Tangles and the Mona Lisa

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Abstract

We show how an image can, in principle, be described by the tangles of the graph of its pixels.

The tangle-tree theorem provides a nested set of separations that efficiently distinguish all the distinguishable tangles in a graph. This translates to a small data set from which the image can be reconstructed.

The tangle duality theorem says that a graph either has a certain-order tangle or a tree-structure witnessing that this cannot exist. This tells us the maximum resolution at which the image contains meaningful information.

1   Introduction

Image recognition and tangles in a graph have a common theme: they seek to identify regions of high coherence, and to separate these from each other. In both cases, the regions are fuzzy and therefore hard to nail down explicitly.

Let K be a complete subgraph of order n in some graph G. For every separation $\{A, B\}$ of G of order $k < n/2$ (say), the entire subgraph K lies squarely on one side or the other: either A or B contains all its vertices. If it lies in B, say, we can *orient* the separation $\{A, B\}$ as (A, B) to indicate this.

Now let H be an $n \times n$ grid in the same graph G. Given a separation $\{A, B\}$ of order $k < n/2$, neither of the sides A, B must contain all of H. But one of them will contain more than $9/10$ of the vertices of H. If B is that side, we can therefore still orient $\{A, B\}$ as (A, B) to encode this information, even though there is a certain fuzziness about the fact that H lives only essentially in B, not entirely.

An orientation of all the separations of G of order less than some given integer k that H induces in this way is called a *tangle of order k*, or *k-tangle* in G.[1] A graph can have many distinct tangles of a given order – e.g., as induced by different complete subgraphs or grids.

Tangles were first introduced by Robertson and Seymour [5]. The shift of paradigm that they have brought to the study of connectivity in graphs is that while large coherent objects such as grids orient all the low-order separations of G in a consistent way (namely, towards them), it is meaningful to consider consistent orientations of all the low-order separations of G as 'objects' in their own right: one lays down some axioms saying what 'consistent' shall mean, and then various sensible notions of consistency give rise to different types of tangle, each a way of 'consistently' orienting all the low-order separations, but in slightly different senses of consistency.

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[1]Precise definitions will be given in Section 2.

1

If one tangle orients a given separation $\{A, B\}$ as (A, B) and another tangle orients it as (B, A), we say that $\{A, B\}$ *distinguishes* these tangles. Distinct tangles of the same order k are always distinguished by some separation: otherwise they would orient all the separations of order $< k$ in the same way, and thus be identical as tangles.

Since a given tangle, by definition, orients *every* separation of G of order $< k$ for some k, it is a complicated object that is not cheap to store, let alone easy to find. The set of *all* the tangles of a given graph, therefore, comes as a formidable data set: there can be exponentially many separations of given order, and this order can take all values between 0 and the order of the graph.

The *tangle-tree* theorem [5, 3], however, asserts that a linear number of separations of G – in fact, no more than $|G|$ – suffices to distinguish all the distinguishable tangles in G: whenever two tangles of G differ at all (i.e., on some separation), they also differ on a separation in this small set. Moreover, the separations in this set are nested, and hence squeeze G into a tree-structure: in the terminology of graph minors, there exists a tree-decomposition of G such that every maximal tangle 'lives in' a node of this tree, and the tree is minimal in that each of its nodes is home to a maximal tangle.

The main features of an image – such as the face in a portrait – are not unlike tangles: they are coherent regions which, given any way of cutting the picture in two along natural lines, would lie on one side or the other. Less blatant but still natural cutting lines might refine this division into regions, delineating a nose or an eye inside the face, for example. If we think of cutting lines as separations, whose order is the lower the more natural the line is to cut along, then the large and obvious features of the image (such as the faces in it) would orient all the low-order separations (towards them), while the more refined features such as eyes and noses would orient also some higher-order separations towards them. Thus, all it needs in order to capture the natural regions of an image as tangles is to define the order of separations in such a way that natural cutting lines have low order. We shall do this in Section 3.

What does the tangle-tree theorem offer when applied to this scenario? The nested set of separations which it provides translates to a set of cutting lines that do not cross each other. In other words, it finds exactly the right lines that we need in order to divide the picture into smaller and smaller regions without cutting through coherent features prematurely. The nodes of the decomposition tree will correspond to the regions of the image that are left when all the lines are drawn. Just as the tangles can be identified from just the nested set of separations provided by the tangle-tree theorem, the distinctive regions – the features – of the image can be recovered from these carefully chosen cutting lines.

These regions are likely to differ in importance: some are carved out early, by particularly natural lines, and would correspond to a tree node that is home to a tangle that has low order but is nevertheless maximal (is not induced by a higher-order tangle). Others are smaller, and appear only as the refining feature (such as an eye) of what was a region cut out by some obvious early lines (such as a face). For example, a portrait showing a face before a black backdrop would, at a modest resolution, capture the backdrop as a low-order tangle that does not get refined further, while the face is another low-order tangle that will be refined by several higher-order tangles (one each for an eye, the nose, an ear etc), which in turn may get refined further: maybe just one or two levels further in the case of the nose, but several levels for the eyes, so as to capture

the details of the iris if these are clearly distinct.)

There is another theorem about tangles that can throw a light on the features of an image: the tangle duality theorem [5, 4]. This theorem tells us something about the structure of the graph if it has *no* tangle of some given order k: it then has an overall tree structure, a *tree-decomposition* into small parts. In our application, this tree structure would once more come as a set of non-crossing cutting lines. But this time the regions left by these lines are small: too small to count as features.

For example, a picture showing a pile of oranges might have many tangles up to some order k, one for each orange, which are carved out by cutting lines around those oranges. But as soon as we allow less obvious cutting lines than these, we would admit lines cutting right through an orange. Since a single orange does not have interesting internal features, these lines would criss-cross through the oranges indiscriminately, without any focus giving them consistency. In tangle terms, this would be becauses there simply is no tangle of higher order inducing (i.e., refining) the tangle of an orange. The tangle duality theorem would then point this out but cutting up the entire image into tiny disjoint regions corresponding to the nodes of the tree-decomposition it affords. In the language of complexity theory: the tree-decomposition which the theorem assures us to exist if there is no high-order tangle will be an efficiently checkable witness to the fact there *cannot* be such a tangle: that we were not just not clever enough to spot it.

This paper is organised as follows. In Section 2 we give a reasonably self-contained introduction to the theory of separation systems and abstract tangles in discrete structures (not just graphs), which leads up to formally precise statements of the versions of the tangle-tree theorem and the tangle duality theorem that we shall wish to apply. In Section 3 we translate the scenario of image recognition and image compression into the language of abstract separation systems and tangles, and apply the two theorems.

We wish to stress that we see this paper in no way as a serious contribution to any practical image recognition or compression problem. Rather, it is our hope to inspire those that know more about these things than we do, by offering some perhaps unexpected and novel ideas from the world of tangles. Its character, therefore, is deliberately one of 'proof of concept'. In particular, the correspondence that we shall describe between the features of an image and the tangles in a suitable abstract separation system is the simplest possible one that we could think of: it works, but it can doubtless be improved. But this we leave to more competent hands.

2 Abstract separation systems and tangles

Our aim in this section is to give a brief introduction to so-called abstract separation systems and their tangles, just enough to state precisely the two theorems we wish to apply to images recognition: the tangle-tree theorem and the tangle duality theorem. In the interest of brevity, we describe the technical setup more or less directly, without much additional motivation of why things are defined the way they are. Readers interested in this will find such discussions in [1, 2, 4, 3].

