

A TALE OF STARS AND CLIQUES

TOMASZ ŁUCZAK, JOANNA POLCYN, AND CHRISTIAN REIHER

ABSTRACT. We show that for an infinitely many natural numbers k there are k -uniform hypergraphs which admit a ‘rescaling phenomenon’ as described in [9]. More precisely, let $\mathcal{A}(k, I, n)$ denote the class of k -graphs on n vertices in which the sizes of all pairwise intersections of edges belong to a set I . We show that if $k = rt^2$ for some $r \geq 1$ and $t \geq 2$, and I is chosen in some special way, the densest graphs in $\mathcal{A}(rt^2, I, n)$ are either dominated by stars of large degree, or basically, they are ‘ t -thick’ rt^2 -graphs in which vertices are partitioned into groups of t vertices each and every edge is a union of tr such groups. It is easy to see that, unlike in stars, the maximum degree of t -thick graphs is of a lower order than the number of its edges. Thus, if we study the graphs from $\mathcal{A}(rt^2, I, n)$ with a prescribed number of edges m which minimize the maximum degree, around the value of m which is the number of edges of the largest t -thick graph, a rapid, discontinuous phase transition can be observed. Interestingly, these two types of k -graphs determine the structure of all hypergraphs in $\mathcal{A}(rt^2, I, n)$. Namely, we show that each such hypergraph can be decomposed into a t -thick graph H_T , a special collection H_S of stars, and a sparse ‘left-over’ graph H_R .

§1. INTRODUCTION

By a *set system* we mean a pair $S = (V, \mathcal{E})$ such that \mathcal{E} is a collection of subsets of V . The members of V are usually referred to as the *vertices* of the set system, whilst the members of \mathcal{E} are called *edges*. If all members of \mathcal{E} are of the same cardinality $k \geq 0$ we call S a *k -uniform hypergraph* or, more briefly, a *k -graph*.

Occasionally we identify a hypergraph H with its set of edges, denoting, for example, by $|H|$ the number of edges in H . For a given set I of nonnegative integers, we say that a k -graph H is *I -intersecting* if $|e \cap f| \in I$ holds for all distinct $e, f \in H$. Starting with the seminal work [4] of Erdős, Ko, and Rado, the study of I -intersecting hypergraphs and set systems has a long tradition in extremal combinatorics (see, e.g., [2, 3, 6, 7, 10, 11] for some milestones). Let us remark that sometimes in the literature (e.g., [2, 3, 6]) an I -intersecting k -graph on n vertices is called an (n, k, I) -system.

Motivated by the stability of extremal graphs for 3-uniform loose path of length 3 the first two authors studied $\{0, 2, 3\}$ -intersecting 4-graphs in [9]. The aim of the present

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article is to extend their results to the more general family $\mathcal{J}(r, t)$ which consists of all I -intersecting rt^2 -graphs, where $r \geq 1$ and $t \geq 2$ are arbitrary integers and

$$I = \{s : t \mid s \text{ or } s \geq rt(t-1)\}.$$

This choice of the set of permissible intersections may look bizarre at first and our main incentive to study it came from the aesthetical merits of the results we hoped to obtain: to explain those, we start from the observation that there are two quite different examples of dense rt^2 -graphs $H \in \mathcal{J}(r, t)$ on n vertices with $\Theta(n^{rt})$ edges.

The most obvious one is the *full* $(rt(t-1))$ -*star*, i.e., a hypergraph H with a distinguished $rt(t-1)$ -set S of vertices, called the *centre* of the star, such that the edges of H are precisely the rt^2 -supersets of S . Clearly such a star has exactly $\binom{n-rt^2+rt}{rt}$ edges and it can be shown that, for large n , it is the unique graph which maximises the number of edges among all graphs from $\mathcal{J}(r, t)$ on n vertices (see Proposition 2.3 below).

However, there exists another natural construction of dense rt^2 -graphs $H \in \mathcal{J}(r, t)$ with n vertices and $\Theta(n^{rt})$ edges. It proceeds by splitting the vertex set into $\lfloor n/t \rfloor$ subsets of size t called *teams* (and a small number of left-over vertices) and to declare an rt^2 -set to be an edge if and only if it is a union of rt teams. We call the resulting hypergraph a *thick clique* and to its subgraphs we refer as *thick hypergraphs*. Note that each thick hypergraph has the property that for any two edges e and f the number $|e \cap f|$ is a multiple of t and, hence, it indeed belongs to $\mathcal{J}(r, t)$.

The point that interests us here is that even though both the star and the thick clique have $\Theta(n^{rt})$ edges, their maximum vertex degrees are of different orders of magnitude. In fact, while the vertices belonging to the centre of a star have degree $\Omega(n^{rt})$, the maximum degree of a thick clique is easily seen to be only $O(n^{rt-1})$. Perhaps surprisingly, it turns out that this phenomenon arises in a very ‘‘discontinuous’’ manner: As soon as a graph from $\mathcal{J}(r, t)$ has one edge more than the thick clique, it needs to contain a vertex of degree $\Omega(n^{rt})$.

This is the main result of the present work which, crudely, can be stated as follows (for further structural results see Theorems 2.2 and 3.5 below).

Theorem 1.1. *For $r \geq 1$ and $t \geq 2$ there exists an n_0 such that for every rt^2 -graph $H \in \mathcal{J}(r, t)$ with $n \geq n_0$ vertices and at least $\binom{\lfloor n/t \rfloor}{rt} + 1$ edges we have $\Delta(H) \geq m/(3t)$.*

On the other hand, for every $n \geq rt^2$ a thick clique $H_0 \in \mathcal{J}(r, t)$ on n vertices has $\binom{\lfloor n/t \rfloor}{rt}$ edges and $\Delta(H_0) = \binom{\lfloor n/t \rfloor - 1}{rt-1}$.

The main step in the proof of Theorem 1.1 is a somewhat surprising structural result (see Theorem 2.2 below). It turns out that stars and thick hypergraphs which naturally emerge when we study graphs from $\mathcal{J}(r, t)$ whose density is close to the maximum density $\Theta(n^{rt})$, are natural building blocks for all ‘not too sparse’ graphs from $\mathcal{J}(r, t)$. More specifically,

we show that up to an ‘error of lower order,’ i.e., up to at most $O(n^{rt-1})$ edges, any such hypergraph arises by attaching “non-overlapping” stars to a thick hypergraph.

This article is organised as follows. In the next section we state a precise version of the structure theorem mentioned above. Then, in Section 3, we show that it does indeed imply Theorem 1.1 and give more structural characterisations of dense graphs from $\mathcal{J}(r, t)$ with small maximum degree. Section 4 collects some tools needed for the proof of this structure theorem including a ‘decomposition lemma’ (see Lemma 4.4 below) that might have some other applications as well. Finally, in Section 5, we prove the structure theorem.

§2. THE STRUCTURE THEOREM

We begin this section with some definitions allowing us to formulate a precise version of the structure theorem for rt^2 -graphs $H \in \mathcal{J}(r, t)$.

Let $k \geq s \geq 0$ be integers. A k -uniform hypergraph $H = (V, E)$ with a set $S \subseteq V$ of distinguished vertices of size $|S| = s$ is an s -star if $S \subseteq e$ holds for all edges $e \in E$. We call S the *centre* of the star and $\bigcup_{h \in E} (h \setminus S)$ is referred to as the *body* of the star. A collection of stars is said to be *semi-disjoint* if their centres are distinct and their bodies are mutually disjoint. Of course, an s -star H on $|V| = n$ vertices can have at most $\binom{n-s}{k-s}$ edges. If this happens we say that H is a *full s -star* and denote it by $S_{n,s}^k$.

Next, for a given hypergraph $H = (V, E)$, we say that a subset $W \subseteq V$ of its vertex set is *inseparable* in H , if for all edges $h \in E$ we have $W \cap h \in \{\emptyset, W\}$. Now consider three natural numbers k, t , and n satisfying $t \mid k$, and suppose that a set V of n vertices is partitioned into $\lfloor n/t \rfloor$ many t -subsets called *teams* and fewer than t further vertices. By $\tilde{K}_{n,t}^k$ we denote the *thick (k, n, t) -clique*, i.e., the k -graph on n vertices whose $\binom{\lfloor n/t \rfloor}{k/t}$ edges are all possible unions of some k/t of these teams. We refer to its subgraphs as *t -thick* or just *thick hypergraphs*. Evidently the teams are inseparable in $\tilde{K}_{n,t}^k$ and a k -graph H on n vertices possessing $\lfloor n/t \rfloor$ mutually disjoint inseparable t -sets of vertices is a subhypergraph of the thick clique $\tilde{K}_{n,t}^k$.

Finally, for positive integers t, ℓ , and a , we define a class $\mathcal{F}(t, \ell, a)$ of ℓt -graphs as follows.

Definition 2.1. For given natural numbers t, ℓ , and a , we say that a ℓt -graph $H = (V, E)$ belongs to the class $\mathcal{F}(t, \ell, a)$ if and only if there are partitions

$$V = V_T \cup V_S \cup V_R \quad \text{and} \quad H = H_T \cup H_S \cup H_R,$$

such that

- (i) V_T is a union of inseparable t -subsets of V , and $H_T = H[V_T]$;
- (ii) $H_S = \{h \in H : |h \cap V_S| = \ell\}$ consists of semi-disjoint $(\ell(t-1))$ -stars with their centres in $V_T \cup V_R$ and their bodies in V_S ;
- (iii) any edge of H that intersects the body of a star $S^* \subseteq H_S$ contains the centre of S^* ;
- (iv) and $|H_R| \leq |V_T||V_S|n^{\ell-3} + |V_R|an^{\ell-2}$.

