

MAXIMUM STAR DENSITIES

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ABSTRACT. Given an integer $k \geq 2$ and a real number $\gamma \in [0, 1]$, which graphs of edge density γ contain the largest number of k -edge stars? For $k = 2$ Ahlswede and Katona proved that asymptotically there cannot be more such stars than in a clique or in the complement of a clique (depending on the value of γ). Here we extend their result to all integers $k \geq 2$.

§1. INTRODUCTION

Ahlsweide and Katona [2] wondered about the maximal number of cherries (3-vertex stars) a graph may contain when the numbers of its vertices and edges are known. They obtained the complete answer to this question describing the extremal graphs for any pair consisting of a number of vertices and a number of edges. Roughly speaking, they found that the extremal graphs are in some sense close to being either cliques or complements of cliques.

For their precise statement, we need to define quasi-complete graphs and quasi-stars. Given nonnegative integers n and m with $0 \leq m \leq \binom{n}{2}$, the *quasi-complete* graph with n vertices and m edges is constructed as follows:

- Write $m = \binom{a}{2} + b$, where $0 \leq b < a$.
- Take a complete graph with a vertices.
- Add another vertex and attach it to b of the previous vertices.
- Add $n - a - 1$ isolated vertices.

On the other hand, the *quasi-star* with n vertices and m edges can be obtained as the complement of the quasi-complete graph with n vertices and $\binom{n}{2} - m$ edges.

Theorem 1.1 ([2]). *Among all graphs G with a given number n of vertices and a given number m of edges, the number of cherries is either maximized by the quasi-complete graph or the quasi-star.*

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Moreover, Ahlswede and Katona showed that there exists a nonnegative integer R (depending on n in a rather nontrivial way, see [1, 16] for details) such that the quasi-star is extremal for $0 \leq m \leq \frac{1}{2}\binom{n}{2} - R$ or $\frac{1}{2}\binom{n}{2} \leq m < \frac{1}{2}\binom{n}{2} + R$, while the quasi-complete graph is extremal for $\frac{1}{2}\binom{n}{2} - R < m \leq \frac{1}{2}\binom{n}{2}$ or $\frac{1}{2}\binom{n}{2} + R \leq m \leq \binom{n}{2}$.

Note that the number of cherries in a graph G can be expressed as

$$\sum_{v \in V(G)} \binom{d(v)}{2}.$$

By the handshake lemma, this is equal to

$$\frac{1}{2} \sum_{v \in V(G)} d(v)^2 - |E(G)|. \quad (1.1)$$

Thus maximizing the number of cherries, given the number of vertices and edges, is equivalent to maximizing the sum of the squared degrees.

It is not difficult to deduce the following asymptotic version of Theorem 1.1:

Theorem 1.2. *Given nonnegative integers n and m , the maximum number of cherries in a graph with n vertices and m edges is*

$$\max(\gamma^{3/2}, \eta + (1 - \eta)\eta^2) \frac{n^3}{2} + O(n^2),$$

where $\gamma = m/\binom{n}{2}$ is the edge density and $\eta = 1 - \sqrt{1 - \gamma}$.

Our aim in this paper is to obtain an analogue of this asymptotic statement for arbitrary stars. While there is no exact identity of the form (1.1), maximizing the number of k -leaf stars S_k is still asymptotically equivalent to maximizing the k^{th} degree moment: this is because the number of k -leaf stars (to be precise, the number of—not necessarily induced—subgraphs isomorphic to S_k) in a graph G can be expressed as

$$\sum_{v \in V(G)} \binom{d(v)}{k}.$$

Up to a factor $k!$, this is the number of injective homomorphisms from S_k to G . The total number of homomorphisms from S_k to G is exactly the k^{th} degree moment

$$\sum_{v \in V(G)} d(v)^k.$$

Natural languages to talk about asymptotic graph theoretical statements are provided by Razborov's theory of flag algebra homomorphisms [12], and by Lovász et al.'s theory of graphons, which is nicely explained in Lovász' recent research monograph [9]. For concreteness we shall work with the latter approach in this article.

Recall that a *graphon* is a measurable symmetric function $W: [0, 1]^2 \rightarrow [0, 1]$; here the word “measurable” should be understood either in the sense of the Borel σ -algebra, or with respect to Lebesgue measurable sets, but it is immaterial for our concerns which one of these two interpretations one actually adopts; the demand that W be “symmetric” just means that $W(x, y) = W(y, x)$ is required to hold for all $x, y \in [0, 1]$. The space of all graphons is denoted by \mathcal{W}_0 . The main quantities studied in extremal graph theory are so-called homomorphism densities, defined as follows: Given a graph H and a graphon W one stipulates

$$t(H, W) = \int_{[0,1]^{V(H)}} \prod_{ij \in E(H)} W(x_i, x_j) \prod_{i \in V(H)} dx_i,$$

and calls $t(H, W)$ the *homomorphism density* from H to W .

In order to formulate the result alluded to above we call a graphon W a *clique* if modulo null sets it is the characteristic function of a quadratic set of the form $A \times A$ for some measurable $A \subseteq [0, 1]$, and W is said to be an *anticlique* if $1 - W$ is a clique; further, we let the pictorial symbols “|” and “ \wedge ” denote the graphs on two vertices with one edge, and the graph on three vertices with two edges, respectively. Now what Ahlswede and Katona proved yields in the limit that among all graphons W for which $t(|, W)$ has some fixed value, those for which $t(\wedge, W)$ is maximal are either cliques or anticliques. Thus

$$t(\wedge, W) \leq \max(\gamma^{3/2}, \eta + (1 - \eta)\eta^2),$$

where $\gamma = t(|, W)$ and $\eta = 1 - \sqrt{1 - \gamma}$, is the best possible inequality in this regard; one may observe that the clique yields a larger value of $t(\wedge, \cdot)$ if $\gamma > \frac{1}{2}$, while the anticlique is better if $\gamma < \frac{1}{2}$; interestingly, if $\gamma = \frac{1}{2}$, there are, up to weak isomorphism in the sense of [9, Chap 7.3], exactly two extremal graphons.