2.1 Separations of sets

Given an arbitrary set V, a *separation* of V is a set $\{A, B\}$ of two subsets A, B such that $A \cup B = V$. Every such separation $\{A, B\}$ has two *orientations*: (A, B) and (B, A). Inverting these is an involution $(A, B) \mapsto (B, A)$ on the set of these *oriented separations* of V.

The oriented separations of a set are partially ordered as

$$(A, B) \leq (C, D) :\Leftrightarrow A \subseteq C \text{ and } B \supseteq D.$$

Our earlier involution reverses this ordering:

$$(A, B) \leq (C, D) \Leftrightarrow (B, A) \geq (D, C).$$

The oriented separations of a set form a lattice under this partial ordering, in which $(A \cap C, B \cup D)$ is the infimum of (A, B) and (C, D), and $(A \cup C, B \cap D)$ is their supremum. The infima and suprema of two separations of a graph or matroid are again separations of that graph or matroid, so these too form lattices under $\leq$.

Their induced posets of all the oriented separations of order $< k$ for some fixed k, however, need not form such a lattice: when (A, B) and (C, D) have order $< k$, this need not be the case for $(A \cap C, B \cup D)$ and $(A \cup C, B \cap D)$.

2.2 Abstract separation systems

A *separation system* $(\vec{S}, \leq, ^*)$ is a partially ordered set $\vec{S}$ with an order-reversing involution *.

The elements of a separation system $\vec{S}$ are called *oriented separations*. When a given element of $\vec{S}$ is denoted as $\vec{s}$, its *inverse* $\vec{s}^*$ will be denoted as $\overleftarrow{s}$, and vice versa. The assumption that * be *order-reversing* means that, for all $\vec{r}, \vec{s} \in \vec{S}$,

$$\vec{r} \leq \vec{s} \ \Leftrightarrow \ \overleftarrow{r} \geq \overleftarrow{s}. \tag{1}$$

For subsets $R \subseteq \vec{S}$ we write $R^* := \{ \overleftarrow{r} \mid \vec{r} \in R \}$.

An (unoriented) *separation* is a set of the form $\{ \vec{s}, \overleftarrow{s} \}$, and then denoted by s.[2] We call $\vec{s}$ and $\overleftarrow{s}$ the *orientations* of s.

When a separation is introduced ahead of its elements and denoted by a single letter s, its elements will then be denoted as $\vec{s}$ and $\overleftarrow{s}$. Given a set $R \subseteq S$ of separations, we write $\vec{R} := \bigcup R \subseteq \vec{S}$ for the set of all the orientations of its elements. With the ordering and involution induced from $\vec{S}$, this is again a separation system.

A separation $\vec{r} \in \vec{S}$ is *trivial in* $\vec{S}$, and $\overleftarrow{r}$ is *co-trivial*, if there exists $s \in S$ such that $\vec{r} < \vec{s}$ as well as $\vec{r} < \overleftarrow{s}$. Note that if $\vec{r}$ is trivial in $\vec{S}$ then so is every $\vec{r}' \leq \vec{r}$. A separation $\vec{s}$ is *small* if $\vec{s} \leq \overleftarrow{s}$. An unoriented separation is *proper* if it has no small orientation.

If there are binary operations $\vee$ and $\wedge$ on a separation system $(\vec{U}, \leq, ^*)$ such that $\vec{r} \vee \vec{s}$ is the supremum and $\vec{r} \wedge \vec{s}$ the infimum of $\vec{r}$ and $\vec{s}$ in $\vec{U}$, we call $(\vec{U}, \leq, ^*, \vee, \wedge)$ a *universe* of (oriented) separations. By (1), it satisfies De Morgan's law:

$$(\vec{r} \vee \vec{s})^* = \overleftarrow{r} \wedge \overleftarrow{s}. \tag{2}$$

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[2]To smooth the flow of the narrative we usually also refer to oriented separations simply as 'separations' if the context or use of the arrow notation $\vec{s}$ shows that they are oriented.

The universe $\vec{U}$ is *submodular* if it comes with a submodular *order function*, a real function $\vec{s} \mapsto |\vec{s}|$ on $\vec{U}$ that satisfies $0 \le |\vec{s}| = |\overleftarrow{s}|$ and

$$|\vec{r} \vee \vec{s}| + |\vec{r} \wedge \vec{s}| \le |\vec{r}| + |\vec{s}|$$

for all $\vec{r}, \vec{s} \in \vec{U}$. We call $|s| := |\vec{s}|$ the *order* of s and of $\vec{s}$. For every integer $k > 0$, then,

$$\vec{S}_k := \{\, \vec{s} \in \vec{U} : |\vec{s}| < k \,\}$$

is a separation system – but not necessarily a universe, because $\vec{S}_k$ is not normally closed under $\wedge$ and $\vee$.

Separations of a set V, and their orientations, are clearly an instance of this abstract setup if we identify $\{A, B\}$ with $\{(A, B), (B, A)\}$. The small separations of V are those of the form (A, V), the proper ones those of the form $\{A, B\}$ with $A \smallsetminus B$ and $B \smallsetminus A$ both nonempty.

The separations of V even form a universe: if $\vec{r} = (A, B)$ and $\vec{s} = (C, D)$, say, then $\vec{r} \vee \vec{s} := (A \cup C, B \cap D)$ and $\vec{r} \wedge \vec{s} := (A \cap C, B \cup D)$ are again separations of V, and are the supremum and infimum of $\vec{r}$ and $\vec{s}$, respectively.

2.3 Orienting a separation system

Given a separation system $(\vec{S}, \le, {}^*)$, a subset $O \subseteq \vec{S}$ is an *orientation* of S (and of $\vec{S}$) if $O \cup O^* = \vec{S}$ and $|O \cap \{\vec{s}, \overleftarrow{s}\}| = 1$ for all $s \in S$. Thus, O contains exactly one orientation of every separation in S. For subsets $R \subseteq S$ we say that O *induces* and *extends* the orientation $O \cap \vec{R}$ of R, and thereby *orients* R.

A set $O \subseteq \vec{S}$ is *consistent* if there are no distinct $r, s \in S$ with orientations $\vec{r} < \vec{s}$ such that $\overleftarrow{r}, \vec{s} \in O$. Consistent orientations of $\vec{S}$ contain all its trivial separations.

We say that $s \in S$ *distinguishes* two orientations O, O' of subsets of S if s has orientations $\vec{s} \in O$ and $\overleftarrow{s} \in O'$. (We then also say that $\vec{s}$ and $\overleftarrow{s}$ themselves distinguish O from O'.) The sets O, O' are *distinguishable* if neither is a subset of the other: if there exists some $s \in S$ that distinguishes them. A set $T \subseteq S$ *distinguishes* O from O' if some $s \in T$ distinguishes them, and T *distinguishes* a set $\mathcal{O}$ of orientations of subsets of S if it distinguishes its elements pairwise.

Distinct orientations of the same separation system S are distinguished by every $s \in S$ on which they disagree, i.e. for which one of them contains $\vec{s}$, the other $\overleftarrow{s}$. Distinct orientations of different separation systems, however can be indistinguishable. For $R \subsetneqq S$, for example, the orientations of S will be indistinguishable from the orientations of R they induce.