Now the structure theorem for rt^2 -graphs $H \in \mathcal{J}(r, t)$ promised in the introduction can be stated as follows.

Theorem 2.2 (Structure Theorem). *For all integers $r \geq 1$ and $t \geq 2$ we have*

$$\mathcal{J}(r, t) \subseteq \mathcal{F}(t, rt, (rt^2)r^{3t^6}).$$

The proof of this result is deferred to Section 5. We conclude this section by pointing out that the structure theorem quickly allows us to determine the extremal rt^2 -graphs from $\mathcal{J}(r, t)$. The following statement shows that the extremal graph, the full $(\ell(t-1))$ -star, is unique and stable for this problem.

Proposition 2.3. *Given natural numbers $\ell \geq t \geq 2$, a , and c , there exists an integer n_* such that every ℓt -graph $H = (V, E) \in \mathcal{F}(t, \ell, a)$ with $|V| = n \geq n_*$ and $m \geq \binom{n-c}{\ell} - \frac{n^{\ell-1}}{2(\ell-1)!}$ edges is obtained from an $(\ell(t-1))$ -star by adding at most $\binom{c}{\ell t}$ further edges.*

Moreover, if $m \geq \binom{n-\ell t}{\ell} - \frac{n^{\ell-1}}{2(\ell-1)!}$, then H is an $(\ell(t-1))$ -star.

In particular, each $H \in \mathcal{F}(t, \ell, a)$ has at most $\binom{n-\ell(t-1)}{\ell}$ edges, and this maximum is achieved only if H is isomorphic to the full $(\ell(t-1))$ -star $S_{n, \ell(t-1)}^{\ell t}$.

Proof. Let us choose n_* and D such that $n_* \gg D \gg \max(t, \ell, a) + c$, so all inequalities below hold for $n \geq n_*$. Moreover, let $H = (V, E)$ with

$$|V| = n \geq n_* \quad \text{and} \quad |E| = m \geq \binom{n-c}{\ell} - \frac{n^{\ell-1}}{2(\ell-1)!}$$

be a ℓt -graph from $\mathcal{F}(t, \ell, a)$, and take partitions

$$V = V_T \cup V_S \cup V_R \quad \text{as well as} \quad H = H_T \cup H_S \cup H_R$$

exemplifying this.

Clearly

$$m \leq \binom{|V_T|/t}{\ell} + \binom{|V_S|}{\ell} + |H_R|, \tag{2.1}$$

where

$$|H_R| \leq |V_T||V_S|n^{\ell-3} + |V_R|an^{\ell-2} \leq (|V_T| + |V_R|)an^{\ell-2} \leq an^{\ell-1},$$

and so

$$m \leq \binom{|V_S| + |V_T|/t}{\ell} + an^{\ell-1}. \tag{2.2}$$

Assume first that $|V_S| < n - 2D$. Then

$$|V_S| + \frac{1}{t}|V_T| \leq |V_S| + \frac{1}{2}(n - |V_S|) = \frac{1}{2}(n + |V_S|) \leq n - D$$

and so, by (2.2),

$$m \leq \binom{n-D}{\ell} + an^{\ell-1} < \binom{n-c}{\ell} - \frac{n^{\ell-1}}{2(\ell-1)!}.$$

As this contradicts our assumption, we may conclude that $|V_S| \geq n - 2D$.

In particular, we have $|V_T| + |V_R| \leq 2D$, and $|H_T \cup H_R| \leq (2Da + 1)n^{\ell-2}$. Consider the largest star $S^* = (V^*, E^*)$ in H_S and let s be its centre. Then $|V^*| \geq n/2$, since otherwise, by (2.1),

$$m = |H_S| + |H_T| + |H_R| \leq \binom{n/2}{\ell} + \binom{n/2}{\ell} + (2Da + 1)n^{\ell-2} < \binom{n-c}{\ell} - \frac{n^{\ell-1}}{2(\ell-1)!}.$$

Now assuming $|V^* \cap V_S| \leq n - c - 1$ we could use (2.1) again and infer

$$m \leq \binom{n-c-1}{\ell} + \binom{c+1}{\ell} + (2Da + 1)n^{\ell-2} < \binom{n-c}{\ell} - \frac{n^{\ell-1}}{2(\ell-1)!},$$

which, again, contradicts our assumption on m .

This proves that $|V^* \cap V_S| \geq n - c$. By Definition 2.1(iii), all edges of H intersecting $V^* \cap V_S$ contain s and therefore they form an $(\ell(t-1))$ -star. Since $|V \setminus (V^* \cap V_S)| \leq c$, there can be at most $\binom{c}{\ell t}$ edges not belonging to this star, which establishes our first assertion. The moreover-part follows from the observation that in case $c = \ell t$ the only potential further edge, $V \setminus (V^* \cap V_S)$, would still contain s and could thus be adjoined to the star. \square

§3. MINIMUM MAXIMUM DEGREE

Let us first start with the proof of Theorem 1.1 which, let us recall, states that in each graph from $\mathcal{J}(r, t)$ with $m > \binom{\lfloor n/t \rfloor}{rt}$ edges there exists a big star which contains a positive fraction of all edges; moreover thick cliques show that this result is sharp. We prove this result in a slightly stronger form, which gives a better estimate for the size of the biggest star for dense graphs. Besides, it states that each graph from $\mathcal{J}(r, t)$ which has almost $\binom{\lfloor n/t \rfloor}{rt}$ edges and small maximum degree is thick. Here, for $H \in \mathcal{J}(r, t)$ by H_S we denote the subgraph consisting of $(rt(t-1))$ -stars whose existence is assured by the Structure Theorem 2.2.

Theorem 3.1. *For $r \geq 1$ and $t \geq 2$ there exists an n_0 such that, for every rt^2 -graph $H \in \mathcal{J}(r, t)$ with $n \geq n_0$ vertices and at least $\binom{\lfloor n/t \rfloor}{rt} + 1$ edges, H_S contains an $(rt(t-1))$ -star with at least \hat{n} vertices in the body and at least $m\hat{n}/n - n^{rt-1} > m/(3t)$ edges, where*

$$\hat{n} = \hat{n}(n, m) = \min \left\{ N : \binom{N-1}{rt-1} \geq \frac{rtm}{n} - rtn^{rt-2} \right\} \geq \frac{n}{t^{rt/(rt-1)}} - (3rt)^{3rt} > \frac{2n}{5t}. \quad (3.1)$$

On the other hand, if $\binom{\lfloor n/t \rfloor}{rt} - \frac{n^{rt-1}}{2^{t^{rt-1}(rt-1)!}} \leq m \leq \binom{\lfloor n/t \rfloor}{rt}$, then each hypergraph $H \in \mathcal{J}(r, t)$ with m edges and $\Delta(H) \leq m/(3t)$ is a subgraph of a thick clique $\tilde{K}_{n,t}^{rt^2}$; in particular, $\Delta(H) \leq \binom{\lfloor n/t \rfloor - 1}{rt-1}$.

Proof. For given integers $t \geq 2$, and $r \geq 1$, choose n_0 so large that all inequalities below hold for $n \geq n_0$. Moreover, let $H \in \mathcal{J}(r, t)$, where the number of edges m satisfies

$$m \geq \binom{\lfloor n/t \rfloor}{rt} - \frac{n^{rt-1}}{2t^{rt-1}(rt-1)!} > \frac{n^{rt}}{t^{rt}(rt)!} - \frac{2rt^2n^{rt-1}}{t^{rt}(rt-1)!} > \left(\frac{n}{rt^2}\right)^{rt}. \quad (3.2)$$

By Theorem 2.2, $H \in \mathcal{F}(t, rt, (rt^2)^{r^3t^6})$, so let us take partitions $V = V_T \cup V_S \cup V_R$ and $H = H_T \cup H_S \cup H_R$ exemplifying this. Then

$$|H| = |H_T| + |H_S| + |H_R| \leq \binom{|V_T|/t}{rt} + \binom{|V_S|}{rt} + |V_T||V_S|n^{rt-3} + |V_R|Cn^{rt-2}, \quad (3.3)$$

where $C = (rt^2)^{r^3t^6}$. As a straightforward consequence of the above inequality we get the following claim.

Claim 3.2. *If $V_S = \emptyset$ then $H_R = \emptyset$ and $m \leq \binom{\lfloor n/t \rfloor}{rt}$.*

Proof. Since $V_S = \emptyset$, the vertex set of H is partitioned into sets V_T and V_R , where $|V_T|$ is divisible by t and $|V_R| = n - |V_T|$. Recall that V_T consists of t -tuples that are inseparable in H . Therefore, if $H_R \neq \emptyset$ then $|V_R| \geq t$ and consequently, by (3.3),

$$\begin{aligned} m &= |H_T| + |H_R| \leq \binom{(n - |V_R|)/t}{rt} + 0 + |V_R|Cn^{rt-2} \leq \binom{\lfloor n/t \rfloor - 1}{rt} + 2tCn^{rt-2} \\ &< \binom{\lfloor n/t \rfloor}{rt} - \frac{n^{rt-1}}{2t^{rt-1}(rt-1)!}, \end{aligned}$$

contrary to (3.2).

