil 4 \ln(8/\varepsilon)/\varepsilon^2 \rceil + 1, \quad T = 2^m, \quad \text{and} \quad L = 2^T + T. \quad (4)$$

We will show that for any  $n > L$  and for every maximal  $K_r$ -free graph  $G = (V, E)$  from  $\mathcal{F}(r, \varepsilon)$  there exists some  $H \in \text{Forb}_L(K_r)$  such that  $G \xrightarrow{\text{hom}} H$ , which clearly suffices to prove the theorem.

In the first part we consider a random subset  $X \subseteq V$  of size  $m$  chosen uniformly at random from all  $m$ -element subsets of  $V$  and we consider the random set

$$U_X = \left\{ v \in V : |N(v) \cap X| < \left( \frac{2r-5}{2r-3} + \frac{\varepsilon}{2} \right) m \right\}$$

of vertices with ‘‘small’’ degree in  $X$ . We show that with positive probability  $|U_X| \leq \varepsilon n/4$  and  $|X \cap U_X| \leq \varepsilon m/4$ .

It follows from Chernoff’s inequality for the hypergeometric distribution (see, e.g., [9, Theorem 2.10, eq. (2.6)] that for a given vertex  $v \in V$  we have

$$\mathbb{P}(v \in U_X) \leq \exp(-\varepsilon^2 m/4). \quad (5)$$

Consequently,

$$\mathbb{E}[|U_X|] \leq \exp(-\varepsilon^2 m/4) \cdot n \stackrel{(4)}{<} \varepsilon n/8$$

and by Markov’s inequality we have

$$\mathbb{P}(|U_X| \leq \varepsilon n/4) > 1/2. \quad (6)$$

In other words, with probability more than  $1/2$  all but at most  $\varepsilon n/4$  vertices inherit approximately the minimum degree condition on the randomly chosen set  $X$ .

Next we show that with probability at least  $1/2$  the intersection of  $X$  with  $U_X$  is small. This follows from a standard double counting argument. In fact, the same argument giving (5) shows that for every  $v \in V$  there are at most  $\exp(-\varepsilon^2(m-1)/4) \binom{n-1}{m-1}$  different  $(m-1)$ -element subsets  $Y$  of  $V$  for which

$$|N(v) \cap Y| \leq \left(\frac{2r-5}{2r-3} + \frac{\varepsilon}{2}\right) \cdot (m-1). \quad (7)$$

Hence, there are at most  $n \exp(-\varepsilon^2(m-1)/4) \binom{n-1}{m-1}$  pairs  $(v, Y)$  such that (7) holds. Therefore, there are at most

$$\frac{n \cdot \exp(-\varepsilon^2(m-1)/4) \binom{n-1}{m-1}}{\varepsilon m/4} \stackrel{(4)}{\leq} \frac{1}{2} \binom{n}{m}$$

$m$ -element subsets  $X \subseteq V$  that contain at least  $\varepsilon m/4$  vertices  $v$  such that  $v$  and  $Y = X \setminus \{v\}$  satisfy (7). Combining this with (6) shows that there exists an  $m$ -element set  $X \subseteq V$  with the promised properties

$$|U_X| \leq \frac{\varepsilon}{4}n \quad \text{and} \quad |X \cap U_X| < \frac{\varepsilon}{4}m.$$

Finally, we set

$$Y = X \setminus U_X \quad \text{and} \quad U_Y = \left\{v \in V : |N(v) \cap Y| < \left(\frac{2r-5}{2r-3} + \frac{\varepsilon}{4}\right)|Y|\right\}$$

and we note that the induced subgraph on  $Y$  satisfies

$$G[Y] \in \mathcal{F}(r, \varepsilon/4)$$

and since  $U_Y \subseteq U_X$  we also have

$$|U_Y| \leq |U_X| \leq \varepsilon n/4.$$

Next we define a vertex partition of  $V \setminus U_Y$  given by the neighbourhoods in  $Y$ . For that we say two vertices  $v, w \in V \setminus U_Y$  are equivalent w.r.t.  $Y$ , if they have the same neighbours in  $Y$ , i.e.,  $N(v) \cap Y = N(w) \cap Y$ . Let  $V_1 \dot{\cup} \dots \dot{\cup} V_t = V \setminus U_Y$  be the corresponding partition given by the equivalence classes and let  $Y_i$  be the neighbourhood of the vertices from  $V_i$  in  $Y$ , i.e., for any  $v_i \in V_i$  we have

$$N(v_i) \cap Y = Y_i.$$

Clearly,  $t \leq 2^{|Y|} \leq 2^{|X|} = 2^m = T$ .