2.4 Tree sets of separations

Given a separation system $(\vec{S}, \le, {}^*)$, two separations $r, s \in S$ are *nested* if they have comparable orientations; otherwise they *cross*. Two oriented separations $\vec{r}, \vec{s}$ are *nested* if r and s are nested. We say that $\vec{r}$ *points towards* s, and $\vec{r}$ *points away from* s, if $\vec{r} \le \vec{s}$ or $\vec{r} \le \overleftarrow{s}$. Then two nested oriented separations are either comparable, or point towards each other, or point away from each other.

A set of separations is *nested* if every two of its elements are nested. Two sets of separations are *nested* if every element of the first set is nested with every element of the second. A *tree set* is a nested separation system without trivial

or degenerate elements. A tree set is *regular* if none of its elements is small [2]. When $\vec{T} \subseteq \vec{S}$ is a tree set, we also call $T \subseteq S$ a *tree set* (and *regular* if $\vec{T}$ is regular).

For example, the set of orientations (u, v) of the edges uv of a tree T form a regular tree set with respect to the involution $(u, v) \mapsto (v, u)$ and the *natural partial ordering* on $\vec{E}(T)$: the ordering in which $(x, y) < (u, v)$ if $\{x, y\} \neq \{u, v\}$ and the unique $\{x, y\}$–$\{u, v\}$ path in T joins y to u. The oriented bipartitions of $V(T)$ defined by deleting an edge of T form a tree set isomorphic to this one.

Note that, in this latter example, the nodes of the tree T correspond bijectively to the consistent orientations of its edge set: orienting every edge towards some fixed node t is consistent, and conversely, every consistent orientation of the edges of a tree has a unique sink t towards which every edge is oriented. Given an arbitrary tree set of separations, we may thus think of its consistent orientations as its 'nodes', and of its elements as edges between these nodes, joining them up into a graph-theoretical tree.

Every consistent orientation of a separation system $\vec{S}$ can be recovered from the set σ of its maximal elements: it is precisely the down-closure of σ in $\vec{S}$. Hence the consistent orientations of a tree set can be represented uniquely by just their sets of maximal elements. Such a set σ will always be a star of separations. The stars $\sigma \subseteq \vec{T}$ in a tree set $\vec{T}$ of this form, those for which $\vec{T}$ has a consistent orientation whose set of maximal elements is precisely σ, are the *splitting stars* of the tree set $\vec{T}$. The splitting stars in the set of oriented edges of a graph-theoretical tree are precisely the stars σ_t of all the edges at a fixed node t, oriented towards t. [2]

2.5   Profiles of separation systems

Let $(\vec{S}, \leq, ^*)$ be a separation system inside a universe $(\vec{U}, \leq, ^*, \vee, \wedge)$. An orientation P of S is a *profile* (of S or $\vec{S}$) if it is consistent and satisfies

> For all $\vec{r}, \vec{s} \in P$ the separation $\overleftarrow{r} \wedge \overleftarrow{s} = (\vec{r} \vee \vec{s})^*$ is not in P.   (P)

Thus if P contains $\vec{r}$ and $\vec{s}$ it also contains $\vec{r} \vee \vec{s}$, unless $\vec{r} \vee \vec{s} \notin \vec{S}$.

Most natural orientations of a separation system that orient all its elements 'towards' some large coherent structure (see the Introduction) will be both consistent and satisfy (P): two oriented separations pointing away from each other (as in the definition of consistency) will hardly both point towards that structure, and similarly if $\vec{r}$ and $\vec{s}$ both point to the structure then their supremum $\vec{r} \vee \vec{s}$ will do so too if it is in $\vec{S}$: otherwise the structure would be squeezed between $\vec{r}$ and $\vec{s}$ on the one hand, and $(\vec{r} \vee \vec{s})^* = \overleftarrow{r} \wedge \overleftarrow{s}$ on the other, and could hardly be large. In our intended application to images, all orientations of some set of dividing lines of the picture towards one of its features will trivially satisfy (P) and thus be a profile.

A profile is *regular* if it contains no separation whose inverse is small. Since all our separations will be proper, all our profiles will be regular.

Note that every subset Q of a profile of $\vec{S}$ is a profile of $\vec{R} = Q \cup Q^* \subseteq \vec{S}$. Put another way, if P is a profile of S and $R \subseteq S$, then $P \cap \vec{R}$ is a profile of R, which we say is *induced* by P.

When $\vec{U}$ is a universe of separations, we call the profiles of the separation systems $\vec{S}_k \subseteq \vec{U}$ of its separations of order $< k$ the *k-profiles in $\vec{U}$*. For $\ell < k$, then, every k-profile P in $\vec{U}$ induces an ℓ-profile.

2.6 The tangle-tree theorem for profiles

We are nearly ready to state the tangle-tree theorems for profiles. These come as a pair: in a canonical version, and in a refined, but non-canonical, version.

The canonical version finds, in any submodular universe $\vec{U}$ of separations, a 'canonical' tree set of separations that distinguishes all the 'robust and regular' profiles in $\vec{U}$. Here, a tree set $\vec{T} \subseteq \vec{U}$ is *canonical* if the map $\vec{U} \mapsto \vec{T}$ commutes with isomorphisms of universes of separations, which are defined in the obvious way [3]. The definition of 'robust' is more technical, but we do not need it: all we need is the following immediate consequence of its definition:

Lemma 2.1. [3] *Every profile of a set of bipartitions of a set is robust.* □

Finally, the tree set T found by the tangle-tree theorems will will distinguish the relevant profiles P, P' *efficiently*: from among all the separation in S that distinghish P from P', the set T will one of minimum order.

Here, then, is the canonical tangle-tree theorem for profiles in its simplest form [3, Corollary 2.10]:

Theorem 2.2. (The canonical tangle-tree theorem for profiles)
Let $\vec{U} = (\vec{U}, \leq, ^*, \vee, \wedge, |\ |)$ be a submodular universe of separations. There is a canonical regular tree set $T \subseteq \vec{U}$ that efficiently distinguishes all the distinguishable robust and regular profiles in $\vec{U}$.

Note that Theorem 2.2 finds a tree set of separations that distinguishes *all* the profiles in $\vec{U}$ that we can ever hope to distinguish. Sometimes, however, we only need to distinguish some of them, say those in some particular set $\mathcal{P}$ of profiles. In this case a smaller tree set T of separations will suffice. The non-canonical tangle-tree theorem says that, if we do not care about canonicity, we can make T as small as we theoretically can: so small that deleting any one separation from T will lose it its property of distinguishing $\mathcal{P}$.

As a nice consequence of this minimality, the non-canonical tangle-tree theorem assigns the profiles it distinguishes, those in $\mathcal{P}$, to the 'nodes of the tree' which T defines. Since T is just a tree set these are, of course, undefined. However, we saw in Section 2.4 that *if* a tree set of separations is the edge set of a tree, or equivalently the set of nested bipartition of the vertices of a tree induced by its edges, then the nodes of this tree correspond to exactly the consistent orientations of its edges.