Thus we must have $H_R = \emptyset$ and, hence,

$$m = |H_T| \leq \binom{\lfloor n/t \rfloor}{rt}. \quad \square$$

It turns out that if $V_S \neq \emptyset$, then the largest degree must be large.

Claim 3.3. *If $V_S \neq \emptyset$, then H_S contains an $(rt(t-1))$ -star with at least \hat{n} vertices and at least $m\hat{n}/n - n^{rt-1}$ edges, where \hat{n} is defined as in (3.1).*

Proof. We start with bounding from below the average degree $\text{ad}(G)$ of the rt -graph $G_S = (V_S, E_S)$ with the set of vertices V_S and the set of edges $E_S = \{h \cap V_S : h \in H_S\}$. Using the upper bound on $|H_R|$ we get

$$\begin{aligned} \text{ad}(G_S) &= \frac{rt|H_S|}{|V_S|} = \frac{rt(m - |H_R| - |H_T|)}{n - |V_R| - |V_T|} \\ &\geq \frac{rtm - rt|V_R|Cn^{rt-2} - rt\binom{|V_T|/t}{rt}}{n - |V_R| - |V_T|} - \frac{rt|V_T||V_S|n^{rt-3}}{|V_S|} \\ &= \frac{rtm}{n} + \frac{\frac{rtm}{n}(|V_R| + |V_T|) - rt|V_R|Cn^{rt-2} - rt\binom{|V_T|/t}{rt}}{n - |V_R| - |V_T|} - rt|V_T|n^{rt-3}. \end{aligned}$$

Here the numerator of the second fraction is, due to (3.2), at least

$$rt|V_R|n^{rt-2} \left(\frac{n}{(rt^2)^{rt}} - C \right) + rt|V_T| \left(\frac{1}{n} \binom{\lfloor n/t \rfloor}{rt} - \frac{n^{rt-2}}{2^{rt-1}(rt-1)!} - \frac{1}{|V_T|} \binom{\lfloor |V_T|/t \rfloor}{rt} \right)$$

and because of $|V_T| \leq n - t$ and the largeness of n this term is positive. For these reasons we have

$$\text{ad}(G_S) \geq \frac{rtm}{n} - rtn^{rt-2} > \binom{\hat{n} - 2}{rt - 1}.$$

Now, each vertex v of degree at least $\text{ad}(G_S)$ must be contained in a component with at least \hat{n} vertices and, since G_S must contain a component whose average degree is at least $\text{ad}(G_S)$, each such component must have at least $\text{ad}(G_S)\hat{n}/(rt)$ edges. \square

Finally, we can complete the proof of Theorem 3.1. If $m > \binom{\lfloor n/t \rfloor}{rt}$, then, by Claim 3.2, we have $V_S \neq \emptyset$, and the first part of Theorem 3.1 follows directly from Claim 3.3. On the other hand, if $H \in \mathcal{J}(r, t)$ has m edges, where

$$\binom{\lfloor n/t \rfloor}{rt} - \frac{n^{rt-1}}{2^{rt-1}(rt-1)!} \leq m \leq \binom{\lfloor n/t \rfloor}{rt},$$

and $\Delta(H) \leq m/(3t)$, then Claim 3.3 implies that V_S is empty and, by Claim 3.2, H_R is empty as well. Thus, H must be a subgraph of a thick (rt^2, n, t) -clique $\tilde{K}_{n,t}^{rt^2}$. \square

Once we know that dense graphs from $\mathcal{J}(r, t)$ with $m > \binom{\lfloor n/t \rfloor}{rt}$ contain vertices of large degree one may ask about the structure of graphs which, for a given $m = m(n)$, minimise the maximum degree. A natural conjecture is that they can be expressed as a union of large disjoint $(rt(t-1))$ -stars with, perhaps, some limited number of extra edges like those which intersect the centres of these stars in sets whose sizes are multiples of t .

In [9] such a result is proved for the family $\mathcal{J}(1, 2)$. Namely, it is shown that from each $\{0, 2, 3\}$ -intersecting 4-graph with $m > \binom{\lfloor n/2 \rfloor}{2}$ edges that minimises the maximum degree one can remove at most 470 edges to get a 4-graph which consists of at most 4 stars and, perhaps, some number of isolated vertices. The number 4 here is optimal, while 470 is clearly not and most likely it can be replaced either by 11, if we want each of the stars to be a 2-star, or just by 8, if we are satisfied that each of the four stars is just a 1-star (for details and discussions of this result see the final remarks in [9]).

The remaining part of this section is devoted to the proof of an analogous result for $\mathcal{J}(r, t)$ in the general case. As we will see shortly, a similar result hold whenever $r = 1$, while for $r \geq 2$ a weaker yet quite satisfactory characterisation of the extremal graphs can be shown. Nevertheless, in order to state our theorem more precisely, let us introduce some notation, analogous to those used in [9].

Thus, let us define the *minimum maximum-degree function* of $\mathcal{J}(r, t)$ by setting

$$f(\mathcal{J}(r, t); n, m) = \min\{\Delta(H) : H = (V, E) \in \mathcal{J}(r, t), |V| = n, \text{ and } |E| = m\}$$

for all nonnegative integers n and m . The corresponding collection of extremal hypergraphs is denoted by $\mathfrak{E}(\mathcal{J}(r, t); n, m)$.

Note that the function $f(\mathcal{J}(r, t); n, m)$ is always bounded from below by the average degree rt^2m/n ; on the other hand, one can always find a thick graph from $\mathcal{J}(r, t)$ such that the degrees of all vertices, except at most $t - 1$, are within distance one from each other. Hence, from Theorem 3.1 it follows that whenever $m \leq \binom{\lfloor n/t \rfloor}{rt}$ and n is large enough we have

$$\lceil rt^2m/n \rceil \leq f(\mathcal{J}(r, t); n, m) \leq \lfloor rt^2m/n \rfloor + 1,$$

i.e., in this range of m the function $f(\mathcal{J}(r, t); n, m)$ is determined up to a constant term. Thus, it remains to study the value of $f(\mathcal{J}(r, t); n, m)$ and the structure of the extremal graphs from $\mathfrak{E}(\mathcal{J}(r, t); n, m)$ for $m > \binom{\lfloor n/t \rfloor}{rt}$. For this we require one more concept.

Let us say that an $(rt(t - 1))$ -star S with some number N of vertices in its body is *heavy* if its minimum vertex degree is at least $2r^2t^3N^{rt-2}$. The standard process of repeatedly removing vertices of low degree allows us to find in any $(rt(t - 1))$ -star S a maximal heavy substar $S' \subseteq S$ (which is possibly empty). This star S' is called the *core of S* , its set of edges is denoted by $\text{cr}_e(S)$, and by $\text{cr}_v(S)$ we mean the set of vertices forming its body.

The first two parts of the following fact list standard properties of the process by means of which the core is constructed, while its third part explains that cores have a property reminiscent of condition (iii) in Definition 2.1.

Fact 3.4. *Let $r \geq 1$ and $t \geq 2$.*

- (a) *There are integers n_0 and c_0 such that if a hypergraph $H \in \mathcal{J}(r, t)$ has $n \geq n_0$ vertices and $m > \binom{\lfloor n/t \rfloor}{rt}$ edges, then there is a heavy star $S \subseteq H$ with*

$$|S| \geq \frac{m\hat{n}(n, m)}{n} - 3r^2t^3n^{rt-1} \quad \text{and} \quad |\text{cr}_v(S)| \geq \hat{n}(n, m) - c_0 > \frac{2n}{5t},$$

where $\hat{n}(n, m)$ is the number introduced in (3.1).

- (b) *For every positive integer a there exists an integer b such that every $(rt(t - 1))$ -star S with N vertices in its body and $|S| \geq \binom{N-a}{rt}$ satisfies $|\text{cr}_v(S)| \geq N - b$.*
- (c) *If $H \in \mathcal{J}(r, t)$ and $S \subseteq H$ is a heavy $(rt(t - 1))$ -star, then every edge of H intersecting the body of S needs to contain the centre of S .*

Proof. For the proof of part (a) we take n_0 to be at least as large as the number provided by Theorem 3.1. We then know that for any $H \in \mathcal{J}(r, t)$ as above there exists a star $\hat{S} \subseteq H$ with $|\hat{S}| \geq \frac{m\hat{n}(n, m)}{n} - n^{rt-1}$. Throughout the process yielding $S = \text{cr}_e(\hat{S})$ we remove at most $2r^2t^3 \sum_{i=1}^n i^{rt-2} < 2r^2t^3n^{rt-1}$ edges and thus S has at least the size we claimed. To obtain the desired lower bound on $|\text{cr}_v(S)|$ we use that the definition of $\hat{n} = \hat{n}(n, m)$ implies $\frac{(\hat{n}-1)m}{n} > \binom{\hat{n}-1}{rt}$, whence

$$\binom{|\text{cr}_v(S)|}{rt} \geq |S| > \binom{\hat{n}-1}{rt} - 3r^2t^3n^{rt-1} > \binom{\hat{n}-c_0}{rt}$$

holds for sufficiently large c_0 and n_0 .

For the verification of part (b) we may take two large constants b, b' with $b \gg b' \gg a$. There is nothing to prove in case $N \leq b$, so let us assume $N > b$ from now on. As above we have $|S \setminus \text{cr}_e(S)| \leq 2r^2t^3N^{rt-1}$ and, hence,

$$|\text{cr}_e(S)| \geq \binom{N-a}{rt} - 2r^2t^3N^{rt-1} \geq \binom{N-b'}{rt},$$

which is only possible if $|\text{cr}_v(S)| \geq N - b'$.