We observe that the vertex classes  $V_1, \dots, V_t$  are independent sets in  $G$ , i.e., for every  $i = 1, \dots, t$  we have

$$E_G(V_i) = \emptyset. \quad (8)$$

In fact, since every vertex  $v \in V \setminus U_Y$  has at least  $(\frac{2r-3}{2r-5} + \varepsilon/4)|Y|$  neighbours in  $Y$  and since  $G[Y] \in \mathcal{F}(r, \varepsilon/4)$  it follows from Proposition 2.3 applied to  $G[Y]$  and  $Z = Y_i$  that  $Y_i$  induces a  $K_{r-2}$ . Consequently, the  $K_r$ -freeness of  $G$  implies that no two vertices  $v_i, w_i \in V_i$  can be adjacent in  $G$  and (8) follows.

Next we observe that the induced bipartite graphs given by the partition of equivalence classes contain no or all edges, i.e., for every  $1 \leq i < j \leq t$  we have

$$e_G(V_i, V_j) = 0 \quad \text{or} \quad e_G(V_i, V_j) = |V_i||V_j|. \quad (9)$$

Suppose for a contradiction that there are (not necessarily distinct) vertices  $v_i, w_i \in V_i$  and  $v_j, w_j \in V_j$  such that  $v_i v_j \in E(V_i, V_j)$  and  $w_i w_j \notin E(V_i, V_j)$ . Due to the edge  $v_i v_j$  the intersection  $Y_i \cap Y_j$  must be  $K_{r-2}$ -free and, hence, in view of Proposition 2.3 applied to  $G[Y]$  and  $Z = Y_i \cap Y_j$  we have

$$|Y_i \cap Y_j| < \left( \frac{2r-6}{2r-3} + \frac{\varepsilon}{4} \right) |Y|$$

and, therefore,

$$|Y_i \cup Y_j| = |Y_i| + |Y_j| - |Y_i \cap Y_j| > \left( 2 \frac{2r-5}{2r-3} - \frac{2r-6}{2r-3} + \frac{\varepsilon}{4} \right) |Y| = \left( \frac{2r-4}{2r-3} + \frac{\varepsilon}{4} \right) |Y|. \quad (10)$$

Next we use that  $w_i \in V_i$  and  $w_j \in V_j$  are non-adjacent. Owing to the maximality of  $G$  we can apply Proposition 2.2 to  $G$  and  $U_Y$  and obtain a clique  $K_{r-2}$  outside  $U_Y$  in the joint neighbourhood of  $w_i$  and  $w_j$ . Let  $R$  be the vertex set of this  $K_{r-2}$ . Since  $R \subseteq V \setminus U_Y$  and since the sets  $V_k$  are independent for every  $k = 1, \dots, t$  the set  $R$  intersects  $r-2$  classes  $V_{k_1}, \dots, V_{k_{r-2}}$  different from  $V_i$  and  $V_j$ . We consider the joint neighbourhood of  $R$  in  $Y$

$$N(R) \cap Y = Y_{k_1} \cap \dots \cap Y_{k_{r-2}}$$

and note that

$$|N(R) \cap Y| \geq (r-2) \left( \frac{2r-5}{2r-3} + \frac{\varepsilon}{4} \right) |Y| - (r-3)|Y| = \left( \frac{1}{2r-3} + \frac{\varepsilon}{4} \right) |Y|.$$

However, combined with (10) this implies that either  $Y_i \cap N(R) \neq \emptyset$  or  $Y_j \cap N(R) \neq \emptyset$ . In either case this gives rise to a  $K_r$  in  $G$ , which yields the desired contradiction and (9) follows.

Note that (8) shows that  $G[V \setminus U_Y]$  is homomorphic to a graph  $H'$  on  $t \leq T$  and it follows from (9) that  $G[V \setminus U_Y]$  is a blow-up of  $H'$ . So in particular  $H'$  is  $K_r$ -free.