As $T \subseteq U$, every profile in $\vec{U}$ will orient some of the separations in T, and it will do so consistently. Such a consistent partial orientation of T will, in general, extend to many consistent orientations of all of T. The partial orientations induced by the profiles in $\mathcal{P}$, however, extend uniquely, and thus lead to a bijection mapping $\mathcal{P}$ to the 'nodes' of T, its consistent orientations:

Theorem 2.3. (The non-canonical tangle-tree theorem for profiles [3])
Let $\vec{U} = (\vec{U}, \leq, ^*, \vee, \wedge, |\ |)$ be a submodular universe of separations. For every robust set $\mathcal{P}$ of regular profiles in $\vec{U}$ there is a regular tree set $T \subseteq \vec{U}$ of separations such that:

(i) *every two profiles in $\mathcal{P}$ are efficiently distinguished by some separation in T;*

(ii) *no proper subset of T distinguishes $\mathcal{P}$;*

(iii) *for every $P \in \mathcal{P}$ the set $P \cap \vec{T}$ extends to a unique consistent orientation O_P of T. This map $P \mapsto O_P$ from $\mathcal{P}$ to the set $\mathcal{O}$ of consistent orientations of T is bijective.*

2.7 Tangles of separation systems, and duality

A set σ of oriented separations in some given separation system $(\vec{S}, \leq, ^*)$ is a *star of separations* if they point towards each other: if $\vec{r} \leq \overleftarrow{s}$ for all distinct $\vec{r}, \vec{s} \in \sigma$. In particular, stars of separations are nested. They are also consistent: if $\overleftarrow{r}, \vec{s}$ lie in the same star we cannot have $\vec{r} < \vec{s}$, since also $\vec{s} \leq \vec{r}$ by the star property.

An *S-tree* is a pair (T, α) of a tree T and a function $\alpha \colon \vec{E}(T) \to \vec{S}$ such that, for every edge xy of T, if $\alpha(x, y) = \vec{s}$ then $\alpha(y, x) = \overleftarrow{s}$. It is an *S-tree over* $\mathcal{F} \subseteq 2^{\vec{S}}$ if for every node t of T we have $\alpha(\vec{F}_t) \in \mathcal{F}$, where

$$\vec{F}_t := \{(x, t) : xt \in E(T)\}.$$

We shall call the set $\vec{F}_t \subseteq \vec{E}(T)$ the *oriented star at t* in T. Usually, $\mathcal{F}$ will be a set of stars in $\vec{S}$.

An *$\mathcal{F}$-tangle* of S is a consistent orientation of S that *avoids* $\mathcal{F}$, that is, has no subset $\sigma \in \mathcal{F}$. Since all consistent orientations of $\vec{S}$ contain all its trivial elements, any $\mathcal{F}$-tangle of S will also be an $\mathcal{F}'$-tangle if $\mathcal{F}'$ is obtained from $\mathcal{F}$ by adding all singletons $\{\overleftarrow{r}\}$ with $\vec{r}$ trivial in S. Sets $\mathcal{F}$ that already contain all these singleton stars will be called *standard*.

The tangle duality theorem for abstract separation systems $\vec{S}$ says that, under certain conditions, if $\mathcal{F} \subseteq 2^{\vec{S}}$ is a standard set of stars there is either an $\mathcal{F}$-tangle of S or an S-tree over $\mathcal{F}$, but never both. The conditions, one on $\vec{S}$ and another on $\mathcal{F}$, are as follows.

A separation $\vec{s_0} \in \vec{S}$ is *linked to* another separation $\vec{r} \leq \vec{s_0}$ if every[3] $\vec{s} \geq \vec{r}$ satisfies $\vec{s} \vee \vec{s_0} \in \vec{S}$. The separation system $\vec{S}$ is *separable* if for every two nontrivial $\vec{r}, \vec{r}' \in \vec{S}$ such that $\vec{r} \leq \vec{r}'$ there exists an $s_0 \in S$ with an orientation $\vec{s_0}$ linked to $\vec{r}$ and its inverse $\overleftarrow{s_0}$ linked to $\vec{r}'$. This may sound complicated, but we do not need to understand the complexity of this here: the separation system in our intended application will easily be seen to be separable, which will allow us to apply the tangle-tree duality theorem and benefit from what it says, which does not involve the notion of separability.

Similarly, $\mathcal{F}$ has to satisfy a complicated condition, which we shall have to check but do not need to understand. Let $\vec{r} \leq \vec{s_0} \leq \vec{r}'$ be as above, with $\vec{s_0}$ linked to $\vec{r}$ and $\overleftarrow{s_0}$ linked to $\vec{r}'$. For every $\vec{s} \in \vec{S}$ with $\vec{r} \leq \vec{s} \ (\neq \overleftarrow{r})$, let

$$f{\downarrow}_{\vec{s_0}}^{\vec{r}}(\vec{s}) := \vec{s} \vee \vec{s_0} \quad \text{and} \quad f{\downarrow}_{\vec{s_0}}^{\vec{r}}(\overleftarrow{s}) := (\vec{s} \vee \vec{s_0})^*.$$

We say that $\mathcal{F}$ is *closed under shifting* if, for all such $\vec{r}, \vec{r}'$ and $\vec{s_0}$, the image under $f{\downarrow}_{\vec{s_0}}^{\vec{r}}$ of every star $\sigma \in \mathcal{F}$ containing a separation $\vec{s} \geq \vec{r}$ again lies in $\mathcal{F}$. (Note that $f{\downarrow}_{\vec{s_0}}^{\vec{r}}(\vec{s}')$ is defined for all $\vec{s}' \in \sigma$, since $\vec{r} \leq \vec{s} \leq \vec{s}'$ by the star property.)

The premise of the tangle duality theorem for abstract separation systems contains a condition that combines these two assumptions into one; it requires $\vec{S}$

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[3]Technically, we also need this only for $\vec{s} \neq \vec{r}$, and only when $\vec{r} \neq \overleftarrow{r}$. Since all the separations we consider in this paper are proper, however, both these will hold automatically: if $\vec{s} = \vec{r}$ then $\vec{r} \leq \vec{s} = \vec{r}$ is small, and likewise if $\vec{r} = \overleftarrow{r}$.

to be '$\mathcal{F}$-separable'. As before, we do not need to understand what this means, since it is an easy consequence of the two assumptions just discussed:

Lemma 2.4. [4] *If $\vec{S}$ is separable and $\mathcal{F}$ is closed under shifting, then $\vec{S}$ is $\mathcal{F}$-separable.*

Here, then, is the tangle duality theorem for separation systems:

Theorem 2.5. (Strong tangle duality theorem [4, Theorem 4.4])
Let $(\vec{U}, \leq, {}^*, \vee, \wedge)$ be a universe of separations containing a separation system $(\vec{S}, \leq, {}^*)$. Let $\mathcal{F} \subseteq 2^{\vec{S}}$ be a standard set of stars. If $\vec{S}$ is $\mathcal{F}$-separable, exactly one of the following assertions holds:

(i) *There exists an S-tree over $\mathcal{F}$.*

(ii) *There exists an $\mathcal{F}$-tangle of S.*

3 Images as collections of tangles

We finally come to apply our theory to image recognition and compression. The basic idea is as follows. In our attempt to identify, or perhaps just to store, the main features of a given image, we consider lines cutting the canvas in two. Lines that cut across a highly coherent region, a region whose pixels are all similar, will be a bad line to cut along, so we assign such lines a high 'order' when we view it as a separation of the picture.

The coherent regions of our image will thus correspond to the tangles, or profiles,[4] of some order, the higher the greater their coherence. Or more precisely: we *define* 'regions' as tangles in the universe of separations given by those cuts. The novelty of this approach is that, as with tangles in a graph, we make no attempt to assign individual pixels to one region or another.