Finally, let H and S be as in (c), write B for the set of vertices forming the body of S , set $N = |B|$, and consider any $e \in H$ intersecting B in some vertex v . The minimum degree condition satisfied by S yields $|S| \geq 2rt^2N^{rt-1}$. As at most rt^2N^{rt-1} edges of S can intersect $e \cap B$, there is an edge $f \in S$ disjoint to $e \cap B$ and, consequently, we have $|e \cap s| \in I$, where s denotes the centre of S . Similarly, for every $w \in e \cap B$ distinct from v there are at most N^{rt-2} edges of S containing both v and w , and thus there is an edge $f' \in S$ with $(e \cap B) \cap f' = \{v\}$, which proves $|e \cap s| + 1 \in I$. But the only possibility for the consecutive integers $|e \cap s|$ and $|e \cap s| + 1$ to belong to I is that $s \subseteq e$, as desired. \square

The following result describes the structure of $H \in \mathfrak{E}(\mathcal{J}(r, t); n, m)$ quite precisely.

Theorem 3.5. *For all integers $t \geq 2$ and $r \geq 1$, there exist an integer $n_* = n_*(r, t)$ and constants $c_i = c_i(r, t)$, $i = 1, 2, 3$, such that from every rt^2 -graph $H = (V, E)$ in $\mathfrak{E}(\mathcal{J}(r, t); n, m)$ with $n \geq n_*$ vertices and $m > \binom{[n/t]}{rt}$ edges one can remove at most c_1 edges to get a graph which consists of $\ell \leq 7t^2$ many $(rt(t-1))$ -stars S^1, S^2, \dots, S^ℓ and, perhaps, some number of isolated vertices.*

Moreover, we can also assume that

- (i) $|\text{cr}_v(S^i)| \geq n/(7t^2)$, for $i = 1, 2, \dots, \ell - 1$;
- (ii) $|V \setminus \bigcup_{i=1}^{\ell} \text{cr}_v(S^i)| \leq c_2$;
- (iii) the centres of the stars S^1, \dots, S^ℓ are pairwise disjoint.

In particular, we can delete from H at most $c_3n^{(r-1)t}$ edges to get a union of at most ℓ vertex disjoint $(rt(t-1))$ -stars.

Proof. Let us assume that n is sufficiently large, $m > \binom{[n/t]}{rt}$, and that $H \in \mathfrak{E}(\mathcal{J}(r, t); n, m)$. The idea for constructing the first $\ell - 1$ of the desired stars is to apply Fact 3.4 (a) iteratively, pulling these stars out of H one by one. This process comes to an end when we cannot guarantee anymore to find a star with a sufficiently large core in the remaining part of H .

More precisely, we start by setting $H^1 = H$. Due to Fact 3.4 (a) there exists a maximal star $S^1 \subseteq H^1$ with $|\text{cr}_v(S^1)| \geq 2n/(5t)$ and $|\text{cr}_e(S^1)| \geq m\hat{n}(n, m)/n - 3r^2t^3n^{rt-1}$. By H^2 we denote the hypergraph arising from H^1 by the deletion of all vertices in $\text{cr}_v(S^1)$ and all edges in S^1 . If for some integer $i \geq 2$ we have just chosen a star $S^{i-1} \subseteq H^{i-1}$ and

constructed a hypergraph $H^i \subseteq H^{i-1}$ with n_i vertices and m_i edges, we check whether the conditions

- (a) $n_i \geq \hat{n}(n, m)$,
- (b) and $m_i > \binom{n_i/t}{rt}$

are satisfied. If at least one of them fails we set $\ell = i$ and terminate the procedure, the last constructed objects being $S^{\ell-1}$ and H^ℓ . On the other hand, if both conditions hold, the assumptions of Fact 3.4 (a) are satisfied by H^i . Thus we find a maximal star $S^i \subseteq H^i$ with $|\text{cr}_v(S^i)| \geq 2n_i/(5t)$ and $|\text{cr}_e(S^i)| \geq m_i \hat{n}(n_i, m_i)/n_i - 3r^2 t^3 n_i^{rt-1}$. Moreover, we let H^{i+1} denote the hypergraph with vertex set $V(H^i) \setminus \text{cr}_v(S^i)$ and edge set $H^i \setminus S^i$.

Notice that for each $i \in [\ell - 1]$ property (a) of the above process entails

$$|\text{cr}_v(S^i)| \geq \frac{2n_i}{5t} \geq \frac{2\hat{n}(n, m)}{5t} \geq \frac{4n}{25t^2} > \frac{n}{7t^2},$$

meaning that condition (i) of the theorem holds. Besides, since $\text{cr}_v(S^1), \dots, \text{cr}_v(S^{\ell-1})$ are mutually disjoint, it also follows that $\ell \leq 7t^2$. Denote the centre of S^i by s_i for $i \in [\ell - 1]$.

Now we investigate in detail the structure of the last hypergraph H^ℓ . The following result is crucial for our argument. It shows, in particular, that our statement about c_1 holds.

Claim 3.6. *There are constants $c_1 = c_1(r, t)$ and $c_4 = c_4(r, t)$ such that after removing at most c_1 edges from H^ℓ this hypergraph becomes the union of an $(rt(t-1))$ -star with at least $\binom{n_\ell - c_4}{rt}$ edges and, perhaps, some number of isolated vertices.*

Proof. Suppose that the assertion does not hold. Our aim is to get a contradiction with the assumption that $H \in \mathfrak{E}(\mathcal{J}(r, t); n, m)$ by constructing a graph $H' \in \mathcal{J}(r, t)$ having the same number of vertices and edges as H but a smaller maximum degree. Choose some absolute constants c, c', c'', c_1 , and c_4 depending only on r and t , sufficiently large so that all arguments below will work, and obeying the hierarchy

$$c_1 \gg c_4 \gg c'' \gg c' \gg c.$$

First case: $n_\ell < \hat{n}(n, m)$

Notice that we may assume $n_\ell \geq c_4$, since otherwise an appropriate choice of c_1 would show that the claim holds with an empty star. Moreover, if $m_\ell > \binom{n_\ell - c''}{rt}$ the desired conclusion can be drawn from Proposition 2.3 (and Theorem 2.2). So we may suppose $m_\ell \leq \binom{n_\ell - c''}{rt}$ from now on.

The hypergraph H' will have three kinds of edges. First, there will be stars $\hat{S}^1, \dots, \hat{S}^{\ell-1}$, where each \hat{S}^i is obtained from $\text{cr}_e(S^i)$ by the omission of a single edge.

Second, there will be edges serving as “substitutes” for the edges in $F^i = S^i \setminus \text{cr}_e(S^i)$ for $i \in [\ell - 1]$. The reason for this substitution is that it “cleans up some space” so that in the end $H' \in \mathcal{J}(r, t)$ will be true. Let us recall that $|F^i| \leq O(n^{rt-1})$ follows from the

construction of cores. Hence, there are disjoint subsets $U^1, \dots, U^{\ell-1}$ of the vertex set of H^ℓ with $|U^i| = \lceil |F^i| / \binom{\text{cr}_v(S^i)}{rt-1} \rceil \leq c$ for $i \in [\ell-1]$. Now instead of F^i we put the same number of edges of the type $\{v\} \cup s_i \cup f$ into H' , where $v \in U^i$, and f is a subset of the body $\text{cr}_v(S^i)$ with $rt-1$ elements.

Third, we include a star with $m_\ell + (\ell-1)$ edges into H' that uses only vertices of H^ℓ that are not occupied by the sets s_i and U^i for $i \in [\ell-1]$. There is enough space for such a star, as at most $\ell(rt^2 + c) \leq c'$ vertices are occupied, $m_\ell \leq \binom{n_\ell - c''}{rt}$, and n_ℓ is sufficiently large.

It remains to check that we have indeed $\Delta(H') < \Delta(H)$. The only vertices of H' that might be problematic are in the centre of the new star that has just been created.

However, working carefully with the estimates provided by Theorem 3.1 and exploiting that we are in the first case, one checks easily that

$$\begin{aligned} \Delta(H) &\geq |S^1| \geq \frac{m\hat{n}(n, m)}{n} - 3r^2t^3n^{rt-1} \geq \binom{\hat{n}(n, m) - c'}{rt} \\ &> \binom{n_\ell - c''}{rt} + (\ell-1) \geq m_\ell + (\ell-1). \end{aligned}$$

Thus H' contradicts indeed our assumption that $H \in \mathfrak{E}(\mathcal{J}(r, t); n, m)$.

Second case: $n_\ell \geq \hat{n}(n, m)$

This means that the iterative procedure that led us to the stars $S^1, \dots, S^{\ell-1}$ stopped owing to the failure of condition (b), i.e., that

$$m_\ell \leq \binom{\lfloor n_\ell/t \rfloor}{rt}. \quad (3.4)$$

One can deal with this case in a very similar way as with the previous one but instead of replacing the edges of H^ℓ by one large star we replace them by t smaller and mutually disjoint stars with roughly n_ℓ/t vertices and m_ℓ/t edges.

The inequality $\frac{m_\ell}{t} \leq \binom{n_\ell/t - c'}{rt}$, which is a direct consequence of (3.4), shows that there is indeed enough space for such stars.

Finally, it remains to check that the hypergraph H' generated as above really has a smaller maximum degree than H .