It remains to deal with the vertices in  $U_Y$ . For that we first observe that for every vertex  $u \in U_Y$  and  $i = 1, \dots, t$  we have

$$N(u) \cap V_i = \emptyset \quad \text{or} \quad N(u) \cap V_i = V_i. \quad (11)$$

In fact, suppose for a contradiction, that for some  $v_i, w_i \in V_i$  we have  $uv_i \in E$  while  $u$  and  $w_i$  are not adjacent. Again the maximality of  $G$  and Proposition 2.2 shows that  $N(u) \cap N(w_i)$  contains a  $K_{r-2}$  avoiding  $U_Y$ . However, since by (8) and (9) the vertices  $v_i$  and  $w_i$  have the same neighbourhood in  $V \setminus U_Y$  the same  $K_{r-2}$  is also in the neighbourhood of  $v_i$ , which together with  $v_i$  and  $u$  yields a  $K_r$  in  $G$ . This contradicts  $K_r \not\subseteq G$  and (11) follows.

Next we partition  $U_Y$  according to the neighbourhoods of its vertices in  $V \setminus U_Y$ . For every  $S \subseteq [t] = \{1, \dots, t\}$  we set

$$V_S = \left\{ u \in U_Y : N(u) \setminus U_Y = \bigcup_{s \in S} V_s \right\},$$

which yields a partition of  $U_Y$  into at most  $2^t \leq 2^T$  classes. Similar as in (9) and (11) we next observe that for any  $S, S' \subseteq [t]$  with  $S \neq S'$  we have

$$e_G(V_S, V_{S'}) = 0 \quad \text{or} \quad e_G(V_S, V_{S'}) = |V_S||V_{S'}|. \quad (12)$$

The proof is very similar to the proof of (11). Suppose for a contradiction without loss of generalisation there exist vertices  $v_S, w_S \in V_S$  and  $u \in V_{S'}$  such that  $uv_S \in E$  while  $u$  and  $w_S$  are not adjacent. Then by the maximality of  $G$  Proposition 2.2 yields a  $K_{r-2}$  in  $N(u) \cap N(w_S)$  avoiding  $U_Y$ . Owing to (11) the vertices  $v_S$  and  $w_S$  have the same neighbourhood in  $V \setminus U_Y$  and, hence, the same  $K_{r-2}$  is also in the neighbourhood of  $v_S$ , which together with  $v_S$  and  $u$  yields a  $K_r$  in  $G$ . This contradicts  $K_r \not\subseteq G$  and (12) follows.

The last thing we have to show is that  $V_S$  is independent in  $G$ , i.e., for every  $S \subseteq [t]$  we have

$$E_G(V_S) = \emptyset. \quad (13)$$

This is a direct consequence of (11) and Proposition 2.3. In fact, it follows from (11) that any two vertices  $u, v \in V_S$  have the same neighbourhood in  $V \setminus U_Y$ . Hence, their joint neighbourhood has size at least  $(\frac{2r-5}{2r-3} + \frac{3\varepsilon}{4})n$  and Proposition 2.3 yields a  $K_{r-2}$  in the joint neighbourhood of  $u$  and  $v$ . Therefore,  $u$  and  $v$  cannot be adjacent in  $G$  and (13) follows.

Summarising, we have shown that there exists a vertex partition

$$\bigcup_{i=1}^t V_i \cup \bigcup_{S \subseteq [t]} V_S = V$$

of  $V$  into independent sets (see (8) and (13)) such that all naturally induced bipartite graphs are either complete or empty (see (9), (11), and (12)). Hence, there exists a graph  $H$  on  $2^T + T \leq L$  vertices such that  $G$  is a blow-up of  $H$  and, therefore,  $G \xrightarrow{\text{hom}} H$  and  $H$  itself must be  $K_r$ -free. This concludes the proof of Theorem 1.1.  $\square$

We remark that the size  $H$  is doubly exponential in  $\text{poly}(1/\varepsilon)$ , i.e., there exists some universal constant  $c$  such that

$$|V(H)| \leq L = 2^{2^{c \ln(\varepsilon)/\varepsilon^2}}$$

holds.

We close by noting that the same approach used in the proof of Theorem 1.1 can be used to show Thomassen's result from [15] that the chromatic threshold of odd cycles of at least five is 0. However, it remains open, if this approach can also be used to address Questions 1 and 2.