However, we shall be able to outline those regions. If we wish to outline some particular set R of regions, we apply the non-canonical tangle-tree theorem to the set $\mathcal{P}$ of profiles given by the regions in R. The theorem will return a nested set T of 'separations', which corresponds to a set L of cutting lines. As T is nested, these lines will not cross each other. There are enough of them that for every pair of regions in R we shall find one that runs between them, separating them in the picture. Since T will distinguish $\mathcal{P}$ efficiently, we shall even find such a line in L whose order is as small as that of any line separating these two regions in the entire picture: it will cut through as few 'like' pixels as possible for any line separating these two regions. Finally, since the profiles in $\mathcal{P}$ correspond bijectively to the consistent orientations, and hence to the splitting stars, of T, the regions in R will correspond to what these splitting stars translate to: the set of all lines in L that are closest to that region, thus outlining it.

Finally, we have the tangle duality theorem. It tells us that if a separation system has no k-profile for some k, it has an S-tree over the set $\mathcal{F}$ of stars violating the profile condition (P). This means that if our picture has no region whose coherence is so high that lines of order $< k$ cannot cut across it, we can

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[4]In informal discourse, we often say 'tangle' instead of 'profile', although formally tangles are just a particular type of profile. The sloppiness is justified, however, by the fact that every k-profile in a universe of bipartitions of a set, as considered here, is an $\mathcal{F}$-tangle of S_k for the set $\mathcal{F}$ of stars violating the profile condition (P). See the proof of Theorem 3.4.

cut up the entire picture into single pixels, using just such lines. This lends some definiteness to our definion of 'regions' as tangles: although it is fuzzy in that single pixels cannot be assigned to regions, it is also hard in the sense that the picture will, or will not, contain a region of coherence k: there is nothing vague about the existence or non-existence of a k-tangle. Up to a certain 'resolution' (value of k) we shall see more and more tangles, each refining earlier ones; but at some point there will be none – the quality of the image will not support any higher resolution. And the tangle duality theorem will tell us at which resolution this happens. As a spin-off, we obtain a mathematically rigorous definition of the adequacy of a given resolution for a given pixellated image.

So here is our formal setup. Consider a *flat* 2-dimensional cell complex X *of squares*: a CW-complex whose 2-cells are each bounded by a set of exactly four 1-cells, and in which every 1-cell lies on the boundary of at most two 2-cells. Think of X as a quadrangulation of a closed surface with or without boundary.

We shall call the 2-cells of X *pixels* and its 1-cells *edges*. We write Π for the set of pixels of X, and E for the set of its edges. An edge that lies on the boundary of two pixels is said to *join* these pixels.

A *picture* on X is a map $\pi\colon \Pi \to 2^n$, where we think of the elements of $2^n = \{0,1\}^n$ as giving each of n parameters a value 0 or 1. Think of these parameters as brightness, colour and so on. We use addition on 2^n modulo 2 and denote your favourite norm on 2^n by $\|\ \|$. To an edge e joining two pixels p and q we assign the number

$$\delta(e) := \|\pi(q) - \pi(p)\|;$$

note that this is well defined, regardless of which of the two pixels is p and which is q. We call $\delta(e)$ the *weight* of the edge e, and δ a *weighting* of the picture π.

The *boundary* of a set $A \subseteq \Pi$ of pixels is the set

$$\partial A := \{\, e \in E \mid e \text{ joins a pixel } p \in A \text{ to a pixel } q \notin A \,\}.$$

We are now ready to set up our universe $\vec{U}$ of separations of a given picture π. We let $\vec{U} := 2^\Pi \smallsetminus \{\Pi, \emptyset\}$. Thus, every set A of pixels is a 'separation'; if desired we can think of it as the oriented bipartition (A^*, A) of Π, where $A^* := \Pi \smallsetminus A$ denotes the involution on $\vec{U}$. The partial ordering on $\vec{U}$ will be $\subseteq$, the join $\vee$ will be $\cup$, and the meet $\wedge$ will be $\cap$.

Note that $\vec{U}$ contains no small separations, since $A^* \not\subseteq A$ for $A \neq \Pi$. Every tree set and every profile in $\vec{U}$ will therefore be regular. Also, all profiles and sets of pairwise distinguishable profiles will be robust in the sense of [3]; see the definitions there.

The *order* of a separation $A \in \vec{U}$ we define as

$$|A| := \sum \{\, N - \delta(e) \mid e \in \partial A \,\},$$

where N is some fixed integer just large enough to make all $|A|$ positive. Formally, we fix not just X but the pair (X, N) at the start, calling it a *canvas*.

Notice that $|A| = |A^*|$, as required for an order function. Moreover,

Lemma 3.1. *The order function $A \mapsto |A|$ is submodular.*

Proof. We have to show that $|A \cap B| + |A \cup B| \leq |A| + |B|$ for all $A, B \subseteq \Pi$. We shall prove that every e such that $\delta(e)$ is counted on the left will also have

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its $\delta(e)$ counted on the right, and if it is counted in both sums on the left it is also counted in both sums on the right.

If $e \in \partial(A \cap B)$, then e joins a pixel $p \in A \cap B$ to a pixel q that fails to lie in A or which fails to lie in B. In the first case, $e \in \partial A$, in the latter case we have $e \in \partial B$. Since $\partial(A \cup B) = \partial(A^* \cap B^*)$, the same holds for edges in $\partial(A \cup B)$: every such edge lies in $\partial(A^*) = \partial A$ or in $\partial(B^*) = \partial B$.

Finally, if e is counted twice on the left, i.e., if $e \in \partial(A \cap B)$ as well as $e \in \partial(A \cup B) = \partial(A^* \cap B^*)$, then e joins some pixesl $p \in A \cap B$ to some other pixel, and it also joins some $q \in A^* \cap B^*$ to some other pixel. As $A \cap B$ and $A^* \cap B^*$ are disjoint, we have $p \neq q$, so p and q are precisely the two pixels on whose boundary e lies. But this means that $e \in \partial A$ as well as $e \in \partial B$, so e is counted twice also on the right. □

It remains to translate the terminology of tangles into some more suitable language for our application.

By a *region* ρ of our picture π we mean a profile in $\vec{U}$. The *complexity* of ρ is the least k for which ρ is a k-profile; its *coherence* is the greatest k for which it extends to a unique k-profile; and its *visibility* is the positive difference between its coherence and its complexity. A region ρ is *trivial* if there exists a pixel p such that $\rho = \{ A \subseteq \Pi \mid p \in A \} \cap \vec{S}_k$ for some k such that ρ is a k-profile.

A *line* is an unoriented separation, an element of U. Thus, formally, a line ℓ is an unordered pair $\{A, A^*\}$, for A a set of pixels. Since $\partial A = \partial A^*$, the line ℓ determines the set $\partial A = \partial A^*$ of edges 'between' A and A^*, and it can be helpful to imagine lines as boundaries. However, boundaries can have more than one connected component, and we make no claim as to whether or not the pair $\{A, A^*\}$ can be reconstructed from this common boundary.[5]

The *order* of a line $\ell = \{A, A^*\}$ is the number $\ell := |A| = |A^*|$. A nested set of lines will be called *laminar*. It is *canonical* if it is invariant under the automorphisms of the universe $\vec{U} = (\vec{U}, \leq, ^*, \vee, \wedge, |\,|)$, which clearly act naturally on its lines.