If $n_\ell \leq \frac{tn}{t+1}$ this follows from

$$\begin{aligned} \left\lceil \frac{m_\ell + (\ell-1)}{t} \right\rceil &\leq \frac{1}{t} \binom{n/(t+1)}{rt} + O(1) \leq \frac{1}{t} \left(\frac{t}{t+1} \right)^{rt} \binom{n/t}{rt} + O(n^{rt-1}) \\ &\leq \frac{4m}{9t} + O(n^{rt-1}) < \frac{m\hat{n}(n, m)}{n} - 3r^2t^3n^{rt-1} \leq |S^1|. \end{aligned}$$

The case $\frac{tn}{t+1} < n_\ell$, however, is impossible, because due to $n_\ell \leq n - \text{cr}_v(S^1) < (1 - \frac{2}{5t})n$ it would entail

$$\begin{aligned} m &= \sum_{i=1}^{\ell-1} (|\text{cr}_e(S^i)| + |F^i|) + m_\ell \leq \sum_{i=1}^{\ell-1} \binom{|\text{cr}_v(S^i)|}{rt} + \binom{\lfloor n_\ell/t \rfloor}{rt} + O(n^{rt-1}) \\ &\leq \binom{n - n_\ell}{rt} + \binom{n_\ell/t}{rt} + O(n^{rt-1}) < \binom{\lfloor n/t \rfloor}{rt}, \end{aligned}$$

where for the last estimate we used that $(1-x)^{rt} + (x/t)^{rt} < (1/t)^{rt}$ holds for all real $x \in [\frac{t}{t+1}, 1 - \frac{2}{5t}]$. But the above estimate contradicts our initial hypothesis about m . \square

Notice that taken together Fact 3.4 (b) and Claim 3.6 show that part (ii) of the theorem holds for a sufficiently large choice of c_2 and an appropriate star $S^\ell \subseteq H^\ell$. Let s_ℓ be the centre of S^ℓ .

In order to verify part (iii) let us observe first that each star S^i consists of at most

$$\binom{|\text{cr}_v(S^i)|}{rt} + O(n^{rt-1})$$

edges. It may be helpful to recall each of the sets $\text{cr}_v(S^1), \dots, \text{cr}_v(S^{\ell-1})$ has size $\Omega(n)$. We do not know the same about the last star, but at least we may suppose that $\text{cr}_v(S^\ell)$ is sufficiently large for otherwise we may ignore this star and proceed. Now let us assume that there are two stars, S^i and S^j , $1 \leq i < j \leq \ell$, which do not have disjoint centres. Then we construct a new hypergraph $H' \in \mathcal{J}(r, t)$ out of $H \in \mathfrak{E}(\mathcal{J}(r, t); n, m)$ in the following way. We delete all the edges of the stars S^i and S^j , say of m'_i and m'_j edges respectively, and on the vertex set $\text{cr}_v(S^i) \cup \text{cr}_v(S^j)$ we create an $(rt(t-1))$ -star S^{ij} which has $m'_i + m'_j - 1 < \Delta(H)$ edges and which uses as few vertices as possible. Due to $|\text{cr}_v(S^i)| \geq \Omega(n)$ and $|\text{cr}_v(S^j)| \geq O(1)$ we have

$$\binom{|\text{cr}_v(S^i)|}{rt} + \binom{|\text{cr}_v(S^j)|}{rt} + O(n^{rt-1}) < \binom{|\text{cr}_v(S^i)| + |\text{cr}_v(S^j)| - 8rt^4}{rt},$$

and hence S^{ij} uses fewer than $|\text{cr}_v(S^i)| + |\text{cr}_v(S^j)| - 7rt^4$ vertices. Now we remove one edge from each of the existing stars S^t , where $t = 1, 2, \dots, \ell$, and $t \neq i, j$, and add $\ell - 1 \leq 7t^2$ disjoint edges to the hypergraph. Such a hypergraph $H' \in \mathcal{J}(r, t)$ has a smaller maximum degree than H , which contradicts the fact that $H \in \mathfrak{E}(\mathcal{J}(r, t); n, m)$. Thus, the centres of the stars S^i , $i = 1, 2, \dots, \ell$, are pairwise disjoint, as claimed in clause (iii) of the theorem. For later use we record that using Fact 3.4 (c) one can show that $s_i \cap \text{cr}_v(S^j) = \emptyset$ holds whenever $i, j \in [\ell]$.

Finally, we need to argue that we can delete from $H \in \mathfrak{E}(\mathcal{J}(r, t); n, m)$ at most $c_3 n^{(r-1)t}$ additional edges to make S^1, \dots, S^ℓ vertex disjoint. We contend that it suffices for this purpose to delete all edges intersecting the set $W = \bigcup_{i=1}^{\ell} \text{cr}_v(S^i)$ in at most $(r-1)t$ vertices. Notice that due to (ii) the resulting hypergraph \tilde{H} differs in at most $c_2^{rt^2} n^{(r-1)t}$ edges from H . Owing to of Fact 3.4 (c) and (iii) there is for every edge $e \in \tilde{H}$ a unique $i \in [\ell]$

with $e \in S^i$; for this i we have $|e \cap \text{cr}_v(S^i)| \geq (r-1)t + 1$ and, hence, e has at most $t-1$ vertices outside $s_i \cup \text{cr}_v(S^i)$. These vertices cannot belong to the centre s_j of another star, for then e would have a forbidden intersection with every edge in $\text{cr}_e(S^j)$. Now assume that one of the vertices in $e \setminus (s_i \cup \text{cr}_v(S^i))$, say v , would belong in \tilde{H} to another star as well. This means that there are an index $j \neq i$ and an edge $f \in S^j \cap \tilde{H}$ with $v \in e \cap f$. Due to $f \in \tilde{H}$ at most $t-1$ vertices of f are outside $s_j \cup \text{cr}_v(S^j)$ and thus we have $1 \leq |e \cap f| \leq t-1$, which is absurd. Thus \tilde{H} is indeed a union of ℓ vertex disjoint stars. \square

Let us comment briefly on the structure of $H \in \mathfrak{E}(\mathcal{J}(r, t); n, m)$ described in Theorem 3.5. Once we know Theorem 3.5 the estimate for the number of stars ℓ can be easily improved to the optimal $\ell \leq \lceil t^{rt/(rt-1)} \rceil$ (see [9], where a similar argument is used for $r = 1, t = 2$). However, we cannot significantly decrease the number of edges needed to make the stars vertex disjoint. To see this, let us consider the rt^2 -graph $\tilde{H} \in \mathfrak{E}(\mathcal{J}(r, t); n, m)$ with vertex set $V = U_1 \cup U_2 \cup C_1 \cup C_2 \cup T$, where $|U_1| = |U_2| = u$, $|C_1| = |C_2| = rt(t-1)$, $|T| = t$, whose set of edges consists of:

- $\binom{u}{rt}$ subsets which are unions of C_1 and some rt -element subset of U_1 ,
- $\binom{u}{rt}$ subsets which are unions of C_2 and some rt -element subset of U_2 ,
- $\binom{u}{(r-1)t}$ subsets which are unions T, C_1 and some $(r-1)t$ -element subset of U_1 ,
- $\binom{u}{(r-1)t}$ subsets which are unions T, C_2 and some $(r-1)t$ -element subset of U_2 ,
- and a thick clique on $T \cup C_1 \cup C_2$ whose teams are T and partitions of C_1, C_2 .

It is easy to see that $\tilde{H} \in \mathfrak{E}(\mathcal{J}(r, t); n, m)$ with $n = 2u + 2rt(t-1) + t$ and the appropriate m . On the other hand \tilde{H} consists of two stars with centres C_1 and C_2 and to make them vertex disjoint one must delete at least $\Omega(n^{(r-1)t})$ edges.

Finally, let us notice that from Theorem 3.5 it follows that almost all the edges of dense extremal graphs from $\mathfrak{E}(\mathcal{J}(r, t); n, m)$ are contained in at most ℓ stars among which $\ell-1$ are roughly equal and only one can be a bit smaller than the others. Having this in mind one can easily compute the scaled extremal function

$$f_{rt}(x) = \lim_{n \rightarrow \infty} \frac{f(\mathcal{J}(r, t); n, x \binom{n-rt(t-1)}{rt})}{\binom{n-rt(t-1)}{rt}}.$$

From Theorems 3.1 and 3.5 we know that the function is well defined for $x \in [0, 1] \setminus \{t^{-tr}\}$. Furthermore, besides the point $x = t^{-tr}$ where it jumps from 0 to some value which is at least $t^{-t^2r^2/(rt-1)}$, it is continuous everywhere. It is also smooth everywhere except the points $x = j^{1-tr}$ for $j = 2, 3, \dots, \lceil t^{tr/(rt-1)} \rceil - 1$ (see [9] where details are worked out for the case $r = 1, t = 2$).

§4. TOOLS

The purpose of this section is to gather three statements that will turn out to be useful in the proof of the Structure Theorem. While the first two of them are fairly well known, the third one (see Lemma 4.4 below) could very well be new.

4.1. Divisible set systems. Given a natural number $t \geq 2$ we shall say that a set system (V, \mathcal{E}) is *t-divisible* if for any two distinct edges $e, e' \in \mathcal{E}$ the size $|e \cap e'|$ of their intersection is a multiple of t . The problem to study upper bounds on the size of such set systems with additional assumptions on the behaviour of the sizes of the edges modulo t was first studied, in the particular case $t = 2$, by Berlekamp [1], who realised that ideas from linear algebra can be applied in such contexts. At a later occasion we will need a variant of one of his result that was first observed, in more general form, by Babai and Frankl (see [2, Theorem 1]).