The question to find for a fixed graph H and a fixed $\gamma \in [0, 1]$ the *maximal* value of $t(H, W)$ as W varies through \mathcal{W}_0 under the constraint $t(|, W) = \gamma$ is, of course, interesting in general; we recall that if H is a clique, the answer is well known by a theorem proved independently by Kruskal [7] and by Katona [4]. In its full generality their theorem speaks about hypergraphs; in the 2-uniform case it tells us that

$$t(K_r, W) \leq t(|, W)^{r/2}$$

holds for all integers $r \geq 2$ and all graphons W .

The opposite question about the minimum possible value of $t(H, W)$ for fixed $t(|, W)$ has also been studied in the literature. There are many such results for bipartite graphs H making partial progress on Sidorenko’s conjecture (see e.g. [3, 6, 8, 15] for some recent contributions). For non-bipartite graphs, the answer only seems to be known when H is a

clique. In this case the answer is given by the clique density theorem from [14], which was proved earlier for triangles by Razborov [13] and for cliques of order four by Nikiforov [11].

We shall prove in this article that replacing “ \wedge ” by the star S_k with k edges one can still get the same qualitative conclusion, while the case distinction on whether a clique or an anticlique is better will depend in a different manner on γ . A reason as to why the case $H = S_k$ should be easier than the general case is that the homomorphism density $t(S_k, W)$ may be interpreted as the k^{th} moment of the vertex degree function. Recall that associated with each graphon W one has its degree function $d_W: [0, 1] \rightarrow [0, 1]$ defined by $d_W(x) = \int_0^1 W(x, y) dy$ for all $x \in [0, 1]$; clearly

$$t(S_k, W) = \int_0^1 d_W^k(x) dx.$$

So one may ask the more general question to bound $\int_0^1 F(d_W(x)) dx$ from above for any given function $F: [0, 1] \rightarrow \mathbb{R}$. We shall identify in Section 2 a slightly artificial condition (see Definition 2.3 below) that, when imposed on F , guarantees that the answer will again be that the extremal graphons are either cliques or anticliques. In other words, this means that

$$\int_0^1 F(d_W(x)) dx \leq \max \left((1 - \sqrt{\gamma})F(0) + \sqrt{\gamma}F(\sqrt{\gamma}), (1 - \eta)F(\eta) + \eta F(1) \right)$$

will hold for all graphons W with $t(\cdot, W) = \gamma$, where again $\eta = 1 - \sqrt{1 - \gamma}$. The verification of this will occupy Section 3. It is not entirely obvious that power functions $x \mapsto x^k$ do indeed satisfy our condition; we shall confirm this in Section 4, thus obtaining the following:

Theorem 1.3. *Let W be a graphon and let k be a positive integer. Set $\gamma = t(\cdot, W)$ and $\eta = 1 - \sqrt{1 - \gamma}$; we have the inequality*

$$t(S_k, W) \leq \max \left(\gamma^{(k+1)/2}, \eta + (1 - \eta)\eta^k \right).$$

For $k \leq 30$, this was proven very recently by Kenyon, Radin, Ren and Sadun [5] using a somewhat different approach; they also conjectured it to be true for arbitrary k . In another recent paper, Nagy [10] obtained an analogous result for the density of another graph, namely the 4-edge path: here it also turns out that cliques or anticliques (depending on γ) are extremal.

§2. A SUFFICIENT CONDITION

Let us say that a measurable function $F: [0, 1] \rightarrow \mathbb{R}$ is *good* for a graphon W if

$$\int_0^1 F(d_W(x)) dx \leq \max \left((1 - \sqrt{\gamma})F(0) + \sqrt{\gamma}F(\sqrt{\gamma}), (1 - \eta)F(\eta) + \eta F(1) \right)$$

holds, where $\gamma = t(|, W)$ and $\eta = 1 - \sqrt{1 - \gamma}$. With this terminology we are interested in a condition for F implying that it will be good for all graphons.

A natural demand on the function F is that it should be convex. Indeed this makes it more likely that quantities such as the left side of the above formula attain their maxima in fairly extreme situations, as we wish. Conversely, convexity already allows us to deal with an easy case.

Observation 2.1. *Convex function are good for all constant graphons.*

Indeed, if W is constant always attaining the value $\gamma \in [0, 1]$, then $t(|, W) = \gamma$ and

$$\int_0^1 F(d_W(x)) dx = F(\gamma) \leq (1 - \sqrt{\gamma})F(0) + \sqrt{\gamma}F(\sqrt{\gamma})$$

follows from Jensen's inequality. Notice that we could have verified

$$F(\gamma) \leq (1 - \eta)F(\eta) + \eta F(1)$$

in the same way, for $\gamma = 2\eta - \eta^2$.