A line ℓ *separates* two regions ρ, ρ' if it distinguishes them as profiles. It separates them *efficiently* if no line in U of lower order than $|\ell|$ separates ρ from ρ'.

An *outline* of a region ρ by a nested set L of lines is a splitting star σ of L that is contained in ρ. Recall that a splitting star of a tree set of separations is the set of maximal elements of some consistent orientation of that tree set. We are assuming here that such a splitting star σ of L is a subset of ρ; this can happen even if ρ does not orient all the lines in L. (But every line in L will have an orientation below an element of $\sigma \subseteq \rho$, even if its order is to large for it to be itself oriented by ρ.) We may think of such an outline $\sigma \subseteq \vec{L}$ of the region ρ as the set of lines in L closest this region, and oriented towards it.

Here, then, are our tangle-tree theorems translated to the world of pictures. First, the canonical version:

Theorem 3.2. *For every picture π on a canvas (X, N) there is a canonical laminar set L of lines that efficiently separates all distinguishable regions of π.* □

Next, the refined but non-canonical version:

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[5]It can, but we do not need this – it just complicates matters to reduce lines to boundaries.

Theorem 3.3. *For every set R of pairwise distinguishable regions of a picture π on a canvas (X, N) there is a laminar set L of lines that efficiently separates all the regions of π, and which is miminal in the sense that no proper subset of L separates all the regions in R (efficiently or not).*

The splitting stars of L are precisely its outlines of the regions in R. □

For the tangle-tree duality theorem we need one more definition. A *3-star* $\sigma \subseteq \vec{U}$ is a star with exactly 3 elements. It is *void* if $\bigcap \sigma = \emptyset$. A star $\sigma = \{\{p\}\}$ for some $p \in \Pi$ is a *single pixel*.

Theorem 3.4. *For every picture π on a canvas (X, N) and every integer $k > 0$, either π has a non-trivial region of coherence at least k, or there exists a laminar set L of lines of order $< k$ all whose splitting stars are void 3-stars or single pixels. For no picture do both these happen at once.*

Proof. Let $\mathcal{F}$ be the set of all void 3-stars and all single pixels. Let us show that S_k has an $\mathcal{F}$-tangle if and only if π has a non-trivial region of coherence at least k.

Assume first that S_k has an $\mathcal{F}$-tangle, P say. To show that P satisfies (P), let $\vec{r}, \vec{s} \in P$ be given. By submodularity, one of $\vec{r} \wedge \overleftarrow{s}$ and $\vec{s} \wedge \overleftarrow{r}$ lies in $\vec{S}_k$. Assume that $\vec{r} \wedge \overleftarrow{s}$ does; the other case is analogous. As $\vec{r} \wedge \overleftarrow{s} \leq \vec{r} \in P$, the consistency of P implies that also $\vec{r} \wedge \overleftarrow{s} \in P$. But $\{\vec{r} \wedge \overleftarrow{s}, \vec{s}, \overleftarrow{r} \wedge \overleftarrow{s}\}$ is a void 3-star, and hence in $\mathcal{F}$. Since P avoids $\mathcal{F}$, we must therefore have $\overleftarrow{r} \wedge \overleftarrow{s} \notin P$, as required by (P). This shows that every $\mathcal{F}$-tangle of S_k is a k-profile, and hence a region of coherence at least k. It is non-trivial as a region, since if it has the form $\{A \subseteq \Pi \mid p \in A\}$, it will in particular contain the single pixel $\{\{p\}\}$, which it does not since single pixels lie in $\mathcal{F}$.

Conversely, assume that π has a non-trivial region of coherence at least k. This is an ℓ-profile extending to a unique k-profile P, where $\ell \leq k$. Clearly, P is again non-trivial as a region. Since any void 3-star in P constitutes a violation of (P), which P satisfies, there are no void 3-stars in P. If P contained a single pixel $\{\{p\}\}$, then by consistency it would also contain every $A \ni p$ of order $< k$. Then $P = \{A \subseteq \Pi \mid p \in A\} \cap \vec{S}_k$ would be trivial as a region, which it is not. Hence P contans neither void 3-stars nor single pixels, and thus is an $\mathcal{F}$-tangle of S_k.

If (T, α) is an S_k-tree over $\mathcal{F}$, the unoriented separations in the image of α form a laminar set L of lines of order $< k$ whose splitting stars lie in $\mathcal{F}$, and are thus void 3-stars or single pixels. Conversely, it is shown in [2] that from any tree set S of separations without splitting stars in $\mathcal{F}$ one can construct an S-tree (T, α) over $\mathcal{F}$ with S the image of α.

In order to apply the tangle duality theorem, we still need to check that $\vec{S}_k$ is separable and that $\mathcal{F}$ is closed under shifting (Lemma 2.4). The first of these is a direct consequence of submodularity and proved in [4, Lemma 5.1]. The fact that $\mathcal{F}$ is closed under shifting is an easy consequence of the fact that $\vec{U}$ consists of bipartitions of a set, Π. Indeed, consider a star $\sigma = \{A_0, A_1, A_2\} \in \mathcal{F}$ as in the definition of shifting. If $\vec{s_0} = C$ and $\vec{r} \leq A_0$, say (see there), then A_1 and A_2 shift to supersets $A_1 \cup C^*$ and $A_2 \cup C^*$, respectively, but A_0 shifts to $A_0 \cap C$. Hence if A_0, A_1, A_2 have an empty overall intersection then so do their shifts: void 3-stars shift to void 3-stars. Similarly, a single pixel $\{\{p\}\}$ shifts to $\{\{p\} \cap C\}$, which is either equal to $\{p\}$ or empty and thus not in $\vec{U}$, a contradiction: since $\vec{S}$ is separable, the shift of σ will be a subset of $\vec{S}_k \subseteq \vec{U}$. So single pixels do not change when shifted. □

We remark that the choice of $\mathcal{F}$, which led to the theorem above, is but a minimal one that makes the tangle duality theorem applicable. We could choose to add more stars of separations to $\mathcal{F}$: any stars $\{A_1, \ldots, A_n\}$ whose 'interiors' $A_1^* \cap \ldots \cap A_n^*$ are deemed to be too small to be home to an interesting tangle.

4   Potential applications

As mentioned in the introduction, tangles are intrinsically large objects from a complexity point of view: a tangle of order k, or k-profile, has to orient every separation of order less than k, of which there can be many and which also have to be found. But while tangles are complex things, the laminar sets of lines given by Theorems 3.2 and 3.3 are not. So even if generating them turned out to be computationally hard (which is not clear; see below), once they are found they offer a much compressed version of the essence of the picture.

Just how well a few lines separating the main regions of a picture can convey its essence can be gleaned, for example, from cartoons: it seems that such lines, more than anything else, trigger our visual understanding and recognition.

Moreover, there is no reason to believe that these laminar sets of lines cannot be generated efficiently from a given picture, taking advantage of the additional structure that pictures offer over abstract separation systems.