Lemma 4.1. *Let (V, \mathcal{E}) be a t -divisible set system for some natural number $t \geq 2$. If $|e| \equiv 1 \pmod{t}$ holds for all $e \in \mathcal{E}$, then $|\mathcal{E}| \leq |V|$.*

Proof. Let p denote a prime factor of t . We identify the members of \mathcal{E} with vectors from the $|V|$ -dimensional vector space \mathbb{F}_p^V via characteristic functions and contend that the stronger conclusion that \mathcal{E} is linearly independent holds. To see this, one looks at a hypothetical linear dependency $\alpha_1 e_1 + \dots + \alpha_n e_n = 0$ with distinct $e_i \in \mathcal{E}$ and certain numbers $\alpha_i \in \mathbb{F}_p \setminus \{0\}$, where $n \geq 1$. Taking the standard scalar product with e_1 we obtain

$$0 = \langle e_1, 0 \rangle = \langle e_1, \alpha_1 e_1 + \dots + \alpha_n e_n \rangle = \sum_{i=1}^n \alpha_i \langle e_1, e_i \rangle = \alpha_1,$$

which is absurd. □

4.2. Delta systems. A set system \mathcal{F} is called a *sunflower* (or a Δ -system) if there exists a (possibly empty) set S of vertices such that the intersection of any two distinct edges of \mathcal{F} is equal to S . This constant intersection S is called the *kernel* of the sunflower.

In 1960 Erdős and Rado [5] proved their “sunflower lemma” saying that any sufficiently large collection of finite sets of bounded size contains big sunflowers.

Theorem 4.2. *For all positive integers a and b , any collection of more than $b!a^{b+1}$ sets of cardinality at most b contains a Δ -system with more than a elements.*

It should perhaps be pointed out that $b!a^{b+1}$ is not the least number $f(a, b)$ for which this statement is true. In fact, Erdős and Rado themselves stated a marginally better but less clean upper bound on this number in [5, Theorem III], but despite the considerable attention that the problem to improve our understanding of the growth behaviour of this function has received (see e.g., [8]) the progress on this problem has been rather slow. For

the purposes of the present article, however, even knowing the exact value of $f(a, b)$ would be quite immaterial.

4.3. Divisible pairs of set systems. The next result makes use of the following concept.

Definition 4.3. Let \mathcal{F} and \mathcal{G} denote two set systems with the same vertex set V and let q be a positive integer. We say that the pair $(\mathcal{F}, \mathcal{G})$ is q -divisible if for all $f \in \mathcal{F}$ and $g \in \mathcal{G}$ the size $|f \cap g|$ of their intersection is divisible by q .

The result that follows will often help us to analyse the structure of divisible pairs.

Lemma 4.4 (Decomposition lemma). *Suppose that k, q are positive integers and that \mathcal{F}, \mathcal{G} are two set systems with the same vertex set V such that*

- *all members of $\mathcal{F} \cup \mathcal{G}$ have size at most k ,*
- *and the pair $(\mathcal{F}, \mathcal{F} \cup \mathcal{G})$ is q -divisible.*

Then there is a set system \mathcal{H} on V with the following properties:

- (i) $\Delta(\mathcal{H}) \leq k^{2k}$;
- (ii) \mathcal{H} is an antichain (that is, $x \not\subseteq y$ holds for all distinct $x, y \in \mathcal{H}$);
- (iii) Every edge of \mathcal{F} is a disjoint union of edges from \mathcal{H} ;
- (iv) The pair $(\mathcal{H}, \mathcal{H} \cup \mathcal{G})$ is q -divisible.

Proof. Without loss of generality we may assume that given k, q , and \mathcal{G} , the set system \mathcal{F} is maximal with respect to inclusion, i.e., that for every set system $\mathcal{F}^* \supsetneq \mathcal{F}$ with $|f| \leq k$ for all $f \in \mathcal{F}^*$ the pair $(\mathcal{F}^*, \mathcal{F}^* \cup \mathcal{G})$ fails to be q -divisible.

Now we define \mathcal{H} to be the collection of those members of $\mathcal{F} \setminus \{\emptyset\}$ that are minimal with respect to inclusion, i.e., we set

$$\mathcal{H} = \{h \in \mathcal{F} \setminus \{\emptyset\} : \text{if } f \in \mathcal{F} \text{ and } f \neq \emptyset, h, \text{ then } f \not\subseteq h\}.$$

This choice of \mathcal{H} makes part (ii) obvious and (iv) follows directly from $\mathcal{H} \subseteq \mathcal{F}$.

Assuming that (iii) would be false let $f \in \mathcal{F}$ be chosen with $|f|$ minimum such that f is not expressible as a disjoint union of appropriate edges from \mathcal{H} . Since the empty set is equal to the empty union, we have $f \neq \emptyset$. Moreover, f cannot belong to \mathcal{H} and, consequently, there exists some $h \in \mathcal{H}$ with $h \subseteq f$ and $0 < |h| < |f|$. Notice that for every $g \in \mathcal{F} \cup \mathcal{G}$ the number $|(f \setminus h) \cap g| = |f \cap g| - |h \cap g|$ is divisible by q . Owing to the maximality of \mathcal{F} it follows that $(f \setminus h) \in \mathcal{F}$. But in view of our minimal choice of f this means that $f \setminus h$ is a disjoint union of edges from \mathcal{H} and, hence, so is f . Hence, \mathcal{H} must satisfy (iii).

Now it remains to show that \mathcal{H} has bounded maximum degree. Assume for the sake of contradiction, that there exists a vertex $v \in V$ contained in more than k^{2k} edges of \mathcal{H} and look at the set system

$$\mathcal{F}_v = \{h \in \mathcal{H} : v \in h\}.$$

Then $|\mathcal{F}_v| > k^{2k} \geq k!k^{k+1}$ and Theorem 4.2 reveals that \mathcal{F}_v contains a Δ -system \mathcal{F}^* with more than k elements. Denote the kernel of \mathcal{F}^* by e and observe that, since $|\mathcal{F}^*| > k$, for all edges $g \in \mathcal{F} \cup \mathcal{G}$, the size of the intersection $|e \cap g|$ is divisible by q . Together with the maximality of \mathcal{F} this implies $e \in \mathcal{F}$. Due to $|\mathcal{F}^*| \geq 2$ there must exist some $h \in \mathcal{F}^* \subseteq \mathcal{H}$ properly containing e and by our definition of \mathcal{H} this is only possible if $e = \emptyset$. But, on the other hand, we certainly have $v \in e$. This contradiction concludes the proof (i) and, hence, the proof of the decomposition lemma. \square

§5. PROOF OF THE STRUCTURE THEOREM

This entire section is dedicated to the proof of Theorem 2.2. Let integers $r \geq 1$ and $t \geq 2$ as well as an rt^2 -uniform hypergraph $H = (V, E)$ with $|V| = n$ be given such that the size of the intersection of any two edges of H belongs to the set

$$I = \{s : t \mid s \text{ or } s \geq rt(t-1)\},$$

which means $H \in \mathcal{J}(r, t)$. We will show that $H \in \mathcal{F}(t, rt, (rt^2)^{r^3t^6})$.

Let us start by colouring all those subsets $f \subseteq V$ with $|f| \leq rt(t-1) + 1$ red that are kernels of sunflowers consisting of at least rt^2 edges of H . Recall that the latter condition means that there are to exist at least rt^2 disjoint sets $f_1, \dots, f_{rt^2} \subseteq V$ of size $rt^2 - |f|$ such that $f \cup f_i \in H$ holds for every $i \in [rt^2]$. We denote the set system on V whose edges are the red sets by H_{red} . By H_{red}^* we mean the $(rt(t-1) + 1)$ -uniform hypergraph on V whose edges are the red $(rt(t-1) + 1)$ -sets and finally we put $\widehat{H}_{\text{red}} = H_{\text{red}} \setminus H_{\text{red}}^*$.

Observe that

$$\text{for any } f, f' \in H \cup H_{\text{red}} \text{ we have } |f \cap f'| \in I. \quad (5.1)$$

This is because we can first extend f to an edge e of H with $e \cap f' = f \cap f'$ and proceeding similarly with f' we get an edge $e' \in H$ with $e \cap e' = f \cap f'$, so that $|f \cap f'| = |e \cap e'| \in I$ follows from the assumption that H be I -intersecting.

As a consequence of this observation we learn that for any distinct $f, f' \in H_{\text{red}}^*$ the number $|f \cap f'|$ is divisible by t and in view of Lemma 4.1 it follows that

$$|H_{\text{red}}^*| \leq n. \quad (5.2)$$

Moreover, (5.1) reveals that the pair $(\widehat{H}_{\text{red}}, \widehat{H}_{\text{red}} \cup H_{\text{red}}^* \cup H)$ is t -divisible, which allows us to apply the decomposition lemma (Lemma 4.4) to rt^2 , t , \widehat{H}_{red} , and $H_{\text{red}}^* \cup H$ here in place of k , q , \mathcal{F} , and \mathcal{G} there. We thus infer the existence of a set system G on V with the following properties:

- (a) $\Delta(G) \leq (rt^2)^{2rt^2}$;
- (b) G is an antichain;
- (c) Every edge of \widehat{H}_{red} is a disjoint union of edges from G ;
- (d) The pair $(G, G \cup H_{\text{red}}^* \cup H)$ is t -divisible.