Optimistically one might hope that all convex functions are good for all graphons, but unfortunately this is not the case, as the following construction demonstrates:

Example 2.2. Let F satisfy $F(0) = F(\frac{1}{5}) = 0$, $F(\frac{3}{5}) = 1$, $F(1) = 3$, and let F be piecewise linear in between. Note that this function is convex. Now look at $\gamma = \frac{9}{25}$, for which we have

$$\max((1 - \sqrt{\gamma})F(0) + \sqrt{\gamma}F(\sqrt{\gamma}), (1 - \eta)F(\eta) + \eta F(1)) = \frac{3}{5}.$$

Let W be the characteristic function of $A \times A \cup A \times B \cup B \times A$, where $A, B \subseteq [0, 1]$ are disjoint. If the Lebesgue measures $y = \lambda(A)$ and $z = \lambda(B)$ are chosen in such a way that $y + z \leq 1$ and $y^2 + 2yz = \gamma = \frac{9}{25}$, then the graphon W satisfies $t(|, W) = \gamma$. Thus we should have

$$\int_0^1 F(d_W(x)) dx = yF(y + z) + zF(y) \leq \frac{3}{5}$$

for all possible choices of y and z , but this is wrong, except for boundary cases.

Indeed, this would mean that if $\frac{1}{5} \leq y \leq \frac{3}{5}$ and $z = \frac{\gamma - y^2}{2y}$, we should have (note here that $\frac{3}{5} \leq y + z \leq 1$)

$$yF(y + z) + zF(y) = y \cdot (5y + 5z - 2) + z \cdot \frac{5y - 1}{2} \leq \frac{3}{5}.$$

It is plain that this fails e.g. for $y = \frac{2}{5}$ and $z = \frac{1}{4}$, for then the left side of the inequality is $\frac{5}{8}$, which is greater than $\frac{3}{5}$. One could show that any other choice of $y \in (\frac{1}{5}, \frac{3}{5})$ leads to a counterexample as well. Also, one could modify F slightly, replacing it by another

function F^* that is differentiable infinitely often while $\|F - F^*\|_\infty$ is kept small. In this way one can construct smoother counterexamples.

So we have to impose stronger conditions on F than just convexity. The following definition, which might look artificial at first glance, provides us exactly with what we need:

Definition 2.3. Let \mathcal{C} denote the class of all twice differentiable convex functions $F: [0, 1] \rightarrow \mathbb{R}$ satisfying the following two conditions.

(C1) For all $a, b, y \in [0, 1]$ with $a < y < b$ and

$$\frac{F(b) - F(y)}{b - y} - \frac{F(y) - F(a)}{y - a} = F'(b) - F'(y),$$

we have

$$\begin{aligned} & 2(b - y) \left[F(b) - (b - y)F'(b) + \frac{1}{2}(b - y)^2 F''(b) - F(y) \right] \\ & + (y - a) \left[F(b) - (b - y)F'(y) + (b - y)^2 F''(y) - F(y) \right] > 0, \end{aligned}$$

and

(C2) for all $a, b, y \in [0, 1]$ with $a < y < b$ and

$$\frac{F(b) - F(y)}{b - y} - \frac{F(y) - F(a)}{y - a} = F'(y) - F'(a),$$

we have

$$\begin{aligned} & 2(y - a) \left[F(a) + (y - a)F'(a) + \frac{1}{2}(y - a)^2 F''(a) - F(y) \right] \\ & + (b - y) \left[F(a) + (y - a)F'(y) + (y - a)^2 F''(y) - F(y) \right] > 0. \end{aligned}$$

As we shall see in the next section, functions in \mathcal{C} are good for all step graphons. Here, a *step graphon* is a symmetric function $W: [0, 1]^2 \rightarrow [0, 1]$ for which there exists a partition $\mathcal{P} = \{P_1, \dots, P_k\}$ of the unit interval into a finite number of measurable pieces such that W is constant on each rectangle of the form $P_i \times P_j$. It is known that the collection of all step graphons is dense in \mathcal{W}_0 with respect to various natural topologies. So if F is sufficiently well behaved and good for all step graphons, then by standard approximation arguments F is automatically good for all graphons. E.g., it suffices to assume that F be continuously differentiable on $[0, 1]$.

§3. THE MAIN RESULT ON THE CLASS \mathcal{C}

The principal goal of this section is to understand why the functions in \mathcal{C} are good for all step graphons, cf. Proposition 3.7 below. To prepare the proof of this assertion, we

collect several lemmata about the functions in this class. The first of them informs us that \mathcal{C} is closed under several operations naturally appearing in our argument.

Lemma 3.1. *Let $F: [0, 1] \rightarrow \mathbb{R}$ belong to \mathcal{C} .*

- (i) *For all $A, B \in \mathbb{R}$, the function $x \mapsto F(x) + A + Bx$ belongs to \mathcal{C} as well.*
- (ii) *The function $G: [0, 1] \rightarrow \mathbb{R}$ given by $x \mapsto F(1 - x)$ is also in \mathcal{C} .*
- (iii) *For all real numbers r and s with $0 \leq r < s \leq 1$, the function $H: [0, 1] \rightarrow \mathbb{R}$ given by $x \mapsto F(r + (s - r)x)$ is in \mathcal{C} .*

Proof. The first part follows from the fact that neither the assumption nor the conclusion of (C1) or (C2) change when a linear function is added to F . Further, G and H are convex and satisfy the requested differentiability condition. To see that G satisfies (C1) for all numbers $a < y < b$ we apply (C2) for F to $1 - b < 1 - y < 1 - a$ and vice versa. This shows that G is indeed in \mathcal{C} . To check similarly that H satisfies (C1) or (C2) for all numbers $a < y < b$, one applies the same property of F to the numbers $r + (s - r)a < r + (s - r)y < r + (s - r)b$. \square

Our next steps are directed towards showing that the class of all graphons for which all functions in \mathcal{C} are good is likewise closed under some operations that occur later on. The first of these assertions, a rather direct consequence of part (ii) from the foregoing lemma, does not carry the induction further by itself, but it will allow us to reduce one of two seemingly different cases to the other one.