#### 4. REFERENCES

- [1] P. Allen, J. Böttcher, S. Griffiths, Y. Kohayakawa, and R. Morris, *The chromatic thresholds of graphs*, Adv. Math. **235** (2013), 261–295, DOI 10.1016/j.aim.2012.11.016. MR3010059 [↑1.2](#)
- [2] B. Andrásfai, *Über ein Extremalproblem der Graphentheorie*, Acta Math. Acad. Sci. Hungar. **13** (1962), 443–455 (German). MR0145503 (26 #3034) [↑1.2](#)
- [3] St. Brandt and St. Thomassé, *Dense triangle-free graphs are four-colorable: A solution to the Erdős-Simonovits problem*. to appear. [↑1.2](#)
- [4] R. Diestel, *Graph theory*, 4th ed., Graduate Texts in Mathematics, vol. 173, Springer, Heidelberg, 2010. MR2744811 (2011m:05002) [↑1.1](#)
- [5] P. Erdős and M. Simonovits, *On a valence problem in extremal graph theory*, Discrete Math. **5** (1973), 323–334. MR0342429 (49 #7175) [↑1.2](#)
- [6] P. Erdős and A. H. Stone, *On the structure of linear graphs*, Bull. Amer. Math. Soc. **52** (1946), 1087–1091. MR0018807 (8,333b) [↑1.2](#)
- [7] W. Goddard and J. Lyle, *Dense graphs with small clique number*, J. Graph Theory **66** (2011), no. 4, 319–331, DOI 10.1002/jgt.20505. MR2791450 (2012f:05148) [↑1.2](#), [1.3](#)
- [8] R. Häggkvist, *Odd cycles of specified length in nonbipartite graphs*, Graph theory (Cambridge, 1981), North-Holland Math. Stud., vol. 62, North-Holland, Amsterdam-New York, 1982, pp. 89–99. MR671908 (84h:05079) [↑1.2](#)
- [9] S. Janson, T. Łuczak, and A. Ruciński, *Random graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000. MR1782847 (2001k:05180) [↑3](#)
- [10] T. Łuczak, *On the structure of triangle-free graphs of large minimum degree*, Combinatorica **26** (2006), no. 4, 489–493, DOI 10.1007/s00493-006-0028-8. MR2260851 (2007e:05077) [↑1.3](#), [1.3](#)
- [11] T. Łuczak and St. Thomassé, *Coloring dense graphs via VC-dimension*. submitted. [↑1.2](#)

- [12] V. Nikiforov, *Chromatic number and minimum degree of  $K_r$ -free graphs*. preprint. [↑1.2](#)
- [13] E. Szemerédi, *Regular partitions of graphs*, Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), Colloq. Internat. CNRS, vol. 260, CNRS, Paris, 1978, pp. 399–401 (English, with French summary). [MR540024 \(81i:05095\)](#) [↑1.3](#)
- [14] C. Thomassen, *On the chromatic number of triangle-free graphs of large minimum degree*, *Combinatorica* **22** (2002), no. 4, 591–596, DOI 10.1007/s00493-002-0009-5. [MR1956996 \(2004a:05058\)](#) [↑1.2, 1.3](#)
- [15] ———, *On the chromatic number of pentagon-free graphs of large minimum degree*, *Combinatorica* **27** (2007), no. 2, 241–243, DOI 10.1007/s00493-007-0054-1. [MR2321926 \(2008a:05101\)](#) [↑1.2, 3](#)
- [16] P. Turán, *Eine Extremalaufgabe aus der Graphentheorie*, *Mat. Fiz. Lapok* **48** (1941), 436–452 (Hungarian, with German summary). [MR0018405 \(8,284j\)](#) [↑1.2](#)

INSTITUT FÜR INFORMATIK, UNIVERSITÄT AUGSBURG, UNIVERSITÄTSSTRASSE 6A, D-86135 AUGSBURG, GERMANY

*E-mail address:* `heiner.oberkampf@informatik.uni-augsburg.de`

FACHBEREICH MATHEMATIK, UNIVERSITÄT HAMBURG, BUNDESSTRASSE 55, D-20146 HAMBURG, GERMANY

*E-mail address:* `schacht@math.uni-hamburg.de`