We imagine that the edges of G have been coloured *green*. The green sets of cardinality t will be referred to as *teams*. Notice that due to condition (d) the teams are inseparable in H and, moreover, by (b) and (d) each team is disjoint to any other green set.

Now we are ready to decompose V and H in the envisioned way. We start by defining V_T to be the union of all teams and setting $H_T = H[V_T]$, which guarantees part (i) of Definition 2.1.

Preparing the definition of H_S we colour a set consisting of $rt(t-1)$ vertices *purple* if it is the kernel of a Delta system in H_{red}^* of size at least rt^2 . Imitating the proof of (5.1) one checks easily that

$$\text{if } Y, Y' \text{ are purple and } f \in H_{\text{red}}^* \cup H, \text{ then } |Y \cap Y'|, |Y \cap f| \in I. \quad (5.3)$$

Now we define H_S to be the collection of all edges $h \in H$ which contain a purple set $Y_h \subseteq h$ such that $Y_h \cup \{v\} \in H_{\text{red}}^*$ holds for each $v \in h \setminus Y_h$. Moreover, we set

$$V_S = \bigcup_{h \in H_S} (h \setminus Y_h),$$

which is clearly a subset of $V \setminus V_T$. Provided we can show

$$Y_h \cap V_S = \emptyset \quad \text{for each } h \in H_S \quad (5.4)$$

it will be clear that H_S is a union of stars with centres $Y_h \subseteq (V \setminus V_S)$ and their bodies in V_S , as required by Definition 2.1 (ii).

For the proof of (5.4) we assume indirectly that for some $h \in H_S$ there is a vertex $v \in Y_h \cap V_S$. This means that there exists an edge $h' \in H_S$ with $v \in h' \setminus Y_{h'}$. But now $|Y_h \cap Y_{h'}|$ and $|Y_h \cap (Y_{h'} \cup \{v\})|$ are two consecutive integers belonging to I by (5.3) and both are at most $|Y_h| = rt(t-1)$, contrary to $t \geq 2$. Thereby (5.4) is proved.

Next we observe that if for some $f \in H$ and $h \in H_S$ there is a vertex $v \in f \cap h \cap V_S$, then the consecutive integers $|f \cap Y_h|$ and $|f \cap (Y_h \cup \{v\})|$, again by (5.3), are both in I .

Consequently,

$$\text{if } f \in H \text{ and } h \in H_S \text{ satisfy } f \cap h \cap V_S \neq \emptyset, \text{ then } Y_h \subseteq f. \quad (5.5)$$

Hence, all stars in H_S must be semi-disjoint and we may associate with each vertex $v \in V_S$ the set $Y_v \in H_{\text{red}}^*$ containing v and the centre of the star to which v belongs. With this notation, (5.5) rewrites as

$$\text{if } f \in H \text{ and } v \in f \cap V_S, \text{ then } Y_v \subseteq f. \quad (5.6)$$

Condition (iii) of Definition 2.1 is an immediate consequence of this statement and it also follows that $H_S \supseteq \{h \in H : |h \cap V_S| = rt\}$. The reverse inclusion is implied by (5.4) and thereby condition (ii) is proved as well.

It remains to establish (iv), i.e., that for

$$V_R = V \setminus (V_T \cup V_S) \quad \text{and} \quad H_R = H \setminus (H_T \cup H_S)$$

we have

$$|H_R| \leq |V_T||V_S|n^{rt-3} + (rt^2)^{r^3t^6}|V_R|n^{rt-2}. \quad (5.7)$$

The first step in the proof of this result is to split H_R into the two subhypergraphs H_{ST} and \widehat{H}_R with the intention of proving $|H_{ST}| \leq |V_T||V_S|n^{rt-3}$ and $|\widehat{H}_R| \leq (rt^2)^{r^3t^6}|V_R|n^{rt-2}$.

The family H_{ST} is defined as

$$H_{ST} = \{h \in H_R : \text{there are } v \in h \cap V_S \text{ and a team } g \subseteq h \text{ with } Y_v \cap g = \emptyset\}.$$

Observe that if $h \in H_{ST}$ and v, g are as in the above definition, then $Y_v \subseteq h$ follows from (5.6) and we have $|h \setminus (Y_v \cup g)| = rt^2 - 1 - t - rt(t-1) \leq rt - 3$. As there are at most $|V_S|$ possibilities for v , $|V_T|$ possibilities for g , and n^{rt-3} possibilities for the set $h \setminus (Y_v \cup g)$, it follows that we have indeed

$$|H_{ST}| \leq |V_T||V_S|n^{rt-3}.$$

Thus to conclude the argument we need to show that the hypergraph $\widehat{H}_R = H_R \setminus H_{ST}$ satisfies

$$|\widehat{H}_R| \leq (rt^2)^{r^3t^6}|V_R|n^{rt-2}. \quad (5.8)$$

In the special case $V_R = \emptyset$ this can only be true if $\widehat{H}_R = \emptyset$ holds as well. For that reason it will certainly help us to establish

$$H_R \setminus V_R \subseteq H_{ST}. \quad (5.9)$$

To verify this, consider any edge $f \in H_R$ not meeting V_R . Owing to $f \notin H_T$ there must exist a vertex $v \in f \cap V_S$ and (5.6) tells us that $Y_v \subseteq f$. Now $f \setminus Y_v$ cannot be a subset of V_S because (5.6) would then yield $f \in H_S$. Together with $f \subseteq (V_S \cup V_T)$ this shows that there must be a vertex $x \in V_T \cap (f \setminus Y_v)$. This vertex must in turn belong to some team $g \in G$, which is in fact a subset of $f \setminus Y_v$. Thereby (5.9) is proved.

Due to the discussion preceding (5.9) we may henceforth suppose that $V_R \neq \emptyset$. Now the idea for proving (5.8) is that we can mark in every edge $h \in \widehat{H}_R$ at least one vertex from $h \cap V_R$ in such a way that every vertex in V_R gets marked at most $(rt^2)^{r^3t^6}n^{rt-2}$ many times. The marking procedure we use depends on the red and green sets contained in h and thus it involves several case distinctions.

In view of property (c) of the green sets, we may write

$$\widehat{H}_R = \widehat{H}_R^1 \cup \widehat{H}_R^2 \cup \widehat{H}_R^3 \quad (5.10)$$

with

$$\widehat{H}_R^1 = \{h \in \widehat{H}_R : h \text{ cannot be written as a union of red and green sets}\},$$

$$\widehat{H}_R^2 = \{h \in \widehat{H}_R : h \text{ is the union of its green subsets}\},$$

$$\text{and } \widehat{H}_R^3 = \{h \in \widehat{H}_R : h \notin \widehat{H}_R^1 \text{ and there is some } f \in H_{\text{red}}^* \text{ with } f \subseteq h\}.$$

Regarding the first of these three hypergraphs, we note that if $h \in \widehat{H}_R^1$ and $v \in h$ is not contained in any red or green subset of h , then $v \in V_T$ is impossible due to the inseparability of the teams, $v \in V_S$ is impossible by (5.6), and hence we must have $v \in V_R$. In other words, if we set

$$H_v = \{h \in \widehat{H}_R: v \in h\}$$

and

$$\widehat{H}^1(v) = \{h \in H_v: \text{there is no } f \in H_{\text{red}} \text{ with } v \in f \subseteq h\}$$

for every $v \in V_R$, then

$$\widehat{H}_R^1 \subseteq \bigcup_{v \in V_R} \widehat{H}^1(v). \quad (5.11)$$

According to our plan the hypergraphs $\widehat{H}^1(v)$ should be of size at most $O(n^{rt-2})$ and this is indeed what we prove next.

Fact 5.1. *For every $v \in V_R$ we have*

$$|\widehat{H}^1(v)| < (rt^2)^{2rt^2} n^{rt-2}.$$

Proof. Assume for the sake of contradiction that $v \in V_R$ violates this claim. Then $x = \{v\}$ is an example of a subset of V with $v \in x$ and

$$|\{h \in \widehat{H}^1(v): x \subseteq h\}| \geq (rt^2)^{2(rt^2-|x|)} n^{rt-2}. \quad (5.12)$$

Now let $x \subseteq V$ be a maximal set of vertices with $v \in x$ that satisfies (5.12). As $x \subseteq h$ for some $h \in H$, we must have $|x| \leq rt^2$. Thus

$$n^{rt-2} \leq (rt^2)^{2(rt^2-|x|)} n^{rt-2} \leq |\{h \in \widehat{H}^1(v): x \subseteq h\}| \leq \binom{n-|x|}{rt^2-|x|} < n^{rt^2-|x|}$$

and it follows that $|x| \leq rt(t-1) + 1$. But owing to the definition of $\widehat{H}^1(v)$ it is not possible for x to be red. This means, in particular, that there is a maximal Δ -system $\mathcal{G} \subseteq \widehat{H}^1(v)$ with kernel x and $|\mathcal{G}| < rt^2$. The size of the set $B = \bigcup_{h \in \mathcal{G}} (h \setminus x)$ can be bounded by $|B| \leq \sum_{h \in \mathcal{G}} |h| < (rt^2)^2$ and the maximality of \mathcal{G} implies that every edge $h \in \widehat{H}^1(v)$ with $x \subseteq h$ intersects B . So by averaging and (5.12) there exists a vertex $w \in B$ with

$$|\{h \in \widehat{H}^1(v): (x \cup \{w\}) \subseteq h\}| \geq \frac{(rt^2)^{2(rt^2-|x|)}}{(rt^2)^2} n^{rt-2} = (rt^2)^{2(rt^2-|x \cup \{w\}|)} n^{rt-2}.$$