Lemma 3.2. *If all functions in \mathcal{C} are good for a graphon W , then the same is true for the graphon $1 - W$.*

Proof. Let $F \in \mathcal{C}$ be a function that we want to prove good for $1 - W$. Using the fact that the function G defined in Lemma 3.1 (ii) is good for W we find that the numbers

$$\gamma = t(\cdot, 1 - W) = 1 - t(\cdot, W) \quad \text{and} \quad \eta = 1 - \sqrt{1 - \gamma}$$

satisfy

$$\begin{aligned} \int_0^1 F(d_{1-W}(x)) dx &= \int_0^1 G(d_W(x)) dx \\ &\leq \max\left(\left(1 - \sqrt{1 - \gamma}\right)G(0) + \sqrt{1 - \gamma}G(\sqrt{1 - \gamma}), \sqrt{\gamma}G(1 - \sqrt{\gamma}) + (1 - \sqrt{\gamma})G(1)\right) \\ &= \max\left(\left(1 - \eta\right)F(\eta) + \eta F(1), \left(1 - \sqrt{\gamma}\right)F(0) + \sqrt{\gamma}F(\sqrt{\gamma})\right), \end{aligned}$$

as desired. \square

The content of the next lemma is that one cannot construct “ L -shaped counterexamples” for functions in \mathcal{C} as in Example 2.2. It is actually the only place in the entire proof where

we really have to work in an essential way with (C1) and (C2). Everything else follows by iterating this case by means of Lemma 3.1 using convexity alone.

Lemma 3.3. *If $F: [0, 1] \rightarrow \mathbb{R}$ is in \mathcal{C} and $x, y, z \in [0, 1]$ satisfy $x + y + z = 1$, then*

$$xF(0) + yF(y+z) + zF(y) \leq \max\left(\left(1 - \sqrt{\gamma}\right)F(0) + \sqrt{\gamma}F(\sqrt{\gamma}), (1 - \eta)F(\eta) + \eta F(1)\right),$$

where $\gamma = y^2 + 2yz$ and $\eta = 1 - \sqrt{1 - \gamma}$.

Remark 3.4. Observe here that $xF(0) + yF(y+z) + zF(y)$ is the value of $\int_0^1 F(d_W(x))dx$ for a graphon W as constructed in Example 2.2.

Proof of Lemma 3.3. By Lemma 3.1 (i) we may assume $F(0) = 0$ for simplicity. If $\gamma = 0$, then $y = z = 0$, and if $\gamma = 1$, then $y = 1$ and $z = 0$. In both cases the claim is clear, so we may suppose $0 < \gamma < 1$ from now on. As one easily confirms, the closed interval $C = [\eta, \sqrt{\gamma}]$ is then non-trivial. Since

$$1 - \gamma = (x + y + z)^2 - (y^2 + 2yz) \geq (x + z)^2,$$

we have $\eta \leq 1 - (x + z) = y$. Moreover $y^2 \leq \gamma$ entails $y \leq \sqrt{\gamma}$, so that altogether we get $y \in C$. Conversely, if for any $t \in C$ one sets $z(t) = \frac{\gamma - t^2}{2t}$, then $z(t) \geq 0$ and

$$t + z(t) = \frac{\gamma + t^2}{2t} = 1 - \frac{1 - \gamma - (1 - t)^2}{2t} \leq 1,$$

for which reason the numbers $1 - t - z(t)$, t , and $z(t)$ satisfy the hypothesis on x , y and z in the statement of the lemma. Notice that $z(\eta) = 1 - \eta$ and $z(\sqrt{\gamma}) = 0$. Defining the function $J: C \rightarrow \mathbb{R}$ by

$$J(t) = tF(t + z(t)) + z(t)F(t)$$

for all $t \in C$ we are to prove that $J(t) \leq \max(J(\eta), J(\sqrt{\gamma}))$, i.e., that J attains its maximum at a boundary point of C . If this failed, there would exist an interior point t_0 of C such that $J'(t_0) = 0$ but $J''(t_0) \leq 0$. Since

$$z'(t) = -\frac{z(t)}{t} - 1$$

and thus

$$J'(t) = F(t + z(t)) - z(t)F'(t + z(t)) + z(t)F'(t) - F(t) - \frac{z(t)}{t}F(t),$$

the equation $J'(t_0) = 0$ can be rewritten as follows (recall that $F(0) = 0$ and $z(t_0) > 0$):

$$\frac{F(t_0 + z(t_0)) - F(t_0)}{z(t_0)} - \frac{F(t_0) - F(0)}{t_0} = F'(t_0 + z(t_0)) - F'(t_0).$$

In other words, the numbers $0 < t_0 < t_0 + z(t_0)$ are as a, y, b in (C1), and by the assumption that $F \in \mathcal{C}$ we have

$$\begin{aligned} & 2z(t_0) \left[F(t_0 + z(t_0)) - z(t_0)F'(t_0 + z(t_0)) + \frac{1}{2}z(t_0)^2F''(t_0 + z(t_0)) - F(t_0) \right] \\ & + t_0 \left[F(t_0 + z(t_0)) - z(t_0)F'(t_0) + z(t_0)^2F''(t_0) - F(t_0) \right] > 0. \end{aligned}$$