For a start, we could economise by generating the regions together with the lines that distinguish (separate) them, perhaps in increasing order of those lines.[6] Once we have generated all the lines up to some desired order, their regions can be obtained relatively easily from the poset they form. Indeed, recall that a region (or profile) is a consistent orientation of the set S_k of lines (separations) of order less than k that satisfies condition (P). With some book-keeping it should be possible to generate the poset $\vec{S}_k$ together with the lines in S_k. Its regions are then easily computed: we iteratively orient a line s not yet oriented, as $\vec{s}$ say, adding at the same time its down-closure in $\vec{S}_k$ and all the oriented lines $\vec{r} \vee \vec{s} \in \vec{S}_k$ it generates with any $\vec{r}$ selected earlier. This region can then be stored, if desired, by remembering just its maximal elements, since it is precisely their down-closure in $\vec{S}_k$.

4.1   Image recognition

The canonical tangle-tree theorem derives from a given image a tree set of lines each labelled by its order. These tree sets are tantamount to trees with labelled edges [2]. Since they are generated canonically, trees coming from images of the same object will have large subtrees that are isomorphic, or nearly so, as edge-labelled trees. Hence, an image recognition algorithm might test for the presence of such subtrees, and return a 'different' verdict for images of objects where such subtrees are not found. Those rare pairs of objects where such trees are found could be analysed in more detail and at greater cost.

The proof of the canonical tangle-tree theorem generates its tree set of separations in increasing order: low-order separations are included in the final tree set at an early stage, before higher-order separations are even considered. When

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[6]There will be a trade-off between the synergy of generating lines of different orders together and the benefits of possibly not even having to look at lines of high order; see Section 4.1.

we compare two pictures, and these differ already by their low-order regions, we will know this early in the algorithm and can terminate it at that stage.

4.2   Compression

The non-canonical tangle-tree theorem returns an even smaller tree set of lines that still separate all the regions of an image. Again, these lines can be labelled by their order and thus give rise to an edge-labelled tree, which will be cheap to store.

In addition, we might reduce the number of lines further by limiting the regions in R we aim to distinguish (see Theorem 3.3) to those of high visibility. These regions will correspond to the most distinctive features of a picture, and it may well be enough to demarcate just these.

To re-create an approximation of the original picture from these few laminar lines, one can draw them on a canvas and fill the areas between them – those corresponding to the splitting stars of the tree set, the nodes of the tree – with pixels similar to those in a very small sample taken from the corresponding areas in the original picture when these lines were computed. For every such area, its pixels are likely to be similar, so even a small sample should suffice to smooth out the gradual differences that can still occur within such an area.[7]

4.3   Supported resolution: telling objects from noise

The tangle-tree duality theorem allows us to offer a mathematically rigorous definition of the maximum resolution that a set $A \subseteq \Pi$ of pixels supports: the *largest k for which it admits a k-profile, i.e., has a region of coherence k.*

For if we are interested in the potential features of an image handed to us as just a data set of pixels, then real features are likely to correspond to regions of coherence at least some k that we may specify, while areas not containing such a region will be unimportant background, or 'noise', at this *resolution k*.

We shall discuss this in more detail in Section 5, in the context of the example shown in Figure 4.

5   Examples

In this section we collect some examples that illustrate some of the ideas discussed in this paper.

Consider the black and white picture of the letter L shown in Figure 1. The function π assigns 1 to the black pixels on the canvas, and 0 to the white ones. Let δ be the weighting that assigns 1 to an edge e if e joins a black pixel to a white one, and 0 otherwise. The line $\ell = \{A, A^*\}$, where A is the set of black pixels, separates the L from its background. If $N = 1$, both A and ℓ have order 0.

Since no line of order 0 has an edge in its boundary that joins two like pixels (both black or both white), because such an edge e would contribute 1 to the

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[7]The footprint under π of the pixels in a large such area may shift gradually. The fact that it comes from a region only means that, locally, we cannot cut through it by a line of low order: adjacent pixels tend to be similar, but they may change gradually, and a few atypical pixels can occur. Still, a small set of pixels sampled from the area should give a good representation.
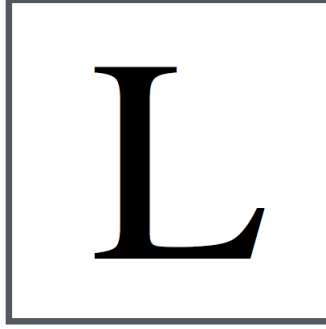
Figure 1: A simple shape with one main region

order of such a line, ℓ is the only line of order 0. It separates two regions of complexity 1: the region $\rho = \{A\}$ representing the letter L, and the region $\{A^*\}$ representing its background.

Assuming that the L is 10 pixels wide where it is thinnest at the bottom, its right serif is represented by the following 11-profile: the region

$$\sigma = \{A, B_1, \ldots, B_n, C_1, \ldots, C_m\},$$

where $B_1 \supset \ldots \supset B_n$ are the subsets of A that contain the right serif and have order 10, i.e., whose boundary contains only 10 (vertical) edges joining like pixels (which are both black), and the C_i are the complements in Π of the sets $A^* \cup D_i$ where D_i is a 'small' non-empty subset of A that has order at most 10 (when viewed as a separation) and is not of the form B_i or $A \cap B_i^*$. The serif region σ thus has complexity 11. Its outline consists of the separation B_n and all the separations $C = \Pi \smallsetminus (D \cup A^*)$ with D maximal among the D_i.

The vertical shaft of the L is represented by another region of complexity 11; let us call it τ. The lines $\{B_i, B_i^*\}$ separate τ from the serif region σ. As both σ and τ extend our 1-profile ρ, this implies that ρ has coherence at most 10. In fact, it has coherence exactly 10: it extends to k-profiles for $k = 2, \ldots, 10$ (add all separations of the form C_i that have order $< k$), and for every such k this is the only k-profile extending ρ. The visibility of the region ρ, then, is $10 - 1 = 9$.

Our serif region σ does not extend to distinct k-profiles for any k. But it does extend to unique k-profiles for some $k > 11$. Roughly, if the diagonal line cutting through it where it is thickest consists of k edges joining like pairs of pixels (both black), then σ will extend to k-profiles but not beyond. This k is about 20, which is also the thickness of the shaft of the L. Both the serif region σ and the shaft region τ thus have complexity 11 and coherence about 20, which would give them a relatively high visibility of 9, the same as that of ρ.

Now consider Figure 2: a square with four quadrants, of which the left two and the right two are similar to each other.

With a natural weighting taking into account the similarity of colours, there is a unique line of lowest order k, which runs vertically down the middle. (We here ignore all the lines $\{A, A^*\}$ with A a very small set.) This line separates the green region A from the red region A^*, in the sense that it distinguishes their corresponding profiles: the $(k+1)$-profile that orients all separations of
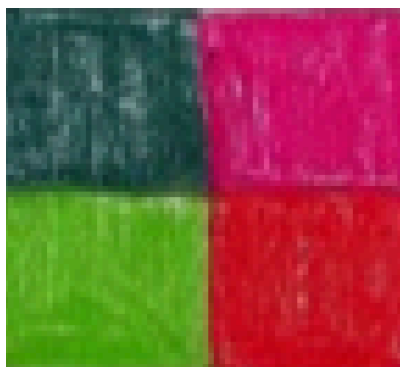
Figure 2: Two regions, each refined by two smaller regions

order at most k towards where most of the green pixels are orients this line as A, whereas the corresponding 'red' $(k+1)$-profile orients it as A^*.