This inequality tells us that $x \cup \{w\}$ contradicts the maximality of x . Thereby Fact 5.1 is proved. \square

This completes our analysis of \widehat{H}_R^1 and we proceed with \widehat{H}_R^2 . To this end, we shall use the trivial decomposition

$$\widehat{H}_R^2 = \bigcup_{v \in V_R} \widehat{H}^2(v), \quad (5.13)$$

where

$$\widehat{H}^2(v) = \{h \in H_v : h \text{ is the union of its green subsets}\}.$$

Fact 5.2. *If $v \in V_R$, then*

$$|\widehat{H}^2(v)| < (rt^2)^{r^3t^6 - 7rt^2} n^{rt-2}.$$

Proof. Consider the auxiliary set system

$$\mathcal{G} = \{x \subseteq V : |x| \leq rt^2 \text{ and } G|x \text{ is connected}\}.$$

Utilising property (a) of the green sets and the fact that for every $x \in \mathcal{G}$ there is a spanning sub-setsystem of $G|x$ consisting of at most $|x|$ sets we obtain

$$\Delta(\mathcal{G}) < (\Delta(G)rt^2)^{rt^2} \leq (rt^2)^{2(rt^2)^2 + rt^2}.$$

As every edge of G has at least t vertices, the same is true about \mathcal{G} . Moreover, the only possibility for $x \in \mathcal{G}$ to have size exactly t is that it is a team.

Now any given $h \in \widehat{H}^2(v)$ can be expressed as a disjoint union of edges of \mathcal{G} by looking at the connected components of $G|h$. The number of edges from \mathcal{G} appearing in such a decomposition can be at most $rt - 1$ because of the remarks from the previous paragraph and as v cannot belong to a team.

Representing each edge $h \in \widehat{H}^2(v)$ by a selection of one vertex from each of its at most $rt - 2$ green components not containing v , we learn that indeed

$$|\widehat{H}^2(v)| \leq \Delta(\mathcal{G})^{rt-1} n^{rt-2} < (rt^2)^{r^3t^6 - 7rt^2} n^{rt-2}. \quad \square$$

It remains to deal with the hypergraph \widehat{H}_R^3 , which may be further decomposed as

$$\widehat{H}_R^3 = \widehat{H}_R^{3,x} \cup \widehat{H}_R^{3,y}, \quad (5.14)$$

where

$$\widehat{H}_R^{3,x} = \{h \in \widehat{H}_R^3 : h \text{ is the union of its subsets belonging to } H_{\text{red}}^*\}$$

and $\widehat{H}_R^{3,y} = \widehat{H}_R^3 \setminus \widehat{H}_R^{3,x}$. We will estimate the sizes of these two hypergraphs in the two facts that follow. In both proofs we will frequently use the inequality $|H_{\text{red}}^*| \leq n$ obtained in (5.2) above without referencing it.

Fact 5.3. *We have $|\widehat{H}_R^{3,x}| \leq (rt^2)^{rt^2} |V_R| n^{rt-2}$.*

Proof. In the light of (5.1) there are only two possibilities for an edge $h \in \widehat{H}_R^{3,x}$. Either

- (i) there are $f, f' \subseteq h$ in H_{red}^* such that $|f \cap f'| \leq rt(t-1) - t$,
- (ii) or there is some $Y_h \subseteq h$ of size $rt(t-1)$ such that $Y_h \cup \{v\} \in H_{\text{red}}^*$ holds for every vertex $v \in h \setminus Y_h$.

If h is of type (i) we have $|f \cup f'| = |f| + |f'| - |f \cap f'| \geq rt^2 - rt + t + 2$ and hence $|h - (f \cup f')| \leq rt - t - 2 \leq rt - 4$. As there are at most n^2 possibilities to choose a pair f, f' of two edges from H_{red}^* and at most n^{rt-4} possibilities to choose at most $rt - 4$ further vertices in V , there can be at most n^{rt-2} edges in $\widehat{H}_R^{3,x}$ to which the description (i) applies.

Next we note that if h and Y_h are as in (ii), then Y_h cannot be purple for otherwise h would satisfy the requirements for belonging to H_S . We will prove below that there are at most $9|V_R|$ such edges in the special case $r = 1$ and $t = 2$, and at most $\binom{rt^2}{rt} \cdot rt^2 n$ such edges if $rt \geq 3$. Due to $V_R \neq \emptyset$ this suffices to establish Fact 5.3 in both cases.

Let us consider the case that $r = 1$ and $t = 2$ first. If h denotes an edge of type (ii), then $h \setminus Y_h \subseteq V_R$ by (5.6), and we may mark any vertex $v \in h \setminus Y_h$. Since the triples $Y_h \cup \{v\}$ and $h \setminus \{v\}$ are both in H_{red}^* , (5.1) implies that v is contained in at most 3 red sets $f \in H_{\text{red}}^*$. For none of them $f \setminus \{v\}$ is purple (because $v \notin V_S$), which in turn means that each of them can be involved at most 3 times in the marking of v . Altogether each $v \in V_R$ gets marked at most 9 times due to edges of type (ii), wherefore there are indeed at most $9|V_R|$ such edges.

Now suppose that $rt \geq 3$ and let h again denote an edge of type (ii). As Y_h arises from a member of H_{red}^* by the deletion of a vertex, there are at most $rt^2 n$ candidates for this set and each of them can be used in at most $\binom{rt^2}{rt}$ edges of type (ii), for otherwise it would be purple. This proves the upper bound of $\binom{rt^2}{rt} \cdot rt^2 n$ on the number of such edges h and the proof of Fact 5.3 is complete. \square

Fact 5.4. *We have $|\widehat{H}_R^{3,y}| \leq ((rt^2)^{2rt^2} + 1)|V_R|n^{rt-2}$.*

Proof. Consider any edge $h \in \widehat{H}_R^{3,y}$. Since $h \in \widehat{H}_R^3$, there is a set $f \in H_{\text{red}}^*$ with $f \subseteq h$. If possible let f be chosen in such a way that it contains a vertex from V_S .

By $h \notin \widehat{H}_R^{3,x}$ there exists a vertex $v \in h \setminus f$ that is not contained in any member of H_{red}^* which at the same time happens to be a subset of h . Therefore $h \notin \widehat{H}_R^1$ tells us that there exists a set $\widehat{g} \in \widehat{H}_{\text{red}} \cup G$ with $v \in \widehat{g} \subseteq h$. Due to property (c) of G this leads us to a green set g with $v \in g \subseteq h$. From (d) and $v \in g \setminus f$ it follows that $|g \setminus f| \geq t$ and hence we have

$$|h \setminus (f \cup g)| \leq rt - t - 1 \leq rt - 3. \quad (5.15)$$

By (5.6) and the choice of v we have $v \notin V_S$ and, hence, v is either in V_R or in V_T . Let us analyse these two possibilities separately.

If $v \in V_R$, then we mark it. Property (a) of G tells us that v is contained in at most $(rt^2)^{2rt^2}$ green sets and, using (5.15), one can conclude that in this way each vertex of V_R is marked at most $(rt^2)^{2rt^2} n^{rt-2}$ many times.

On the other hand, if $v \in V_T$, then g is a team. Appealing to $h \notin H_{ST}$ and the choice of f we learn that $h \cap V_S = \emptyset$. But $|h \setminus f| \equiv -1 \pmod{t}$ and therefore it is not possible that $h \setminus f$ is entirely covered by teams. Consequently there is a vertex $w \in (h \setminus (f \cup g)) \cap V_R$

that can be marked. Now there are at most n possibilities for f , for g , and for each of the remaining vertices in $h \setminus (g \cup f \cup \{w\})$. Using (5.15) again, we get that in this way each vertex is marked at most n^{rt-2} further times.

Summarising the above estimations one obtains

$$|\widehat{H}_R^{3,y}| \leq |V_R|(rt^2)^{2rt^2} n^{rt-2} + |V_R|n^{rt-2}. \quad \square$$

Collecting all the above results we get

$$|\widehat{H}_R^1| \leq (rt^2)^{2rt^2} |V_R| n^{rt-2}$$

from (5.11) and Fact 5.1,

$$|\widehat{H}_R^2| \leq (rt^2)^{r^3t^6-7rt^2} |V_R| n^{rt-2}$$

from (5.13) and Fact 5.2,

$$|\widehat{H}_R^3| \leq 2(rt^2)^{2rt^2} |V_R| n^{rt-2}$$

from (5.14), Fact 5.3, Fact 5.4, and finally

$$|\widehat{H}_R| \leq (3(rt^2)^{2rt^2} + (rt^2)^{r^3t^6-7rt^2}) |V_R| n^{rt-2} \leq (rt^2)^{r^3t^6} |V_R| n^{rt-2}$$

from (5.10) and the three previous estimates. This concludes the proof of (5.8) and, hence, the proof of the Structure Theorem 2.2.

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ADAM MICKIEWICZ UNIVERSITY, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, POZNAŃ,
POLAND

E-mail address: tomasz@amu.edu.pl

E-mail address: joaska@amu.edu.pl

FACHBEREICH MATHEMATIK, UNIVERSITÄT HAMBURG, HAMBURG, GERMANY

E-mail address: Christian.Reiher@uni-hamburg.de