This rewrites as

$$t_0z(t_0)J''(t_0) + (2z(t_0) + t_0)J'(t_0) > 0,$$

contradicting our choice of t_0 as a point for which $J'(t_0) = 0$ and $J''(t_0) \leq 0$. This completes the proof of the lemma. \square

To explain how the preceding lemma may actually be used in an inductive argument we introduce the following notation: given a graphon W and a real number $\lambda \in [0, 1]$ we define $[\lambda, W]$ to be the graphon satisfying

$$[\lambda, W](x, y) = \begin{cases} 0 & \text{if } 0 \leq x < \lambda \text{ or } 0 \leq y < \lambda, \\ W\left(\frac{x-\lambda}{1-\lambda}, \frac{y-\lambda}{1-\lambda}\right) & \text{if } \lambda \leq x \leq 1 \text{ and } \lambda \leq y \leq 1. \end{cases}$$

Lemma 3.5. *If $\lambda \in [0, 1]$ and the graphon W has the property that all functions in \mathcal{C} are good for it, then the same applies to $[\lambda, W]$.*

Proof. Let $F \in \mathcal{C}$ be any function that we want to prove good for $[\lambda, W]$. By Lemma 3.1 (iii) the function $H: [0, 1] \rightarrow \mathbb{R}$ given by $H(x) = F((1-\lambda)x)$ for all $x \in [0, 1]$ is in \mathcal{C} . Thus it is good for W , which tells us that

$$\int_0^1 H(d_W(x)) \, dx \leq \max\left((1-\sqrt{\gamma})H(0) + \sqrt{\gamma}H(\sqrt{\gamma}), (1-\eta)H(\eta) + \eta H(1)\right),$$

where $\gamma = t(|, W)$ and $\eta = 1 - \sqrt{1-\gamma}$. Since

$$\begin{aligned} \int_0^1 F(d_{[\lambda, W]}(x)) \, dx &= \lambda F(0) + \int_\lambda^1 F\left((1-\lambda)d_W\left(\frac{x-\lambda}{1-\lambda}\right)\right) \, dx \\ &= \lambda F(0) + (1-\lambda) \int_0^1 H(d_W(x)) \, dx, \end{aligned}$$

it follows that either

$$\int_0^1 F(d_{[\lambda, W]}(x)) \, dx \leq \lambda F(0) + (1-\lambda)(1-\sqrt{\gamma})F(0) + (1-\lambda)\sqrt{\gamma}F((1-\lambda)\sqrt{\gamma}),$$

or

$$\int_0^1 F(d_{[\lambda, W]}(x)) \, dx \leq \lambda F(0) + (1-\lambda)(1-\eta)F((1-\lambda)\eta) + (1-\lambda)\eta F(1-\lambda).$$

In the former case the right side simplifies to

$$(1-\sqrt{\gamma'})F(0) + \sqrt{\gamma'}F(\sqrt{\gamma'}),$$

where $\gamma' = (1 - \lambda)^2\gamma = t(|, [\lambda, W])$, meaning that F is, in particular, good for $[\lambda, W]$.

So from now on we may assume that the second alternative occurs. Setting $x = \lambda$, $y = (1 - \lambda)\eta$ and $z = (1 - \lambda)(1 - \eta)$ we thus get

$$\int_0^1 F(d_{[\lambda, W]}(x)) \, dx \leq xF(0) + zF(y) + yF(y + z).$$

Since $y^2 + 2yz = (1 - \lambda)^2(2\eta - \eta^2) = (1 - \lambda)^2\gamma = \gamma'$, it follows in view of Lemma 3.3 that

$$\int_0^1 F(d_{[\lambda, W]}(x)) \, dx \leq \max\left(\left((1 - \sqrt{\gamma'})F(0) + \sqrt{\gamma'}F(\sqrt{\gamma'}), (1 - \eta')F(\eta') + \eta'F(1)\right),\right),$$

where $\eta' = 1 - \sqrt{1 - \gamma'}$. This tells us that F is indeed good for $[\lambda, W]$. \square

A second construction we use is that of a graphon $[W, \lambda]$ defined for any real $\lambda \in [0, 1]$ and graphon W by

$$[W, \lambda](x, y) = \begin{cases} W\left(\frac{x}{1-\lambda}, \frac{y}{1-\lambda}\right) & \text{if } 0 \leq x \leq 1 - \lambda \text{ and } 0 \leq y \leq 1 - \lambda, \\ 1 & \text{if } 1 - \lambda < x \leq 1 \text{ or } 1 - \lambda < y \leq 1. \end{cases}$$

Lemma 3.6. *If all functions in \mathcal{C} are good for the graphon W and $\lambda \in [0, 1]$, then all functions in \mathcal{C} are good for $[W, \lambda]$ as well.*

Proof. Since $[W, \lambda]$ is isomorphic to $1 - [\lambda, 1 - W]$, this follows from Lemma 3.2 and Lemma 3.5. \square

Now we come to the main result of this section.

Proposition 3.7. *Every function in \mathcal{C} is good for every step graphon.*

Proof. We prove this statement by contradiction. If it does not hold, let W be a step graphon and $F: [0, 1] \rightarrow \mathbb{R}$ a function in \mathcal{C} such that W fails to be good for F . Let W be a step function with respect to the partition $\mathcal{P} = \{P_1, \dots, P_k\}$ of the unit interval, write $\alpha_i = \lambda(P_i)$ for each $i \in [k] = \{1, 2, \dots, k\}$, and let β_{ij} be the value attained by W on $P_i \times P_j$ for $i, j \in [k]$. Let T denote the number of pairs $(i, j) \in [k]^2$ for which $\beta_{ij} \in \{0, 1\}$. We may assume that among all possibilities W has been chosen in such a way that k is as small as possible and subject to this T is as large as possible. It is plain that the numbers $\alpha_1, \dots, \alpha_k$ are positive under this assumption.