The (profile of the) green region extends to distinct k'-profiles for some $k' > k$ that correspond to the two green quadrants, and similarly for red. While the four quadrants are pairwise distinguishable, the green quadrants are not distinguishable from the entire green region, and similarly for red. If the two greens are quite similar to each other, the entire green region will have higher visibility than the two green squares. If the two greens differ, in hue, nearly as much from each other as they differ from the reds, the entire 'green' region A will be less visible than the two 'green' squares.

The proof of the canonical tangle-tree theorem will produce five nested separations to distinguish these tangles: the vertical line separating green from red, and in addition one L-shaped line around each of the four quadrants.

The non-canonical version of the theorem, applied to the set R of the four quadrants (which are pairwise distinguishable, as the premise of the theorem requires), will then discard two of these latter four separations, retaining just the vertical line and a line around one green and one red quadrant. Note that these three lines still separate all four quadrant, but no subset of just two lines will.

Figure 3: Concentric circles have low order, radial lines have high order

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Figure 3 is intended to illustrate which lines have low order and which do not. The lowest-order separations are the innermost and the outermost circular line between differently coloured regions. The latter has low order, because all its edges e join a blue pixel to a red one, making $\delta(e)$ large. The innermost circle has smaller values of $\delta(e)$, but fewer edges in total, making for a similarly low order. The remaining concentric circles mark differences in hue that are about equal in degree, so the longer of these circles have larger order as separations.

The radial lines in Figure 3, by contrast, have maximum values of δ, since every edge joins two like pixels. Hence the blue background, the yellow innermost disc, and the red concentric bands are the only regions in this picture.
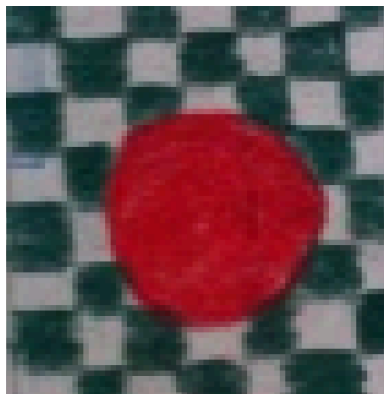

Figure 4: Only one region of high coherence

The inner red disc in picture in Figure 4 is a highly visible region, one of low complexity and large coherence: the order of any straight line that roughly cuts it in half and otherwise runs between differently coloured squares is a lower bound for its coherence.

The checkerboard background as such does not represent a region. Roughly, the reason for this is that the lines around it, such as the boundary of the red disc or any circle indside the red disc, have order no smaller than the lines cutting right through it: if these run between differently coloured squares, they will have similarly low order as the boundary of the red disc.

Each of the green and white squares does represent a region. But both the complexity and coherence of such a 'square' region is low: the order of a line around the square is essentially its complexity, while the order of the shortest line that cuts vertically through it and otherwise runs between diffent squares is essentially an upper bound for its coherence.

Indeed, if that order is k, then the chequered area has no non-trivial region of coherence much greater than k at all. This is because we can subdivide it into single pixels by nested lines of order only slighly greater than k, chosen inductively: since lines of order k can cut right through a square, we may, at each step, subdivide any area still bigger than a single pixel by a line that has order only slightly greater than k and is nested with all previously chosen lines. These nested lines will be separations forming, for some k' only slightly bigger than k, an $S_{k'}$-tree over the set $\mathcal{F}$ of void 3-stars and single pixels, which by the easy direction of the tangle-tree duality theorem witnesses that the chequered area has no non-trivial region of coherence at least k'.
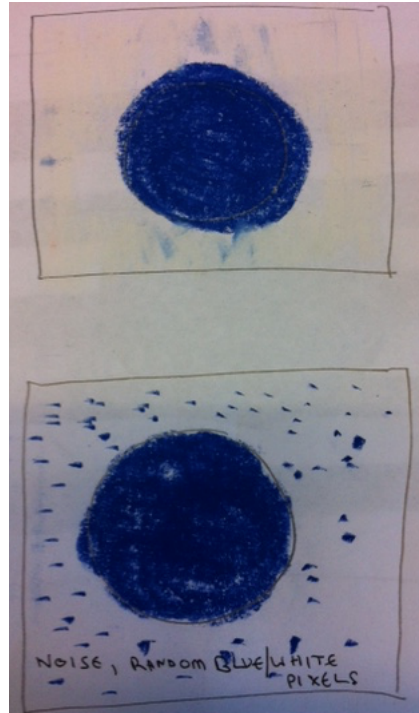
Figure 5: Two regions above, only one below

In the upper image in Figure 5 we see a blue disc against a white background. Here we have two highly visible regions, of low complexity and high coherence, separated by the circular line around the blue disc. In the second image, imagine the background as consisting of blue and white pixels whose colours are chosen independently at random, with equal probability for blue and white. Note that an edge joining two of these random pixels is as likely to join pixels of the same colour as an edge on the circle around the blue disc, which joins a blue pixel to a random pixel. Hence, as in Figure 4, the central blue disc represents a region of high coherence, but – unlike in the upper image – the background does not represent any region at all (with high probability).

6 Outlook

Our aim in these notes was to show that, in principle, tangles in abstract separation systems can be used to model regions of an image, with the consequence that the tangle-tree theorem and the tangle duality theorem can be used to analyse images. We are aware that our concrete model, as defined in Section 3, amounts to no more than a proof of concept, which can be improved and fine-tuned in many ways.

Our hope is that, nonetheless, the experts in the field may find our approach sufficiently inspiring to be tempted to develop it further.
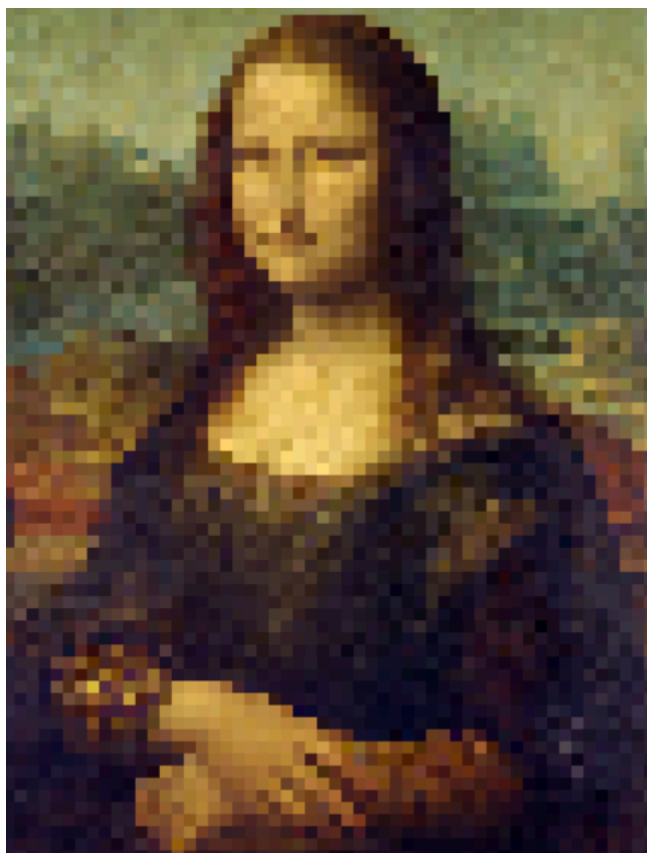
Figure 6: Where are the lines separating the regions of high visibility?

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