Defining $d_i = \sum_{j=1}^k \alpha_j \beta_{ij}$ for each $i \in [k]$ we are to prove

$$\sum_{i=1}^k \alpha_i F(d_i) \leq \max\left(\left((1 - \sqrt{\gamma})F(0) + \sqrt{\gamma}F(\sqrt{\gamma}), (1 - \eta)F(\eta) + \eta F(1)\right),\right),$$

where $\gamma = \sum_{i=1}^k \alpha_i d_i$ and $\eta = 1 - \sqrt{1 - \gamma}$. Without loss of generality we may assume $d_1 \leq d_2 \leq \dots \leq d_k$. Observation 2.1 shows that $k \geq 2$.

Claim 3.8. *If $1 \leq i \leq k$, $1 \leq r < s \leq k$, and $\beta_{ir} > 0$, then $\beta_{is} = 1$.*

Proof. Assume $\beta_{ir} > 0$ and $\beta_{is} < 1$. Define a step function Q with respect to \mathcal{P} as follows: let δ_{ij} denote the Kronecker delta, and set, for $x \in P_m$ and $y \in P_n$,

$$Q(x, y) = \begin{cases} -(1 + \delta_{ir})\alpha_s & \text{if } \{m, n\} = \{i, r\}, \\ (1 + \delta_{is})\alpha_r & \text{if } \{m, n\} = \{i, s\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\varepsilon \geq 0$ be maximal such that $W' = W + \varepsilon Q$ is still a graphon, i.e. maps to the interval $[0, 1]$. By our assumptions on β_{ir} and β_{is} , ε is positive, and the maximality of T implies that F is good for W' (observe that W' is identically 0 or 1 on at least $T + 1$ of the sets $P_i \times P_j$ by construction). Moreover

$$\int_{[0,1]^2} Q(x, y) dx dy = \alpha_i \alpha_r \alpha_s ((1 + \delta_{is})(2 - \delta_{is}) - (1 + \delta_{ir})(2 - \delta_{ir})) = 0,$$

whence we have $t(\cdot, W') = t(\cdot, W)$. So to derive the desired contradiction we just need to check that

$$\int_0^1 F(d_W(x)) dx \leq \int_0^1 F(d_{W'}(x)) dx.$$

For $j \in [k]$ we let d'_j denote the value attained by $d_{W'}(x)$ for $x \in P_j$. Clearly $d'_j = d_j$ holds for all $j \notin \{i, r, s\}$. Further

$$d'_r - d_r = -(1 + \delta_{ir})\alpha_i \alpha_s \varepsilon + \delta_{ir}(1 + \delta_{is})\alpha_i \alpha_s \varepsilon = -\alpha_i \alpha_s \varepsilon,$$

and similarly

$$d'_s - d_s = +\alpha_i \alpha_r \varepsilon.$$

Finally, if $i \notin \{r, s\}$, then $d'_i = d_i$. So altogether we get indeed

$$\begin{aligned} \int_0^1 F(d_{W'}(x)) dx - \int_0^1 F(d_W(x)) dx &= \sum_{j=1}^k \alpha_j (F(d'_j) - F(d_j)) \\ &= \alpha_s (F(d_s + \alpha_i \alpha_r \varepsilon) - F(d_s)) + \alpha_r (F(d_r - \alpha_i \alpha_s \varepsilon) - F(d_r)) \\ &\geq \alpha_i \alpha_r \alpha_s \varepsilon (F'(d_s) - F'(d_r)) \geq 0 \end{aligned}$$

by the convexity of F and because $d_s \geq d_r$. This proves Claim 3.8. \square

Claim 3.9. $\beta_{1k} > 0$.

Proof. If this does not hold, then $\beta_{1k} = 0$ and the previous claim entails $\beta_{1i} = 0$ for all $i \in [k - 1]$. It follows that there is a step graphon W' with $k - 1$ steps such that W is isomorphic to $[\alpha_1, W']$. Due to the minimality of k all functions in \mathcal{C} are good for W' and by Lemma 3.5 the same applies to the graphon W , contrary to its choice. \square

Claim 3.10. $\beta_{1k} < 1$.

Proof. If we had $\beta_{k1} = 1$, then Claim 3.8 would imply $\beta_{ki} = 1$ for all i with $2 \leq i \leq k$. So some step graphon W' has the property that $[W', \alpha_k]$ is isomorphic to W , which yields a contradiction via Lemma 3.6. \square

So we must have $0 < \beta_{1k} < 1$. The conclusions drawn from Claim 3.8 in the two previous proofs are still valid, i.e., we have $\beta_{1i} = 0$ for all $i \in [k-1]$ and $\beta_{jk} = 1$ for all j with $2 \leq j \leq k$. Divide P_k into two measurable subsets Q_k and Q_{k+1} satisfying $\lambda(Q_k) = (1 - \beta_{1k})\alpha_k$ and, consequently, $\lambda(Q_{k+1}) = \beta_{1k}\alpha_k$. Set $Q_i = P_i$ for $i \in [k-1]$ and $\mathcal{Q} = \{Q_1, \dots, Q_{k+1}\}$. Let W' be the step graphon with respect to \mathcal{Q} defined as follows: for $x \in Q_i$ and $y \in Q_j$,

$$W'(x, y) = \begin{cases} \beta_{ij} & \text{if } 2 \leq i \leq k \text{ and } 2 \leq j \leq k, \\ 0 & \text{if } i = 1 \text{ and } j \in [k] \text{ or vice versa,} \\ 1 & \text{if } i = k+1 \text{ or } j = k+1. \end{cases}$$

By the last two clauses W' is isomorphic to a graphon of the form $\left[\left[\frac{\alpha_1}{1 - \beta_{1k}\alpha_k}, W''\right], \beta_{1k}\alpha_k\right]$ for some graphon W'' , and by the first clause W'' is a step graphon with $k-1$ steps. So Lemma 3.5 and Lemma 3.6 show that F is good for W' . Since $t(\cdot, W') = \gamma$, this means

$$\begin{aligned} & \sum_{i=1}^{k-1} \alpha_i F(d_i) + \alpha_k \left((1 - \beta_{1k})F(d') + \beta_{1k}F(d'') \right) \\ & \leq \max \left((1 - \sqrt{\gamma})F(0) + \sqrt{\gamma}F(\sqrt{\gamma}), (1 - \eta)F(\eta) + \eta F(1) \right) \end{aligned}$$

for some real numbers d' and d'' satisfying $(1 - \beta_{1k})d' + \beta_{1k}d'' = d_k$. Now Jensen's inequality implies that F is indeed good for W . \square

§4. VERIFYING THE ASSUMPTION FOR POWER FUNCTIONS

The only thing currently missing from a proof of Theorem 1.3 is that we do not know yet that for $k \geq 2$ the function $x \mapsto x^k$ is indeed in the class \mathcal{C} . To verify this is the main objective of the present section. Fortunately the first half is easy due to the following lemma:

Lemma 4.1. *Suppose that $F: [0, 1] \rightarrow \mathbb{R}$ is thrice continuously differentiable and satisfies $F''(x) > 0$ as well as $F'''(x) \geq 0$ for all $x \in (0, 1)$. Then F has the property (C1).*

Proof. We show something stronger, namely that independent of any further hypothesis all real numbers $a < y < b$ from the unit interval satisfy

$$2(b-y) \left[F(b) - (b-y)F'(b) + \frac{1}{2}(b-y)^2 F''(b) - F(y) \right] \\ + (y-a) \left[F(b) - (b-y)F'(y) + (b-y)^2 F''(y) - F(y) \right] > 0.$$

Notice that the convexity assumptions on F imply $F(b) \geq F(y) + (b-y)F'(y)$ and $F''(y) > 0$, for which reason the second square bracket is positive. Furthermore the general version of the mean value theorem yields the existence of some real $\xi \in (y, b)$ such that

$$F(y) = F(b) + (y-b)F'(b) + \frac{1}{2}(y-b)^2 F''(b) + \frac{1}{6}(y-b)^3 F'''(\xi).$$

Hence the first square bracket is $\frac{1}{6}(b-y)^3 F'''(\xi) \geq 0$. \square

Remark 4.2. One could formulate a similar statement obtaining (C2) from $F'''(x) \leq 0$. Due to the symmetry expressed in Lemma 3.1 (ii) and its proof this is, of course, not surprising.

To handle the second half we will use the following inequality twice:

Lemma 4.3. *If $x \geq 1$ is a real number and $m \geq 0$ an integer, then*

$$\sum_{i=0}^m (m+1-i)(3i-m)x^i \geq 0.$$

Proof. It is obvious that

$$\sum_{i=0}^{m-1} (i+1)(m-i)(m+1-i)x^i \geq 0.$$

Multiplying this by $x-1$ gives us the desired inequality after some simple manipulations. \square

Proposition 4.4. *For each integer $k \geq 2$ the function $x \mapsto x^k$ is in \mathcal{C} .*

Proof. Condition (C1) holds by Lemma 4.1, so it remains to deal with (C2). Omitting the condition $b \leq 1$ we prove that if any nonnegative real numbers $a < y < b$ satisfy

$$\frac{b^k - y^k}{b-y} - \frac{y^k - a^k}{y-a} = k(y^{k-1} - a^{k-1}),$$

then

$$2(y-a) \left[a^k + k(y-a)a^{k-1} + \binom{k}{2}(y-a)^2 a^{k-2} - y^k \right] \\ + (b-y) \left[a^k + k(y-a)y^{k-1} + k(k-1)(y-a)^2 y^{k-2} - y^k \right] > 0.$$

Everything is homogeneous, so we may suppose $y = 1$ for notational simplicity. So we are given that

$$\sum_{i=1}^{k-1} b^i = \sum_{i=1}^{k-1} a^i + k(1 - a^{k-1}). \quad (4.1)$$

Applying Lemma 4.3 to $x = b$ and $m = k - 2$ we get

$$\sum_{i=0}^{k-2} (k-1-i)(3i+2-k)b^i \geq 0.$$

We multiply by $\frac{b-1}{2}$ to infer that

$$\frac{(k-2)(k-1)}{2} + \sum_{i=1}^{k-1} (3i+1-2k)b^i \geq 0.$$

We write $3i+1-2k$ as $3(i+1-k) + (k-2)$, split the sum and divide by 3 to obtain

$$\sum_{i=0}^{k-1} (k-1-i)b^i \leq \frac{(k-1)k}{2} + \frac{k-2}{3} \left\{ \sum_{i=1}^{k-1} b^i - (k-1) \right\}. \quad (4.2)$$

If $a \neq 0$, we can apply Lemma 4.3 to $m = k - 2$ and $x = \frac{1}{a}$ and obtain, upon multiplication with a^{k-2} , the inequality

$$\sum_{i=0}^{k-2} (k-1-i)(3i+2-k)a^{k-2-i} \geq 0.$$

This is certainly also true for $a = 0$, so it holds unconditionally. Reversing the order of summation we find

$$\sum_{i=0}^{k-2} (i+1)(2k-4-3i)a^i \geq 0,$$

which may be weakened to

$$\sum_{i=0}^{k-2} (i+1)(2k+2-3i)a^i > 0.$$

Multiplying by $\frac{1-a}{2}$ we get

$$\sum_{i=0}^{k-2} (k+1-3i)a^i + \frac{(k-1)(k-8)}{2}a^{k-1} > 0.$$

Since $\frac{(k-1)(k-8)}{2} < 2(k-1)(k-2)$, this can again be weakened to

$$\sum_{i=0}^{k-2} (k+1-3i)a^i + 2(k-1)(k-2)a^{k-1} > 0,$$

which in turn can be rearranged in the same way as (4.2) to read

$$\frac{2(k-2)}{3} \left\{ \sum_{i=0}^{k-1} a^i - ka^{k-1} \right\} < \sum_{i=0}^{k-2} (k-1-i)a^i.$$

In combination with (4.1) and (4.2) this yields

$$\begin{aligned} 2 \sum_{i=0}^{k-1} (k-1-i)b^i &\leq (k-1)k + \frac{2(k-2)}{3} \left\{ \sum_{i=1}^{k-1} b^i - (k-1) \right\} \\ &= (k-1)k + \frac{2(k-2)}{3} \left\{ \sum_{i=0}^{k-1} a^i - ka^{k-1} \right\} \\ &< (k-1)k + \sum_{i=0}^{k-2} (k-1-i)a^i. \end{aligned}$$

Now we multiply again by $b-1$ and use (4.1), thus learning that

$$2 \left\{ \sum_{i=0}^{k-1} a^i - ka^{k-1} \right\} < (b-1) \left\{ (k-1)k + \sum_{i=0}^{k-2} (k-1-i)a^i \right\}.$$

The left side contains $1-a$ as a factor:

$$\sum_{i=0}^{k-1} a^i - ka^{k-1} = (1-a) \sum_{i=0}^{k-2} (i+1)a^i.$$

So after a further weakening we find

$$2(1-a) \left\{ \sum_{i=0}^{k-2} (i+1)a^i - \binom{k}{2} a^{k-2} \right\} < (b-1) \left\{ (k-1)k + \sum_{i=0}^{k-2} (k-1-i)a^i \right\}.$$

Now is a good moment to multiply by $(1-a)^2$, because this leads to

$$\begin{aligned} 2(1-a) \left(1 - a^k - k(1-a)a^{k-1} - \binom{k}{2}(1-a)^2 a^{k-2} \right) \\ < (b-1) (a^k + k(1-a) + k(k-1)(1-a)^2 - 1), \end{aligned}$$

which is exactly what we wanted to prove. \square

Proof of Theorem 1.3. The case $k = 1$ is clear, so suppose $k \geq 2$ from now on. If W happens to be a step graphon, the result follows from Proposition 3.7, its main assumption being verified in Proposition 4.4. Now the general case follows from the known facts that both sides of the inequality we seek to prove depend in a manner on W that is continuous with respect to the cut norm, and that the step graphons are dense in \mathcal{W}_0 with respect to the cut norm. \square

The following analogue of Theorem 1.2 is now an immediate consequence of Theorem 1.3:

Corollary 4.5. *Given nonnegative integers n and m and an integer $k \geq 2$, the maximum number of copies of the star S_k in a graph with n vertices and m edges is*

$$\max \left(\gamma^{(k+1)/2}, \eta + (1-\eta)\eta^k \right) \frac{n^{k+1}}{k!} + O(n^k),$$

where $\gamma = m/\binom{n}{2}$ is the edge density and $\eta = 1 - \sqrt{1 - \gamma}$.

Likewise, we also have

Corollary 4.6. *Given nonnegative integers n and m , the maximum of the k^{th} degree moment $\sum_v d(v)^k$ in a graph with n vertices and m edges is*

$$\max \left(\gamma^{(k+1)/2}, \eta + (1 - \eta)\eta^k \right) n^{k+1} + O(n^k),$$

where $\gamma = m/\binom{n}{2}$ is the edge density and $\eta = 1 - \sqrt{1 - \gamma}$.

Remark 4.7. The quasi-complete graph and the quasi-star attain the bound asymptotically, but it is worth pointing out that they are not always the graphs for which the maximum number of copies of S_k is attained. For example, if $k = 3$, $n = 13$ and $m = 61$, then neither the quasi-complete graph nor the quasi-star contains the greatest number of copies of the star S_3 : the quasi-complete graph has 1610 copies, the quasi-star 1620. Now consider the following graph:

- Start with a complete 11-vertex graph and select three of its vertices, v_1, v_2, v_3 .
- Now add two more vertices w_1, w_2 and all six possible edges between v_i and w_j ($1 \leq i \leq 3, 1 \leq j \leq 2$).

This graph has 13 vertices and 61 edges and contains 1622 copies of S_3 , which is in fact the maximum. The same graph also has the greatest third degree moment (number of homomorphisms from S_3) at 13238, as opposed to the quasi-complete graph and the quasi-star with 13202 and 13172 respectively.

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